

CSE 431/531: Algorithm Analysis and Design (Fall 2021)

Dynamic Programming

Lecturer: Shi Li

*Department of Computer Science and Engineering
University at Buffalo*

Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many **independent** sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many **overlapping** sub-problems
- Build solutions for larger and larger sub-problems
- Use a **table** to store solutions for sub-problems for reuse

Recall: Computing the n -th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots

Fib(n)

```
1:  $F[0] \leftarrow 0$   
2:  $F[1] \leftarrow 1$   
3: for  $i \leftarrow 2$  to  $n$  do  
4:    $F[i] \leftarrow F[i - 1] + F[i - 2]$   
5: return  $F[n]$ 
```

- Store each $F[i]$ for future use.

Outline

- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem
- 3 Knapsack Problem
- 4 Longest Common Subsequence
 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Directed Acyclic Graphs
- 6 Matrix Chain Multiplication
- 7 Optimum Binary Search Tree
- 8 Summary

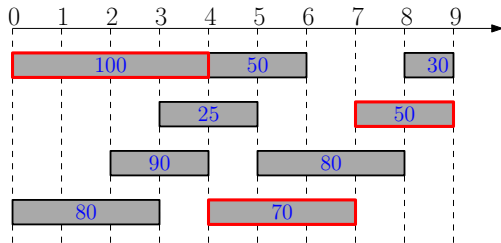
Recall: Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs



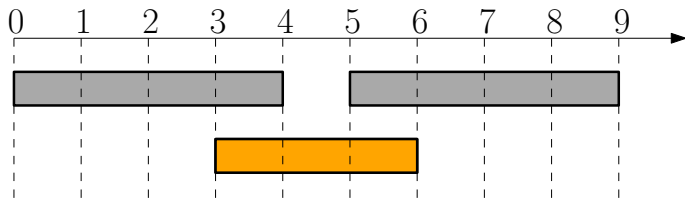
Optimum value = 220

Hard to Design a Greedy Algorithm

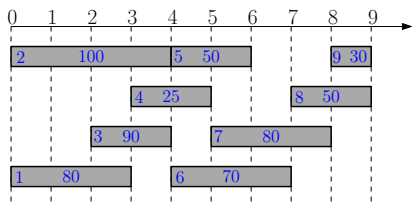
Q: Which job is safe to schedule?

- Job with the earliest finish time? No, we are ignoring weights
- Job with the largest weight? No, we are ignoring times
- Job with the largest $\frac{\text{weight}}{\text{length}}$?

No, when weights are equal, this is the shortest job



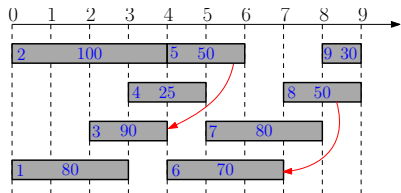
Designing a Dynamic Programming Algorithm



- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \dots, i\}$

i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \dots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \dots, opt[i - 1]$

Q: The value of optimal solution that **does not contain** i ?

A: $opt[i - 1]$

Q: The value of optimal solution that **contains** job i ?

A: $v_i + opt[p_i]$,

$p_i =$ the largest j such that $f_j \leq s_i$

Designing a Dynamic Programming Algorithm

Q: The value of optimal solution that **does not contain** i ?

A: $opt[i - 1]$

Q: The value of optimal solution that **contains** job i ?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

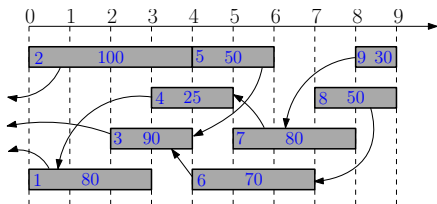
Recursion for $opt[i]$:

$$opt[i] = \max \{opt[i - 1], v_i + opt[p_i]\}$$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max \{opt[i - 1], v_i + opt[p_i]\}$$

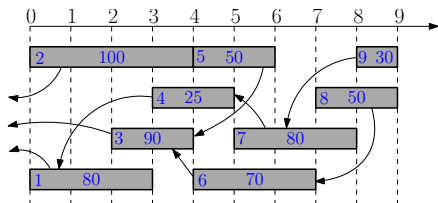


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\} = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max \{opt[i - 1], v_i + opt[p_i]\}$$



- $opt[0] = 0, opt[1] = 80, opt[2] = 100$
- $opt[3] = 100, opt[4] = 105, opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Dynamic Programming

- 1: sort jobs by non-decreasing order of finishing times
- 2: compute p_1, p_2, \dots, p_n
- 3: $opt[0] \leftarrow 0$
- 4: **for** $i \leftarrow 1$ to n **do**
- 5: $opt[i] \leftarrow \max\{opt[i - 1], v_i + opt[p_i]\}$

- Running time sorting: $O(n \lg n)$
- Running time for computing p : $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

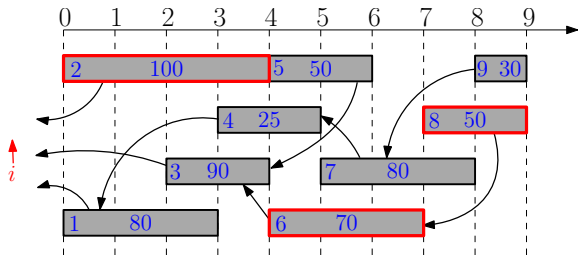
How Can We Recover the Optimum Schedule?

```
1: sort jobs by non-decreasing order of
   finishing times
2: compute  $p_1, p_2, \dots, p_n$ 
3:  $opt[0] \leftarrow 0$ 
4: for  $i \leftarrow 1$  to  $n$  do
5:     if  $opt[i - 1] \geq v_i + opt[p_i]$  then
6:          $opt[i] \leftarrow opt[i - 1]$ 
7:          $b[i] \leftarrow N$ 
8:     else
9:          $opt[i] \leftarrow v_i + opt[p_i]$ 
10:         $b[i] \leftarrow Y$ 
```

```
1:  $i \leftarrow n, S \leftarrow \emptyset$ 
2: while  $i \neq 0$  do
3:     if  $b[i] = N$  then
4:          $i \leftarrow i - 1$ 
5:     else
6:          $S \leftarrow S \cup \{i\}$ 
7:          $i \leftarrow p_i$ 
8: return  $S$ 
```

Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	\perp
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



Dynamic Programming

- Break up a problem into many **overlapping** sub-problems
- Build solutions for larger and larger sub-problems
- Use a **table** to store solutions for sub-problems for reuse

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Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \quad \text{s.t. } \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W , and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Q: What if we change “non-increasing” to “non-decreasing”?

A: No. $W = 100, n = 3, w = (1, 50, 50)$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \dots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that **does not contain** i ?

A: $opt[i - 1, W']$

Q: The value of the optimum solution that **contains** i ?

A: $opt[i - 1, W' - w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \dots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i - 1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{l} opt[i - 1, W'] \\ opt[i - 1, W' - w_i] + w_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

Dynamic Programming

```
1: for  $W' \leftarrow 0$  to  $W$  do  
2:    $opt[0, W'] \leftarrow 0$   
3: for  $i \leftarrow 1$  to  $n$  do  
4:   for  $W' \leftarrow 0$  to  $W$  do  
5:      $opt[i, W'] \leftarrow opt[i - 1, W']$   
6:     if  $w_i \leq W'$  and  $opt[i - 1, W' - w_i] + w_i \geq opt[i, W']$  then  
7:        $opt[i, W'] \leftarrow opt[i - 1, W' - w_i] + w_i$   
8: return  $opt[n, W]$ 
```

Recover the Optimum Set

```
1: for  $W' \leftarrow 0$  to  $W$  do
2:    $opt[0, W'] \leftarrow 0$ 
3: for  $i \leftarrow 1$  to  $n$  do
4:   for  $W' \leftarrow 0$  to  $W$  do
5:      $opt[i, W'] \leftarrow opt[i - 1, W']$ 
6:      $b[i, W'] \leftarrow N$ 
7:     if  $w_i \leq W'$  and  $opt[i - 1, W' - w_i] + w_i \geq opt[i, W']$ 
   then
8:        $opt[i, W'] \leftarrow opt[i - 1, W' - w_i] + w_i$ 
9:        $b[i, W'] \leftarrow Y$ 
10: return  $opt[n, W]$ 
```

Recover the Optimum Set

```
1:  $i \leftarrow n, W' \leftarrow W, S \leftarrow \emptyset$   
2: while  $i > 0$  do  
3:   if  $b[i, W'] = Y$  then  
4:      $W' \leftarrow W' - w_i$   
5:      $S \leftarrow S \cup \{i\}$   
6:    $i \leftarrow i - 1$   
7: return  $S$ 
```


Running Time of Algorithm

```
1: for  $W' \leftarrow 0$  to  $W$  do
2:    $opt[0, W'] \leftarrow 0$ 
3: for  $i \leftarrow 1$  to  $n$  do
4:   for  $W' \leftarrow 0$  to  $W$  do
5:      $opt[i, W'] \leftarrow opt[i - 1, W']$ 
6:     if  $w_i \leq W'$  and  $opt[i - 1, W' - w_i] + w_i \geq opt[i, W']$  then
7:        $opt[i, W'] \leftarrow opt[i - 1, W' - w_i] + w_i$ 
8: return  $opt[n, W]$ 
```

- Running time is $O(nW)$
- Running time is **pseudo-polynomial** because it depends on value of the input integers.

Avoiding Unnecessary Computation and Memory Using Memoized Algorithm and Hash Map

compute-opt(i, W')

```
1: if  $opt[i, W'] \neq \perp$  then return  $opt[i, W']$ 
2: if  $i = 0$  then  $r \leftarrow 0$ 
3: else
4:    $r \leftarrow \text{compute-opt}(i - 1, W')$ 
5:   if  $w_i \leq W'$  then
6:      $r' \leftarrow \text{compute-opt}(i - 1, W' - w_i) + w_i$ 
7:     if  $r' > r$  then  $r \leftarrow r'$ 
8:  $opt[i, W'] \leftarrow r$ 
9: return  $r$ 
```

- Use hash map for opt

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Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \quad \text{s.t. } \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W , and want to buy a subset of items of maximum total value

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \dots, i\}$.
- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \dots, W$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i - 1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{l} opt[i - 1, W'] \\ opt[i - 1, W' - w_i] + v_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

Exercise: Items with 3 Parameters

Input: integer bounds $W > 0$, $Z > 0$,
a set of n items, each with an integer weight $w_i > 0$
a size $z_i > 0$ for each item i
a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\begin{aligned} & \text{maximizes } \sum_{i \in S} v_i && \text{s.t.} \\ & \sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} z_i \leq Z \end{aligned}$$

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Subsequence

- $A = bacdca$
- $C = adca$
- C is a subsequence of A

Def. Given two sequences $A[1 .. n]$ and $C[1 .. t]$ of letters, C is called a **subsequence** of A if there exists integers $1 \leq i_1 < i_2 < i_3 < \dots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \dots, t$.

- Exercise: how to check if sequence C is a subsequence of A ?

Longest Common Subsequence

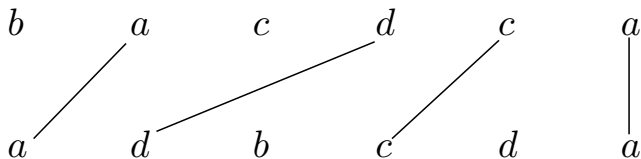
Input: $A[1 .. n]$ and $B[1 .. m]$

Output: the longest common subsequence of A and B

Example:

- $A = 'bacdca'$
 - $B = 'adbcdca'$
 - $LCS(A, B) = 'adca'$
-
- Applications: edit distance (diff), similarity of DNAs

Matching View of LCS



- Goal of LCS: find a maximum-size non-crossing matching between letters in A and letters in B .

Reduce to Subproblems

- $A = \text{'bacdca'}$
- $B = \text{'adbcdca'}$
- either the last letter of A is not matched:
 - need to compute $\text{LCS}(\text{'bacd'}, \text{'adbcd'})$
- or the last letter of B is not matched:
 - need to compute $\text{LCS}(\text{'bacdc'}, \text{'adbc'})$

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} opt[i - 1, j - 1] + 1 & \text{if } A[i] = B[j] \\ \max \begin{cases} opt[i - 1, j] \\ opt[i, j - 1] \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

```
1: for  $j \leftarrow 0$  to  $m$  do
2:    $opt[0, j] \leftarrow 0$ 
3: for  $i \leftarrow 1$  to  $n$  do
4:    $opt[i, 0] \leftarrow 0$ 
5:   for  $j \leftarrow 1$  to  $m$  do
6:     if  $A[i] = B[j]$  then
7:        $opt[i, j] \leftarrow opt[i - 1, j - 1] + 1, \pi[i, j] \leftarrow \swarrow$ 
8:     else if  $opt[i, j - 1] \geq opt[i - 1, j]$  then
9:        $opt[i, j] \leftarrow opt[i, j - 1], \pi[i, j] \leftarrow \leftarrow$ 
10:    else
11:       $opt[i, j] \leftarrow opt[i - 1, j], \pi[i, j] \leftarrow \uparrow$ 
```

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Find Common Subsequence

```
1:  $i \leftarrow n, j \leftarrow m, S \leftarrow ()$ 
2: while  $i > 0$  and  $j > 0$  do
3:   if  $\pi[i, j] = "\searrow"$  then
4:     add  $A[i]$  to beginning of  $S, i \leftarrow i - 1, j \leftarrow j - 1$ 
5:   else if  $\pi[i, j] = "\uparrow"$  then
6:      $i \leftarrow i - 1$ 
7:   else
8:      $j \leftarrow j - 1$ 
9: return  $S$ 
```


Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B ?

Example:

- $A = \text{ocurrance}$, $B = \text{occurrence}$
- 3 operations: insert 'c', remove 'a' and insert 'e'

Obs. $\#OPs = \text{length}(A) + \text{length}(B) - 2 \cdot \text{length}(LCS(A, B))$

Variants of Problem

Edit Distance with Insertions, Deletions and Replacing

Input: a string A ,

each time we can delete a letter from A , insert a letter to A or **change a letter**

Output: how many operations do we need to change A to B ?

Example:

- $A = \text{ocurrance}$, $B = \text{occurrence}$.
- 2 operations: insert 'c', change 'a' to 'e'

- Not related to LCS any more

Edit Distance (with Replacing)

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: edit distance between $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ then $opt[i, j] = j$; if $j = 0$ then $opt[i, j] = i$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} opt[i - 1, j - 1] & \text{if } A[i] = B[j] \\ \min \begin{cases} opt[i - 1, j] + 1 \\ opt[i, j - 1] + 1 \\ opt[i - 1, j - 1] + 1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Exercise: Longest Palindrome

Def. A **palindrome** is a string which reads the same backward or forward.

- example: “racecar”, “wasitacaroracatisaw”, “putitup”

Longest Palindrome Subsequence

Input: a sequence A

Output: the longest subsequence C of A that is a palindrome.

Example:

- Input: **ac**bc**ede**ac**ab**
- Output: **ac**ed**eca**

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Computing the Length of LCS

```
1: for  $j \leftarrow 0$  to  $m$  do
2:    $opt[0, j] \leftarrow 0$ 
3: for  $i \leftarrow 1$  to  $n$  do
4:    $opt[i, 0] \leftarrow 0$ 
5:   for  $j \leftarrow 1$  to  $m$  do
6:     if  $A[i] = B[j]$  then
7:        $opt[i, j] \leftarrow opt[i - 1, j - 1] + 1$ 
8:     else if  $opt[i, j - 1] \geq opt[i - 1, j]$  then
9:        $opt[i, j] \leftarrow opt[i, j - 1]$ 
10:    else
11:       $opt[i, j] \leftarrow opt[i - 1, j]$ 
```

Obs. The i -th row of table only depends on $(i - 1)$ -th row.

Reducing Space to $O(n + m)$

Obs. The i -th row of table only depends on $(i - 1)$ -th row.

Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i - 1)$ -th row and the i -th row.

Linear Space Algorithm to Compute Length of LCS

```
1: for  $j \leftarrow 0$  to  $m$  do
2:    $opt[0, j] \leftarrow 0$ 
3: for  $i \leftarrow 1$  to  $n$  do
4:    $opt[i \bmod 2, 0] \leftarrow 0$ 
5:   for  $j \leftarrow 1$  to  $m$  do
6:     if  $A[i] = B[j]$  then
7:        $opt[i \bmod 2, j] \leftarrow opt[i - 1 \bmod 2, j - 1] + 1$ 
8:     else if  $opt[i \bmod 2, j - 1] \geq opt[i - 1 \bmod 2, j]$  then
9:        $opt[i \bmod 2, j] \leftarrow opt[i \bmod 2, j - 1]$ 
10:    else
11:       $opt[i \bmod 2, j] \leftarrow opt[i - 1 \bmod 2, j]$ 
12: return  $opt[n \bmod 2, m]$ 
```


How to Recover LCS Using Linear Space?

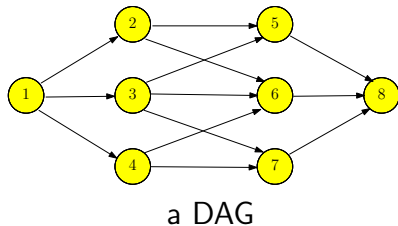
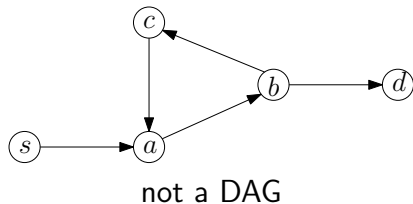
- Only keep the last two rows: only know how to match $A[n]$
- Can recover the LCS using n rounds: time = $O(n^2m)$
- Using **Divide and Conquer** + Dynamic Programming:
 - Space: $O(m + n)$
 - Time: $O(nm)$

Outline

- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem
- 3 Knapsack Problem
- 4 Longest Common Subsequence
 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Directed Acyclic Graphs**
- 6 Matrix Chain Multiplication
- 7 Optimum Binary Search Tree
- 8 Summary

Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



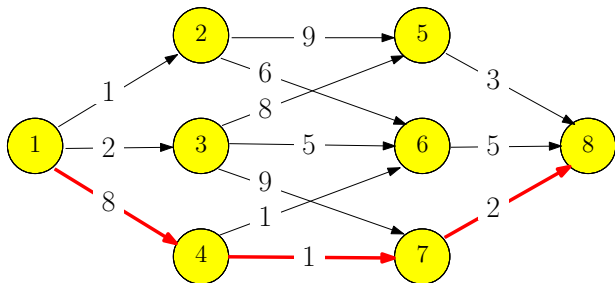
Lemma A directed graph is a DAG if and only if its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \rightarrow \mathbb{R}$.

Assume $V = \{1, 2, 3, \dots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i , for every $i \in V$



Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i) \in E} \{f(j) + w(j,i)\} & i = 2, 3, \dots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

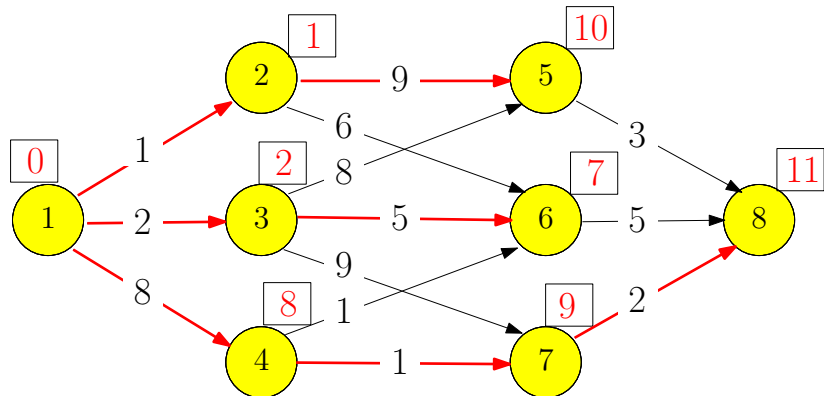
Shortest Paths in DAG

```
1:  $f[1] \leftarrow 0$ 
2: for  $i \leftarrow 2$  to  $n$  do
3:    $f[i] \leftarrow \infty$ 
4:   for each incoming edge  $(j, i)$  of  $i$  do
5:     if  $f[j] + w(j, i) < f[i]$  then
6:        $f[i] \leftarrow f[j] + w(j, i)$ 
7:        $\pi(i) \leftarrow j$ 
```

print-path(t)

```
1: if  $t = 1$  then
2:   print(1)
3:   return
4: print-path( $\pi(t)$ )
5: print(", ",  $t$ )
```

Example



Variant: Heaviest Path in a Directed Acyclic Graph

Heaviest Path in a Directed Acyclic Graph

Input: directed acyclic graph $G = (V, E)$ and $w : E \rightarrow \mathbb{R}$.

Assume $V = \{1, 2, 3, \dots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the path with the **largest** weight (the **heaviest** path) from 1 to n .

- $f[i]$: weight of the **heaviest** path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \max_{j:(j,i) \in E} \{f(j) + w(j, i)\} & i = 2, 3, \dots, n \end{cases}$$

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Matrix Chain Multiplication

Matrix Chain Multiplication

Input: n matrices A_1, A_2, \dots, A_n of sizes

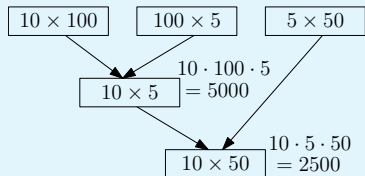
$r_1 \times c_1, r_2 \times c_2, \dots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \dots, n - 1$.

Output: the order of computing $A_1 A_2 \dots A_n$ with the minimum number of multiplications

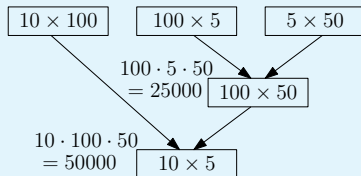
Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100$, $A_2 : 100 \times 5$, $A_3 : 5 \times 50$



$$\text{cost} = 5000 + 2500 = 7500$$



$$\text{cost} = 25000 + 50000 = 75000$$

- $(A_1 A_2) A_3 : 10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1 (A_2 A_3) : 100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Matrix Chain Multiplication: Design DP

- Assume the last step is $(A_1A_2 \cdots A_i)(A_{i+1}A_{i+2} \cdots A_n)$
- Cost of last step: $r_1 \times c_i \times c_n$
- Optimality for sub-instances: we need to compute $A_1A_2 \cdots A_i$ and $A_{i+1}A_{i+2} \cdots A_n$ optimally
- $opt[i, j]$: the minimum cost of computing $A_iA_{i+1} \cdots A_j$

$$opt[i, j] = \begin{cases} 0 & i = j \\ \min_{k:i \leq k < j} (opt[i, k] + opt[k + 1, j] + r_i c_k c_j) & i < j \end{cases}$$

Matrix Chain Multiplication: Design DP

matrix-chain-multiplication($n, r[1..n], c[1..n]$)

```
1: let  $opt[i, i] \leftarrow 0$  for every  $i = 1, 2, \dots, n$ 
2: for  $\ell \leftarrow 2$  to  $n$  do
3:   for  $i \leftarrow 1$  to  $n - \ell + 1$  do
4:      $j \leftarrow i + \ell - 1$ 
5:      $opt[i, j] \leftarrow \infty$ 
6:     for  $k \leftarrow i$  to  $j - 1$  do
7:       if  $opt[i, k] + opt[k + 1, j] + r_i c_k c_j < opt[i, j]$  then
8:          $opt[i, j] \leftarrow opt[i, k] + opt[k + 1, j] + r_i c_k c_j$ 
9:          $\pi[i, j] \leftarrow k$ 
10: return  $opt[1, n]$ 
```

Constructing Optimal Solution

Print-Optimal-Order(i, j)

```
1: if  $i = j$  then  
2:   print("A" $i$ )  
3: else  
4:   print("(")  
5:   Print-Optimal-Order( $i, \pi[i, j]$ )  
6:   Print-Optimal-Order( $\pi[i, j] + 1, j$ )  
7:   print(")
```

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$\text{opt}[1, 2] = \text{opt}[1, 1] + \text{opt}[2, 2] + 3 \times 5 \times 2 = 30, \quad \pi[1, 2] = 1$$

$$\text{opt}[2, 3] = \text{opt}[2, 2] + \text{opt}[3, 3] + 5 \times 2 \times 6 = 60, \quad \pi[2, 3] = 2$$

$$\text{opt}[3, 4] = \text{opt}[3, 3] + \text{opt}[4, 4] + 2 \times 6 \times 9 = 108, \quad \pi[3, 4] = 3$$

$$\text{opt}[4, 5] = \text{opt}[4, 4] + \text{opt}[5, 5] + 6 \times 9 \times 4 = 216, \quad \pi[4, 5] = 4$$

$$\begin{aligned} \text{opt}[1, 3] &= \min\{\text{opt}[1, 1] + \text{opt}[2, 3] + 3 \times 5 \times 6, \\ &\quad \text{opt}[1, 2] + \text{opt}[3, 3] + 3 \times 2 \times 6\} \\ &= \min\{0 + 60 + 90, 30 + 0 + 36\} = 66, \quad \pi[1, 3] = 2 \end{aligned}$$

$$\begin{aligned} \text{opt}[2, 4] &= \min\{\text{opt}[2, 2] + \text{opt}[3, 4] + 5 \times 2 \times 9, \\ &\quad \text{opt}[2, 3] + \text{opt}[4, 4] + 5 \times 6 \times 9\} \\ &= \min\{0 + 108 + 90, 60 + 0 + 270\} = 198, \quad \pi[2, 4] = 2, \end{aligned}$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$\begin{aligned}
 \text{opt}[3, 5] &= \min\{\text{opt}[3, 3] + \text{opt}[4, 5] + 2 \times 6 \times 4, \\
 &\quad \text{opt}[3, 4] + \text{opt}[5, 5] + 2 \times 9 \times 4\} \\
 &= \min\{0 + 216 + 48, 108 + 0 + 72\} = 180,
 \end{aligned}$$

$$\pi[3, 5] = 4,$$

$$\begin{aligned}
 \text{opt}[1, 4] &= \min\{\text{opt}[1, 1] + \text{opt}[2, 4] + 3 \times 5 \times 9, \\
 &\quad \text{opt}[1, 2] + \text{opt}[3, 4] + 3 \times 2 \times 9, \\
 &\quad \text{opt}[1, 3] + \text{opt}[4, 4] + 3 \times 6 \times 9\} \\
 &= \min\{0 + 198 + 135, 30 + 108 + 54, 66 + 0 + 162\} = 192,
 \end{aligned}$$

$$\pi[1, 4] = 2,$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$\begin{aligned}
 \text{opt}[2, 5] &= \min\{\text{opt}[2, 2] + \text{opt}[3, 5] + 5 \times 2 \times 4, \\
 &\quad \text{opt}[2, 3] + \text{opt}[4, 5] + 5 \times 6 \times 4, \\
 &\quad \text{opt}[2, 4] + \text{opt}[5, 5] + 5 \times 9 \times 4\} \\
 &= \min\{0 + 180 + 40, 60 + 216 + 120, 198 + 0 + 180\} = 220,
 \end{aligned}$$

$$\begin{aligned}
 \text{opt}[1, 5] &= \min\{\text{opt}[1, 1] + \text{opt}[2, 5] + 3 \times 5 \times 4, \\
 &\quad \text{opt}[1, 2] + \text{opt}[3, 5] + 3 \times 2 \times 4, \\
 &\quad \text{opt}[1, 3] + \text{opt}[4, 5] + 3 \times 6 \times 4, \\
 &\quad \text{opt}[1, 4] + \text{opt}[5, 5] + 3 \times 9 \times 4\} \\
 &= \min\{0 + 220 + 60, 30 + 180 + 24, \\
 &\quad 66 + 216 + 72, 192 + 0 + 108\} \\
 &= 234,
 \end{aligned}$$

$$\pi[1, 5] = 2.$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, /	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, /	60, 2	198, 2	220, 2
$i = 3$			0, /	108, 3	180, 4
$i = 4$				0, /	216, 4
$i = 5$					0, /

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, /	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, /	60, 2	198, 2	220, 2
$i = 3$			0, /	108, 3	180, 4
$i = 4$				0, /	216, 4
$i = 5$					0, /

Print-Optimal-Order(1,5)

 Print-Optimal-Order(1, 2)

 Print-Optimal-Order(1, 1)

 Print-Optimal-Order(2, 2)

 Print-Optimal-Order(3, 5)

 Print-Optimal-Order(3, 4)

 Print-Optimal-Order(3, 3)

 Print-Optimal-Order(4, 4)

 Print-Optimal-Order(5, 5)

Optimum way for multiplication: $((A_1A_2)((A_3A_4)A_5))$

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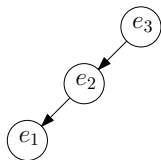
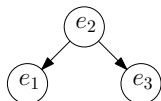
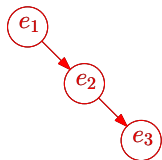
Optimum Binary Search Tree

- n elements $e_1 < e_2 < e_3 < \dots < e_n$
- e_i has frequency f_i
- goal: build a binary search tree for $\{e_1, e_2, \dots, e_n\}$ with the minimum accessing cost:

$$\sum_{i=1}^n f_i \times (\text{depth of } e_i \text{ in the tree})$$

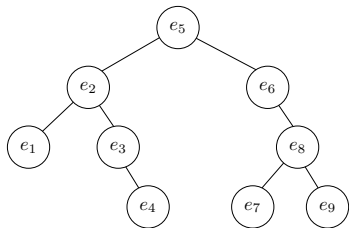
Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$

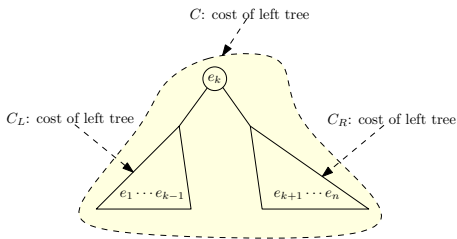


- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

- suppose we decided to let e_k be the root
- e_1, e_2, \dots, e_{k-1} are on left sub-tree
- $e_{k+1}, e_{k+2}, \dots, e_n$ are on right sub-tree
- d_j : depth of e_j in our tree
- C, C_L, C_R : cost of tree, left sub-tree and right sub-tree



- $d_1 = 3, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 1,$
- $d_6 = 2, d_7 = 4, d_8 = 3, d_9 = 4,$
- $C = 3f_1 + 2f_2 + 3f_3 + 4f_4 + f_5 + 2f_6 + 4f_7 + 3f_8 + 4f_9$
- $C_L = 2f_1 + f_2 + 2f_3 + 3f_4$
- $C_R = f_6 + 3f_7 + 2f_8 + 3f_9$
- $C = C_L + C_R + \sum_{j=1}^9 f_j Z$



$$\begin{aligned}
 C &= \sum_{\ell=1}^n f_{\ell} d_{\ell} = \sum_{\ell=1}^n f_{\ell} (d_{\ell} - 1) + \sum_{\ell=1}^n f_{\ell} \\
 &= \sum_{\ell=1}^{k-1} f_{\ell} (d_{\ell} - 1) + \sum_{\ell=k+1}^n f_{\ell} (d_{\ell} - 1) + \sum_{\ell=1}^n f_{\ell} \\
 &= C_L + C_R + \sum_{\ell=1}^n f_{\ell}
 \end{aligned}$$

$$C = C_L + C_R + \sum_{\ell=1}^n f_{\ell}$$

- In order to minimize C , need to minimize C_L and C_R respectively
- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \dots, f_j)$

$$opt[1, n] = \min_{k: 1 \leq k \leq n} (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^n f_{\ell}$$

- In general, $opt[i, j] =$

$$\begin{cases} 0 & \text{if } i = j + 1 \\ \min_{k: i \leq k \leq j} (opt[i, k-1] + opt[k+1, j]) + \sum_{\ell=i}^j f_{\ell} & \text{if } i \leq j \end{cases}$$

Optimum Binary Search Tree

```
1:  $fsum[0] \leftarrow 0$ 
2: for  $i \leftarrow 1$  to  $n$  do  $fsum[i] \leftarrow fsum[i - 1] + f_i$ 
    $\triangleright fsum[i] = \sum_{j=1}^i f_j$ 
3: for  $i \leftarrow 0$  to  $n$  do  $opt[i + 1, i] \leftarrow 0$ 
4: for  $\ell \leftarrow 1$  to  $n$  do
5:   for  $i \leftarrow 1$  to  $n - \ell + 1$  do
6:      $j \leftarrow i + \ell - 1, opt[i, j] \leftarrow \infty$ 
7:     for  $k \leftarrow i$  to  $j$  do
8:       if  $opt[i, k - 1] + opt[k + 1, j] < opt[i, j]$  then
9:          $opt[i, j] \leftarrow opt[i, k - 1] + opt[k + 1, j]$ 
10:         $\pi[i, j] \leftarrow k$ 
11:     $opt[i, j] \leftarrow opt[i, j] + fsum[j] - fsum[i - 1]$ 
```

Printing the Tree

Print-Tree(i, j)

```
1: if  $i > j$  then  
2:   return  
3: else  
4:   print('(')  
5:   Print-Tree( $i, \pi[i, j] - 1$ )  
6:   print( $\pi[i, j]$ )  
7:   Print-Tree( $\pi[i, j] + 1, j$ )  
8:   print('')
```

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Dynamic Programming

- Break up a problem into many **overlapping** sub-problems
- Build solutions for larger and larger sub-problems
- Use a **table** to store solutions for sub-problems for reuse

Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many **independent** sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \dots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \dots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j]$ = value of instance defined by $A[1..i]$ and $B[1..j]$
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Matrix chain multiplication, optimum binary search tree:
 $opt[i, j]$ = value of instances defined by matrices i to j