

CSE 431/531: Algorithm Analysis and Design (Fall 2021)

Graph Algorithms

Lecturer: Shi Li

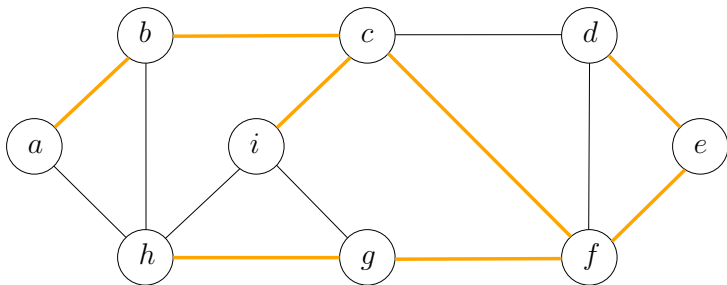
*Department of Computer Science and Engineering
University at Buffalo*

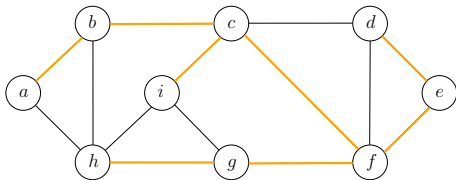
Outline

- 1 Minimum Spanning Tree
 - Kruskal's Algorithm
 - Reverse-Kruskal's Algorithm
 - Prim's Algorithm
- 2 Single Source Shortest Paths
 - Dijkstra's Algorithm
- 3 Shortest Paths in Graphs with Negative Weights
- 4 All-Pair Shortest Paths and Floyd-Warshall

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a **spanning tree** $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V .





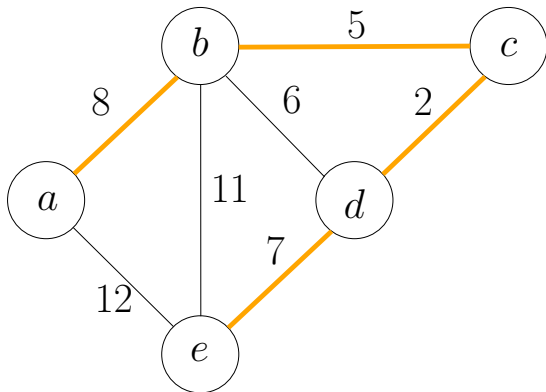
Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G ;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \rightarrow \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight



Recall: Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

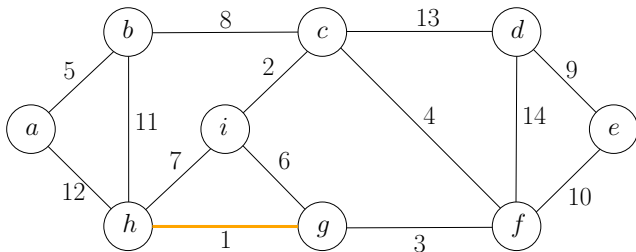
Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

Two Classic Greedy Algorithms for MST

- Kruskal’s Algorithm
- Prim’s Algorithm

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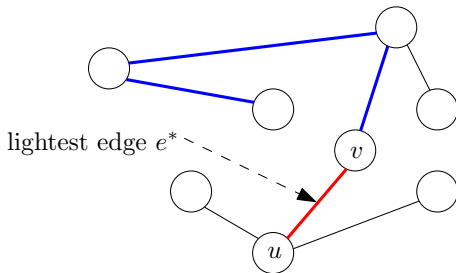
Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

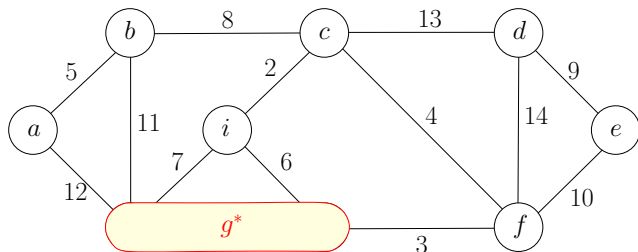
Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v
- Remove any edge e in the path to obtain tree T'
- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: T' is also a MST □

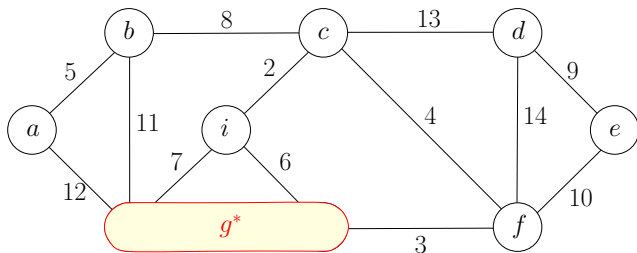


Is the Residual Problem Still a MST Problem?



- Residual problem: find the minimum spanning tree that contains edge (g, h)
- **Contract** the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- **May create parallel edges!** E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

- 1 Choose the lightest edge e^* , add e^* to the spanning tree
- 2 Contract e^* and update G be the contracted graph

Q: What edges are removed due to contractions?

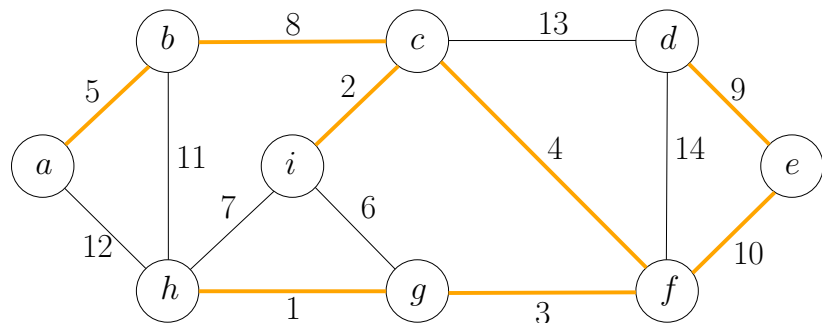
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

- 1: $F \leftarrow \emptyset$
- 2: sort edges in E in non-decreasing order of weights w
- 3: **for** each edge (u, v) in the order **do**
- 4: **if** u and v are not connected by a path of edges in F **then**
- 5: $F \leftarrow F \cup \{(u, v)\}$
- 6: **return** (V, F)

Kruskal's Algorithm: Example



Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

- 1: $F \leftarrow \emptyset$
- 2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
- 3: sort the edges of E in non-decreasing order of weights w
- 4: **for** each edge $(u, v) \in E$ in the order **do**
- 5: $S_u \leftarrow$ the set in \mathcal{S} containing u
- 6: $S_v \leftarrow$ the set in \mathcal{S} containing v
- 7: **if** $S_u \neq S_v$ **then**
- 8: $F \leftarrow F \cup \{(u, v)\}$
- 9: $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
- 10: **return** (V, F)

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

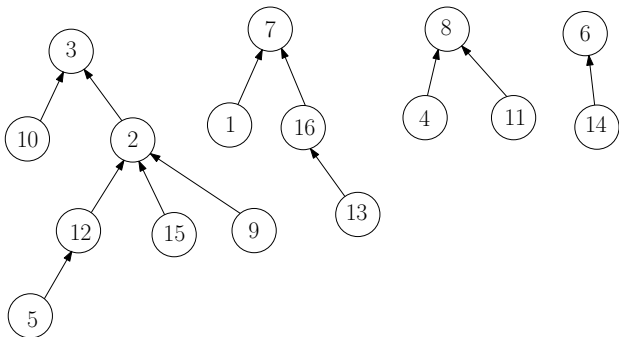
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- 8: $F \leftarrow F \cup \{(u, v)\}$
- 9: $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
- 10: **return** (V, F)

Use **union-find** data structure to support ②, ⑤, ⑥, ⑦, ⑨.

Union-Find Data Structure

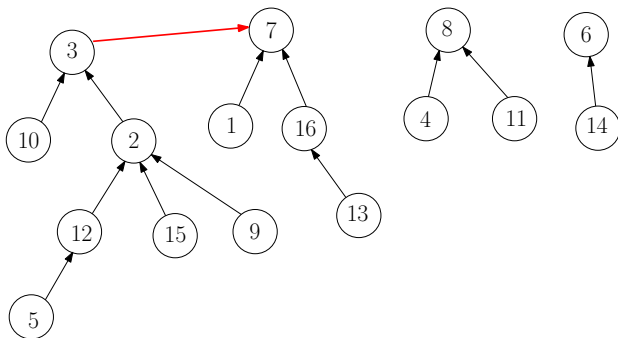
- V : ground set
- We need to maintain a partition of V and support following operations:
 - Check if u and v are in the same set of the partition
 - Merge two sets in partition

- $V = \{1, 2, 3, \dots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}$, $\{1, 7, 13, 16\}$, $\{4, 8, 11\}$, $\{6, 14\}$



- $par[i]$: parent of i , ($par[i] = \perp$ if i is a root).

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if $\text{root}(u) = \text{root}(v)$.
- $\text{root}(u)$: the root of the tree containing u
- Merge the trees with root r and r' : $\text{par}[r] \leftarrow r'$.

Union-Find Data Structure

root(v)

```
1: if  $par[v] = \perp$  then  
2:   return  $v$   
3: else  
4:   return  $root(par[v])$ 
```

root(v)

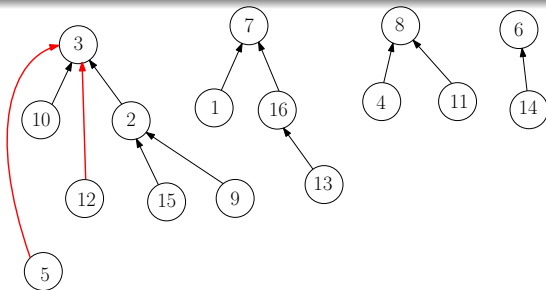
```
1: if  $par[v] = \perp$  then  
2:   return  $v$   
3: else  
4:    $par[v] \leftarrow root(par[v])$   
5: return  $par[v]$ 
```

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.

Union-Find Data Structure

$\text{root}(v)$

- 1: **if** $\text{par}[v] = \perp$ **then**
- 2: **return** v
- 3: **else**
- 4: $\text{par}[v] \leftarrow \text{root}(\text{par}[v])$
- 5: **return** $\text{par}[v]$



MST-Kruskal(G, w)

- 1: $F \leftarrow \emptyset$
- 2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
- 3: sort the edges of E in non-decreasing order of weights w
- 4: **for** each edge $(u, v) \in E$ in the order **do**
- 5: $S_u \leftarrow$ the set in \mathcal{S} containing u
- 6: $S_v \leftarrow$ the set in \mathcal{S} containing v
- 7: **if** $S_u \neq S_v$ **then**
- 8: $F \leftarrow F \cup \{(u, v)\}$
- 9: $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
- 10: **return** (V, F)

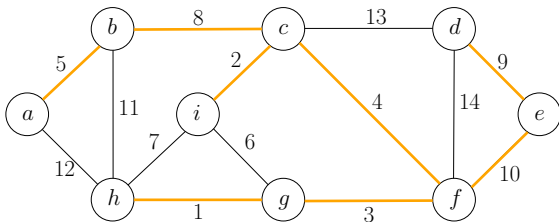
MST-Kruskal(G, w)

```
1:  $F \leftarrow \emptyset$ 
2: for every  $v \in V$  do:  $par[v] \leftarrow \perp$ 
3: sort the edges of  $E$  in non-decreasing order of weights  $w$ 
4: for each edge  $(u, v) \in E$  in the order do
5:    $u' \leftarrow root(u)$ 
6:    $v' \leftarrow root(v)$ 
7:   if  $u' \neq v'$  then
8:      $F \leftarrow F \cup \{(u, v)\}$ 
9:      $par[u'] \leftarrow v'$ 
10: return  $(V, F)$ 
```

- ②, ⑤, ⑥, ⑦, ⑨ takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time = time for ③ = $O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is **not** in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



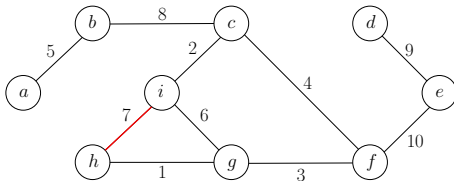
- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

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Two Methods to Build a MST

- 1 Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
- 2 Start from $F \leftarrow E$, and **remove** edges from F one by one until we obtain a spanning tree



Q: Which edge can be safely **excluded** from the MST?

A: The heaviest non-**bridge** edge.

Def. A **bridge** is an edge whose removal disconnects the graph.

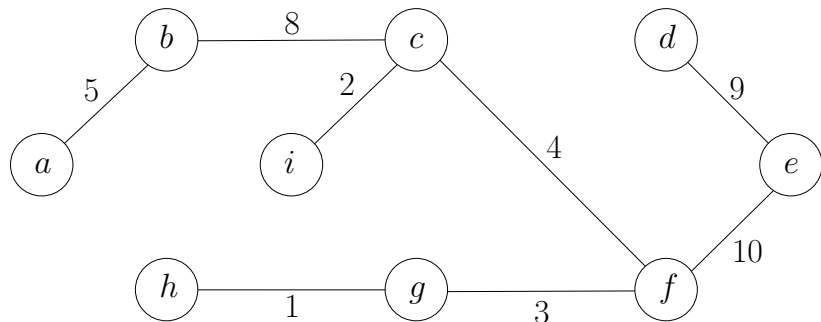
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

- 1: $F \leftarrow E$
- 2: sort E in non-increasing order of weights
- 3: **for** every e in this order **do**
- 4: **if** $(V, F \setminus \{e\})$ is connected **then**
- 5: $F \leftarrow F \setminus \{e\}$
- 6: **return** (V, F)

Reverse Kruskal's Algorithm: Example

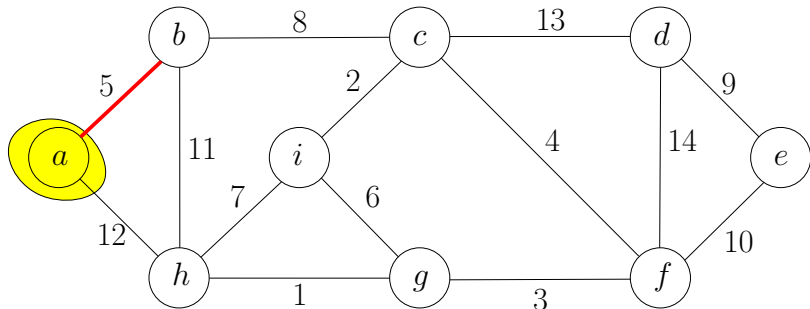


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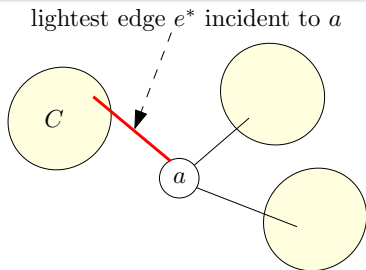
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



- Greedy strategy for Prim's algorithm: choose the lightest edge incident to *a*.

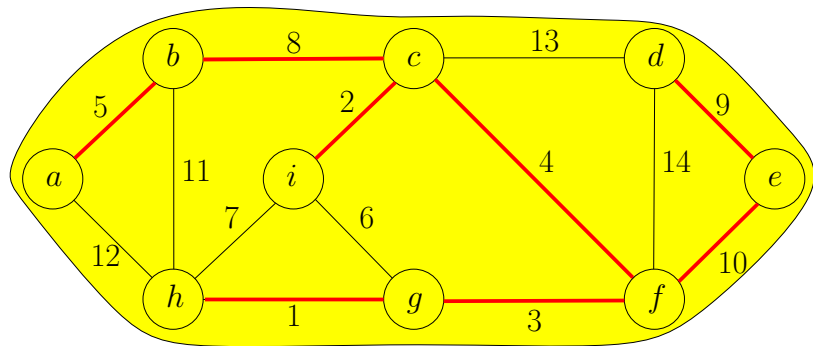
Lemma It is safe to include the lightest edge incident to a .



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ □

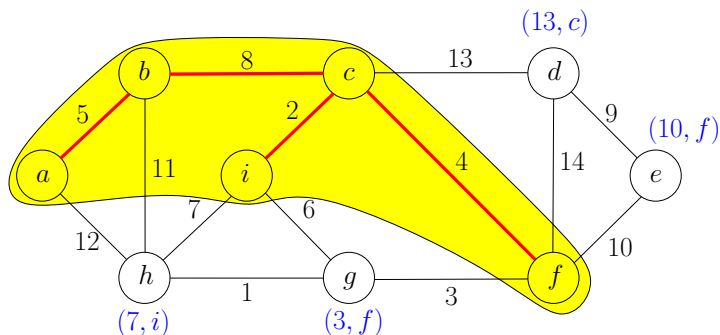
Prim's Algorithm: Example



Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S: (u,v) \in E} w(u, v)$:
the weight of the lightest edge between v and S
- $\pi(v) = \arg \min_{u \in S: (u,v) \in E} w(u, v)$:
 $(\pi(v), v)$ is the lightest edge between v and S



Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S: (u,v) \in E} w(u, v)$:
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 $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

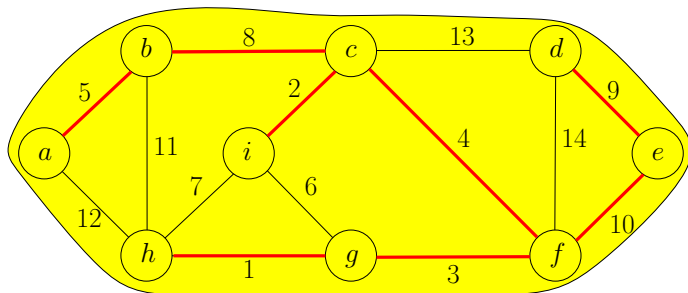
- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to F
- Add u to S , update d and π values.

Prim's Algorithm

MST-Prim(G, w)

```
1:  $s \leftarrow$  arbitrary vertex in  $G$ 
2:  $S \leftarrow \emptyset, d(s) \leftarrow 0$  and  $d(v) \leftarrow \infty$  for every  $v \in V \setminus \{s\}$ 
3: while  $S \neq V$  do
4:    $u \leftarrow$  vertex in  $V \setminus S$  with the minimum  $d(u)$ 
5:    $S \leftarrow S \cup \{u\}$ 
6:   for each  $v \in V \setminus S$  such that  $(u, v) \in E$  do
7:     if  $w(u, v) < d(v)$  then
8:        $d(v) \leftarrow w(u, v)$ 
9:        $\pi(v) \leftarrow u$ 
10: return  $\{(u, \pi(u)) \mid u \in V \setminus \{s\}\}$ 
```

Example



Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S: (u,v) \in E} w(u, v)$:
the weight of the lightest edge between v and S
- $\pi(v) = \arg \min_{u \in S: (u,v) \in E} w(u, v)$:
 $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value extract_min
- Add $(\pi(u), u)$ to F
- Add u to S , update d and π values. decrease_key

Use a **priority queue** to support the operations

Def. A **priority queue** is an **abstract** data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- $\text{insert}(v, \text{key_value})$: insert an element v , whose associated key value is key_value .
- $\text{decrease_key}(v, \text{new_key_value})$: decrease the key value of an element v in queue to new_key_value
- $\text{extract_min}()$: return and remove the element in queue with the smallest key value
- ...

Prim's Algorithm

MST-Prim(G, w)

- 1: $s \leftarrow$ arbitrary vertex in G
- 2: $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
- 3:
- 4: **while** $S \neq V$ **do**
- 5: $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
- 6: $S \leftarrow S \cup \{u\}$
- 7: **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
- 8: **if** $w(u, v) < d(v)$ **then**
- 9: $d(v) \leftarrow w(u, v)$
- 10: $\pi(v) \leftarrow u$
- 11: **return** $\{(u, \pi(u)) \mid u \in V \setminus \{s\}\}$

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

- 1: $s \leftarrow$ arbitrary vertex in G
- 2: $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
- 3: $Q \leftarrow$ empty queue, for each $v \in V: Q.insert(v, d(v))$
- 4: **while** $S \neq V$ **do**
- 5: $u \leftarrow Q.extract_min()$
- 6: $S \leftarrow S \cup \{u\}$
- 7: **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
- 8: **if** $w(u, v) < d(v)$ **then**
- 9: $d(v) \leftarrow w(u, v), Q.decrease_key(v, d(v))$
- 10: $\pi(v) \leftarrow u$
- 11: **return** $\{(u, \pi(u)) \mid u \in V \setminus \{s\}\}$

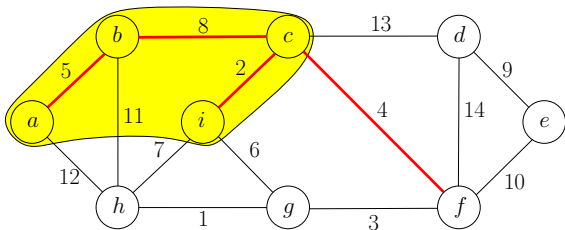
Running Time of Prim's Algorithm Using Priority Queue

$O(n) \times (\text{time for extract_min}) + O(m) \times (\text{time for decrease_key})$

concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a **cut** $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.



- (c, f) is in MST because of cut $(\{a, b, c, i\}, V \setminus \{a, b, c, i\})$
- (i, g) is not in MST because no such cut exists

“Evidence” for $e \in \text{MST}$ or $e \notin \text{MST}$

Assumption Assume all edge weights are different.

- $e \in \text{MST} \leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin \text{MST} \leftrightarrow$ there is a cycle in which e is the heaviest edge

Exactly one of the following is true:

- There is a cut in which e is the lightest edge
- There is a cycle in which e is the heaviest edge

Thus, the minimum spanning tree is unique with assumption.

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algorithm	graph	weights	SS?	running time
Simple DP	DAG	\mathbb{R}	SS	$O(n + m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	\mathbb{R}	SS	$O(nm)$
Floyd-Warshall	U/D	\mathbb{R}	AP	$O(n^3)$

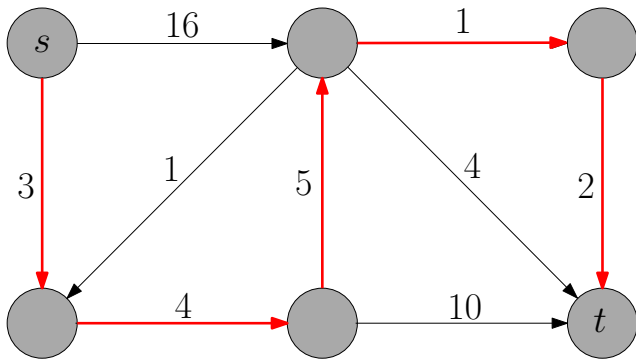
- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

s - t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

Output: shortest path from s to t



Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s - t shortest path problem more efficiently than solving single source shortest path problem
- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

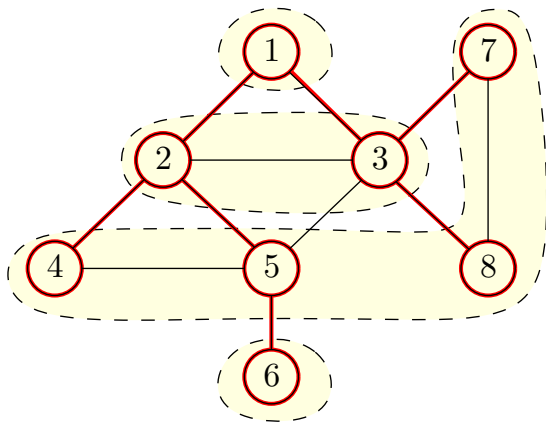
$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

Output: $\pi(v), v \in V \setminus s$: the parent of v in shortest path tree

$d(v), v \in V \setminus s$: the length of shortest path from s to v

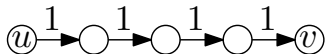
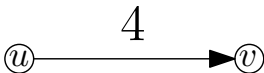
Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s



Assumption Weights $w(u, v)$ are integers (w.l.o.g.).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges



Shortest Path Algorithm by Running BFS

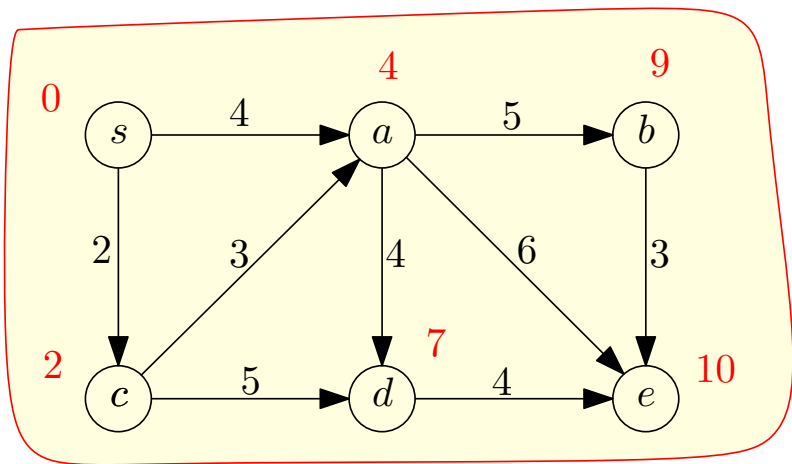
- 1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
- 2: run BFS **virtually**
- 3: $\pi(v) \leftarrow$ vertex from which v is visited
- 4: $d(v) \leftarrow$ index of the level containing v

- Problem: $w(u, v)$ may be too large!

Shortest Path Algorithm by Running BFS Virtually

- 1: $S \leftarrow \{s\}, d(s) \leftarrow 0$
- 2: **while** $|S| \leq n$ **do**
- 3: find a $v \notin S$ that minimizes $\min_{u \in S: (u,v) \in E} \{d(u) + w(u, v)\}$
- 4: $S \leftarrow S \cup \{v\}$
- 5: $d(v) \leftarrow \min_{u \in S: (u,v) \in E} \{d(u) + w(u, v)\}$

Virtual BFS: Example



Time 10

Outline

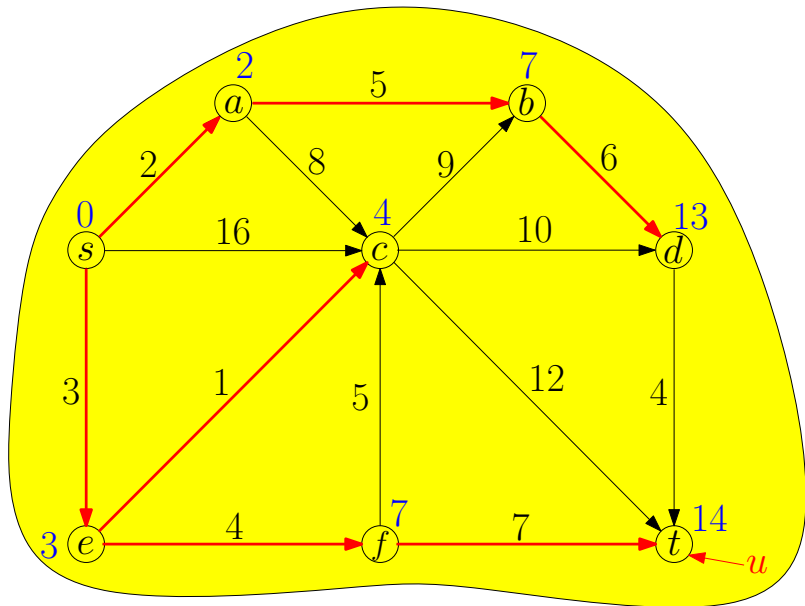
- 1 Minimum Spanning Tree
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Dijkstra's Algorithm

Dijkstra(G, w, s)

- 1: $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
- 2: **while** $S \neq V$ **do**
- 3: $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
- 4: add u to S
- 5: **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
- 6: **if** $d(u) + w(u, v) < d(v)$ **then**
- 7: $d(v) \leftarrow d(u) + w(u, v)$
- 8: $\pi(v) \leftarrow u$
- 9: **return** (d, π)

- Running time = $O(n^2)$



Improved Running Time using Priority Queue

Dijkstra(G, w, s)

- 1: $s \leftarrow$ arbitrary vertex in G
- 2: $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
- 3: $Q \leftarrow$ empty queue, for each $v \in V: Q.insert(v, d(v))$
- 4: **while** $S \neq V$ **do**
- 5: $u \leftarrow Q.extract_min()$
- 6: $S \leftarrow S \cup \{u\}$
- 7: **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
- 8: **if** $d(u) + w(u, v) < d(v)$ **then**
- 9: $d(v) \leftarrow d(u) + w(u, v), Q.decrease_key(v, d(v))$
- 10: $\pi(v) \leftarrow u$
- 11: **return** (π, d)

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

- 1: $s \leftarrow$ arbitrary vertex in G
- 2: $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
- 3: $Q \leftarrow$ empty queue, for each $v \in V: Q.\text{insert}(v, d(v))$
- 4: **while** $S \neq V$ **do**
- 5: $u \leftarrow Q.\text{extract_min}()$
- 6: $S \leftarrow S \cup \{u\}$
- 7: **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
- 8: **if** $w(u, v) < d(v)$ **then**
- 9: $d(v) \leftarrow w(u, v), Q.\text{decrease_key}(v, d(v))$
- 10: $\pi(v) \leftarrow u$
- 11: **return** $\{(u, \pi(u)) \mid u \in V \setminus \{s\}\}$

Improved Running Time

Running time:

$O(n) \times (\text{time for extract_min}) + O(m) \times (\text{time for decrease_key})$

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

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Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

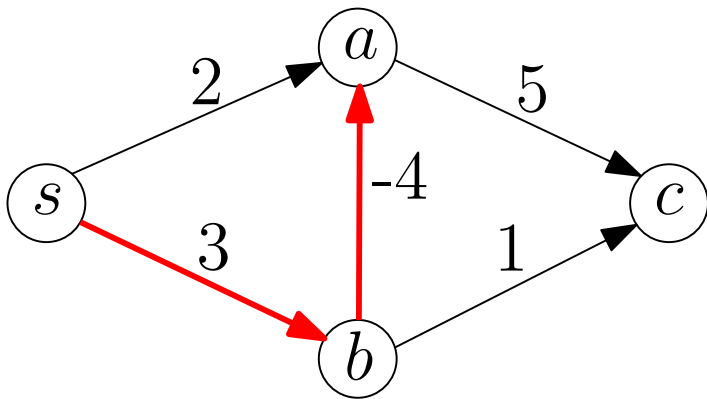
assume all vertices are reachable from s

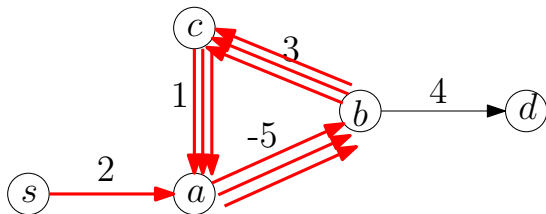
$w : E \rightarrow \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' \rightarrow 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

Dijkstra's Algorithm Fails if We Have Negative Weights





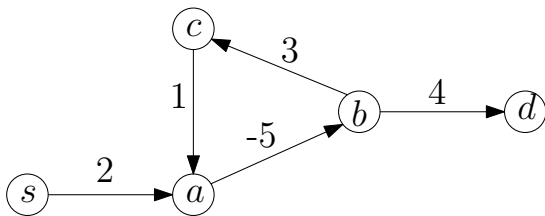
Q: What is the length of the shortest path from s to d ?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or
- allow algorithm to report “negative cycle exists”



Q: What is the length of the shortest **simple** path from s to d ?

A: 1

- Unfortunately, computing the shortest simple path between two vertices is an **NP-hard** problem.

algorithm	graph	weights	SS?	running time
Simple DP	DAG	\mathbb{R}	SS	$O(n + m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	\mathbb{R}	SS	$O(nm)$
Floyd-Warshall	U/D	\mathbb{R}	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

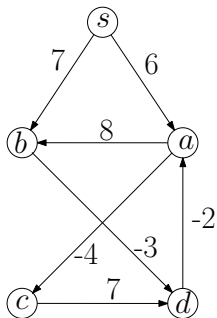
Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \rightarrow \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

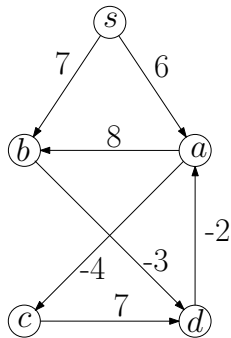
- first try: $f[v]$: length of shortest path from s to v
- issue: do not know in which order we compute $f[v]$'s
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3, \dots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges



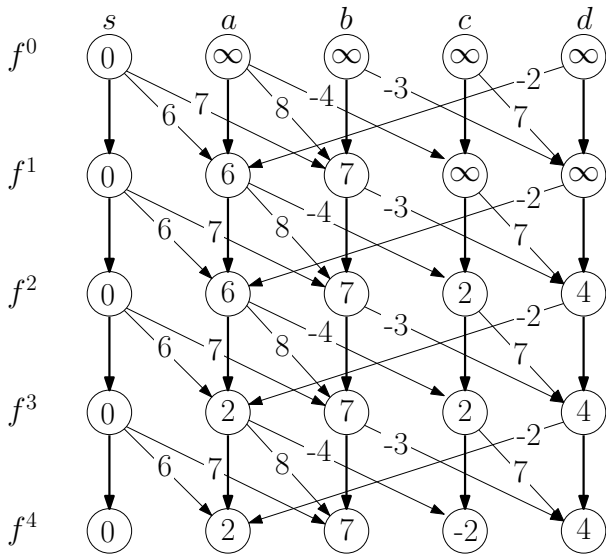
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \dots, n-1\}$, $v \in V$:
length of shortest path from s to v that uses at most ℓ edges
- $f^2[a] = 6$
- $f^3[a] = 2$

$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \left\{ \begin{array}{l} f^{\ell-1}[v] \\ \min_{u:(u,v) \in E} (f^{\ell-1}[u] + w(u, v)) \end{array} \right. & \ell > 0 \end{cases}$$

Dynamic Programming: Example



↓ length-0 edge



dynamic-programming(G, w, s)

- 1: $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
- 2: **for** $\ell \leftarrow 1$ to $n - 1$ **do**
- 3: copy $f^{\ell-1} \rightarrow f^\ell$
- 4: **for** each $(u, v) \in E$ **do**
- 5: **if** $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ **then**
- 6: $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
- 7: **return** $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges

Proof.

If there is a path containing at least n edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. \square

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

```
1:  $f[s] \leftarrow 0$  and  $f[v] \leftarrow \infty$  for any  $v \in V \setminus \{s\}$ 
2: for  $\ell \leftarrow 1$  to  $n - 1$  do
3:   for each  $(u, v) \in E$  do
4:     if  $f[u] + w(u, v) < f[v]$  then
5:        $f[v] \leftarrow f[u] + w(u, v)$ 
6: return  $f$ 
```

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
- After iteration ℓ , $f[v]$ is **at most** the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of **some path** from s to v

Bellman-Ford Algorithm

- After iteration ℓ :

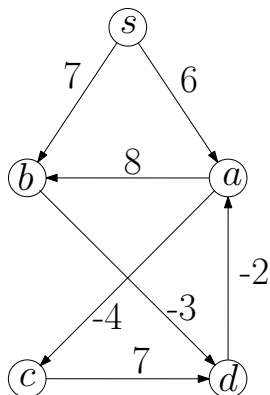
$$\begin{aligned} & \text{length of shortest } s\text{-}v \text{ path} \\ & \leq f[v] \\ & \leq \text{length of shortest } s\text{-}v \text{ path using at most } \ell \text{ edges} \end{aligned}$$

- Assuming there are no negative cycles:

$$\begin{aligned} & \text{length of shortest } s\text{-}v \text{ path} \\ & = \text{length of shortest } s\text{-}v \text{ path using at most } n - 1 \text{ edges} \end{aligned}$$

- So, assuming there are no negative cycles, after iteration $n - 1$:

$$f[v] = \text{length of shortest } s\text{-}v \text{ path}$$



- order in which we consider edges:
 (s, a) , (s, b) , (a, b) , (a, c) , (b, d) ,
 (c, d) , (d, a)

vertices	s	a	b	c	d
f	0	∞ 62	∞ 7	∞ 2-2	∞ 4

- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4
- Algorithm terminates in 3 iterations, instead of 4.

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

- 1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
- 2: **for** $\ell \leftarrow 1$ to n **do**
- 3: $updated \leftarrow \text{false}$
- 4: **for each** $(u, v) \in E$ **do**
- 5: **if** $f[u] + w(u, v) < f[v]$ **then**
- 6: $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
- 7: $updated \leftarrow \text{true}$
- 8: **if not** $updated$, then return f
- 9: output “negative cycle exists”

- $\pi[v]$: the parent of v in the shortest path tree
- Running time = $O(nm)$

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All-Pair Shortest Paths

All Pair Shortest Paths

Input: directed graph $G = (V, E)$,
 $w : E \rightarrow \mathbb{R}$ (can be negative)

Output: shortest path from u to v for **every** $u, v \in V$

- 1: **for** every starting point $s \in V$ **do**
- 2: run Bellman-Ford(G, w, s)

- Running time = $O(n^2m)$

Summary of Shortest Path Algorithms we learned

algorithm	graph	weights	SS?	running time
Simple DP	DAG	\mathbb{R}	SS	$O(n + m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	\mathbb{R}	SS	$O(nm)$
Floyd-Warshall	U/D	\mathbb{R}	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \dots, n\}$
- For simplicity, extend the w values to non-edges:

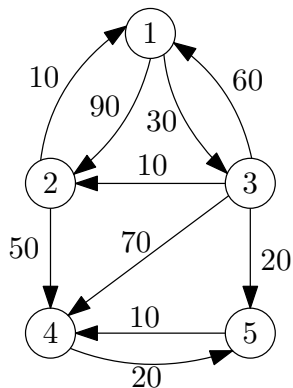
$$w(i, j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\ \infty & i \neq j, (i, j) \notin E \end{cases}$$

- For now assume there are no negative cycles

Cells for Floyd-Warshall Algorithm

- First try: $f[i, j]$ is length of shortest path from i to j
- Issue: do not know in which order we compute $f[i, j]$'s
- $f^k[i, j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \dots, k\}$ as intermediate vertices

Example for Definition of $f^k[i, j]$'s



$$f^0[1, 4] = \infty$$

$$f^1[1, 4] = \infty$$

$$f^2[1, 4] = 140$$

$$f^3[1, 4] = 90$$

$$f^4[1, 4] = 90$$

$$f^5[1, 4] = 60$$

$$(1 \rightarrow 2 \rightarrow 4)$$

$$(1 \rightarrow 3 \rightarrow 2 \rightarrow 4)$$

$$(1 \rightarrow 3 \rightarrow 2 \rightarrow 4)$$

$$(1 \rightarrow 3 \rightarrow 5 \rightarrow 4)$$

$$w(i, j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\ \infty & i \neq j, (i, j) \notin E \end{cases}$$

- $f^k[i, j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \dots, k\}$ as intermediate vertices

$$f^k[i, j] = \begin{cases} w(i, j) & k = 0 \\ \min \begin{cases} f^{k-1}[i, j] \\ f^{k-1}[i, k] + f^{k-1}[k, j] \end{cases} & k = 1, 2, \dots, n \end{cases}$$

Floyd-Warshall(G, w)

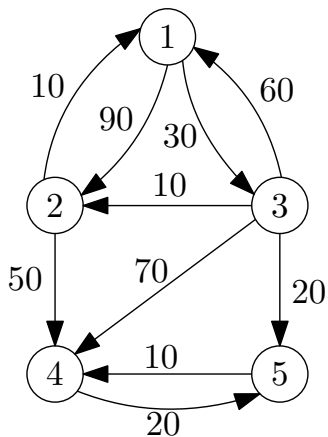
```
1:  $f^0 \leftarrow w$ 
2: for  $k \leftarrow 1$  to  $n$  do
3:   copy  $f^{k-1} \rightarrow f^k$ 
4:   for  $i \leftarrow 1$  to  $n$  do
5:     for  $j \leftarrow 1$  to  $n$  do
6:       if  $f^{k-1}[i, k] + f^{k-1}[k, j] < f^k[i, j]$  then
7:          $f^k[i, j] \leftarrow f^{k-1}[i, k] + f^{k-1}[k, j]$ 
```

Floyd-Warshall(G, w)

```
1:  $f^{\text{old}} \leftarrow w$ 
2: for  $k \leftarrow 1$  to  $n$  do
3:   copy  $f^{\text{old}} \rightarrow f^{\text{new}}$ 
4:   for  $i \leftarrow 1$  to  $n$  do
5:     for  $j \leftarrow 1$  to  $n$  do
6:       if  $f^{\text{old}}[i, k] + f^{\text{old}}[k, j] < f^{\text{new}}[i, j]$  then
7:          $f^{\text{new}}[i, j] \leftarrow f^{\text{old}}[i, k] + f^{\text{old}}[k, j]$ 
```

Lemma Assume there are no negative cycles in G . After iteration k , for $i, j \in V$, $f[i, j]$ is **exactly** the length of shortest path from i to j that only uses vertices in $\{1, 2, 3, \dots, k\}$ as intermediate vertices.

- Running time = $O(n^3)$.



	1	2	3	4	5
1	0	9040	30	∞ 140	∞
2	10	0	∞ 40	50	∞
3	6020	10	0	7060	20
4	∞	∞	∞	0	20
5	∞	∞	∞	10	0

- $i = 1, i = 2, i = 3, k = 1, k = 2,$
 $k = 3, j = 1, j = 2, j = 3, j = 4$

Recovering Shortest Paths

Floyd-Warshall(G, w)

```
1:  $f \leftarrow w, \pi[i, j] \leftarrow \perp$  for every  $i, j \in V$ 
2: for  $k \leftarrow 1$  to  $n$  do
3:   for  $i \leftarrow 1$  to  $n$  do
4:     for  $j \leftarrow 1$  to  $n$  do
5:       if  $f[i, k] + f[k, j] < f[i, j]$  then
6:          $f[i, j] \leftarrow f[i, k] + f[k, j], \pi[i, j] \leftarrow k$ 
```

print-path(i, j)

```
1: if  $\pi[i, j] = \perp$  then then
2:   if  $i \neq j$  then print( $i, "$ ")
3: else
4:   print-path( $i, \pi[i, j]$ ), print-path( $\pi[i, j], j$ )
```

Detecting Negative Cycles

Floyd-Warshall(G, w)

```
1:  $f \leftarrow w, \pi[i, j] \leftarrow \perp$  for every  $i, j \in V$ 
2: for  $k \leftarrow 1$  to  $n$  do
3:   for  $i \leftarrow 1$  to  $n$  do
4:     for  $j \leftarrow 1$  to  $n$  do
5:       if  $f[i, k] + f[k, j] < f[i, j]$  then
6:          $f[i, j] \leftarrow f[i, k] + f[k, j], \pi[i, j] \leftarrow k$ 
7: for  $k \leftarrow 1$  to  $n$  do
8:   for  $i \leftarrow 1$  to  $n$  do
9:     for  $j \leftarrow 1$  to  $n$  do
10:      if  $f[i, k] + f[k, j] < f[i, j]$  then
11:        report "negative cycle exists" and exit
```

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algorithm	graph	weights	SS?	running time
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- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs