CSE 431/531: Algorithm Analysis and Design (Fall 2021)

NP-Completeness

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NP-Completeness Theory

- The topics we discussed so far are positive results: how to design efficient algorithms for solving a given problem.
- NP-Completeness provides negative results: some problems can not be solved efficiently.

Q: Why do we study negative results?

- A given problem X cannot be solved in polynomial time.
- Without knowing it, you will have to keep trying to find polynomial time algorithm for solving X. All our efforts are doomed!

Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5}\log n), O(n^{100})$
- Not polynomial time: $O(2^n), O(n^{\log n})$
- Almost all algorithms we learnt so far run in polynomial time

Reason for Efficient = Polynomial Time

- For natural problems, if there is an $O(n^k)$-time algorithm, then k is small, say 4
- A good cut separating problems: for most natural problems, either we have a polynomial time algorithm, or the best algorithm runs in time $\Omega(2^{n^c})$ for some c
- Do not need to worry about the computational model

Outline

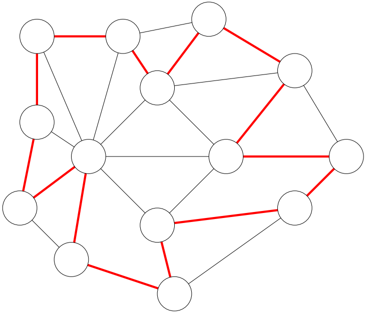
Example: Hamiltonian Cycle Problem

Def. Let G be an undirected graph. A Hamiltonian Cycle (HC) of G is a cycle C in G that passes each vertex of G exactly once.

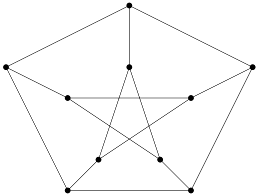
Hamiltonian Cycle (HC) Problem

Input: graph $G = (V, E)$

Output: whether G contains a Hamiltonian cycle



Example: Hamiltonian Cycle Problem



- The graph is called the Petersen Graph. It has no HC.

Example: Hamiltonian Cycle Problem

Hamiltonian Cycle (HC) Problem

Input: graph $G = (V, E)$

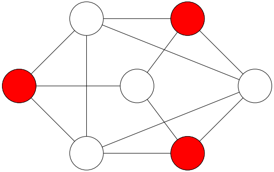
Output: whether G contains a Hamiltonian cycle

Algorithm for Hamiltonian Cycle Problem:

- Enumerate all possible permutations, and check if it corresponds to a Hamiltonian Cycle
- Running time: $O(n!m) = 2^{O(n \lg n)}$
- Better algorithm: $2^{O(n)}$
- Far away from polynomial time
- HC is NP-hard: it is unlikely that it can be solved in polynomial time.

Maximum Independent Set Problem

Def. An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in I are adjacent in G.



Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the size of the maximum independent set of G

- Maximum Independent Set is NP-hard

Formula Satisfiability

Formula Satisfiability

Input: boolean formula with n variables, with $\vee, \wedge, \neg$ operators.

Output: whether the boolean formula is satisfiable

- Example: $\neg((\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3))$ is not satisfiable
- Trivial algorithm: enumerate all possible assignments, and check if each assignment satisfies the formula
- Formula Satisfiablity is NP-hard

Outline

Decision Problem Vs Optimization Problem

Def. A problem X is called a decision problem if the output is either 0 or 1 (yes/no).

- When we define the P and NP, we only consider decision problems.

Fact For each optimization problem X, there is a decision version X' of the problem. If we have a polynomial time algorithm for the decision version X', we can solve the original problem X in polynomial time.

Optimization to Decision

Shortest Path

Input: graph $G = (V, E)$, weight w, s, t and a bound L

Output: whether there is a path from s to t of length at most L

Maximum Independent Set

Input: a graph G and a bound k

Output: whether there is an independent set of size at least k

Encoding

The input of a problem will be encoded as a binary string.

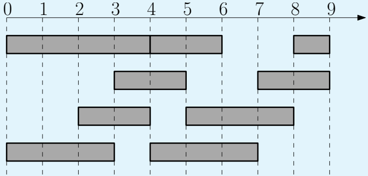
Example: Sorting problem

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)
- String: 111101111100011111000011000001 110000110111111111000001

Encoding

The input of an problem will be encoded as a binary string.

Example: Interval Scheduling Problem



- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$
- Encode the sequence into a binary string as before

Encoding

Def. The size of an input is the length of the encoded string s for the input, denoted as $|s|$.

Q: Does it matter how we encode the input instances?

A: No! As long as we are using a "natural" encoding. We only care whether the running time is polynomial or not

Define Problem as a Set

Def. A decision problem X is the set of strings on which the output is yes. i.e, $s \in X$ if and only if the correct output for the input s is 1 (yes).

Def. An algorithm A solves a problem X if, $A(s) = 1$ if and only if $s \in X$.

Def. A has a polynomial running time if there is a polynomial function $p(\cdot)$ so that for every string s, the algorithm A terminates on s in at most $p(|s|)$ steps.

Complexity Class P

Def. The complexity class P is the set of decision problems X that can be solved in polynomial time.

- The decision versions of interval scheduling, shortest path and minimum spanning tree all in P.

Certifier for Hamiltonian Cycle (HC)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for HC
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

Q: Given a graph $G = (V, E)$ with a HC, how can Alice convince Bob that G contains a Hamiltonian cycle?

A: Alice gives a Hamiltonian cycle to Bob, and Bob checks if it is really a Hamiltonian cycle of G

Def. The message Alice sends to Bob is called a certificate, and the algorithm Bob runs is called a certifier.

Certifier for Independent Set (Ind-Set)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for Ind-Set
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

Q: Given graph $G = (V, E)$ and integer k, such that there is an independent set of size k in G, how can Alice convince Bob that there is such a set?

A: Alice gives a set of size k to Bob and Bob checks if it is really a independent set in G.

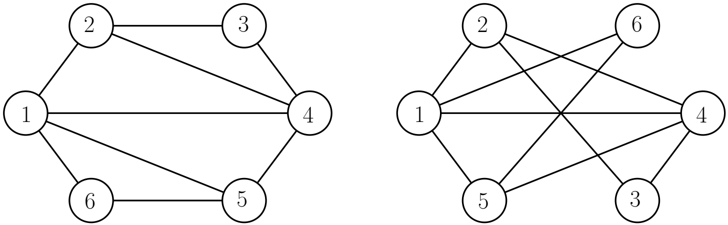
- Certificate: a set of size k
- Certifier: check if the given set is really an independent set

Graph Isomorphism

Graph Isomorphism

Input: two graphs G_1 and G_2,

Output: whether two graphs are isomorphic to each other



- What is the certificate?
- What is the certifier?

The Complexity Class NP

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t
- there is a polynomial function p such that, $s \in X$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

The string t such that $B(s,t) = 1$ is called a certificate.

Def. The complexity class NP is the set of all problems for which there exists an efficient certifier.

Hamiltonian Cycle $\in$ NP

- Input: Graph G
- Certificate: a sequence S of edges in G
- $|\text{encoding}(S)| \le p(|\text{encoding}(G)|)$ for some polynomial function p
- Certifier B: $B(G, S) = 1$ if and only if S is an HC in G
- Clearly, B runs in polynomial time
- $G \in \text{HC} \qquad \Longleftrightarrow \qquad \exists S,\ B(G, S) = 1$

Graph Isomorphism $\in$ NP

- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on V
- Certificate: a 1-1 function $f : V \to V$
- $|\text{encoding}(f)| \leq p(|\text{encoding}(G_1, G_2)|)$ for some polynomial function p
- Certifier B: $B((G_1, G_2), f) = 1$ if and only if for every $u, v \in V$, we have $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.
- Clearly, B runs in polynomial time

- $(G_1, G_2) \in \text{GI} \quad \Longleftrightarrow \quad \exists f,\ B((G_1, G_2), f) = 1$

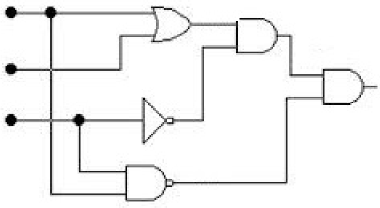
Maximum Independent Set $\in$ NP

- Input: graph $G = (V, E)$ and integer k
- Certificate: a set $S \subseteq V$ of size k
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G, k)|)$ for some polynomial function p
- Certifier B: $B((G, k), S) = 1$ if and only if S is an independent set in G
- Clearly, B runs in polynomial time
- $(G, k) \in$ MIS $\iff$ $\exists S$, $B((G, k), S) = 1$

Circuit Satisfiablity (Circuit-Sat) Problem

Input: a circuit with and/or/not gates

Output: whether there is an assignment such that the output is 1?



- Is Circuit-Sat $\in$ NP?

$\overline{\text{HC}}$

Input: graph $G = (V, E)$

Output: whether G does not contain a Hamiltonian cycle

- Is $\overline{\text{HC}} \in$ NP?
- Can Alice convince Bob that G is a yes-instance (i.e, G does not contain a HC), if this is true.
- Unlikely
- Alice can only convince Bob that G is a no-instance
- $\overline{\text{HC}} \in$ Co-NP

The Complexity Class Co-NP

Def. For a problem X, the problem $\overline{X}$ is the problem such that $s \in \overline{X}$ if and only if $s \notin X$.

Def. Co-NP is the set of decision problems X such that $\overline{X} \in$ NP.

Def. A tautology is a boolean formula that always evaluates to 1.

Tautology Problem

Input: a boolean formula

Output: whether the formula is a tautology

- e.g. $(\neg x_1 \land x_2) \lor (\neg x_1 \land \neg x_3) \lor x_1 \lor (\neg x_2 \land x_3)$ is a tautology
- Bob can certify that a formula is not a tautology
- Thus Tautology $\in$ Co-NP
- Indeed, Tautology $= \overline{\text{Formula-Unsat}}$

Prime

Prime

Input: an integer $q \geq 2$

Output: whether q is a prime

- It is easy to certify that q is not a prime
- Prime $\in$ Co-NP
- [Pratt 1970] Prime $\in$ NP
- P $\subseteq$ NP $\cap$ Co-NP (see soon)
- If a natural problem X is in NP $\cap$ Co-NP, then it is likely that $X \in P$
- [AKS 2002] Prime $\in$ P

P $\subseteq$ NP

- Let $X \in$ P and $s \in X$

Q: How can Alice convince Bob that s is a yes instance?

A: Since $X \in$ P, Bob can check whether $s \in X$ by himself, without Alice's help.

- The certificate is an empty string
- Thus, $X \in$ NP and P $\subseteq$ NP
- Similarly, P $\subseteq$ Co-NP, thus P $\subseteq$ NP $\cap$ Co-NP

Is P = NP?

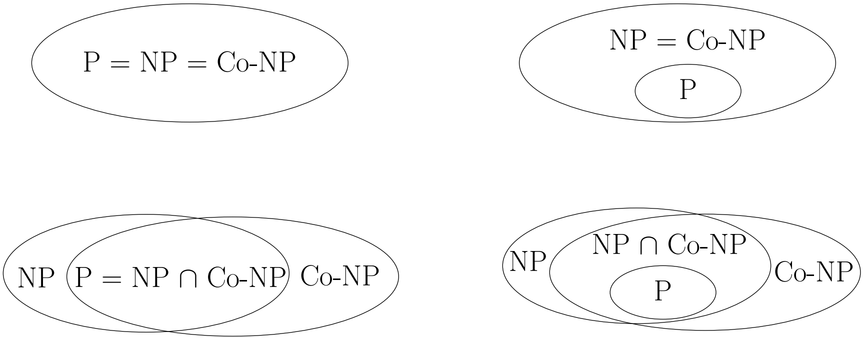
- A famous, big, and fundamental open problem in computer science
- Little progress has been made
- Most researchers believe P $\neq$ NP
- It would be too amazing if P = NP: if one can check a solution efficiently, then one can find a solution efficiently
- Complexity assumption: P $\neq$ NP
- We said it is unlikely that Hamiltonian Cycle can be solved in polynomial time:
  - if P $\neq$ NP, then HC $\notin$ P
  - HC $\notin$ P, unless P = NP

Is NP $=$ Co-NP?

- Again, a big open problem
- Most researchers believe NP $\neq$ Co-NP.

4 Possibilities of Relationships

Notice that $X \in \text{NP} \iff \overline{X} \in \text{Co-NP}$ and $\text{P} \subseteq \text{NP} \cap \text{Co-NP}$



- People commonly believe: we are in the 4th scenario

Outline

Polynomial-Time Reducations

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

To prove positive results:

Suppose $Y \leq_P X$. If X can be solved in polynomial time, then Y can be solved in polynomial time.

To prove negative results:

Suppose $Y \leq_P X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

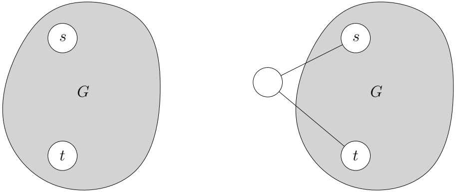
Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

Output: whether there is a Hamiltonian path from s to t in G

Lemma HP $\leq_P$ HC.



Obs. G has a HP from s to t if and only if graph on right side has a HC.

NP-Completeness

Def. A problem X is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_P X$ for every $Y \in \text{NP}$.

Theorem If X is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-complete problems are the hardest problems in NP
- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)
- To prove $\text{P} = \text{NP}$ (if you believe it), you only need to give an efficient algorithm for any NP-complete problem
- If you believe $\text{P} \neq \text{NP}$, and proved that a problem X is NP-complete (or NP-hard), stop trying to design efficient algorithms for X

Outline

1. Some Hard Problems
2. P, NP and Co-NP
3. Polynomial Time Reductions and NP-Completeness
4. NP-Complete Problems
5. Summary

Def. A problem X is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_\text{P} X$ for every $Y \in \text{NP}$.

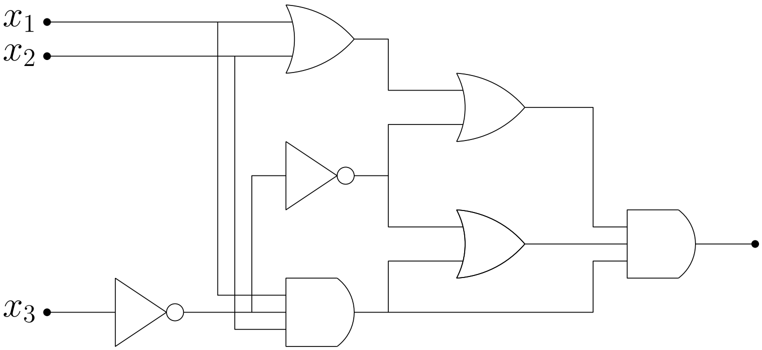
- How can we find a problem $X \in \text{NP}$ such that every problem $Y \in \text{NP}$ is polynomial time reducible to X? Are we asking for too much?
- No! There is indeed a large family of natural NP-complete problems

The First NP-Complete Problem: Circuit-Sat

Circuit Satisfiability (Circuit-Sat)

Input: a circuit

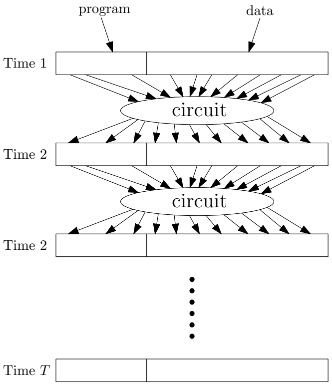
Output: whether the circuit is satisfiable



Circuit-Sat is NP-Complete

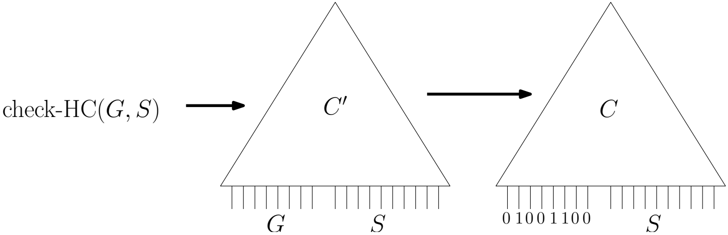
- key fact: algorithms can be converted to circuits

Fact Any algorithm that takes n bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.



- Then, we can show that any problem $Y \in$ NP can be reduced to Circuit-Sat.
- We prove HC $\leq_P$ Circuit-Sat as an example.

HC $\leq_P$ Circuit-Sat



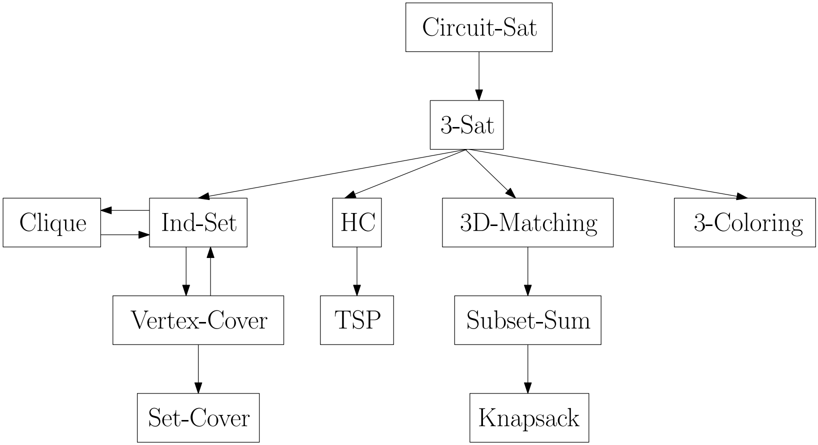
- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.
- G is a yes-instance if and only if there is an S such that check-HC(G, S) returns 1
- Construct a circuit C' for the algorithm check-HC
- hard-wire the instance G to the circuit C' to obtain the circuit C
- G is a yes-instance if and only if C is satisfiable

$Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

- Let check-Y(s, t) be the certifier for problem Y: check-Y(s, t) returns 1 if t is a valid certificate for s.
- s is a yes-instance if and only if there is a t such that check-Y(s, t) returns 1
- Construct a circuit C' for the algorithm check-Y
- hard-wire the instance s to the circuit C' to obtain the circuit C
- s is a yes-instance if and only if C is satisfiable $\square$

Theorem Circuit-Sat is NP-complete.

Reductions of NP-Complete Problems



Outline

Summary

- We consider decision problems
- Inputs are encoded as $\{0,1\}$-strings

Def. The complexity class P is the set of decision problems X that can be solved in polynomial time.

- Alice has a supercomputer, fast enough to run an exponential time algorithm
- Bob has a slow computer, which can only run a polynomial-time algorithm

Def. (Informal) The complexity class NP is the set of problems for which Alice can convince Bob a yes instance is a yes instance

Summary

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t
- there is a polynomial function p such that, $s \in X$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

The string t such that $B(s,t) = 1$ is called a certificate.

Def. The complexity class NP is the set of all problems for which there exists an efficient certifier.

Summary

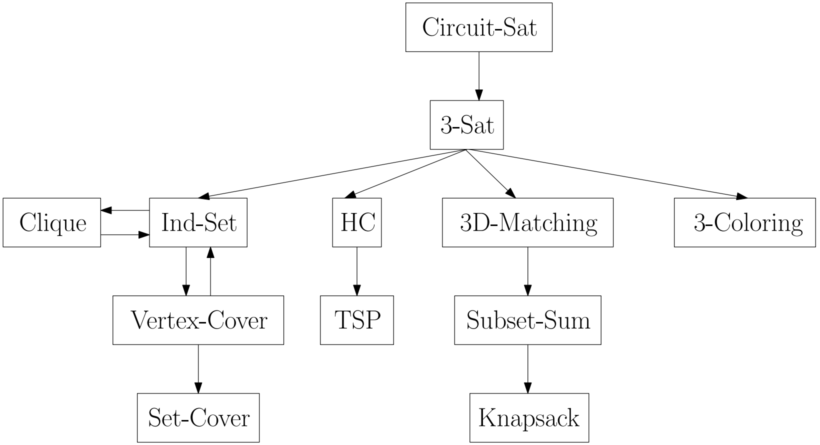
Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \le_P X$.

Def. A problem X is called NP-complete if

1. $X \in$ NP, and
2. $Y \le_\text{P} X$ for every $Y \in$ NP.

- If any NP-complete problem can be solved in polynomial time, then $P = NP$
- Unless $P = NP$, a NP-complete problem can not be solved in polynomial time

Summary


Circuit-Sat
3-Sat
Clique
Ind-Set
HC
3D-Matching
3-Coloring
Vertex-Cover
TSP
Subset-Sum
Set-Cover
Knapsack

Summary

Proof of NP-Completeness for Circuit-Sat

- Fact 1: a polynomial-time algorithm can be converted to a polynomial-size circuit
- Fact 2: for a problem in NP, there is a efficient certifier.
- Given a problem $X \in$ NP, let $B(s, t)$ be the certifier
- Convert $B(s, t)$ to a circuit and hard-wire s to the input gates
- s is a yes-instance if and only if the resulting circuit is satisfiable

- Proof of NP-Completeness for other problems by reductions