CSE 431/531: Analysis of Algorithms

Greedy Algorithms

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Main Goal of Algorithm Design

- Design fast algorithms to solve problems
- Design more efficient algorithms to solve problems

Trivial Algorithm for an Optimization Problem

Enumerate all potential solutions, compare them and output the best one that is valid.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.
- $f(n)$ is polynomial if $f(n) = O(n^k)$ for some constant $k > 0$.
- convention: polynomial time = efficient

Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming

Greedy Algorithm

- Build up the solutions in step
- At each step, make a decision that optimizes some criterion, that is "reasonable"

Outline

Toy Problem 1: Bill Changing

Input: Integer $A \geq 0$

Currency denominations: $\$1, \$2, \$5, \$10, \$20$

Output: A way to pay A dollars using fewest number of bills

Example:

- Input: 48
- Output: 5 bills, $\$48 = \$20 \times 2 + \$5 + \$2 + \$1$

Cashier's Algorithm

1. while $A \geq 0$ do
2. $\quad a \leftarrow \max\{t \in \{1, 2, 5, 10, 20\} : t \leq A\}$
3. $\quad$ pay a \$$a$ bill
4. $\quad A \leftarrow A - a$

Cashier's Algorithm is Optimum

Lemma Cashier's algorithm gives the optimum solution for currency denominations \$1, \$2, \$5, \$10, \$20.

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid
- minimize $n_1 + n_2 + n_5 + n_{10} + n_{20}$ subject to
  $n_1 + 2n_2 + 5n_5 + 10n_{10} + 20n_{20} = A$

Obs.

- $n_1 < 2$ — $2 \leq A < 5$: pay a \$2 bill
- $n_1 + 2n_2 < 5$ — $5 \leq A < 10$: pay a \$5 bill
- $n_1 + 2n_2 + 5n_5 < 10$ — $10 \leq A < 20$: pay a \$10 bill
- $n_1 + 2n_2 + 5n_5 + 10n_{10} < 20$ — $20 \leq A < \infty$: pay a \$20 bill

Toy Example 2: Box Packing

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

Example:

- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: $45 \to 60, 20 \to 40, 19 \to 25$

Box Packing: Design Greedy Strategy

Q: Take box 1 (with capacity c_1). Which item should we put in box 1? Can you design a simple strategy?

A: The item of the largest size that can be put into the box.

- Reason why the strategy is good: putting the item in box 1 will give us the easiest residual problem.

Greedy Algorithm for Box Packing

1. $T \leftarrow \{1, 2, 3, \cdots, m\}$
2. for $i \leftarrow 1$ to n do
3. if some item in T can be put into box i, then
4. $j \leftarrow$ the largest item in T that can be put into box i
5. print("put item j in box i")
6. $T \leftarrow T \setminus \{j\}$

Steps of Designing Greedy Algorithms

1. Design a greedy choice
2. Prove it is "safe" to make the greedy choice
   - Usually done by "exchange argument"
3. Show that the remaining task after applying the greedy choice is to solve a (many) smaller instance(s) of the same problem.
   - The step is usually trivial

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

Exchange argument: let S be an arbitrary optimum solution. If S is consistent with the greedy choice, we are done. Otherwise, modify it to another optimum solution S' such that S' is consistent with the greedy choice.

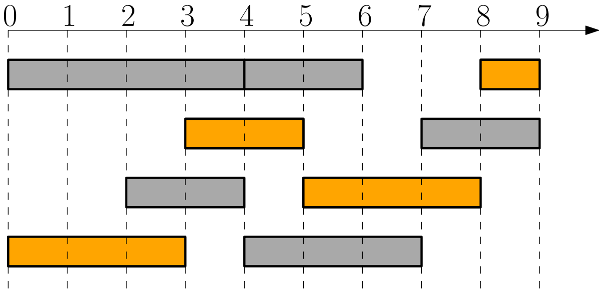
Outline

Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

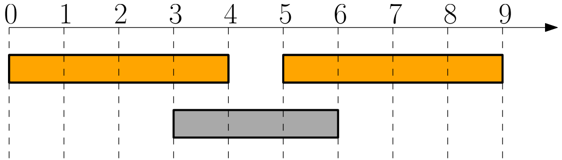
i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs



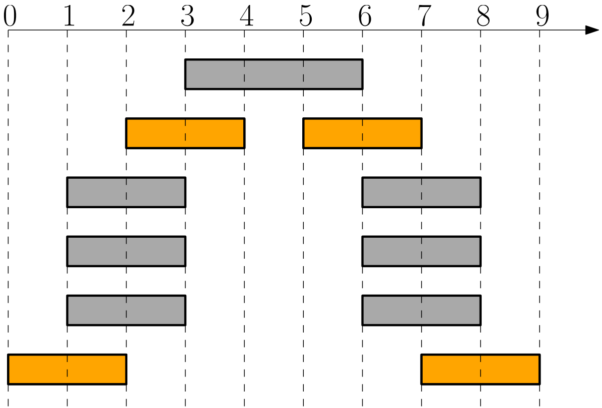
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!



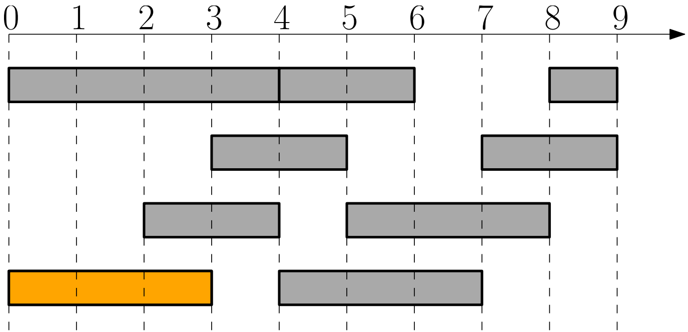
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!



Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time? Yes!

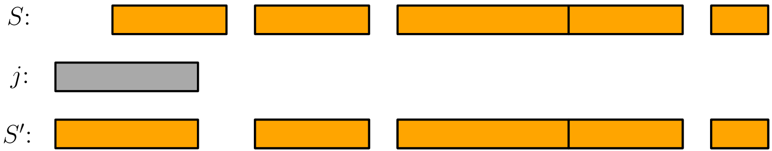


Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

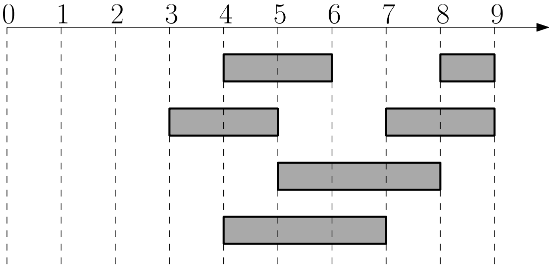
- Take an arbitrary optimum solution S
- If it contains j, done
- Otherwise, replace the first job in S with j to obtain an new optimum schedule S'. $\square$



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

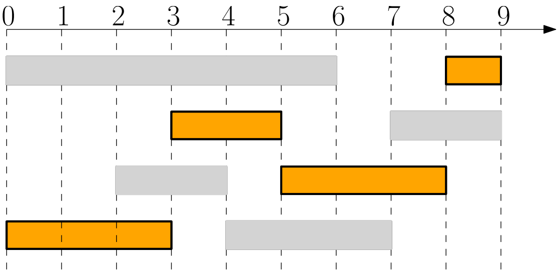
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem? Yes!



Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1. $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2. while $A \neq \emptyset$
3. $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
4. $\quad S \leftarrow S \cup \{j\};\ A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5. return S



Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1. $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2. while $A \neq \emptyset$
3. $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
4. $\quad S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5. return S

Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

sort jobs according to f values

$t \leftarrow 0$, $S \leftarrow \emptyset$

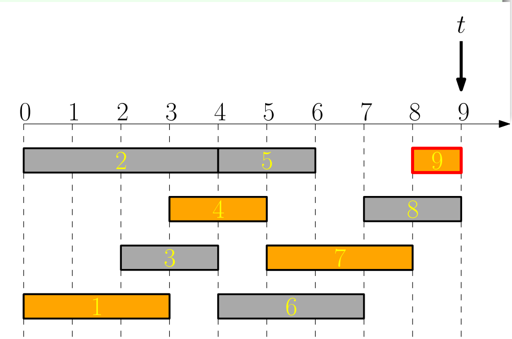
for every $j \in [n]$ according to non-decreasing order of f_j

  if $s_j \geq t$ then

    $S \leftarrow S \cup \{j\}$

    $t \leftarrow f_j$

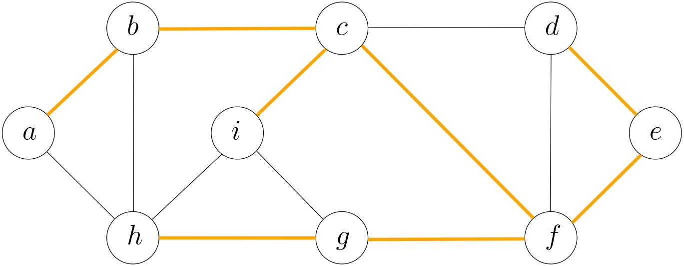
return S

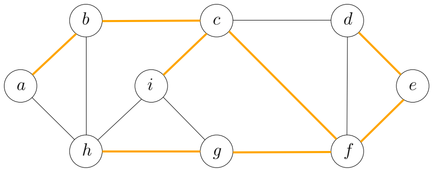


Outline

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





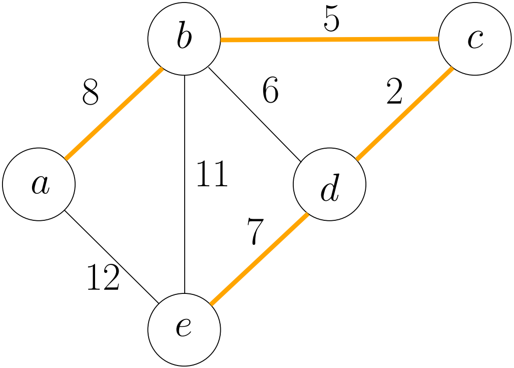
Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight



Recall: Steps for Designing Greedy Algorithms

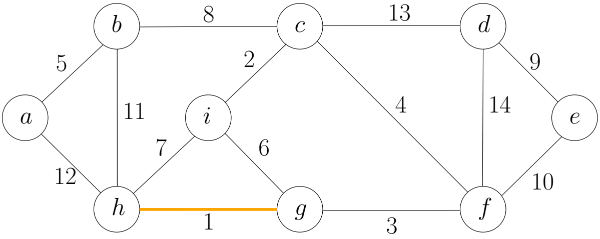
1. Design a greedy choice
2. Prove it is "safe" to make the greedy choice
   - Usually done by "exchange argument"
3. Show that the remaining task after applying the greedy choice is to solve a (many) smaller instance(s) of the same problem.
   - The step is usually trivial

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

Outline



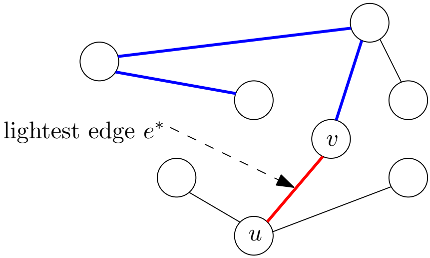
Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

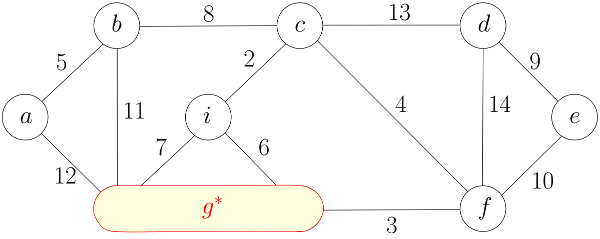
Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v
- Remove any edge e in the path to obtain tree T'
- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: T' is also a MST $\square$

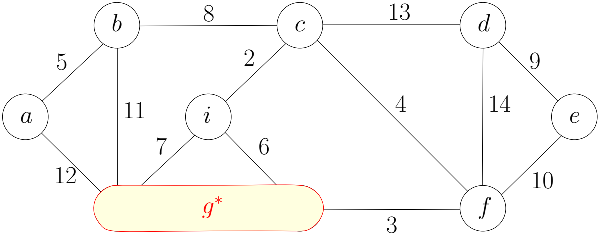


Is the Residual Problem Still a MST Problem?



- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges parallel to (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

Q: What edges are removed due to contractions?

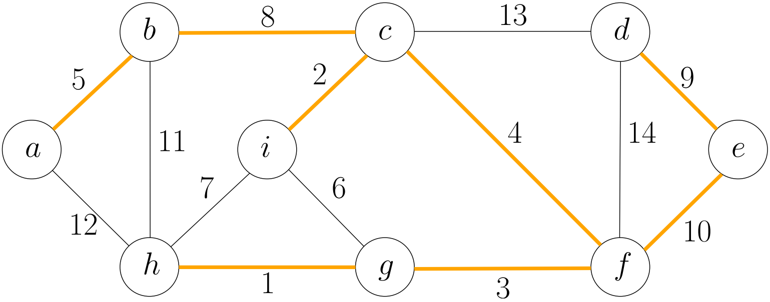
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

1. $F = \emptyset$
2. sort edges in E in non-decreasing order of weights w
3. for each edge (u, v) in the order
4.   if u and v are not connected by a path of edges in F
5. $F = F \cup \{(u, v)\}$
6. return (V, F)

Kruskal's Algorithm: Example



Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

$F \leftarrow \emptyset$

$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$

sort the edges of E in non-decreasing order of weights w

for each edge $(u, v) \in E$ in the order

  $S_u \leftarrow$ the set in $\mathcal{S}$ containing u

  $S_v \leftarrow$ the set in $\mathcal{S}$ containing v

  if $S_u \neq S_v$

    $F \leftarrow F \cup \{(u, v)\}$

    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$

return (V, F)

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

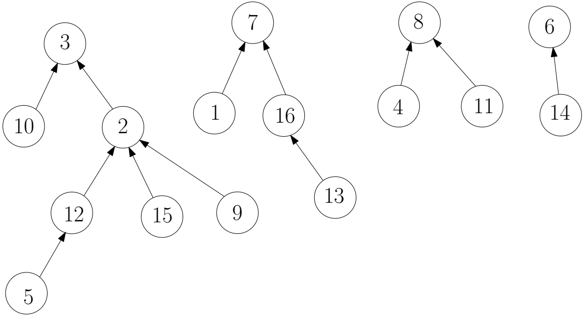
1. $F \leftarrow \emptyset$
2. $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3. sort the edges of E in non-decreasing order of weights w
4. for each edge $(u, v) \in E$ in the order
5. $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6. $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7.   if $S_u \neq S_v$
8. $F \leftarrow F \cup \{(u, v)\}$
9. $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10. return (V, F)

Use union-find data structure to support 2, 5, 6, 7, 9.

Union-Find Data Structure

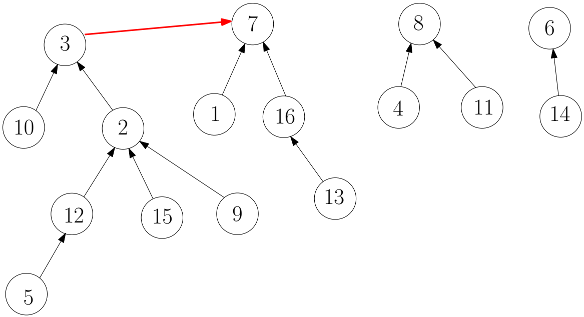
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] =$ nil if i is a root).

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).
- root(u): the root of the tree containing u
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure

root(v)

1. if $par[v] =$ nil then
2.   return v
3. else
4.   return root($par[v]$)

root(v)

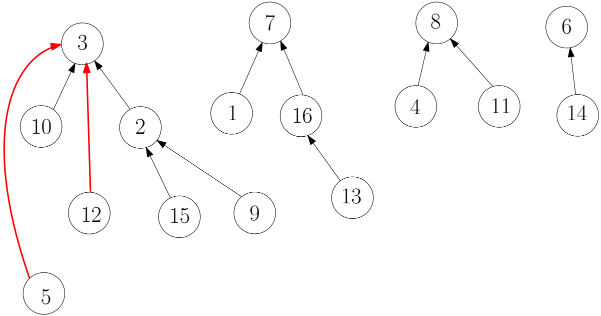
1. if par$[v] =$ nil then
2.   return v
3. else
4. $par[v] \leftarrow$ root($par[v]$)
5.   return $par[v]$

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.

Union-Find Data Structure

root(v)

1. if $par[v] = $ nil then
2. return v
3. else
4. $par[v] \leftarrow$ root($par[v]$)
5. return $par[v]$



MST-Kruskal(G, w)

$F \leftarrow \emptyset$

$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$

sort the edges of E in non-decreasing order of weights w

for each edge $(u, v) \in E$ in the order

 $S_u \leftarrow$ the set in $\mathcal{S}$ containing u

 $S_v \leftarrow$ the set in $\mathcal{S}$ containing v

 if $S_u \neq S_v$

  $F \leftarrow F \cup \{(u, v)\}$

  $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$

return (V, F)

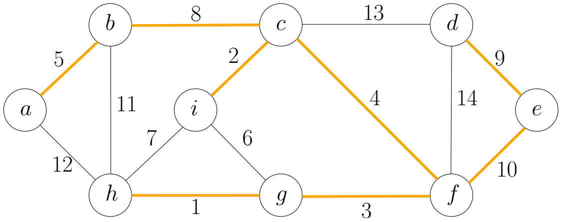
MST-Kruskal(G, w)

1. $F \leftarrow \emptyset$
2. for every $v \in V$: let $par[v] \leftarrow$ nil
3. sort the edges of E in non-decreasing order of weights w
4. for each edge $(u, v) \in E$ in the order
5. $\quad u' \leftarrow$ root(u)
6. $\quad v' \leftarrow$ root(v)
7. $\quad$ if $u' \neq v'$
8. $\quad\quad F \leftarrow F \cup \{(u, v)\}$
9. $\quad\quad par[u'] \leftarrow v'$
10. return (V, F)

- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $=$ time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.

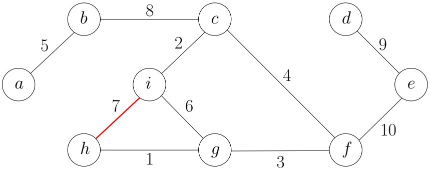


- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from F one by one until we obtain a spanning tree



Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

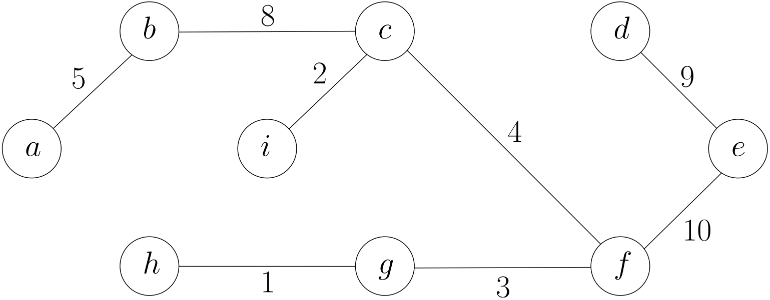
Proof left as a homework exercise.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

1. $F \leftarrow E$
2. sort E in non-increasing order of weights
3. for every e in this order
4. if $(V, F \setminus \{e\})$ is connected then
5. $F \leftarrow F \setminus \{e\}$
6. return (V, F)

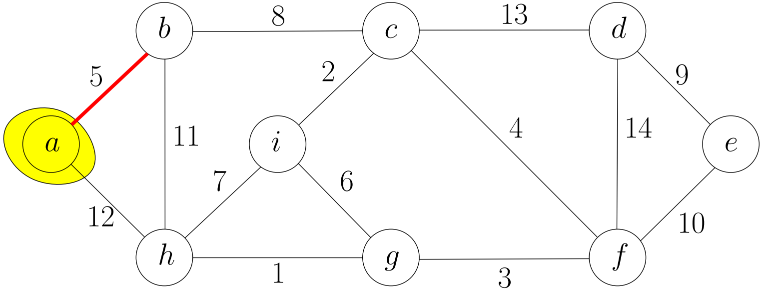
Reverse Kruskal's Algorithm: Example



Outline

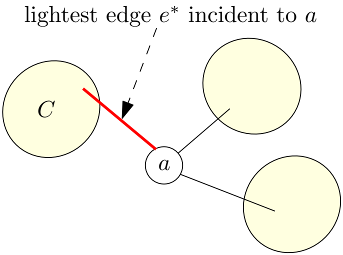
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



- Greedy strategy for Prim's algorithm: choose the lightest edge incident to a.

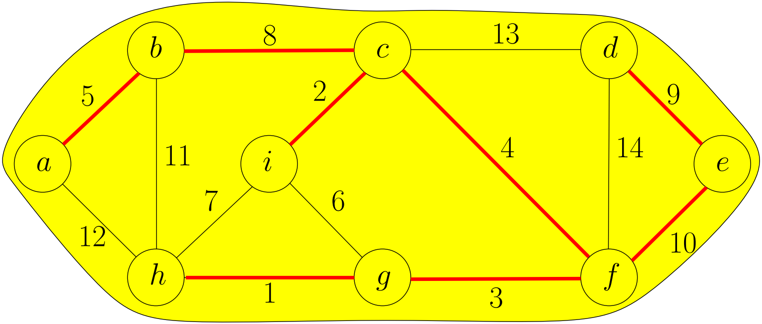
Lemma It is safe to include the lightest edge incident to a.



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus e \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ $\square$

Prim's Algorithm: Example



Greedy Algorithm

MST-Greedy1(G, w)

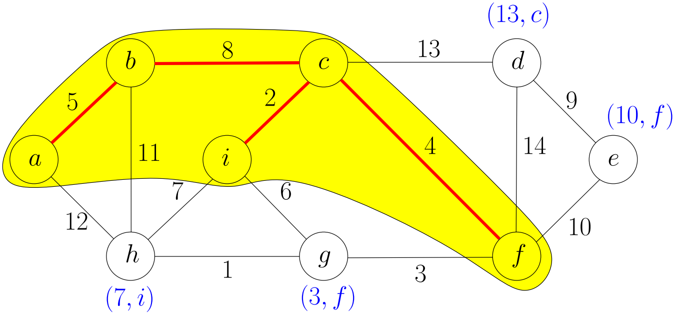
1. $S \leftarrow \{s\}$, where s is arbitrary vertex in V
2. $F \leftarrow \emptyset$
3. while $S \neq V$
4. $\quad (u, v) \leftarrow$ lightest edge between S and $V \setminus S$, where $u \in S$ and $v \in V \setminus S$
5. $\quad S \leftarrow S \cup \{v\}$
6. $\quad F \leftarrow F \cup \{(u, v)\}$
7. return (V, F)

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S



Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$

while $S \neq V$, do

  $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$

  $S \leftarrow S \cup \{u\}$

  for each $v \in V \setminus S$ such that $(u, v) \in E$

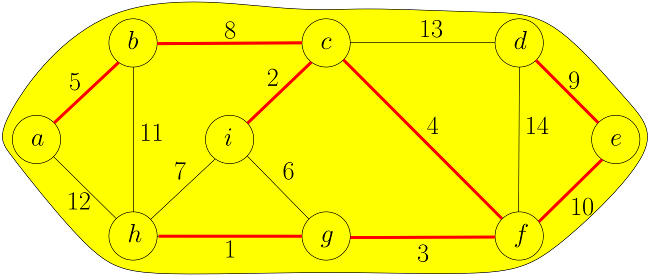
    if $w(u, v) < d(v)$ then

      $d(v) \leftarrow w(u, v)$

      $\pi(v) \leftarrow u$

return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$

Example



Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v)\in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v)\in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value — extract_min
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values. — decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$

while $S \neq V$, do

  $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$

  $S \leftarrow S \cup \{u\}$

  for each $v \in V \setminus S$ such that $(u, v) \in E$

    if $w(u, v) < d(v)$ then

      $d(v) \leftarrow w(u, v)$

      $\pi(v) \leftarrow u$

return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $\big\{(u, \pi(u)) | u \in V \setminus \{s\}\big\}$

Running Time of Prim's Algorithm Using Priority Queue

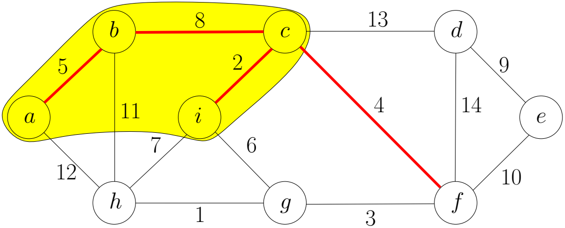
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

We will talk about the heap data structure soon.

Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.



- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

“Evidence” for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

Exactly one of the following is true:

- There is a cut in which e is the lightest edge
- There is a cycle in which e is the heaviest edge

Thus, the minimum spanning tree is unique with assumption.

Outline

- Let V be a ground set of size n.

Def. A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element $v \in V \setminus U$, with associated key value key_value.
- decrease_key(v, new_key_value): decrease the key value of an element $v \in U$ to new_key_value
- extract_min$()$: return and remove the element in U with the smallest key value
- $\cdots$

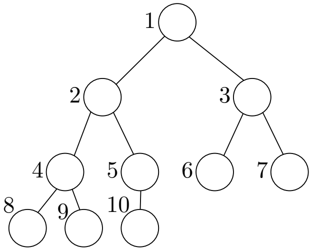
Simple Implementations for Priority Queue

- $n =$ size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Heap

The elements in a heap is organized using a complete binary tree:

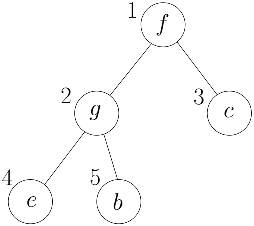


- Nodes are indexed as $\{1, 2, 3, \cdots, s\}$
- Parent of node i: $\lfloor i/2 \rfloor$
- Left child of node i: $2i$
- Right child of node i: $2i + 1$

Heap

A heap H contains the following fields

- s: size of U (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node i of the tree
- $p(v), v \in U$: the index of node containing v
- $key(v), v \in U$: the key value of element v

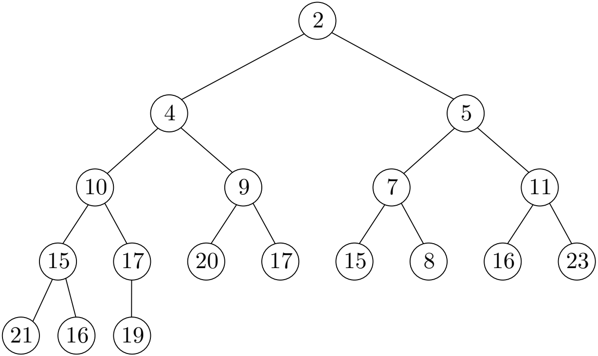


- $s = 5$
- $A = (`f', `g', `c', `e', `b')$
- $p(`f') = 1, p(`g') = 2, p(`c') = 3,$
  $p(`e') = 4, p(`b') = 5$

Heap

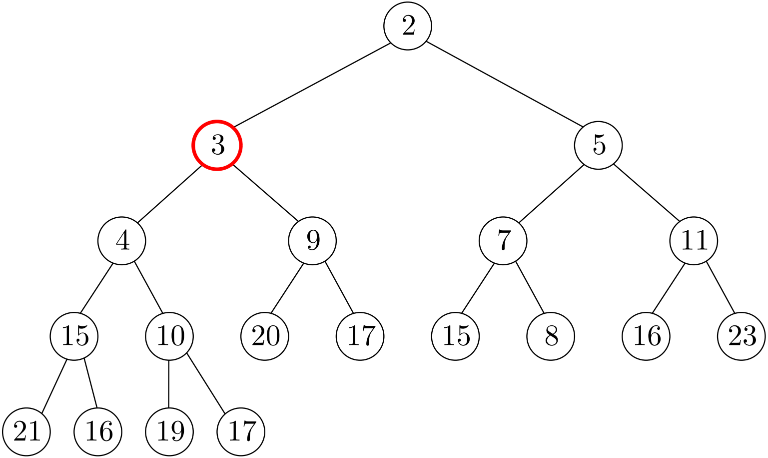
The following heap property is satisfied:

- for any two nodes i, j such that i is the parent of j, we have $key(A[i]) \leq key(A[j])$.



A heap. Numbers in the circles denote key values of elements.

insert(v, key_value)



insert(v, key_value)

$s \leftarrow s + 1$

$A[s] \leftarrow v$

$p(v) \leftarrow s$

$key(v) \leftarrow key\_value$

heapify_up(s)

heapify-up(i)

while $i > 1$

  $j \leftarrow \lfloor i/2 \rfloor$

  if $key(A[i]) < key(A[j])$ then

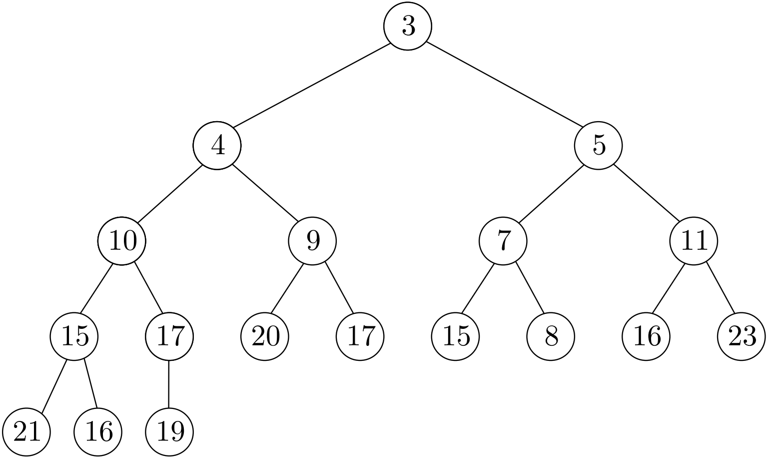
    swap $A[i]$ and $A[j]$

    $p(A[i]) \leftarrow i$, $p(A[j]) \leftarrow j$

    $i \leftarrow j$

  else break

extract_min()



extract_min()

$ret \leftarrow A[1]$
$A[1] \leftarrow A[s]$
$p(A[1]) \leftarrow 1$
$s \leftarrow s - 1$
if $s \geq 1$ then
    heapify_down(1)
return ret

decrease_key(v, key_value)

$key(v) \leftarrow key\_value$
heapify-up$(p(v))$

heapify-down(i)

while $2i \leq s$
    if $2i = s$ or $key(A[2i]) \leq key(A[2i+1])$ then
        $j \leftarrow 2i$
    else
        $j \leftarrow 2i + 1$
    if $key(A[j]) < key(A[i])$ then
        swap $A[i]$ and $A[j]$
        $p(A[i]) \leftarrow i$, $p(A[j]) \leftarrow j$
        $i \leftarrow j$
    else break

- Running time of heapify_up and heapify_down: $O(\lg n)$
- Running time of insert, exact_min and decrease_key: $O(\lg n)$

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key(A[i])$ is too small if we can increase $key(A[i])$ to make H a heap.

Def. We say that H is almost a heap except that $key(A[i])$ is too big if we can decrease $key(A[i])$ to make H a heap.

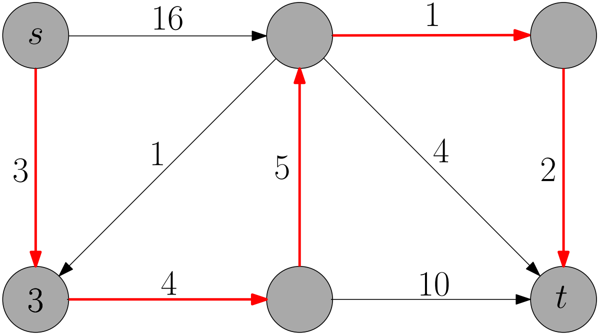
Outline

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest path from s to t



Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

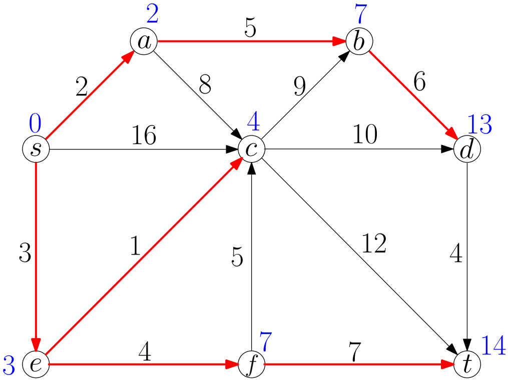
- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

- Shortest path from s to v may contain $\Omega(n)$ edges
- There are $\Omega(n)$ different vertices v
- Thus, printing out all shortest paths may take time $\Omega(n^2)$
- Not acceptable if graph is sparse

Shortest Path Tree

- $O(n)$-size data structure to represent all shortest paths
- For every vertex v, we only need to remember the parent of v: second-to-last vertex in the shortest path from s to v (why?)



Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

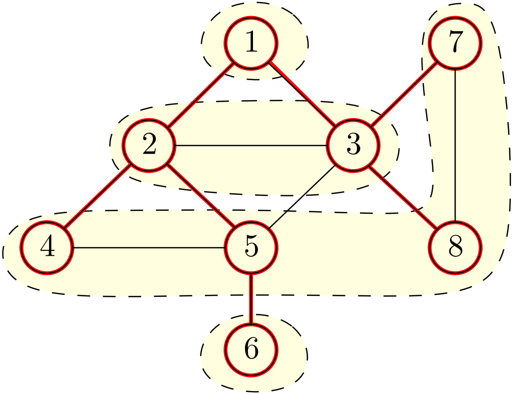
$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi(v), v \in V \setminus s$: the parent of v

$d(v), v \in V \setminus s$: the length of shortest path from s to v

Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s



Assumption Weights $w(u,v)$ are integers (w.l.o.g).

- An edge of weight $w(u,v)$ is equivalent to a pah of $w(u,v)$ unit-weight edges

Shortest Path Algorithm by Running BFS

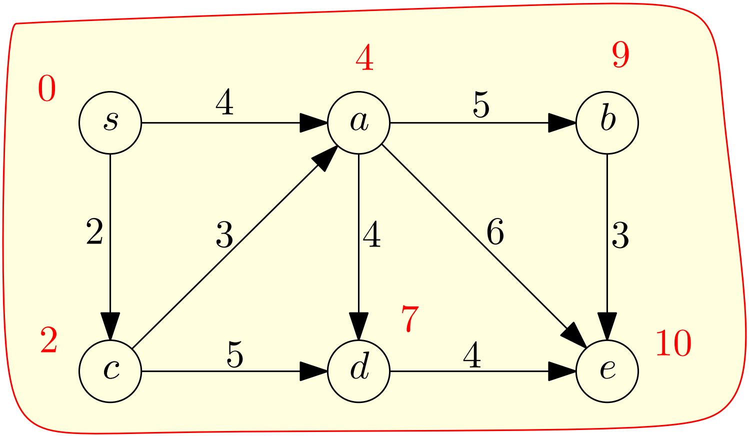
1. replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2. run BFS virtually
3. $\pi(v)$ = vertex from which v is visited
4. $d(v)$ = index of the level containing v

- Problem: $w(u,v)$ may be too large!

Shortest Path Algorithm by Running BFS Virtually

1. $S \leftarrow \{s\}, d(s) \leftarrow 0$
2. while $|S| \leq n$
3. find a $v \notin S$ that minimizes $\displaystyle\min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$
4. $S \leftarrow S \cup \{v\}$
5. $d(v) \leftarrow \min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$

Virtual BFS: Example



Time 10

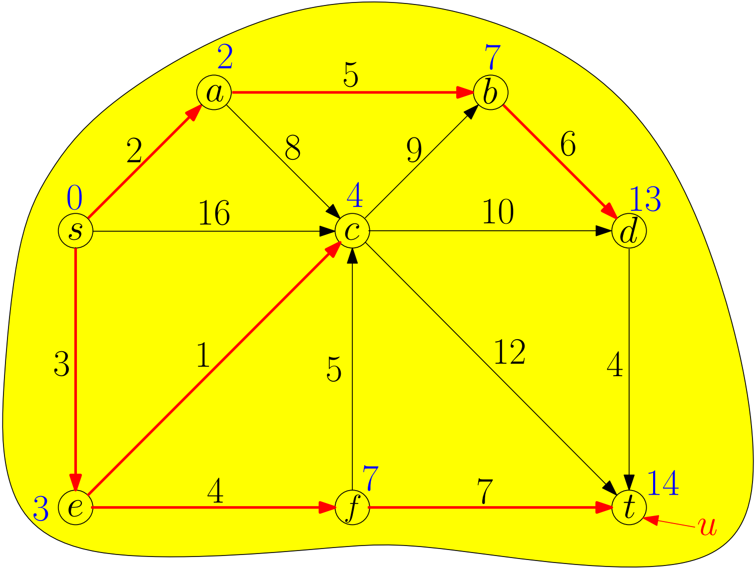
Outline

Dijkstra's Algorithm

Dijkstra(G, w, s)

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    add u to S
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $d(u) + w(u, v) < d(v)$ then
            $d(v) \leftarrow d(u) + w(u, v)$
            $\pi(v) \leftarrow u$
return (d, π)

- Running time $= O(n^2)$



2
7
a
5
b
2
8
9
6
0
16
4
10
13
s
c
d
3
1
5
12
4
7
14
3
e
4
f
7
t
u

Improved Running Time using Priority Queue

Dijkstra(G, w, s)

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $d(u) + w(u, v) < d(v)$ then
            $d(v) \leftarrow d(u) + w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return (π, d)

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $\big\{(u, \pi(u)) | u \in V \setminus \{s\}\big\}$

Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

Summary for Greedy Algorithms

1. Design a greedy choice
   - Interval scheduling problem: schedule the job j^* with the earliest deadline
   - Kruskal's algorithm for MST: select lightest edge e^*
   - Inverse Kruskal's algorithm for MST: drop the heaviest non-bridge edge e^*
   - Prim's algorithm for MST: select the lightest edge e^* incident to a specified vertex s

Summary for Greedy Algorithms

1. Design a greedy choice
2. Prove it is "safe" to make the greedy choice

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

- Usually done by "exchange argument"
- Interval scheduling problem: exchange j^* with the first job in an optimal solution
- Kruskal's algorithm: exchange e^* with some edge e in the cycle in $T \cup \{e^*\}$
- Prim's algorithm: exchange e^* with some other edge e incident to s

Summary for Greedy Algorithms

1. Design a greedy choice
2. Prove it is “safe” to make the greedy choice
3. Show that the remaining task after applying the greedy choice is to solve a (many) smaller instance(s) of the same problem.
   - Interval scheduling problem: remove j^* and the jobs it conflicts with
   - Kruskal and Prim's algorithms: contracting e^*
   - Inverse Kruskal's algorithm: remove e^*

Summary for Greedy Algorithms

- Dijkstra's algorithm does not quite fit in the framework.
- It combines "greedy algorithm" and "dynamic programming"
- Greedy algorithm: each time select the vertex in $V \setminus S$ with the smallest d value and add it to S
- Dynamic programming: remember the d values of vertices in S for future use
- Dijkstra's algorithm is very similar to Prim's algorithm for MST