CSE 431/531: Analysis of Algorithms

Introduction and Syllabus

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Outline

1. Syllabus
2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

CSE 431/531: Analysis of Algorithms

- Course webpage:
  `http://www.cse.buffalo.edu/~shil/courses/CSE531/`
- Please sign up the course on Piazza:
  `http://piazza.com/buffalo/fall2016/cse431531`

- Time and locatiion:
  - MoWeFr, 9:00-9:50am
  - Cooke 121
- Lecturer:
  - Shi Li, shil@buffalo.edu
  - Office hours: TBD
- TAs
  - Di Wang, dwang45@buffalo.edu
  - Minwei Ye, minweiye@buffalo.edu
  - Alexander Stachnik, ajstachn@buffalo.edu

You should know:

- Mathematical Tools
  - Mathematical inductions
  - Probabilities and random variables
- Data Structures
  - Stacks, queues, linked lists
- Some Programming Experience
  - E.g., C, C++ or Java

You may know:

- Asymptotic analysis
- Simple algorithm design techniques such as greedy, divide-and-conquer, dynamic programming

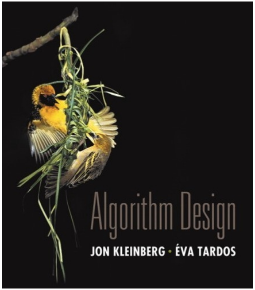
You Will Learn

- Classic algorithms for classic problems
  - Sorting
  - Shortest paths
  - Minimum spanning tree
  - Network flow
- How to analyze algorithms
  - Correctness
  - Running time (efficiency)
  - Space requirement
- Meta techniques to design algorithms
  - Greedy algorithms
  - Divide and conquer
  - Dynamic programming
  - Reductions
- NP-completeness

Textbook

Required Textbook:

- Algorithm Design, 1st Edition, by
  Jon Kleinberg and *Eva Tardos*



Other Reference Books

- Introduction to Algorithms, Third Edition, *Thomas Cormen, Charles Leiserson, Rondald Rivest, Clifford Stein*

Grading

- 20% for homeworks
  - 5 homeworks, each worth 4%
- 20% for projects
  - 2 projects, each worth 10%
- 30% for in-class exams
  - 2 in-class exams, each worth 15%
- 30% for final exam
  - If to your advantage: each in-class exam is worth 5% and final is worth 50%

For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussing
  - Must write down solutions on your own, in your own words
  - Write down names of students you collaborated with

For Homeworks, You Are Not Allowed to

- Use external resources
  - Can't Google or ask questions online for solutions
  - Can't read posted solutions from other algorithm courses
- Copy solutions from other students

If you are not following the rules, you will get an "F" for the course.

Projects

- Need to implement an algorithm for each of the two projects
- Can not copy codes from others or the Internet

If you are not following the rules, you will get an “F” for the course.

Late policy

- You have one late credit
- turn in a homework or a project late for three days using the late credit
- no other late submissions will be accepted

Exams

- Closed-book
- Can bring one A4 handwritten sheet

If you are caught cheating in exams, you will get an "F" for the course.

Questions?

Outline

Outline

What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
- Computational problem: specifies the input/output relationship.
- An algorithm solves a computational problem if it produces the correct output for any given input.

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$
- $(270, 210) \to (210, 60) \to (60, 30) \to (30, 0)$

Examples

Sorting

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

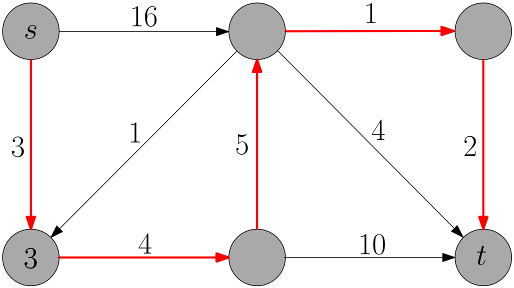
- Algorithms: insertion sort, merge sort, quicksort, ...

Examples

Shortest Path

Input: directed graph $G = (V, E)$, $s, t \in V$

Output: a shortest path from s to t in G



- Algorithm: Dijkstra's algorithm

Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, associated with a particular programming language

Pseudo-Code

Pseudo-Code:

Euclidean(a, b)

1. while $b > 0$
2. $\quad (a, b) \leftarrow (b, a \bmod b)$
3. return a

C++ program:

- int Euclidean(int a, int b){
-     int c;
-     while (b > 0){
-         c = b;
-         b = a % b;
-         a = c;
-     }
-     return a;
- }

Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - readability
  - extensibility
  - user-friendliness
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?

1. feasible vs. infeasible
2. use efficiency to pay for user-friendliness, extensibility, etc.
3. fundamental
4. it is fun!

Outline

Sorting Problem

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \le a'_2 \le \cdots \le a'_n$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

Insertion-Sort

- At the end of j-th iteration, make the first j numbers sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n
    key ← A[j]
    i ← j − 1
    while i > 0 and A[i] > key
        A[i + 1] ← A[i]
        i ← i − 1
    A[i + 1] ← key
```

- $j = 6$
- $key = 15$

12 15 21 35 53 59

$\uparrow$

i

Outline

Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

$$\text{after } j = 3 : 12, 35, 53, 21, 59, 15$$

$$\text{after } j = 4 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 5 : 12, 21, 35, 53, 59, 15$$

$$\text{after } j = 6 : 12, 15, 21, 35, 53, 59$$

Analyze Running Time of Insertion Sort

- Q: Size of input?
- A: Running time as function of size
- possible definition of size : # integers, total length of integers, # vertices in graph, # edges in graph
- Q: Which input?
- A: Worst-case analysis:
  - Worst running time over all input instances of a given size
- Q: How fast is the computer?
- Q: Programming language?
- A: Important idea: asymptotic analysis
  - Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $2^{n/3+100} + 100n^{100} \Rightarrow 2^{n/3+100} \Rightarrow 2^{n/3}$
- $2^{n/3+100} + 100n^{100} = O(2^{n/3})$

Asymptotic Analysis: O-notation

- Ignoring lower order terms
- Ignoring leading constant

O-notation allows us to

- ignore architecture of computer
- ignore programming language

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

for $j \leftarrow 2$ to n
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow key$

- Worst-case running time for iteration j in the outer loop?
  Answer: $O(j)$
- Total running time $= \sum_{j=2}^{n} O(j) = O(n^2)$ (informal)

Computation Model

- Random-Access Machine (RAM) model: read $A[j]$ takes $O(1)$ time.
- Basic operations take $O(1)$ time: addition, subtraction, multiplication, etc.
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
- Precision of real numbers?
  In most scenarios in the course, assuming real numbers are represented exactly

- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort, heap sort, ...

- Remember to sign up for Piazza.

Questions?

Outline

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough n.
- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50$? No
- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-Notation For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some c and large enough n.
- Informally, think of it as "$f \leq g$".

- $3n^2 + 2n \in O(n^3)$
- $3n^2 + 2n \in O(n^2)$
- $n^{100} \in O(2^n)$
- $n^3 \notin O(n^2)$

Conventions

- We use "$f(n) = O(g(n))$" to denote "$f(n) \in O(g(n))$"
- $3n^2 + 2n = O(n^3)$
- $4n^3 + 3n^2 + 2n = 4n^3 + O(n^3)$
  - There exists a function $f(n) \in O(n^3)$, such that $4n^3 + 3n^2 + 2n = 4n^3 + f(n)$.
- $n^2 + O(n) = O(n^2)$
  - For every function $f(n) \in O(n)$, there exists a function $g(n) \in O(n^2)$, such that $n^2 + f(n) = g(n)$.
- Rule: left side $\rightarrow \forall$, right side $\rightarrow \exists$

Conventions

- $3n^2 + 2n = O(n^3)$
- $4n^3 + 3n^2 + 2n = 4n^3 + O(n^3)$
- $n^2 + O(n) = O(n^2)$

- "$=$" is asymmetric! Following statements are wrong:
  - $O(n^3) = 3n^2 + 2n$
  - $4n^3 + O(n^3) = 4n^3 + 3n^2 + 2n$
  - $O(n^2) = n^2 + O(n)$

- Chaining is allowed:
  $4n^3 + 3n^2 + 2n = 4n^3 + O(n^3) = O(n^3) = O(n^4)$

Ω-Notation: Asymptotic Lower Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

Ω-**Notation** For a function $g(n)$,

$$\Omega(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0\}.$$

- In other words, $f(n) \in \Omega(g(n))$ if $f(n) \geq cg(n)$ for some c and large enough n.
- Informally, think of it as "$f \geq g$".

Ω-Notation: Asymptotic Lower Bound

- Again, we use "$=$" instead of $\in$.
  - $4n^2 = \Omega(n)$
  - $3n^2 - n + 10 = \Omega(n^2)$
  - $\Omega(n^2) + n = \Omega(n^2) = \Omega(n)$

Theorem $f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$.

Θ-Notation: Asymptotic Tight Bound

Θ-Notation For a function $g(n)$,

$$\Theta(g(n)) = \{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0\}.$$

- $f(n) = \Theta(g(n))$, then for large enough n, we have "$f(n) \approx g(n)$".
- Informally, think of it as "$f = g$".

- $3n^2 + 2n = \Theta(n^2)$
- $2^{n/3+100} = \Theta(2^{n/3})$

Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Exercise

For each pair of functions f, g in the following table, indicate whether f is O, Ω or Θ of g.

f	g	O	Ω	Θ
$\lg^{10} n$	$n^{0.1}$	Yes	No	No
2^n	$2^{n/2}$	No	Yes	No
$\sqrt{n}$	$n^{\sin n}$	No	No	No
$n^2 - 100n$	$5n^2 + 30n$	Yes	Yes	Yes

Asymptotic Notations	O	Ω	Θ
Comparison Relations	$\leq$	$\geq$	$=$

Trivial Facts on Comparison Relations

- $f \leq g \;\Leftrightarrow\; g \geq f$
- $f = g \;\Leftrightarrow\; f \leq g$ and $f \geq g$
- $f \leq g$ or $f \geq g$

Correct Analogies

- $f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \;\Leftrightarrow\; f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

Incorrect Analogy

- $f(n) = O(g(n))$ or $g(n) = O(f(n))$

Incorrect Analogy

- $f(n) = O(g(n))$ or $g(n) = O(f(n))$

$$f(n) = n^2$$

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases}$$

Recall: informal way to define O-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
- Thus $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2), 3n^2 - 10n - 5 = \Theta(n^2)$

Formally: if $n > 10$, then $n^2 < 3n^2 - 10n - 5 < 3n^2$. So, $3n^2 - 10n - 5 \in \Theta(n^2)$.

o and ω-Notations

o-Notation For a function $g(n)$,

$$o(g(n)) = \{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

ω-Notation For a function $g(n)$,

$$\omega(g(n)) = \{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0\}.$$

Example:

- $3n^2 + 5n + 10 = o(n^2 \lg n)$.
- $3n^2 + 5n + 10 = \omega(n^2 / \lg n)$.

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Questions?

Outline

$O(n)$ (Linear) Running Time

Computing the sum of n numbers

sum(A, n)

1. $S \leftarrow 0$
2. for $i \leftarrow 1$ to n
3. $\quad S \leftarrow S + A[i]$
4. return S

$O(n)$ (Linear) Running Time

- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

3	5	7	8	9	12	20	25	29	32	48

$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\ B and C are sorted, with length n_1 and n_2

```
A ← []; i ← 1; j ← 1
while i ≤ n1 and j ≤ n2
    if (B[i] ≤ C[j]) then
        append B[i] to A; i ← i + 1
    else
        append C[j] to A; j ← j + 1
if i ≤ n1 then append B[i..n1] to A
if j ≤ n2 then append C[j..n2] to A
return A
```

Running time $= O(n)$ where $n = n_1 + n_2$.

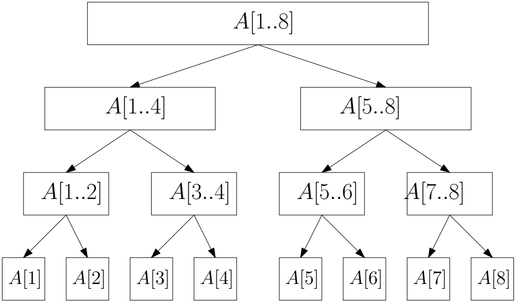
$O(n \lg n)$ Running Time

merge-sort(A, n)

1. if $n = 1$ then
2. return A
3. else
4. $B \leftarrow$ merge-sort$\left(A\left[1..\lfloor n/2 \rfloor\right], \lfloor n/2 \rfloor\right)$
5. $C \leftarrow$ merge-sort$\left(A\left[\lfloor n/2 \rfloor + 1..n\right], n - \lfloor n/2 \rfloor\right)$
6. return merge$(B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$

$O(n \lg n)$ Running Time

- Merge-Sort



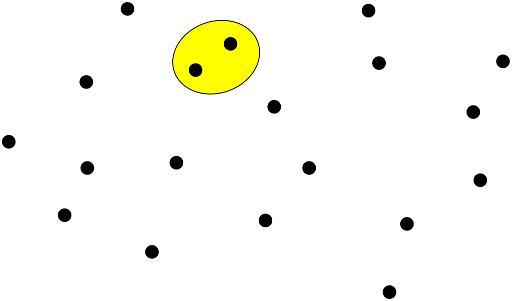
- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair(x, y, n)

```
bestd ← ∞
for i ← 1 to n − 1
    for j ← i + 1 to n
        d ← √((x[i] − x[j])² + (y[i] − y[j])²)
        if d < bestd then
            besti ← i, bestj ← j, bestd ← d
return (besti, bestj)
```

Closest pair can be solved in $O(n \lg n)$ time!

$O(n^3)$ (Cubic) Running Time

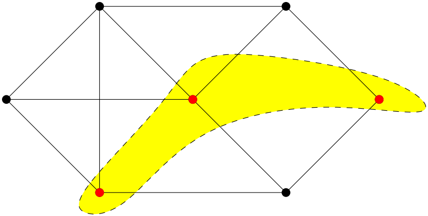
Multiply two matrices of size $n \times n$

matrix-multiplication(A, B, n)

$C \leftarrow$ matrix of size $n \times n$, with all entries being 0
for $i \leftarrow 1$ to n
    for $j \leftarrow 1$ to n
        for $k \leftarrow 1$ to n
            $C[i,k] \leftarrow C[i,k] + A[i,j] \times B[j,k]$
return C

$O(n^k)$ Running Time for Integer $k \geq 4$

Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.



Independent set of size k

Input: graph $G = (V, E)$, an integer k

Output: whether there is an independent set of size k

$O(n^k)$ Running Time for Integer $k \geq 4$

Independent Set of Size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

independent-set$(G = (V, E))$

for every set $S \subseteq V$ of size k
    $b \leftarrow$ true
    for every $u, v \in S$
        if $(u, v) \in E$ then $b \leftarrow$ false
    if b return true
return false

Running time $= O(\frac{n^k}{k!} \times k^2) = O(n^k)$ (assume k is a constant)

Beyond Polynomial Time: $O(2^n)$

Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the maximum independent set of G

max-independent-set($G = (V, E)$)

```
R ← ∅
for every set S ⊆ V
    b ← true
    for every u, v ∈ S
        if (u, v) ∈ E then b ← false
    if b and |S| > |R| then R ← S
return R
```

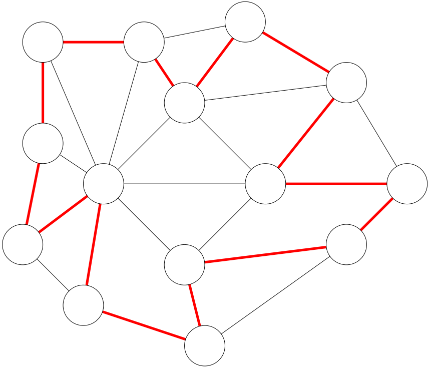
Running time $= O(2^n n^2)$.

Beyond Polynomial Time: $O(n!)$

Hamiltonian Cycle Problem

Input: a graph with n vertices

Output: a cycle that visits each node exactly once, or say no such cycle exists



Beyond Polynomial Time: $n!$

Hamiltonian$(G = (V, E))$

for every permutation $(p_1, p_2, \cdots, p_n)$ of V
    $b \leftarrow true$
    for $i \leftarrow 1$ to $n - 1$
        if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
    if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
    if b then return $(p_1, p_2, \cdots, p_n)$
return "No Hamiltonian Cycle"

Running time $= O(n! \times n)$

$O(\lg n)$ (Logarithmic) Running Time

- Binary search
  - Input: sorted array A of size n, an integer t;
  - Output: whether t appears in A.
- E.g, search 35 in the following array:

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

$O(\lg n)$ (Logarithmic) Running Time

Binary search

- Input: sorted array A of size n, an integer t;
- Output: whether t appears in A.

binary-search(A, n, t)

$i \leftarrow 1, j \leftarrow n$
while $i \leq j$ do
  $k \leftarrow \lfloor (i+j)/2 \rfloor$
  if $A[k] = t$ return true
  if $A[k] < t$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
return false

Running time $= O(\lg n)$

Compare the Orders

- Sort the functions from asymptotically smallest to asymptotically largest (informally, using "$<$" and "$=$")
  $n^{\sqrt{n}}$, $\lg n$, n, n^2, $n \lg n$, $n!$, 2^n, e^n, $\lg(n!)$, n^n
- $\lg n < n^{\sqrt{n}}$
- $\lg n < n < n^{\sqrt{n}}$
- $\lg n < n < n^2 < n^{\sqrt{n}}$
- $\lg n < n < n \lg n < n^2 < n^{\sqrt{n}}$
- $\lg n < n < n \lg n < n^2 < n^{\sqrt{n}} < n!$
- $\lg n < n < n \lg n < n^2 < n^{\sqrt{n}} < 2^n < n!$
- $\lg n < n < n \lg n < n^2 < n^{\sqrt{n}} < 2^n < e^n < n!$
- $\lg n < n < n \lg n = \lg(n!) < n^2 < n^{\sqrt{n}} < 2^n < e^n < n!$
- $\lg n < n < n \lg n = \lg(n!) < n^2 < n^{\sqrt{n}} < 2^n < e^n < n! < n^n$

Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

When we talk about lower bounds:

- Super-linear time: $\omega(n)$
- Super-quadratic time: $\omega(n^2)$
- Super-polynomial time: $\bigcap_{k>0} \omega(n^k)$

Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

Q: Does ignoring the leading constant cause any issues?

- e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

- Sometimes yes
- However, when n is big enough, $1000n < 0.1n^2$
- For "natural" algorithms, constants are not so big!
- For reasonable n, algorithm with lower order running time beats algorithm with higher order running time.