CSE 431/531: Analysis of Algorithms
Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo

Outline

1. Syllabus
2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

- Course webpage:
  `http://www.cse.buffalo.edu/~shil/courses/CSE531/`
- Please sign up the course on Piazza:
  `http://piazza.com/buffalo/fall2016/cse431531`

- Time and locatiion:
  - MoWeFr, 9:00-9:50am
  - Cooke 121
- Lecturer:
  - Shi Li, shil@buffalo.edu
  - Office hours: TBD
- TAs
  - Di Wang, dwang45@buffalo.edu
  - Minwei Ye, minweiye@buffalo.edu
  - Alexander Stachnik, ajstachn@buffalo.edu

You should know:

You should know:

- Mathematical Tools
  - Mathematical inductions
  - Probabilities and random variables

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You may know:

- Asymptotic analysis
- Simple algorithm design techniques such as greedy, divide-and-conquer, dynamic programming

You Will Learn

- Classic algorithms for classic problems
  - Sorting
  - Shortest paths
  - Minimum spanning tree
  - Network flow

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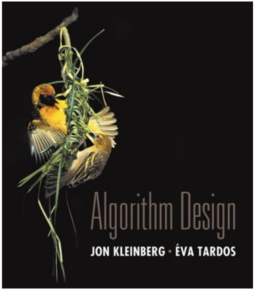
- NP-completeness

Textbook

Required Textbook:

- Algorithm Design, 1st Edition, by *Jon Kleinberg* and *Eva Tardos*



Other Reference Books

- Introduction to Algorithms, Third Edition, *Thomas Cormen, Charles Leiserson, Rondald Rivest, Clifford Stein*

Grading

- 20% for homeworks
  - 5 homeworks, each worth 4%
- 20% for projects
  - 2 projects, each worth 10%
- 30% for in-class exams
  - 2 in-class exams, each worth 15%
- 30% for final exam
  - If to your advantage: each in-class exam is worth 5% and final is worth 50%

For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussing
  - Must write down solutions on your own, in your own words
  - Write down names of students you collaborated with

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Late policy

- You have one late credit
- turn in a homework or a project late for three days using the late credit
- no other late submissions will be accepted

Exams

- Closed-book
- Can bring one A4 handwritten sheet

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Questions?

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- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
- Computational problem: specifies the input/output relationship.
- An algorithm solves a computational problem if it produces the correct output for any given input.

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

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Examples

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- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$
- $(270, 210) \to (210, 60) \to (60, 30) \to (30, 0)$

Examples

Sorting

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \le a'_2 \le \cdots \le a'_n$

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- Algorithms: insertion sort, merge sort, quicksort, ...

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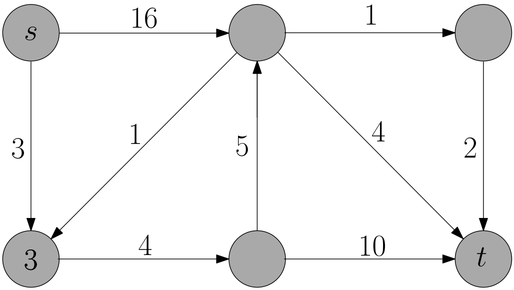

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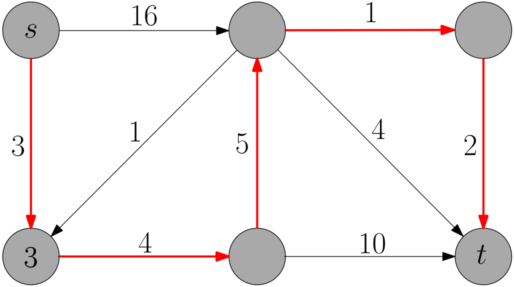

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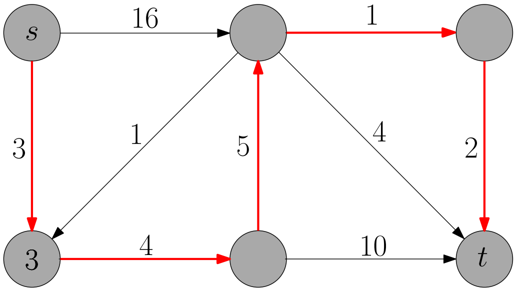

- Algorithm: Dijkstra's algorithm

Algorithm = Computer Program?

- Algorithm: "abstract", can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: "concrete", implementation of algorithm, associated with a particular programming language

Pseudo-Code

Pseudo-Code:

Euclidean(a, b)

1. while $b > 0$
2. $\quad (a, b) \leftarrow (b, a \bmod b)$
3. return a

C++ program:

```
int Euclidean(int a, int b){
    int c;
    while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```

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  2. use efficiency to pay for user-friendliness, extensibility, etc.

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  4. it is fun!

Outline

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Insertion-Sort

- At the end of j-th iteration, make the first j numbers sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

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insertion-sort(A, n)

for $j \leftarrow 2$ to n
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > key$
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Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

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- Q: Programming language?
- A: Important idea: asymptotic analysis
  - Focus on growth of running-time as a function, not any particular value.

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- Ignoring leading constant

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- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
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O-notation allows us to

- ignore architecture of computer
- ignore programming language

Asymptotic Analysis of Insertion Sort

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- Total running time $= \sum_{j=2}^{n} O(j) = O(n^2)$ (informal)

Computation Model

- Random-Access Machine (RAM) model: read $A[j]$ takes $O(1)$ time.
- Basic operations take $O(1)$ time: addition, subtraction, multiplication, etc.
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- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort, heap sort, ...

- Remember to sign up for Piazza.

Questions?

Outline

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

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- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

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- Rule: left side $\rightarrow \forall$, right side $\rightarrow \exists$

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- Chaining is allowed:
  $4n^3 + 3n^2 + 2n = 4n^3 + O(n^3) = O(n^3) = O(n^4)$

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Theorem $f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$.

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Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Exercise

For each pair of functions f, g in the following table, indicate whether f is O, Ω or Θ of g.

f	g	O	Ω	Θ
$\lg^{10} n$	$n^{0.1}$			
2^n	$2^{n/2}$			
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Correct Analogies

- $f(n) = O(g(n)) \;\Leftrightarrow\; g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \;\Leftrightarrow\; f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

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$$f(n) = n^2$$

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases}$$

Recall: informal way to define O-notation

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Formally: if $n > 10$, then $n^2 < 3n^2 - 10n - 5 < 3n^2$. So, $3n^2 - 10n - 5 \in \Theta(n^2)$.

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$$\omega(g(n)) = \{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \ge cg(n), \forall n \ge n_0\}.$$

Example:

- $3n^2 + 5n + 10 = o(n^2 \lg n)$.
- $3n^2 + 5n + 10 = \omega(n^2 / \lg n)$.

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Questions?

Outline

$O(n)$ (Linear) Running Time

Computing the sum of n numbers

sum(A, n)

$S \leftarrow 0$
for $i \leftarrow 1$ to n
    $S \leftarrow S + A[i]$
return S

$O(n)$ (Linear) Running Time

- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

$O(n)$ (Linear) Running Time

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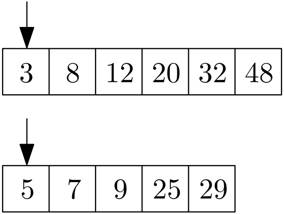

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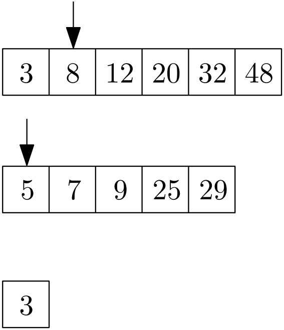

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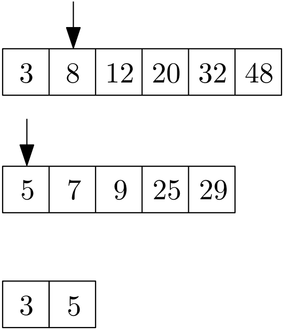
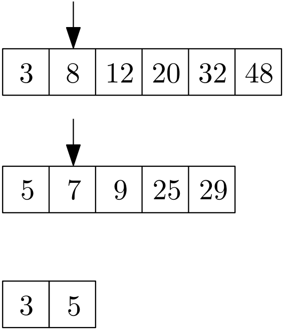


$O(n)$ (Linear) Running Time

- Merge two sorted arrays



$O(n)$ (Linear) Running Time

- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

3	5	7

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$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\\\ B and C are sorted, with length n_1 and n_2

```
A ← []; i ← 1; j ← 1
while i ≤ n1 and j ≤ n2
    if (B[i] ≤ C[j]) then
        append B[i] to A; i ← i + 1
    else
        append C[j] to A; j ← j + 1
if i ≤ n1 then append B[i..n1] to A
if j ≤ n2 then append C[j..n2] to A
return A
```

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if i ≤ n1 then append B[i..n1] to A
if j ≤ n2 then append C[j..n2] to A
return A
```

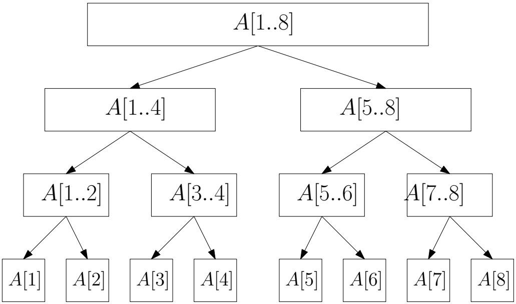
Running time $= O(n)$ where $n = n_1 + n_2$.

$O(n \lg n)$ Running Time

merge-sort(A, n)

if $n = 1$ then
    return A
else
    $B \leftarrow$ merge-sort$\left(A\left[1..\lfloor n/2 \rfloor\right], \lfloor n/2 \rfloor\right)$
    $C \leftarrow$ merge-sort$\left(A\left[\lfloor n/2 \rfloor + 1..n\right], n - \lfloor n/2 \rfloor\right)$
return merge$(B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$

- Merge-Sort



$O(n \lg n)$ Running Time

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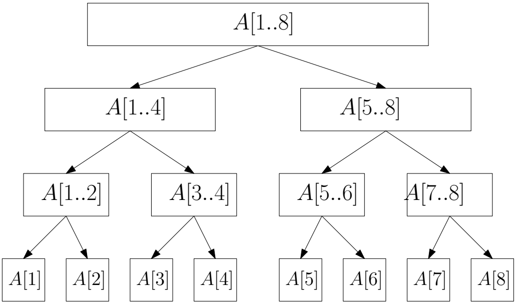
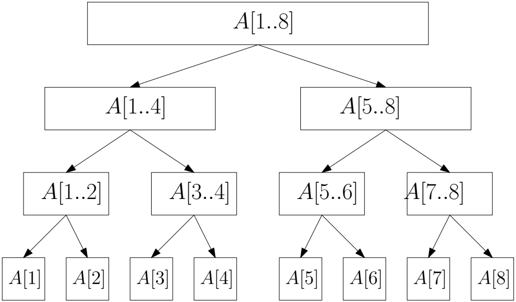


- Each level takes running time $O(n)$

$O(n \lg n)$ Running Time

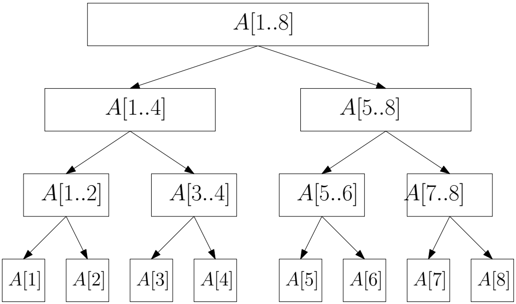
- Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels

$O(n \lg n)$ Running Time

- Merge-Sort



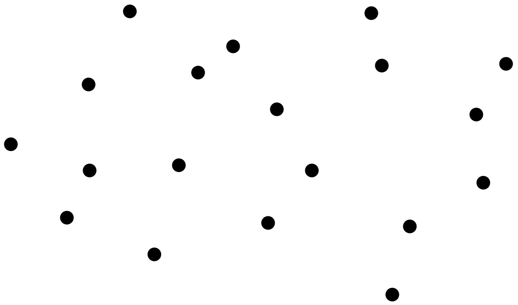
- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



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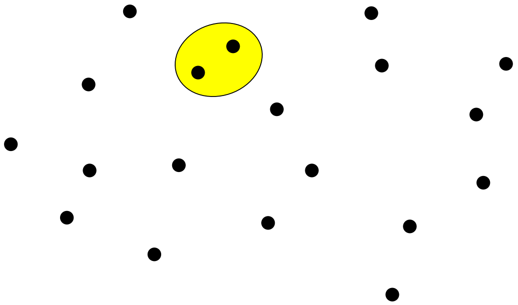

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Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

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closest-pair(x, y, n)

$bestd \leftarrow \infty$

for $i \leftarrow 1$ to $n - 1$

    for $j \leftarrow i + 1$ to n

        $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$

        if $d < bestd$ then

            $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$

return $(besti, bestj)$

$O(n^2)$ (Quardatic) Running Time

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5. if $d < bestd$ then
6. $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
7. return $(besti, bestj)$

Closest pair can be solved in $O(n \lg n)$ time!

$O(n^3)$ (Cubic) Running Time

Multiply two matrices of size $n \times n$

matrix-multiplication(A, B, n)

1. $C \leftarrow$ matrix of size $n \times n$, with all entries being 0
2. for $i \leftarrow 1$ to n
3. for $j \leftarrow 1$ to n
4. for $k \leftarrow 1$ to n
5. $C[i,k] \leftarrow C[i,k] + A[i,j] \times B[j,k]$
6. return C

Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

$O(n^k)$ Running Time for Integer $k \geq 4$

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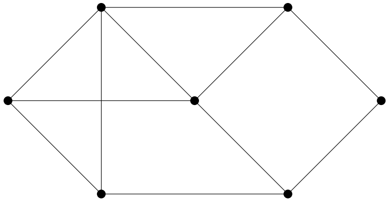

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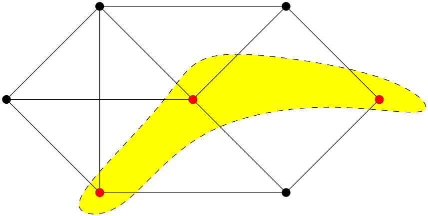

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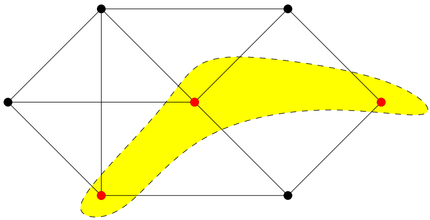


Independent set of size k

Input: graph $G = (V, E)$, an integer k

Output: whether there is an independent set of size k

$O(n^k)$ Running Time for Integer $k \geq 4$

Independent Set of Size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

independent-set$(G = (V, E))$

```
for every set S ⊆ V of size k
    b ← true
    for every u, v ∈ S
        if (u, v) ∈ E then b ← false
    if b return true
return false
```

Running time $= O(\frac{n^k}{k!} \times k^2) = O(n^k)$ (assume k is a constant)

Beyond Polynomial Time: $O(2^n)$

Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the maximum independent set of G

max-independent-set($G = (V, E)$)

```
R ← ∅
for every set S ⊆ V
    b ← true
    for every u, v ∈ S
        if (u, v) ∈ E then b ← false
    if b and |S| > |R| then R ← S
return R
```

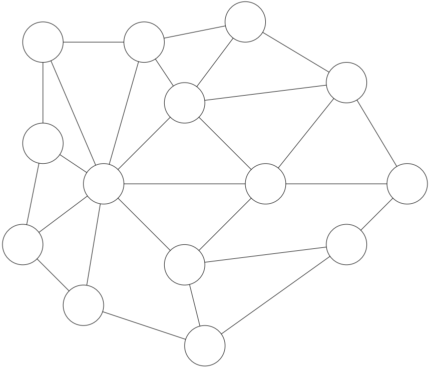
Running time $= O(2^n n^2)$.

Beyond Polynomial Time: $O(n!)$

Hamiltonian Cycle Problem

Input: a graph with n vertices

Output: a cycle that visits each node exactly once, or say no such cycle exists



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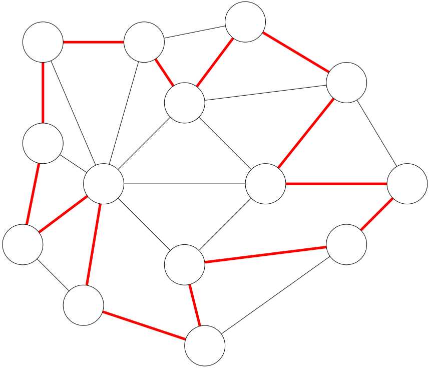

Beyond Polynomial Time: $n!$

Hamiltonian($G = (V, E)$)

for every permutation $(p_1, p_2, \cdots, p_n)$ of V
    $b \leftarrow true$
    for $i \leftarrow 1$ to $n - 1$
        if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
    if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
    if b then return $(p_1, p_2, \cdots, p_n)$
return "No Hamiltonian Cycle"

Running time $= O(n! \times n)$

$O(\lg n)$ (Logarithmic) Running Time

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binary-search(A, n, t)

1. $i \leftarrow 1, j \leftarrow n$
2. while $i \leq j$ do
3. $\quad k \leftarrow \lfloor (i+j)/2 \rfloor$
4. $\quad$ if $A[k] = t$ return true
5. $\quad$ if $A[k] < t$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6. return false

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Running time $= O(\lg n)$

Compare the Orders

- Sort the functions from asymptotically smallest to asymptotically largest (informally, using "$<$" and "$=$")
  $n^{\sqrt{n}}, \;\; \lg n, \;\; n, \;\; n^2, \;\; n \lg n, \;\; n!, \;\; 2^n, \;\; e^n, \;\; \lg(n!), \;\; n^n$

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- $\lg n < n^{\sqrt{n}}$

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Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
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- Polynomial time: $O(n^k)$ for some constant k
- Exponential time: $O(c^n)$ for some $c > 1$
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- Sub-quadratic time: $o(n^2)$

When we talk about lower bounds:

- Super-linear time: $\omega(n)$
- Super-quadratic time: $\omega(n^2)$
- Super-polynomial time: $\bigcap_{k>0} \omega(n^k)$

Goal of Algorithm Design

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- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

Q: Does ignoring the leading constant cause any issues?

- e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

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- e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

- Sometimes yes
- However, when n is big enough, $1000n < 0.1n^2$
- For "natural" algorithms, constants are not so big!
- For reasonable n, algorithm with lower order running time beats algorithm with higher order running time.