

CSE 431/531: Algorithm Analysis and Design (Fall 2021)

Divide-and-Conquer

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Outline

- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 4 Polynomial Multiplication
- 5 Other Classic Algorithms using Divide-and-Conquer
- 6 Solving Recurrences
- 7 Computing n -th Fibonacci Number

Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

Divide-and-Conquer

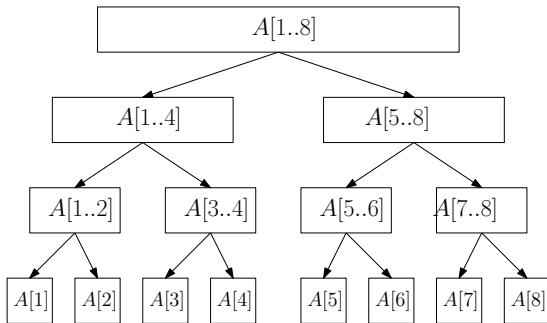
- **Divide:** Divide instance into many smaller instances
- **Conquer:** Solve each of smaller instances recursively and separately
- **Combine:** Combine solutions to small instances to obtain a solution for the original big instance

merge-sort(A, n)

```
1: if  $n = 1$  then  
2:   return  $A$   
3: else  
4:    $B \leftarrow \text{merge-sort}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$   
5:    $C \leftarrow \text{merge-sort}(A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil)$   
6: return  $\text{merge}(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$ 
```

- Divide: trivial
- Conquer: 4, 5
- Combine: 6

Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time = $O(n \lg n)$
- Better than insertion sort

Running Time for Merge-Sort Using Recurrence

- $T(n)$ = running time for sorting n numbers, then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

- With some tolerance of informality:

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases}$$

- Even simpler: $T(n) = 2T(n/2) + O(n)$. (Implicit assumption: $T(n) = O(1)$ if n is at most some constant.)
- Solving this recurrence, we have $T(n) = O(n \lg n)$ (we shall show how later)

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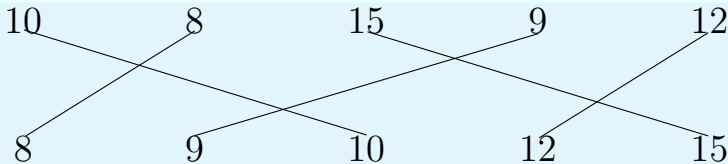
Def. Given an array A of n integers, an inversion in A is a pair (i, j) of indices such that $i < j$ and $A[i] > A[j]$.

Counting Inversions

Input: an sequence A of n numbers

Output: number of inversions in A

Example:



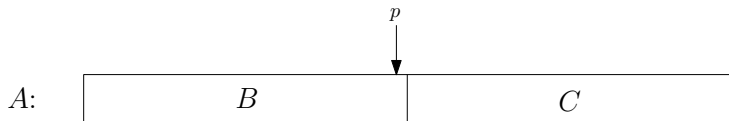
- 4 inversions (for convenience, using numbers, not indices):
(10, 8), (10, 9), (15, 9), (15, 12)

Naive Algorithm for Counting Inversions

count-inversions(A, n)

```
1:  $c \leftarrow 0$ 
2: for every  $i \leftarrow 1$  to  $n - 1$  do
3:   for every  $j \leftarrow i + 1$  to  $n$  do
4:     if  $A[i] > A[j]$  then  $c \leftarrow c + 1$ 
5: return  $c$ 
```

Divide-and-Conquer



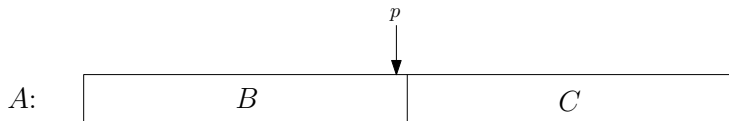
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p + 1..n]$
- $$\#invs(A) = \#invs(B) + \#invs(C) + m$$
$$m = |\{(i, j) : B[i] > C[j]\}|$$

Q: How fast can we compute m , via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

Divide-and-Conquer

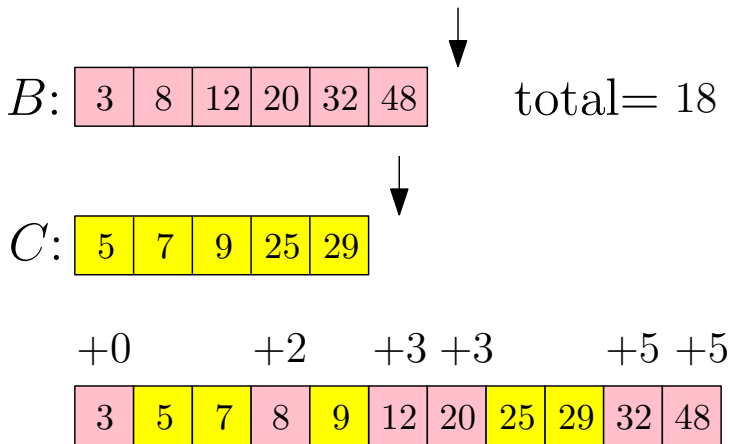


- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p + 1..n]$
- $\#invs(A) = \#invs(B) + \#invs(C) + m$
 $m = |\{(i, j) : B[i] > C[j]\}|$

Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:



Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

```
1:  $count \leftarrow 0$ ;  
2:  $A \leftarrow []$ ;  $i \leftarrow 1$ ;  $j \leftarrow 1$   
3: while  $i \leq n_1$  or  $j \leq n_2$  do  
4:   if  $j > n_2$  or ( $i \leq n_1$  and  $B[i] \leq C[j]$ ) then  
5:     append  $B[i]$  to  $A$ ;  $i \leftarrow i + 1$   
6:      $count \leftarrow count + (j - 1)$   
7:   else  
8:     append  $C[j]$  to  $A$ ;  $j \leftarrow j + 1$   
9: return ( $A, count$ )
```

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A :

sort-and-count(A, n)

1: **if** $n = 1$ **then**

2: **return** ($A, 0$)

3: **else**

4: $(B, m_1) \leftarrow \text{sort-and-count}(A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor)$

5: $(C, m_2) \leftarrow \text{sort-and-count}(A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil)$

6: $(A, m_3) \leftarrow \text{merge-and-count}(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$

7: **return** ($A, m_1 + m_2 + m_3$)

- Divide: trivial

- Conquer: 4, 5

- Combine: 6, 7

sort-and-count(A, n)

```
1: if  $n = 1$  then  
2:   return ( $A, 0$ )  
3: else  
4:    $(B, m_1) \leftarrow$  sort-and-count( $A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor$ )  
5:    $(C, m_2) \leftarrow$  sort-and-count( $A[\lfloor n/2 \rfloor + 1..n], \lceil n/2 \rceil$ )  
6:    $(A, m_3) \leftarrow$  merge-and-count( $B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil$ )  
7:   return ( $A, m_1 + m_2 + m_3$ )
```

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$
- Running time = $O(n \lg n)$

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Quicksort vs Merge-Sort

	Merge Sort	Quicksort
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

29	82	75	64	38	45	94	69	25	76	15	92	37	17	85
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29	38	45	25	15	37	17	64	82	75	94	92	69	76	85
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

25	15	17	29	38	45	37	64	82	75	94	92	69	76	85
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

Quicksort

quicksort(A, n)

- 1: if $n \leq 1$ then return A
- 2: $x \leftarrow$ lower median of A
- 3: $A_L \leftarrow$ elements in A that are less than x \\ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x \\ Divide
- 5: $B_L \leftarrow$ quicksort($A_L, A_L.size$) \\ Conquer
- 6: $B_R \leftarrow$ quicksort($A_R, A_R.size$) \\ Conquer
- 7: $t \leftarrow$ number of times x appear A
- 8: return the array obtained by concatenating B_L , the array containing t copies of x , and B_R

- Recurrence $T(n) \leq 2T(n/2) + O(n)$
- Running time = $O(n \lg n)$

Assumption We can choose median of an array of size n in $O(n)$ time.

Q: How to remove this assumption?

A:

- 1 There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
- 2 Choose a **pivot randomly** and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

quicksort(A, n)

- 1: if $n \leq 1$ then return A
- 2: $x \leftarrow$ a random element of A (x is called a pivot)
- 3: $A_L \leftarrow$ elements in A that are less than x \\ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x \\ Divide
- 5: $B_L \leftarrow$ quicksort($A_L, A_L.size$) \\ Conquer
- 6: $B_R \leftarrow$ quicksort($A_R, A_R.size$) \\ Conquer
- 7: $t \leftarrow$ number of times x appear A
- 8: return the array obtained by concatenating B_L , the array containing t copies of x , and B_R

Randomized Algorithm Model

Assumption There is a procedure to produce a random real number in $[0, 1]$.

Q: Can computers really produce random numbers?

A: No! The execution of a computer programs is deterministic!

- In practice: use **pseudo-random-generator**, a deterministic algorithm returning numbers that “look like” random
- In theory: assume they can.

Quicksort Using A Random Pivot

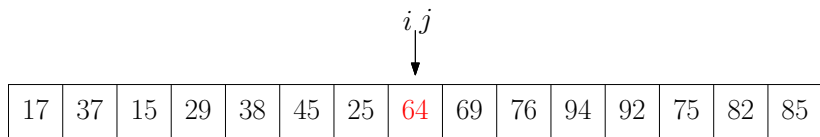
quicksort(A, n)

- 1: if $n \leq 1$ then return A
- 2: $x \leftarrow$ a random element of A (x is called a pivot)
- 3: $A_L \leftarrow$ elements in A that are less than x \\ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x \\ Divide
- 5: $B_L \leftarrow$ quicksort($A_L, A_L.size$) \\ Conquer
- 6: $B_R \leftarrow$ quicksort($A_R, A_R.size$) \\ Conquer
- 7: $t \leftarrow$ number of times x appear A
- 8: return the array obtained by concatenating B_L , the array containing t copies of x , and B_R

Lemma The expected running time of the algorithm is $O(n \lg n)$.

Quicksort Can Be Implemented as an “In-Place” Sorting Algorithm

- In-Place Sorting Algorithm: an algorithm that only uses “small” **extra** space.



- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

- 1: $p \leftarrow$ random integer between ℓ and r , swap $A[p]$ and $A[\ell]$
- 2: $i \leftarrow \ell, j \leftarrow r$
- 3: **while true do**
- 4: **while** $i < j$ and $A[i] < A[j]$ **do** $j \leftarrow j - 1$
- 5: **if** $i = j$ **then break**
- 6: swap $A[i]$ and $A[j]$; $i \leftarrow i + 1$
- 7: **while** $i < j$ and $A[i] < A[j]$ **do** $i \leftarrow i + 1$
- 8: **if** $i = j$ **then break**
- 9: swap $A[i]$ and $A[j]$; $j \leftarrow j - 1$
- 10: **return** i

In-Place Implementation of Quick-Sort

quicksort(A, ℓ, r)

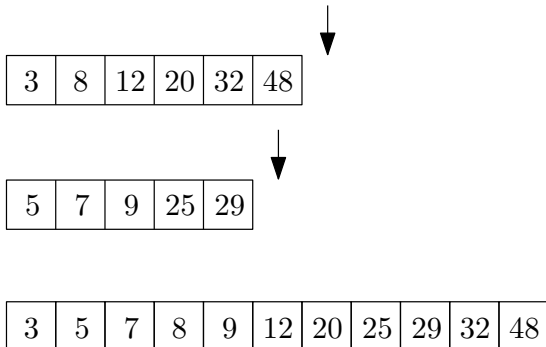
- 1: **if** $\ell \geq r$ **then return**
- 2: $m \leftarrow \text{partition}(A, \ell, r)$
- 3: **quicksort**($A, \ell, m - 1$)
- 4: **quicksort**($A, m + 1, r$)

- To sort an array A of size n , call **quicksort**($A, 1, n$).

Note: We pass the array A by reference, instead of by copying.

Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays



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Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

Comparison-Based Sorting Algorithms

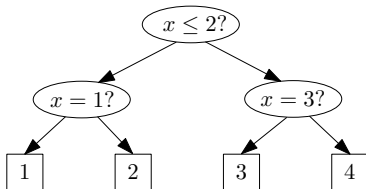
- To sort, we are only allowed to **compare** two elements
- We can not use “internal structures” of the elements

Lemma The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$.

- Bob has one number x in his hand, $x \in \{1, 2, 3, \dots, N\}$.
- You can ask Bob “yes/no” questions about x .

Q: How many questions do you need to ask Bob in order to know x ?

A: $\lceil \log_2 N \rceil$.



Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

- Bob has a permutation π over $\{1, 2, 3, \dots, n\}$ in his hand.
- You can ask Bob “yes/no” questions about π .

Q: How many questions do you need to ask in order to get the permutation π ?

A: $\log_2 n! = \Theta(n \lg n)$

Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

A: No, for comparison-based sorting algorithms.

- Bob has a permutation π over $\{1, 2, 3, \dots, n\}$ in his hand.
- You can ask Bob questions of the form “does i appear before j in π ?”

Q: How many questions do you need to ask in order to get the permutation π ?

A: At least $\log_2 n! = \Theta(n \lg n)$

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Selection Problem

Input: a set A of n numbers, and $1 \leq i \leq n$

Output: the i -th smallest number in A

- Sorting solves the problem in time $O(n \lg n)$.
- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

quicksort(A, n)

- 1: **if** $n \leq 1$ **then return** A
- 2: $x \leftarrow$ lower median of A
- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
- 5: $B_L \leftarrow$ quicksort($A_L, A_L.size$) ▷ Conquer
- 6: $B_R \leftarrow$ quicksort($A_R, A_R.size$) ▷ Conquer
- 7: $t \leftarrow$ number of times x appear A
- 8: **return** the array obtained by concatenating B_L , the array containing t copies of x , and B_R

Selection Algorithm with Median Finder

selection(A, n, i)

```
1: if  $n = 1$  then return  $A$ 
2:  $x \leftarrow$  lower median of  $A$ 
3:  $A_L \leftarrow$  elements in  $A$  that are less than  $x$            ▷ Divide
4:  $A_R \leftarrow$  elements in  $A$  that are greater than  $x$      ▷ Divide
5: if  $i \leq A_L.size$  then
6:   return selection( $A_L, A_L.size, i$ )                       ▷ Conquer
7: else if  $i > n - A_R.size$  then
8:   return selection( $A_R, A_R.size, i - (n - A_R.size)$ )    ▷ Conquer
9: else
10:  return  $x$ 
```

- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

selection(A, n, i)

- 1: **if** $n = 1$ **then return** A
- 2: $x \leftarrow$ **random element** of A (called **pivot**)
- 3: $A_L \leftarrow$ elements in A that are less than x ▷ Divide
- 4: $A_R \leftarrow$ elements in A that are greater than x ▷ Divide
- 5: **if** $i \leq A_L.size$ **then**
- 6: **return** selection($A_L, A_L.size, i$) ▷ Conquer
- 7: **else if** $i > n - A_R.size$ **then**
- 8: **return** selection($A_R, A_R.size, i - (n - A_R.size)$) ▷ Conquer
- 9: **else**
- 10: **return** x

- **expected** running time = $O(n)$

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Polynomial Multiplication

Input: two polynomials of degree $n - 1$

Output: product of two polynomials

Example:

$$\begin{aligned} & (3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\ &= 6x^6 - 9x^5 + 18x^4 - 15x^3 \\ &\quad + 4x^5 - 6x^4 + 12x^3 - 10x^2 \\ &\quad - 10x^4 + 15x^3 - 30x^2 + 25x \\ &\quad + 8x^3 - 12x^2 + 24x - 20 \\ &= 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20 \end{aligned}$$

- **Input:** $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output:** $(-20, 49, -52, 20, 2, -5, 6)$

Naïve Algorithm

polynomial-multiplication(A, B, n)

- 1: let $C[k] \leftarrow 0$ for every $k = 0, 1, 2, \dots, 2n - 2$
- 2: **for** $i \leftarrow 0$ to $n - 1$ **do**
- 3: **for** $j \leftarrow 0$ to $n - 1$ **do**
- 4: $C[i + j] \leftarrow C[i + j] + A[i] \times B[j]$
- 5: **return** C

Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$

$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x - 3)x^2 + (6x - 5)$$

- $p(x)$: degree of $n - 1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

$$\begin{aligned}pq &= (p_Hx^{n/2} + p_L)(q_Hx^{n/2} + q_L) \\ &= p_Hq_Hx^n + (p_Hq_L + p_Lq_H)x^{n/2} + p_Lq_L\end{aligned}$$

Divide-and-Conquer for Polynomial Multiplication

$$\begin{aligned}pq &= (p_H x^{n/2} + p_L)(q_H x^{n/2} + q_L) \\ &= p_H q_H x^n + (p_H q_L + p_L q_H) x^{n/2} + p_L q_L\end{aligned}$$

$$\begin{aligned}\text{multiply}(p, q) &= \text{multiply}(p_H, q_H) \times x^n \\ &\quad + (\text{multiply}(p_H, q_L) + \text{multiply}(p_L, q_H)) \times x^{n/2} \\ &\quad + \text{multiply}(p_L, q_L)\end{aligned}$$

- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

Reduce Number from 4 to 3

$$\begin{aligned}pq &= (p_H x^{n/2} + p_L)(q_H x^{n/2} + q_L) \\ &= p_H q_H x^n + (p_H q_L + p_L q_H) x^{n/2} + p_L q_L\end{aligned}$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

$$r_H = \text{multiply}(p_H, q_H)$$

$$r_L = \text{multiply}(p_L, q_L)$$

$$\begin{aligned} \text{multiply}(p, q) &= r_H \times x^n \\ &+ (\text{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L) \times x^{n/2} \\ &+ r_L \end{aligned}$$

- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

multiply(A, B, n)

- 1: if $n = 1$ then return $(A[0]B[0])$
- 2: $A_L \leftarrow A[0 .. n/2 - 1], A_H \leftarrow A[n/2 .. n - 1]$
- 3: $B_L \leftarrow B[0 .. n/2 - 1], B_H \leftarrow B[n/2 .. n - 1]$
- 4: $C_L \leftarrow \text{multiply}(A_L, B_L, n/2)$
- 5: $C_H \leftarrow \text{multiply}(A_H, B_H, n/2)$
- 6: $C_M \leftarrow \text{multiply}(A_L + A_H, B_L + B_H, n/2)$
- 7: $C \leftarrow$ array of $(2n - 1)$ 0's
- 8: **for** $i \leftarrow 0$ to $n - 2$ **do**
- 9: $C[i] \leftarrow C[i] + C_L[i]$
- 10: $C[i + n] \leftarrow C[i + n] + C_H[i]$
- 11: $C[i + n/2] \leftarrow C[i + n/2] + C_M[i] - C_L[i] - C_H[i]$
- 12: **return** C

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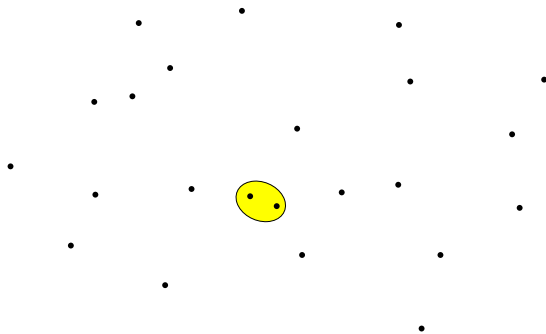
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- Closest pair
- Convex hull
- Matrix multiplication
- FFT(Fast Fourier Transform): polynomial multiplication in $O(n \lg n)$ time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$

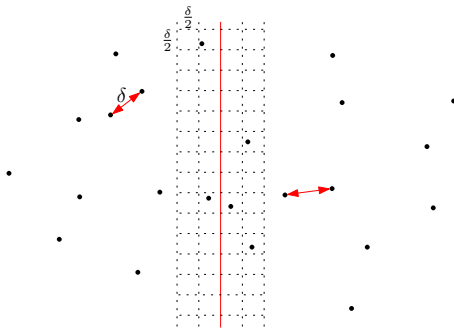
Output: the pair of points that are closest



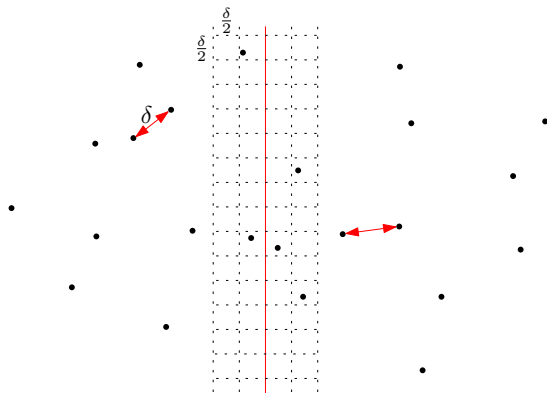
- Trivial algorithm: $O(n^2)$ running time

Divide-and-Conquer Algorithm for Closest Pair

- **Divide:** Divide the points into two halves via a vertical line
- **Conquer:** Solve two sub-instances recursively
- **Combine:** Check if there is a closer pair between left-half and right-half

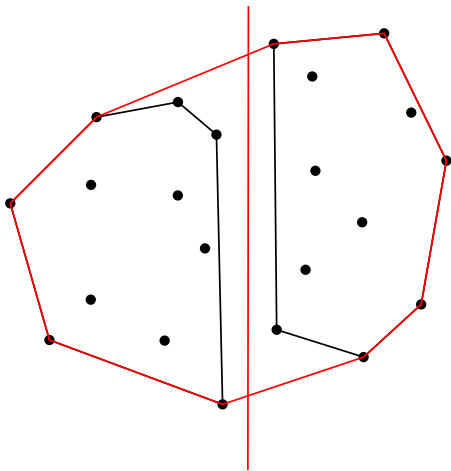


Divide-and-Conquer Algorithm for Closest Pair



- Each box contains at most one pair
- For each point, only need to consider $O(1)$ boxes nearby
- time for combine = $O(n)$ (many technicalities omitted)
- Recurrence: $T(n) = 2T(n/2) + O(n)$
- Running time: $O(n \lg n)$

$O(n \lg n)$ -Time Algorithm for Convex Hull



Strassen's Algorithm for Matrix Multiplication

Matrix Multiplication

Input: two $n \times n$ matrices A and B

Output: $C = AB$

Naive Algorithm: matrix-multiplication(A, B, n)

```
1: for  $i \leftarrow 1$  to  $n$  do
2:   for  $j \leftarrow 1$  to  $n$  do
3:      $C[i, j] \leftarrow 0$ 
4:     for  $k \leftarrow 1$  to  $n$  do
5:        $C[i, j] \leftarrow C[i, j] + A[i, k] \times B[k, j]$ 
6: return  $C$ 
```

- running time = $O(n^3)$

Try to Use Divide-and-Conquer

$$A = \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \quad B = \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}$$

The diagram shows two 2x2 matrices, A and B. Matrix A has elements A₁₁, A₁₂, A₂₁, and A₂₂. Matrix B has elements B₁₁, B₁₂, B₂₁, and B₂₂. Brackets above each matrix indicate that the width of each column is n/2. A bracket to the right of each matrix indicates that the height of each row is n/2.

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- `matrix_multiplication(A, B)` recursively calls `matrix_multiplication(A11, B11)`, `matrix_multiplication(A12, B21)`, ...
- Recurrence for running time: $T(n) = 8T(n/2) + O(n^2)$
- $T(n) = O(n^3)$

Strassen's Algorithm

- $T(n) = 8T(n/2) + O(n^2)$
- Strassen's Algorithm: improve the number of multiplications from 8 to 7!
- New recurrence: $T(n) = 7T(n/2) + O(n^2)$
- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

Outline

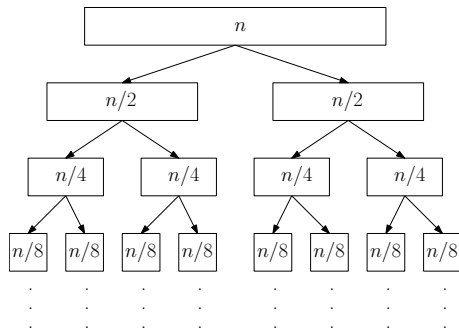
- 1 Divide-and-Conquer
- 2 Counting Inversions
- 3 Quicksort and Selection
 - Quicksort
 - Lower Bound for Comparison-Based Sorting Algorithms
 - Selection Problem
- 4 Polynomial Multiplication
- 5 Other Classic Algorithms using Divide-and-Conquer
- 6 Solving Recurrences**
- 7 Computing n -th Fibonacci Number

Methods for Solving Recurrences

- The recursion-tree method
- The master theorem

Recursion-Tree Method

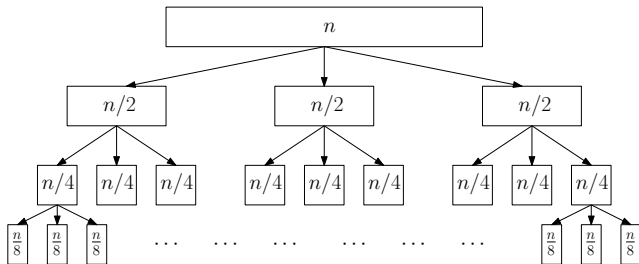
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time = $O(n \lg n)$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

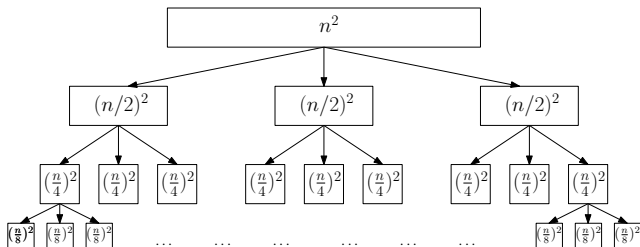


- Total running time at level i ? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{2}\right)^i n = O\left(n \left(\frac{3}{2}\right)^{\lg_2 n}\right) = O(3^{\lg_2 n}) = O(n^{\lg_2 3}).$$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



- Total running time at level i ? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$
- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$	3	2	2	$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} O(n^{\lg_b a}) & \text{if } c < \lg_b a \\ O(n^c \lg n) & \text{if } c = \lg_b a \\ O(n^c) & \text{if } c > \lg_b a \end{cases}$$

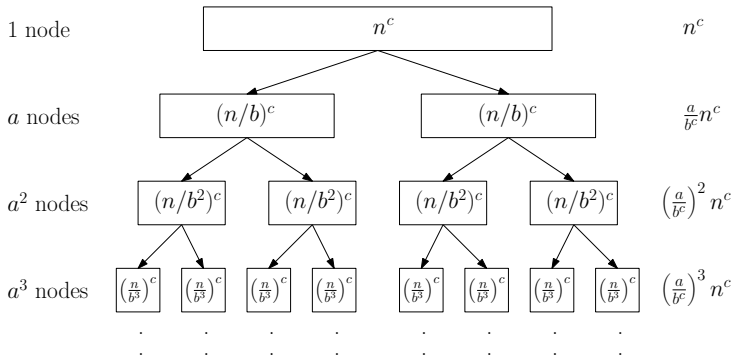
Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} O(n^{\lg_b a}) & \text{if } c < \lg_b a \\ O(n^c \lg n) & \text{if } c = \lg_b a \\ O(n^c) & \text{if } c > \lg_b a \end{cases}$$

- Ex: $T(n) = 4T(n/2) + O(n^2)$. **Case 2.** $T(n) = O(n^2 \lg n)$
- Ex: $T(n) = 3T(n/2) + O(n)$. **Case 1.** $T(n) = O(n^{\lg_2 3})$
- Ex: $T(n) = T(n/2) + O(1)$. **Case 2.** $T(n) = O(\lg n)$
- Ex: $T(n) = 2T(n/2) + O(n^2)$. **Case 3.** $T(n) = O(n^2)$

Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



- $c < \lg_b a$: bottom-level dominates: $(\frac{a}{b^c})^{\lg_b n} n^c = n^{\lg_b a}$
- $c = \lg_b a$: all levels have same time: $n^c \lg_b n = O(n^c \lg n)$
- $c > \lg_b a$: top-level dominates: $O(n^c)$

Outline

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Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots

n -th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

- 1: if $n = 0$ return 0
- 2: if $n = 1$ return 1
- 3: return $\text{Fib}(n - 1) + \text{Fib}(n - 2)$

Q: Is the running time of the algorithm polynomial or exponential in n ?

A: Exponential

- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n : Reasonable Algorithm

Fib(n)

```
1:  $F[0] \leftarrow 0$   
2:  $F[1] \leftarrow 1$   
3: for  $i \leftarrow 2$  to  $n$  do  
4:    $F[i] \leftarrow F[i - 1] + F[i - 2]$   
5: return  $F[n]$ 
```

- Dynamic Programming
- Running time = $O(n)$

Computing F_n : Even Better Algorithm

$$\begin{aligned}\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} \\ \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} \\ &\dots \\ \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} &= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}\end{aligned}$$

power(n)

- 1: if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- 2: $R \leftarrow \text{power}(\lfloor n/2 \rfloor)$
- 3: $R \leftarrow R \times R$
- 4: if n is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
- 5: **return** R

Fib(n)

- 1: if $n = 0$ then return 0
- 2: $M \leftarrow \text{power}(n - 1)$
- 3: **return** $M[1][1]$

- Recurrence for running time? $T(n) = T(n/2) + O(1)$
- $T(n) = O(\lg n)$

Running time = $O(\lg n)$: We Cheated!

Q: How many bits do we need to represent $F(n)$?

A: $\Theta(n)$

- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time
- Even printing $F(n)$ requires time much larger than $O(\lg n)$

Fixing the Problem

To compute F_n , we need $O(\lg n)$ **basic arithmetic operations** on integers

Summary: Divide-and-Conquer

- **Divide:** Divide instance into many smaller instances
- **Conquer:** Solve each of smaller instances recursively and separately
- **Combine:** Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, ...:
 $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \lg n)$
- Integer Multiplication:
 $T(n) = 3T(n/2) + O(n) \Rightarrow T(n) = O(n^{\lg_2 3})$
- Matrix Multiplication:
 $T(n) = 7T(n/2) + O(n^2) \Rightarrow T(n) = O(n^{\lg_2 7})$
- Usually, designing better algorithm for “combine” step is key to improve running time