CSE 431/531: Algorithm Analysis and Design (Fall 2021)
Graph Basics

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo

Outline

1. Graphs
2. Connectivity and Graph Traversal
   - Testing Bipartiteness
3. Topological Ordering
4. Properties of BFS and DFS trees

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

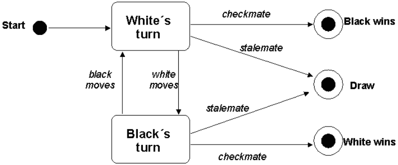
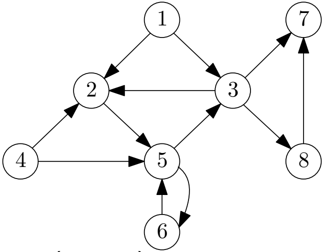


Figure: Transition Graphs

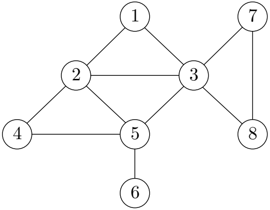
(Undirected) Graph $G = (V, E)$



- V: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
  - $E = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,5\}, \{3,7\}, \{3,8\}, \{4,5\}, \{5,6\}, \{7,8\}\}$

Abuse of Notations

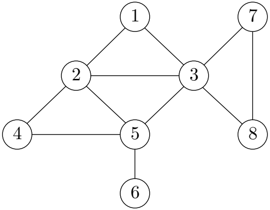
- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.

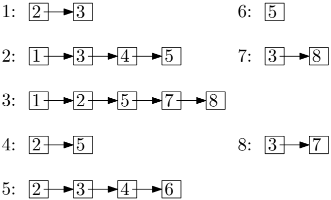


- $E = \{(1,2), (1,3), (2,3), (2,4), (2,5), (3,5), (3,7), (3,8), (4,5), (5,6), (7,8)\}$

- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

Representation of Graphs





- Adjacency matrix
  - $n \times n$ matrix, $A[u,v] = 1$ if $(u,v) \in E$ and $A[u,v] = 0$ otherwise
  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbours of v.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n - 1 \le m \le n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u, v) \in E$	$O(1)$	$O(d_u)$
time to list all neighbours of v	$O(n)$	$O(d_v)$

Outline

Connectivity Problem

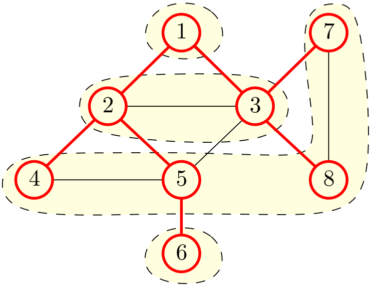
Input: graph $G = (V, E)$, (using linked lists)
two vertices $s, t \in V$

Output: whether there is a path connecting s to t in G

- Algorithm: starting from s, search for all vertices that are reachable from s and check if the set contains t
  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j



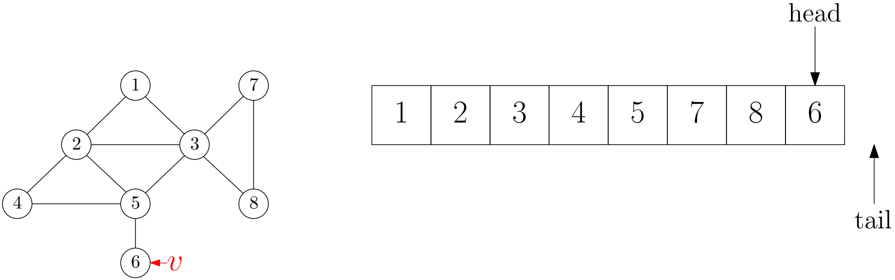
Implementing BFS using a Queue

BFS(s)

```
head ← 1, tail ← 1, queue[1] ← s
mark s as "visited" and all other vertices as "unvisited"
while head ≥ tail do
    v ← queue[tail], tail ← tail + 1
    for all neighbours u of v do
        if u is "unvisited" then
            head ← head + 1, queue[head] = u
            mark u as "visited"
```

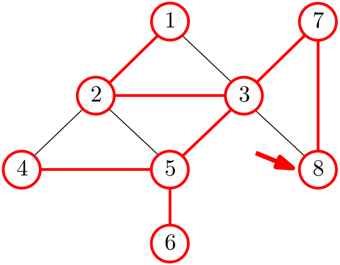
- Running time: $O(n+m)$.

Example of BFS via Queue



Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
- When reach an already-visited vertex ("dead-end"), go back
- Travel through the next edge
- If tried all edges leading out of the current vertex, go back



Implementing DFS using Recurrsion

DFS(s)

```
mark all vertices as "unvisited"
recursive-DFS(s)
```

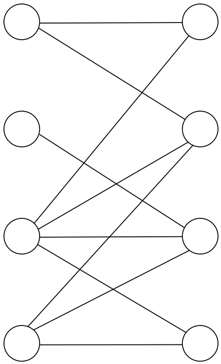
recursive-DFS(v)

```
mark v as "visited"
for all neighbours u of v do
    if u is unvisited then recursive-DFS(u)
```

Outline

Testing Bipartiteness: Applications of BFS

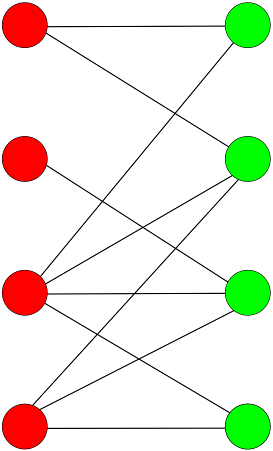
Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, we have either $u \in L, v \in R$ or $v \in L, u \in R$.

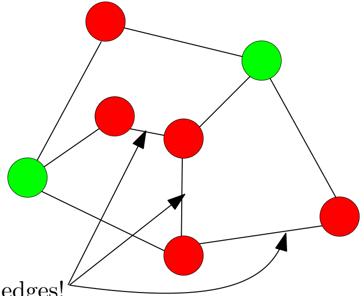


Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L
- $\cdots$
- Report "not a bipartite graph" if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

Test Bipartiteness





Testing Bipartiteness using BFS

BFS(s)

$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$

mark s as "visited" and all other vertices as "unvisited"

$color[s] \leftarrow 0$

while head $\geq$ tail **do**

  $v \leftarrow queue[tail], tail \leftarrow tail + 1$

  for all neighbours u of v **do**

    if u is "unvisited" **then**

      $head \leftarrow head + 1, queue[head] = u$

      mark u as "visited"

      $color[u] \leftarrow 1 - color[v]$

    else if $color[u] = color[v]$ **then**

      print("G is not bipartite") and exit

Testing Bipartiteness using BFS

mark all vertices as "unvisited"
for each vertex $v \in V$ **do**
    if v is "unvisited" **then**
        test-bipartiteness(v)
print("G is bipartite")

Obs. Running time of algorithm $= O(n+m)$

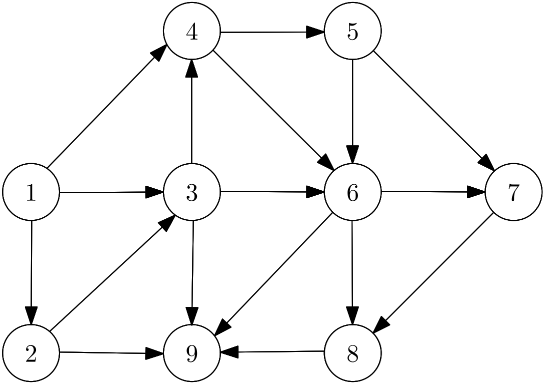
Outline

Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

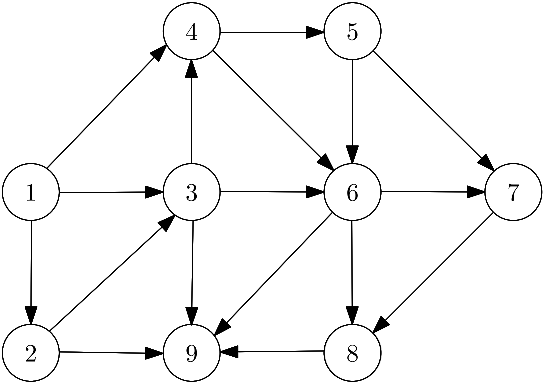
Output: 1-to-1 function $\pi : V \to \{1, 2, 3 \cdots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.

Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

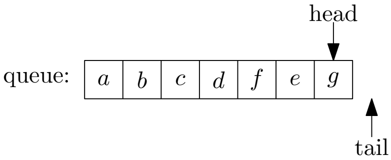
let $d_v \leftarrow 0$ for every $v \in V$
for every $v \in V$ **do**
    for every u such that $(v, u) \in E$ **do**
        $d_u \leftarrow d_u + 1$
$S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
while $S \neq \emptyset$ **do**
    $v \leftarrow$ arbitrary vertex in S, $S \leftarrow S \setminus \{v\}$
    $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
    for every u such that $(v, u) \in E$ **do**
        $d_u \leftarrow d_u - 1$
        if $d_u = 0$ **then** add u to S
if $i < n$ then output "not a DAG"

- S can be represented using a queue or a stack
- Running time $= O(n + m)$

S as a Queue or a Stack

DS	Queue	Stack
Initialization	$head \leftarrow 0$, $tail \leftarrow 1$	$top \leftarrow 0$
Non-Empty?	$head \geq tail$	$top > 0$
Add(v)	$head \leftarrow head + 1$ $S[head] \leftarrow v$	$top \leftarrow top + 1$ $S[top] \leftarrow v$
Retrieve v	$v \leftarrow S[tail]$ $tail \leftarrow tail + 1$	$v \leftarrow S[top]$ $top \leftarrow top - 1$

Example



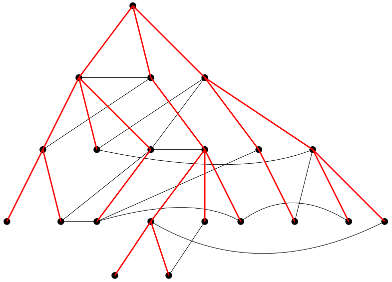
	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Outline

Properties of a BFS Tree

Given a BFS tree T of a connected graph G

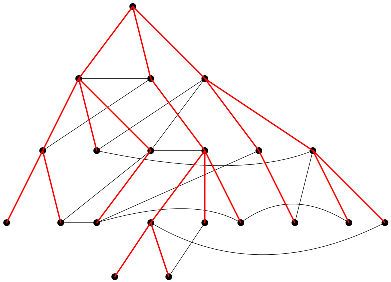
- Can there be a vertical edge (u, v), $u \geq 2$ levels above v?
- No. v should be a child of u
- Can there be a horizontal edge (u, v) $u \geq 2$ levels above v?
- No. v should be a child of u.
- Can there be a horizontal edge (u, v), where u is 1 level above v, but v's parent is to the right of u?
- No. v should be a child of u.



Properties of a BFS Tree

Given a BFS tree T of a connected graph G, other than the tree edges, we only have horizontal edges (u, v), where

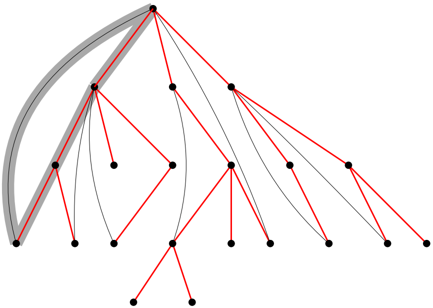
- either u and v are at the same level
- or u is 1 level above v, and v's parent is to the left of u, (or vice versa)



Properties of a DFS Tree

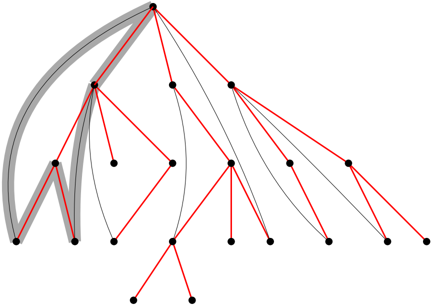
Given a tree DFS tree T of a graph (connected) G,

- Can there be a horizontal edge (u, v)?
- No.
- All non-tree edges are vertical edges.
- A vertical edge (u, v) and its the edges in the path from u to v in T form a cycle; we call it a canonical cycle.



Properties of a DFS Tree

Lemma If G contains a cycle, then it has a canonical cycle.



Proof.

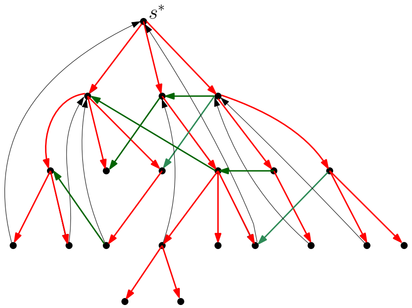
- If G contains a cycle, then it must have at least non-tree edge.
- W.r.t DFS tree T, we can only have vertical + tree edges
- $\exists$ at least one vertical edge
- There is a canonical cycle □

- There might or might not be non-canonical ones.

Properties of a DFS Tree Over a Directed Graph

Given a tree DFS tree T of a directed graph G, assuming all vertices can be reached from the starting vertex s^*

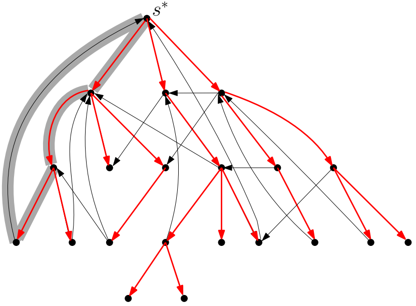
- Can there be a horizontal (directed) edge (u, v) where u is visited before v?
- No.
- However, there can be horizontal edges (u, v) where u is visited after v.



Properties of a DFS Tree Over a Directed Graph

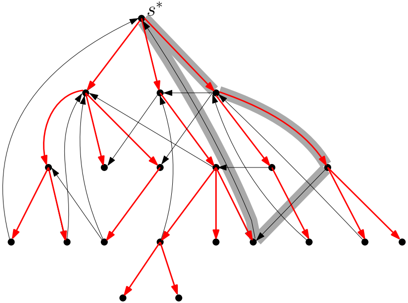
Given a tree DFS tree T of a directed graph G, assuming all vertices can be reached from the starting vertex s^*

- Other than tree edges, there are two types of edges:
  - vertical edges directed to ancestors
  - horizontal edges (u, v) where u is visited after v.
- An vertical edge (u, v) and the tree edges in the tree path from v to u form a cycle, and we call it a canonical cycle.



Properties of a DFS Tree Over a Directed Graph

Lemma If there is a cycle in the directed graph G, then there must be a canonical one.



Proof.

- Focus on tree edges and horizontal edges
- post-order-traversal of T gives a reversed topological ordering
- Without vertical edges, G has no cycles $\square$

- Again, there might be non-canonical cycles.

Cycle Detection Using DFS in Directed Graphs

Algorithm 1 Check-Cycle-Directed

add a source s^* to G and edges from s^* to all other vertices.
$visited \leftarrow$ boolean array over V, with $visited[v] = false, \forall v$
$instack \leftarrow$ boolean array over V, with $instack[v] = false, \forall v$
DFS(s^*)
return "no cycle"

Algorithm 2 DFS(v)

$visited[v] \leftarrow true, instack[v] \leftarrow true$
for every outgoing edge (v, u) of v **do**
    if $inqueue[u]$ **then** ▷ Find a vertical edge
        exit the whole algorithm, by returning "there is a cycle"
    else if $visited[u] = false$ **then**
        DFS(u)
$instack[v] \leftarrow false$