CSE 431/531: Algorithm Analysis and Design (Fall 2021)

NP-Completeness

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NP-Completeness Theory

- The topics we discussed so far are positive results: how to design efficient algorithms for solving a given problem.
- NP-Completeness provides negative results: some problems can not be solved efficiently.

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Q: Why do we study negative results?

- A given problem X cannot be solved in polynomial time.
- Without knowing it, you will have to keep trying to find polynomial time algorithm for solving X. All our efforts are doomed!

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- A good cut separating problems: for most natural problems, either we have a polynomial time algorithm, or the best algorithm runs in time $\Omega(2^{n^c})$ for some c
- Do not need to worry about the computational model

Outline

1. Some Hard Problems
2. P, NP and Co-NP
3. Polynomial Time Reductions and NP-Completeness
4. NP-Complete Problems
5. Summary

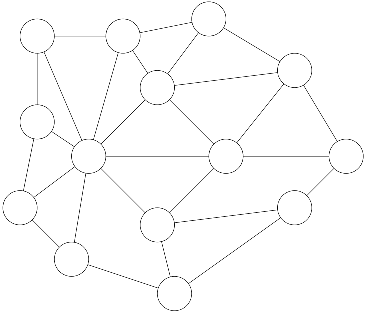
Example: Hamiltonian Cycle Problem

Def. Let G be an undirected graph. A Hamiltonian Cycle (HC) of G is a cycle C in G that passes each vertex of G exactly once.

Hamiltonian Cycle (HC) Problem

Input: graph $G = (V, E)$

Output: whether G contains a Hamiltonian cycle



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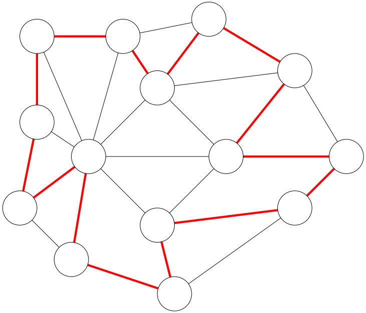

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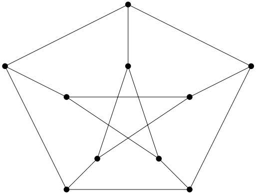

- The graph is called the Petersen Graph. It has no HC.

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- Far away from polynomial time
- HC is NP-hard: it is unlikely that it can be solved in polynomial time.

Maximum Independent Set Problem

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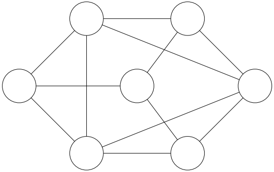

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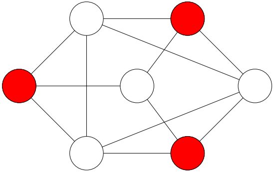

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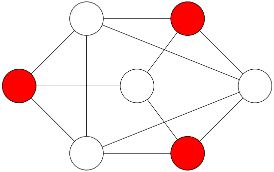

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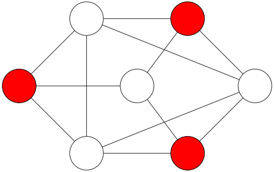

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- Maximum Independent Set is NP-hard

Formula Satisfiability

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Input: boolean formula with n variables, with $\vee, \wedge, \neg$ operators.

Output: whether the boolean formula is satisfiable

- Example: $\neg((\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3))$ is not satisfiable
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Fact For each optimization problem X, there is a decision version X' of the problem. If we have a polynomial time algorithm for the decision version X', we can solve the original problem X in polynomial time.

Optimization to Decision

Shortest Path

Input: graph $G = (V, E)$, weight w, s, t and a bound L

Output: whether there is a path from s to t of length at most L

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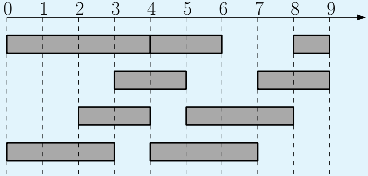

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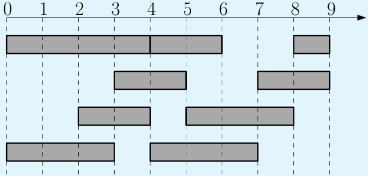

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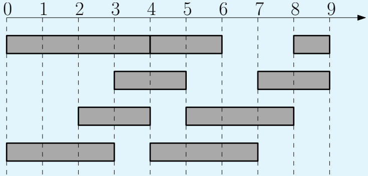

- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$
- Encode the sequence into a binary string as before

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A: No! As long as we are using a “natural” encoding. We only care whether the running time is polynomial or not

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Def. A has a polynomial running time if there is a polynomial function $p(\cdot)$ so that for every string s, the algorithm A terminates on s in at most $p(|s|)$ steps.

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- The decision versions of interval scheduling, shortest path and minimum spanning tree all in P.

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Def. The message Alice sends to Bob is called a certificate, and the algorithm Bob runs is called a certifier.

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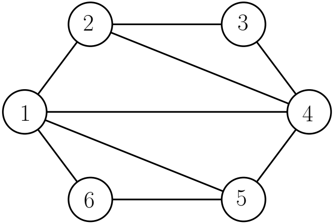

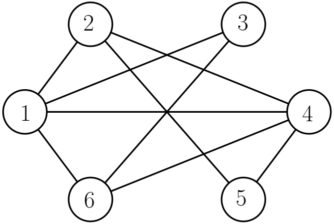

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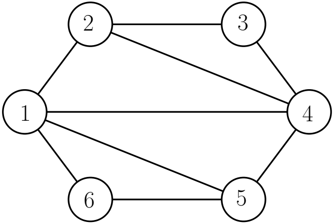

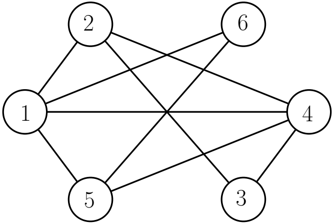

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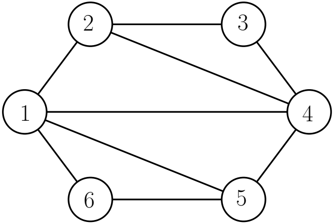

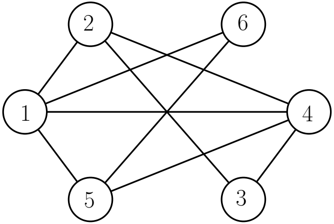

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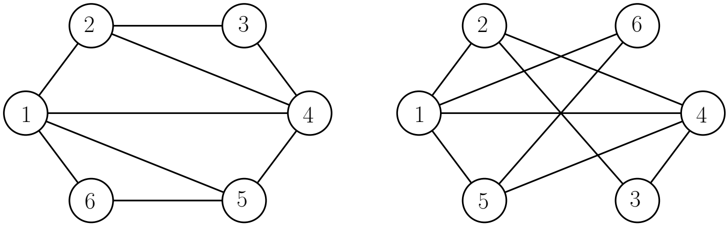

- What is the certificate?
- What is the certifier?

The Complexity Class NP

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t
- there is a polynomial function p such that, $s \in X$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

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- Clearly, B runs in polynomial time
- $G \in \text{HC} \quad \Longleftrightarrow \quad \exists S,\ B(G, S) = 1$

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- Certifier B: $B((G_1, G_2), f) = 1$ if and only if for every $u, v \in V$, we have $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.

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- Clearly, B runs in polynomial time

Graph Isomorphism $\in$ NP

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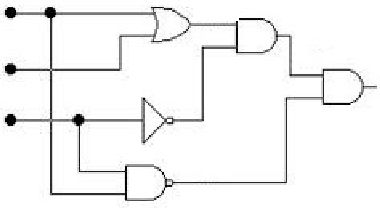
Circuit Satisfiablity (Circuit-Sat) Problem

Input: a circuit with and/or/not gates

Output: whether there is an assignment such that the output is 1?



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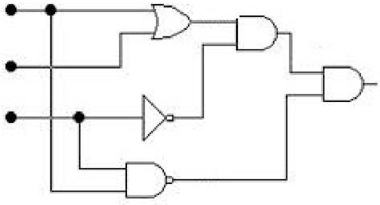

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The Complexity Class Co-NP

Def. For a problem X, the problem $\overline{X}$ is the problem such that $s \in \overline{X}$ if and only if $s \notin X$.

Def. Co-NP is the set of decision problems X such that $\overline{X} \in$ NP.

Def. A tautology is a boolean formula that always evaluates to 1.

Tautology Problem

Input: a boolean formula

Output: whether the formula is a tautology

- e.g. $(\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3)$ is a tautology

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$\mathsf{P} \subseteq \mathsf{NP}$

P ⊆ NP

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- Similarly, P $\subseteq$ Co-NP, thus P $\subseteq$ NP $\cap$ Co-NP

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  - HC $\notin$ P, unless P = NP

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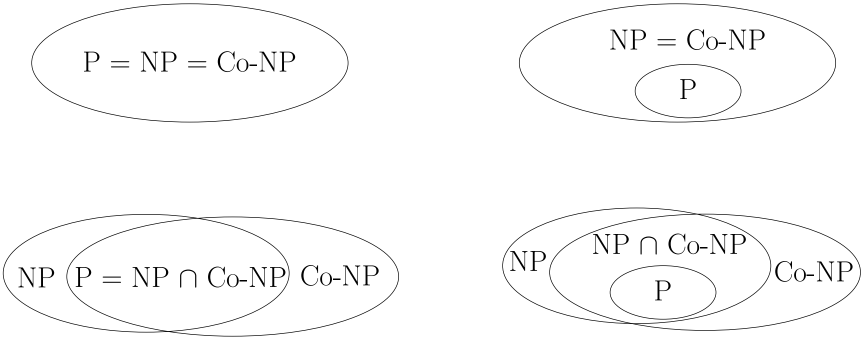
- Most researchers believe NP $\neq$ Co-NP.

4 Possibilities of Relationships

Notice that $X \in \text{NP} \iff \overline{X} \in \text{Co-NP}$ and $\text{P} \subseteq \text{NP} \cap \text{Co-NP}$



- People commonly believe: we are in the 4th scenario

Outline

1. Some Hard Problems
2. P, NP and Co-NP
3. Polynomial Time Reductions and NP-Completeness
4. NP-Complete Problems
5. Summary

Polynomial-Time Reducations

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

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To prove negative results:

Suppose $Y \leq_P X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

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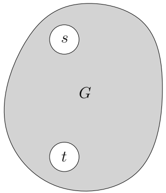

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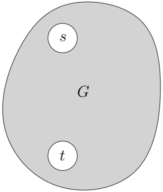

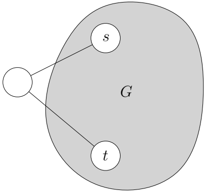

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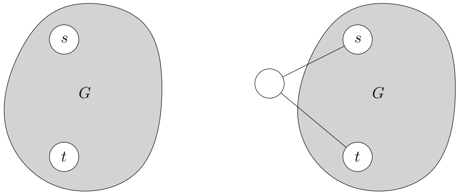

Obs. G has a HP from s to t if and only if graph on right side has a HC.

NP-Completeness

Def. A problem X is called NP-complete if

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- To prove $\mathrm{P} = \mathrm{NP}$ (if you believe it), you only need to give an efficient algorithm for any NP-complete problem
- If you believe $\mathrm{P} \neq \mathrm{NP}$, and proved that a problem X is NP-complete (or NP-hard), stop trying to design efficient algorithms for X

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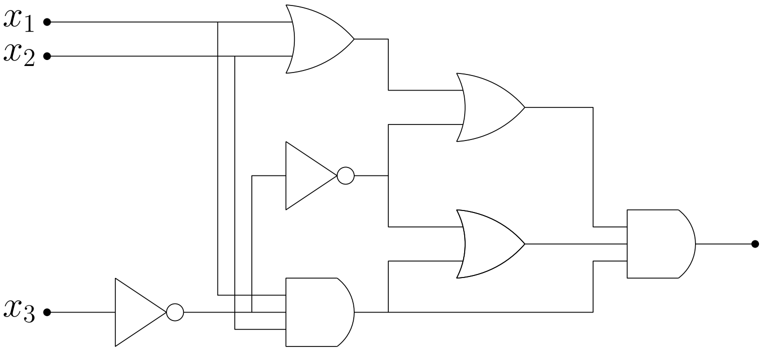
- No! There is indeed a large family of natural NP-complete problems

The First NP-Complete Problem: Circuit-Sat

Circuit Satisfiability (Circuit-Sat)

Input: a circuit

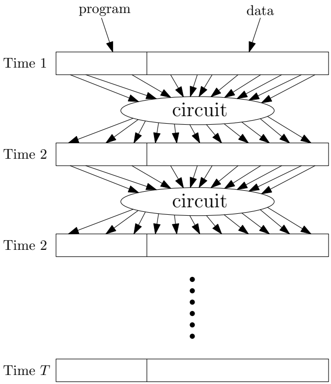
Output: whether the circuit is satisfiable



Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

Fact Any algorithm that takes n bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.



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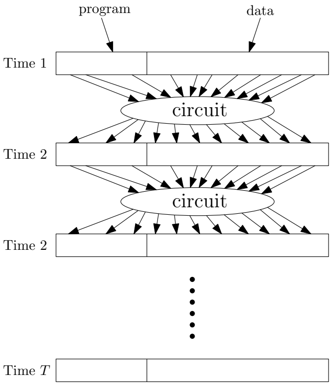

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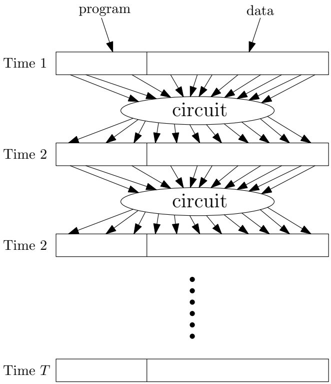

- Then, we can show that any problem $Y \in$ NP can be reduced to Circuit-Sat.
- We prove HC $\leq_P$ Circuit-Sat as an example.

HC $\leq_P$ Circuit-Sat

check-HC(G, S)

- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.

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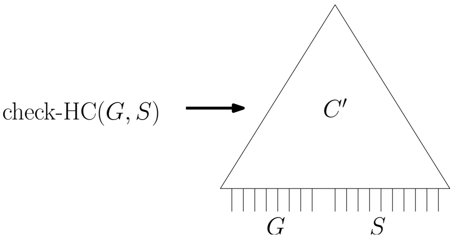

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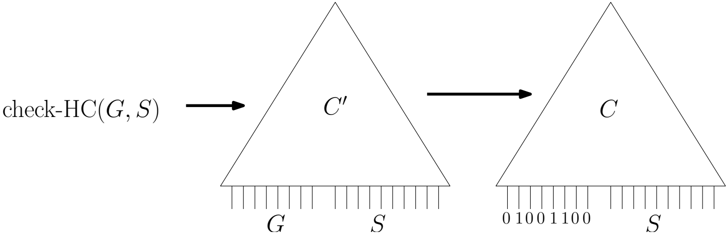

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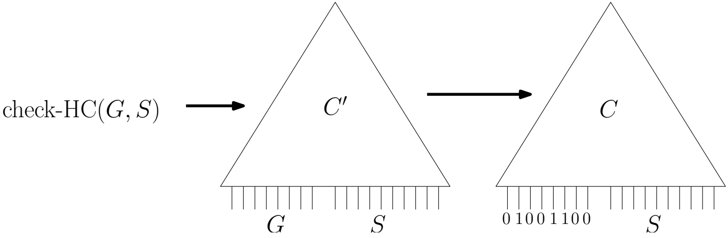

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- G is a yes-instance if and only if C is satisfiable

$Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

- Let check-Y(s, t) be the certifier for problem Y: check-Y(s, t) returns 1 if t is a valid certificate for s.
- s is a yes-instance if and only if there is a t such that check-Y(s, t) returns 1
- Construct a circuit C' for the algorithm check-Y
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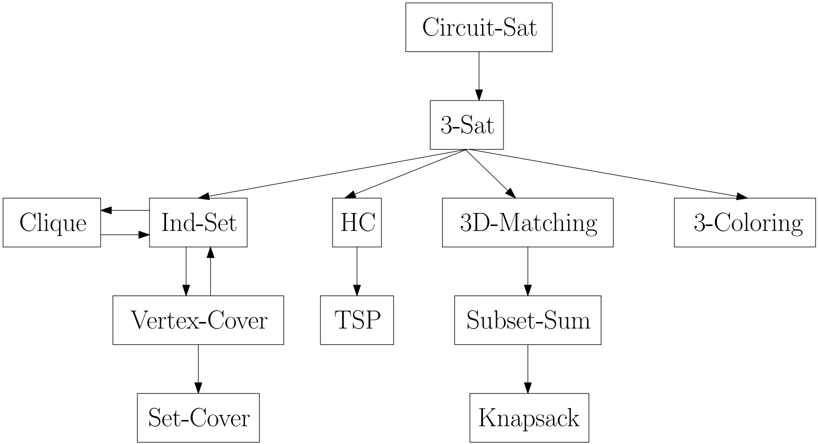
Theorem Circuit-Sat is NP-complete.

Reductions of NP-Complete Problems



Outline

1. Some Hard Problems
2. P, NP and Co-NP
3. Polynomial Time Reductions and NP-Completeness
4. NP-Complete Problems
5. Summary

Summary

- We consider decision problems
- Inputs are encoded as $\{0, 1\}$-strings

Def. The complexity class P is the set of decision problems X that can be solved in polynomial time.

- Alice has a supercomputer, fast enough to run an exponential time algorithm
- Bob has a slow computer, which can only run a polynomial-time algorithm

Def. (Informal) The complexity class NP is the set of problems for which Alice can convince Bob a yes instance is a yes instance

Summary

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t
- there is a polynomial function p such that, $s \in X$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

The string t such that $B(s,t) = 1$ is called a certificate.

Def. The complexity class NP is the set of all problems for which there exists an efficient certifier.

Summary

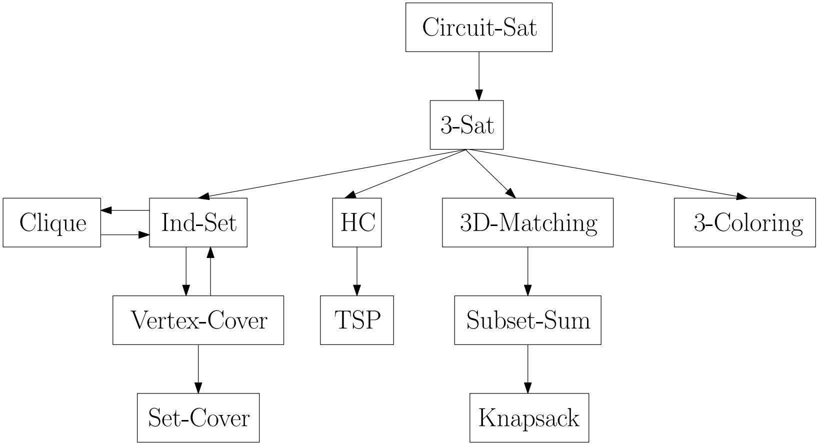
Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

Def. A problem X is called NP-complete if

1. $X \in$ NP, and
2. $Y \leq_{\mathrm{P}} X$ for every $Y \in$ NP.

- If any NP-complete problem can be solved in polynomial time, then $P = NP$
- Unless $P = NP$, a NP-complete problem can not be solved in polynomial time

Summary


Circuit-Sat
3-Sat
Clique
Ind-Set
HC
3D-Matching
3-Coloring
Vertex-Cover
TSP
Subset-Sum
Set-Cover
Knapsack

Summary

Proof of NP-Completeness for Circuit-Sat

- Fact 1: a polynomial-time algorithm can be converted to a polynomial-size circuit
- Fact 2: for a problem in NP, there is a efficient certifier.
- Given a problem $X \in$ NP, let $B(s, t)$ be the certifier
- Convert $B(s, t)$ to a circuit and hard-wire s to the input gates
- s is a yes-instance if and only if the resulting circuit is satisfiable

- Proof of NP-Completeness for other problems by reductions