CSE 431/531: Algorithm Analysis and Design (Spring 2018)

Divide-and-Conquer

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Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
6. Solving Recurrences
7. Computing n-th Fibonacci Number

Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

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Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

```
merge-sort$(A, n)$
if $n = 1$ then
    return $A$
else
    $B \leftarrow$ merge-sort$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
    $C \leftarrow$ merge-sort$\Big(A\big[\lfloor n/2 \rfloor + 1..n\big], \lceil n/2 \rceil\Big)$
return merge$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
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merge-sort(A, n)

```
① if $n = 1$ then
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③ else
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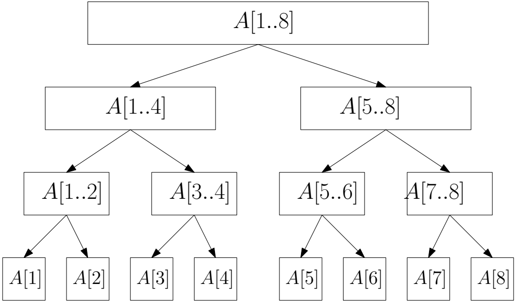
```

- Divide: trivial
- Conquer: ④, ⑤
- Combine: ⑥

Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$
- Better than insertion sort

Running Time for Merge-Sort Using Recurrence

- $T(n) =$ running time for sorting n numbers,then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

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- Solving this recurrence, we have $T(n) = O(n \lg n)$ (we shall show how later)

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$$10 \qquad 8 \qquad 15 \qquad 9 \qquad 12$$

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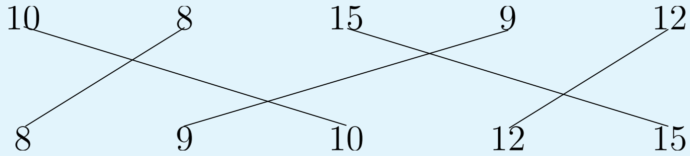

10 8 15 9 12

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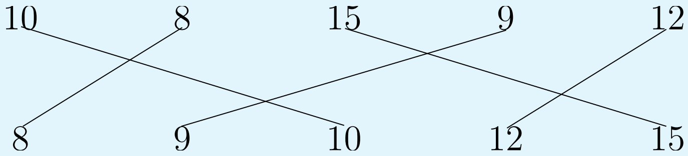
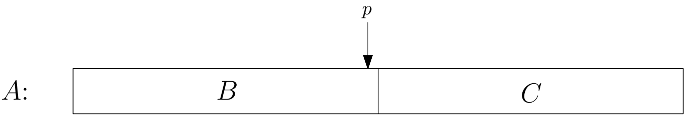


- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

Naive Algorithm for Counting Inversions

```
count-inversions($(A, n)$)
$c \leftarrow 0$
for every $i \leftarrow 1$ to $n - 1$
    for every $j \leftarrow i + 1$ to $n$
        if $A[i] > A[j]$ then $c \leftarrow c + 1$
return $c$
```

Divide-and-Conquer



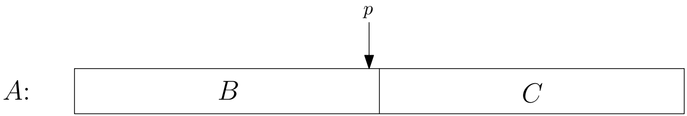
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$
  $$m = \Big|\big\{(i, j) : B[i] > C[j]\big\}\Big|$$

Q: How fast can we compute m, via trivial algorithm?

A: $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

Divide-and-Conquer



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Lemma If both B and C are sorted, then we can compute m in $O(n)$ time!

Counting Inversions between B and C

Count pairs i, j such that $B[i] > C[j]$:

B:	3	8	12	20	32	48

total$=$ 0

C:	5	7	9	25	29

Counting Inversions between B and C

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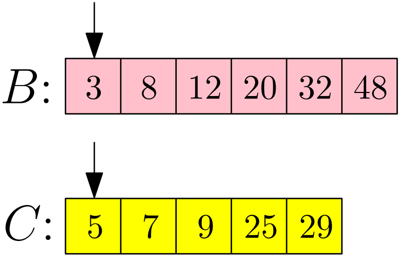

B: 3 | 8 | 12 | 20 | 32 | 48

C: 5 | 7 | 9 | 25 | 29

total= 0

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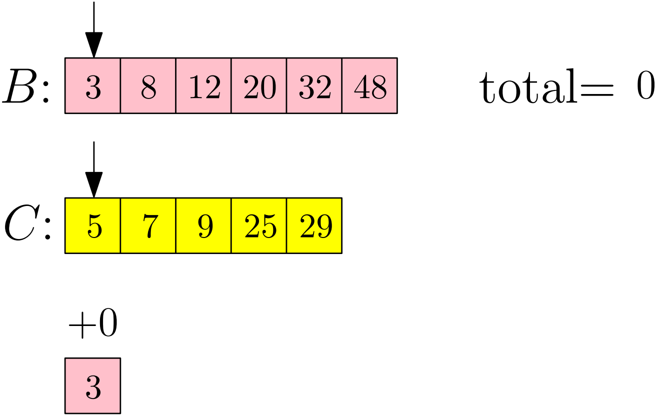

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total$=$ 0

$+0$

3

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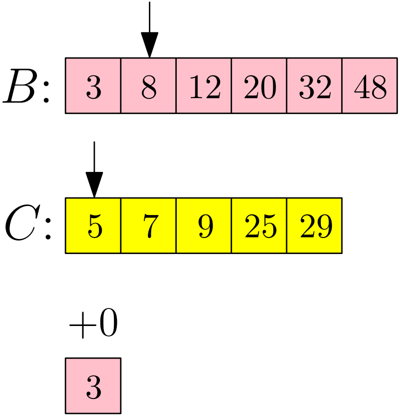

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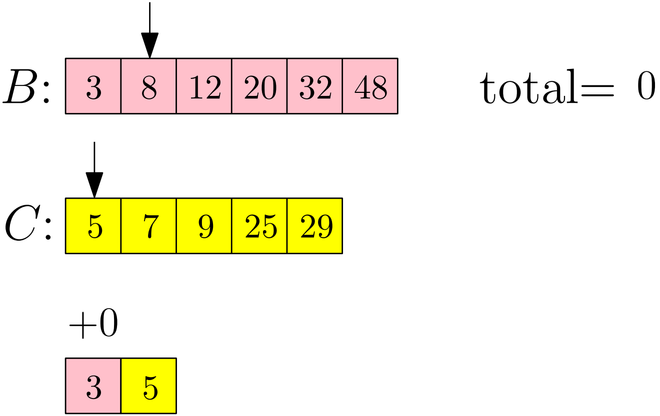

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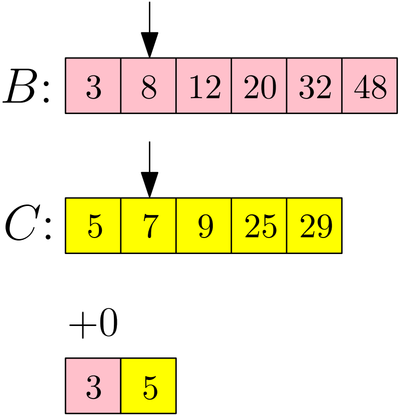

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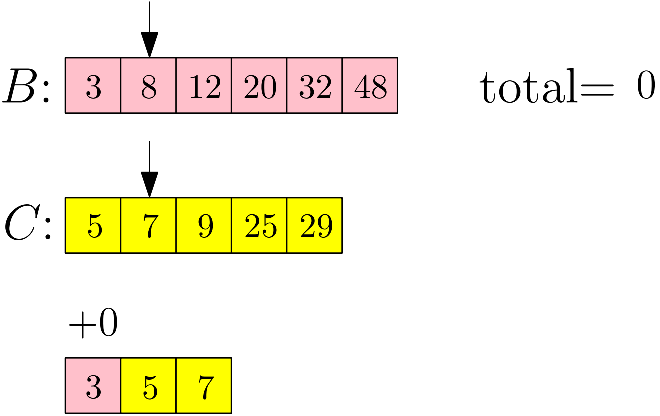

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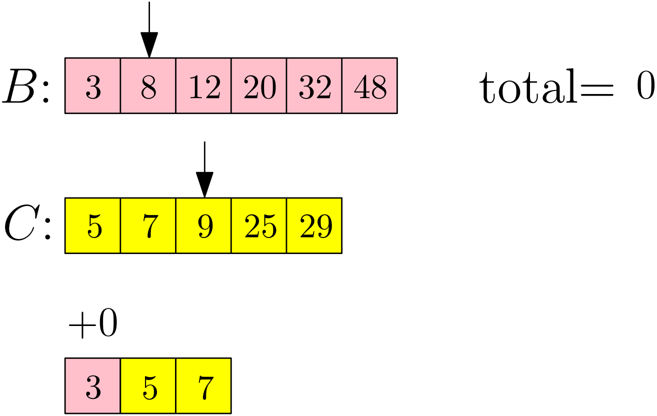

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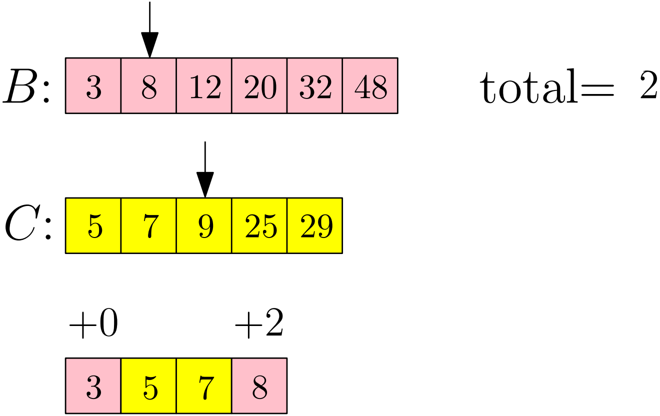

B: 3 | 8 | 12 | 20 | 32 | 48

total= 2

C: 5 | 7 | 9 | 25 | 29

$+0$ $+2$

3 | 5 | 7 | 8

Counting Inversions between B and C

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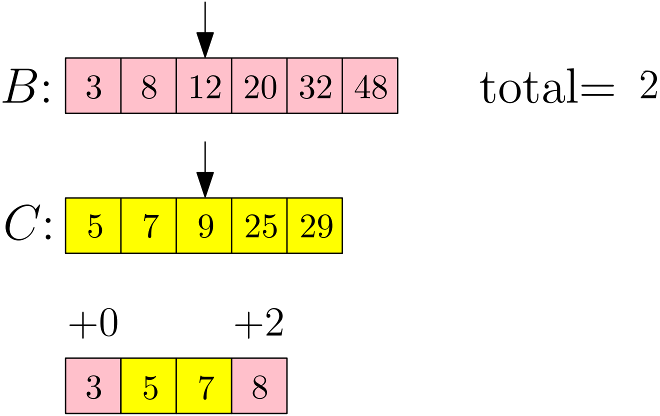

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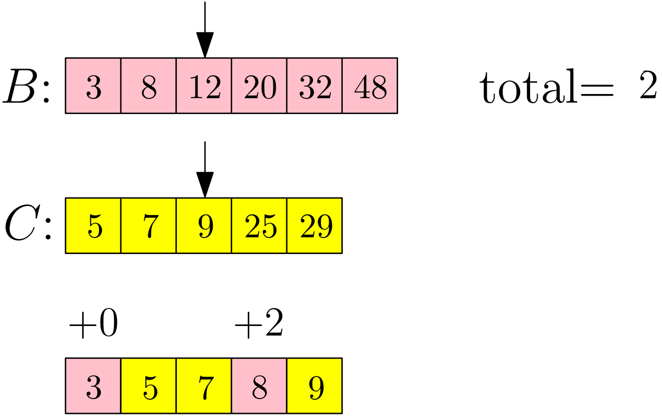

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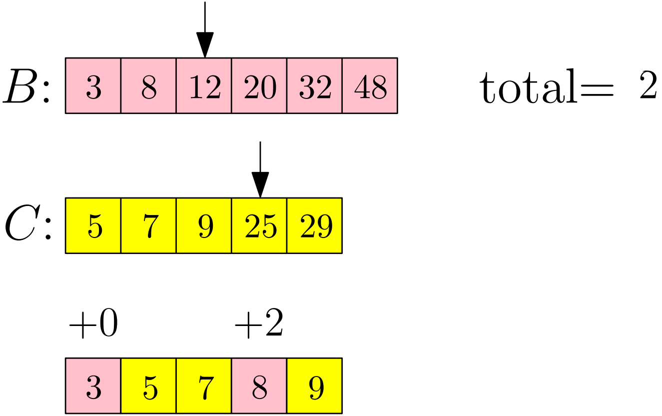

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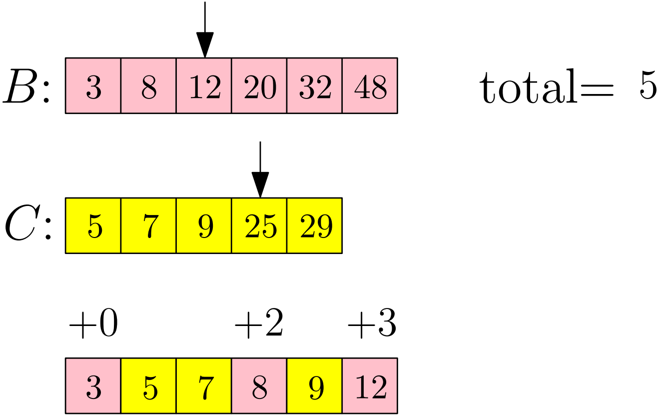

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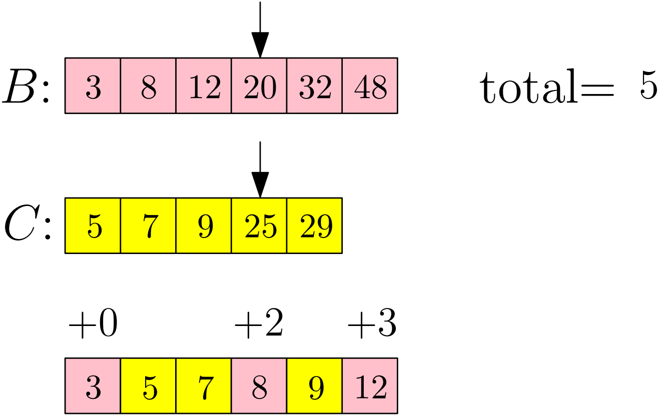

B: 3 8 12 20 32 48

total$=$ 5

C: 5 7 9 25 29

$+0$ $+2$ $+3$

3 5 7 8 9 12

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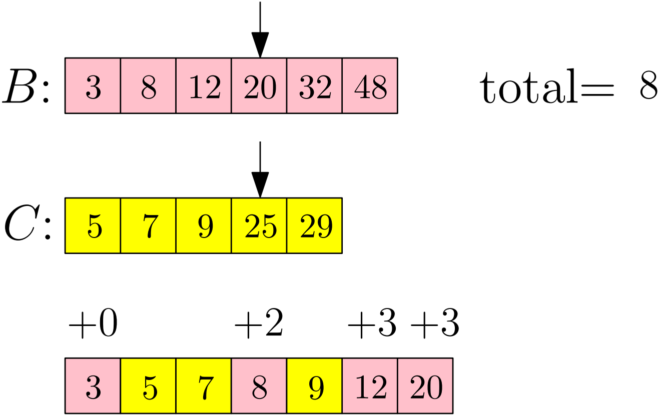

B:

3	8	12	20	32	48

total$=$ 8

C:

5	7	9	25	29

$+0$ $+2$ $+3$ $+3$

3	5	7	8	9	12	20

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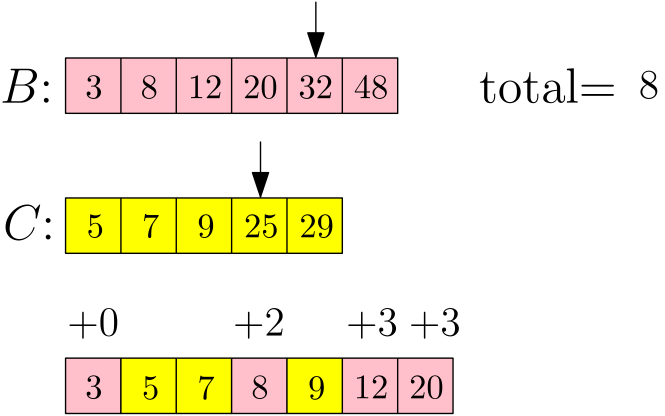

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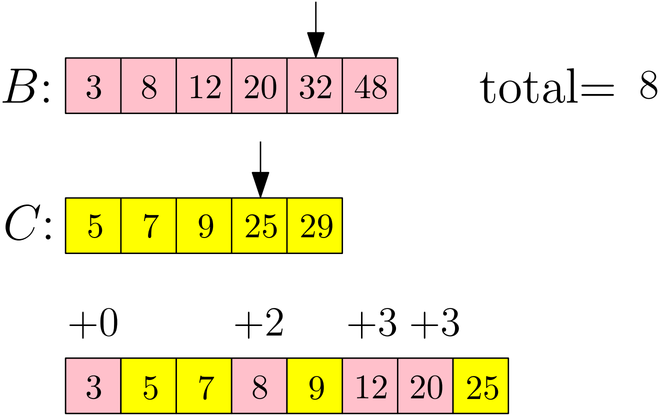

B:

3	8	12	20	32	48

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$+0$			$+2$		$+3$	$+3$	
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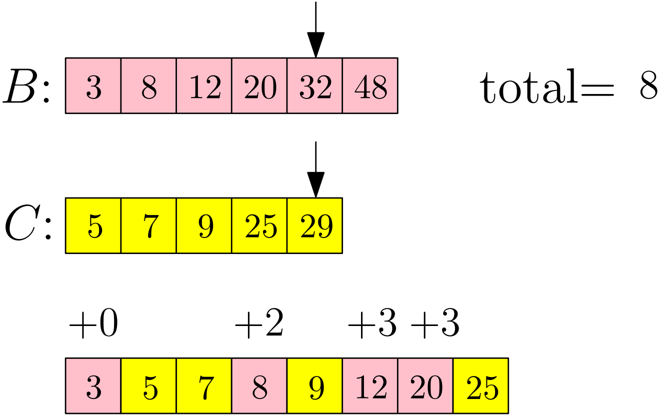

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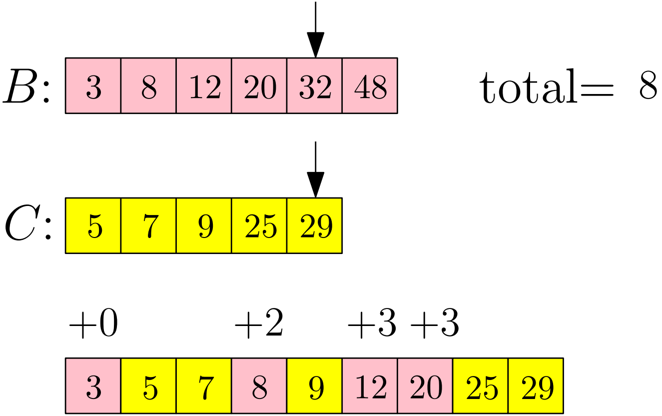

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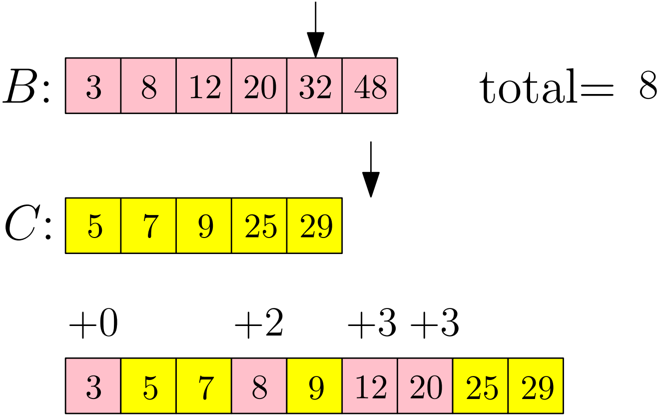

B: 3 | 8 | 12 | 20 | 32 | 48

total$=$ 8

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+0 +2 +3 +3

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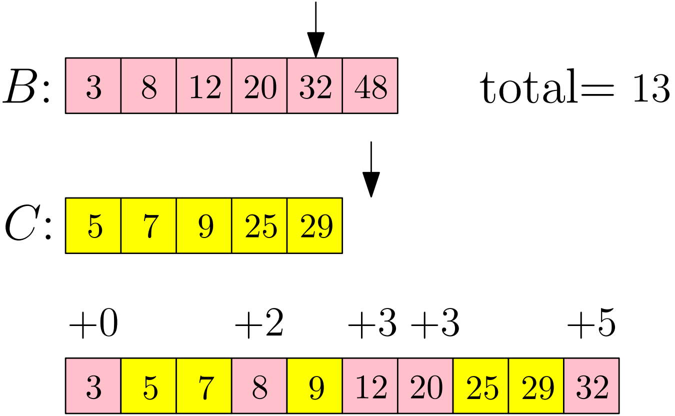

B: 3 | 8 | 12 | 20 | 32 | 48

total$=13$

C: 5 | 7 | 9 | 25 | 29

$+0$ $+2$ $+3$ $+3$ $+5$

3 | 5 | 7 | 8 | 9 | 12 | 20 | 25 | 29 | 32

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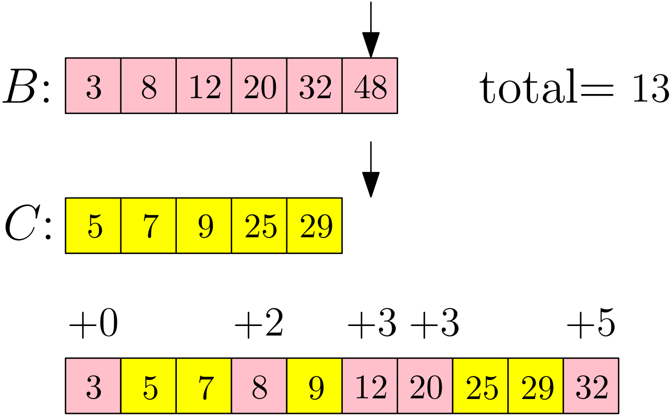
Count pairs i, j such that $B[i] > C[j]$:



B:

3	8	12	20	32	48

total$= 13$

C:

5	7	9	25	29

+0			+2		+3	+3			+5
3	5	7	8	9	12	20	25	29	32

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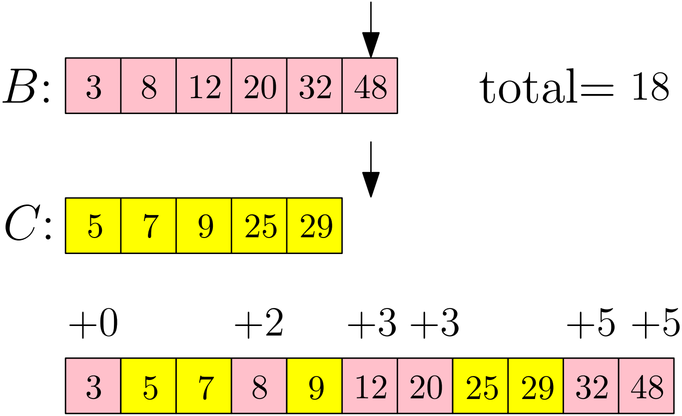

B:	3	8	12	20	32	48

total$= 18$

C:	5	7	9	25	29

$+0$			$+2$		$+3$	$+3$			$+5$	$+5$
3	5	7	8	9	12	20	25	29	32	48

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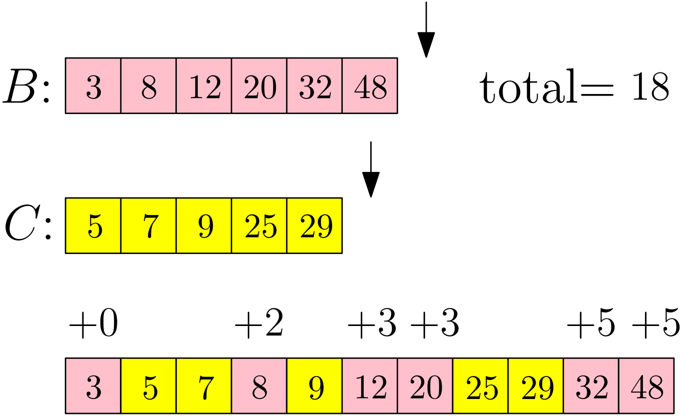

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total$=18$

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$+0$			$+2$		$+3$	$+3$			$+5$	$+5$
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Count Inversions between B and C

- Procedure that merges B and C and counts inversions between B and C at the same time

merge-and-count(B, C, n_1, n_2)

```
$count \leftarrow 0$;
$A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ or $j \leq n_2$
    if $j > n_2$ or $(i \leq n_1$ and $B[i] \leq C[j])$ then
        append $B[i]$ to $A$; $i \leftarrow i + 1$
        $count \leftarrow count + (j - 1)$
    else
        append $C[j]$ to $A$; $j \leftarrow j + 1$
return $(A, count)$
```

Sort and Count Inversions in A

- A procedure that returns the sorted array of A and counts the number of inversions in A:

sort-and-count(A, n)

```
if $n = 1$ then
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\left(A\left[1..\lfloor n/2 \rfloor\right], \lfloor n/2 \rfloor\right)$
    $(C, m_2) \leftarrow$ sort-and-count$\left(A\left[\lfloor n/2 \rfloor + 1..n\right], \lceil n/2 \rceil\right)$
    $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
    return $(A, m_1 + m_2 + m_3)$
```

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- A procedure that returns the sorted array of A and counts the number of inversions in A:

```
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2      return $(A, 0)$
3  else
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7      return $(A, m_1 + m_2 + m_3)$
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- Divide: trivial
- Conquer: ④, ⑤
- Combine: ⑥, ⑦

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    return $(A, m_1 + m_2 + m_3)$
```

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$

sort-and-count(A, n)

```
if $n = 1$ then
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
    $(C, m_2) \leftarrow$ sort-and-count$\Big(A\big[\lfloor n/2 \rfloor + 1..n\big], \lceil n/2 \rceil\Big)$
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Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
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4. Polynomial Multiplication
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Quicksort vs Merge-Sort

	Merge Sort	**Quicksort**
Divide	Trivial	Separate small and big numbers
Conquer	Recurse	Recurse
Combine	Merge 2 sorted arrays	Trivial

Quicksort Example

Assumption We can choose median of an array of size n in $O(n)$ time.

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Quicksort

quicksort(A, n)

```
if $n \leq 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ elements in $A$ that are less than $x$          \\ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$       \\ Divide
$B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$                     \\ Conquer
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$t \leftarrow$ number of times $x$ appear $A$
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2. Choose a pivot randomly and pretend it is the median (it is practical)

Quicksort Using A Random Pivot

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quicksort($A, n$)
if $n \leq 1$ then return $A$
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- In theory: assume they can.

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Lemma The expected running time of the algorithm is $O(n \lg n)$.

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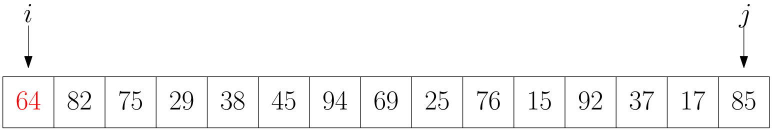

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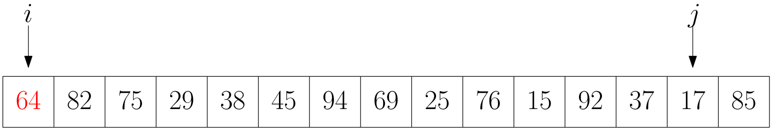

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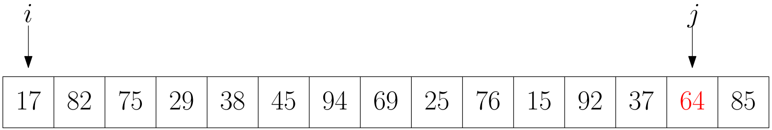

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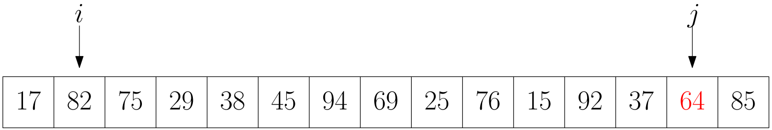

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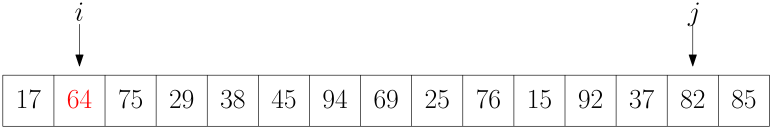

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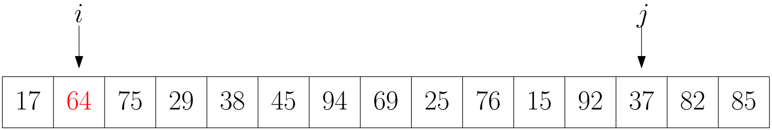

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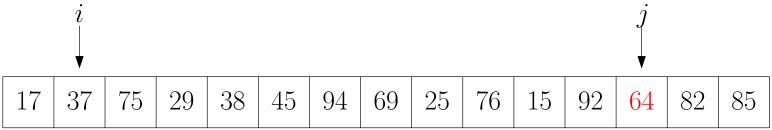

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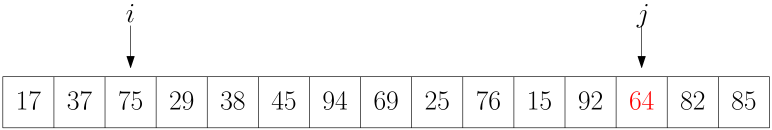

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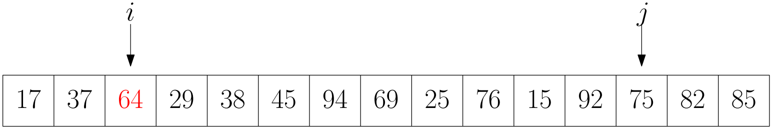

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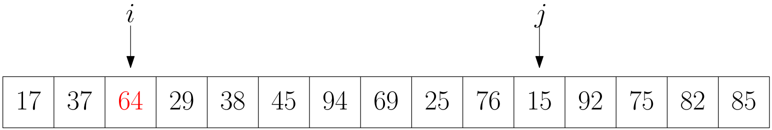

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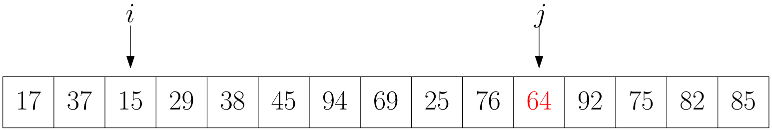

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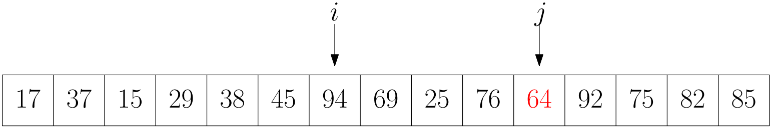

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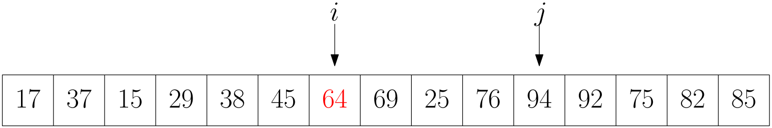

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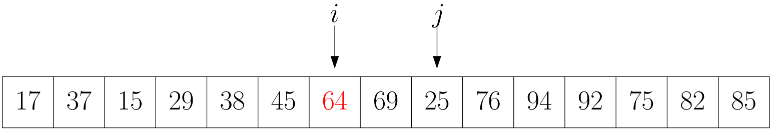

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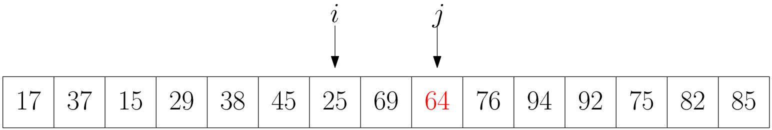

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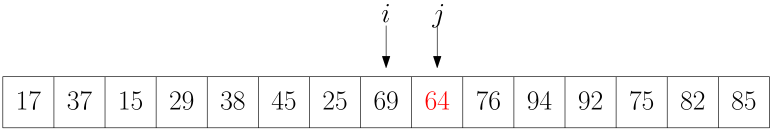

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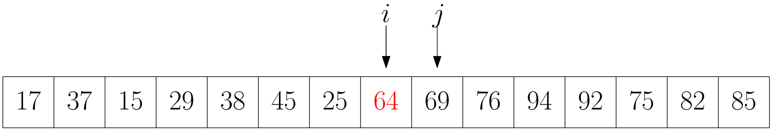

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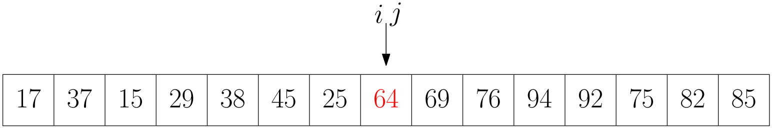

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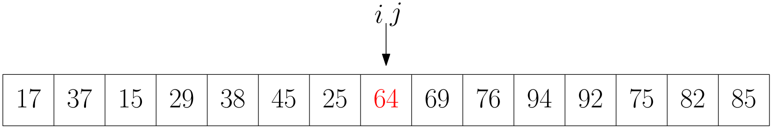

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- To partition the array into two parts, we only need $O(1)$ extra space.

partition(A, ℓ, r)

```
$p \leftarrow$ random integer between $\ell$ and $r$, swap $A[p]$ and $A[\ell]$
$i \leftarrow \ell, j \leftarrow r$
while $i < j$ do
    while $i < j$ and $A[i] \le A[j]$ do $j \leftarrow j - 1$
    swap $A[i]$ and $A[j]$
    while $i < j$ and $A[i] \le A[j]$ do $i \leftarrow i + 1$
    swap $A[i]$ and $A[j]$
$\ell' \leftarrow i, r' \leftarrow i$
for $j \leftarrow i - 1$ down to $\ell$
    if $A[j] = A[i]$ then $\ell' \leftarrow \ell' - 1$ and swap $A[\ell']$ and $A[j]$
for $j \leftarrow i + 1$ to $r$
    if $A[j] = A[i]$ then $r' \leftarrow r' + 1$ and swap $A[r']$ and $A[j]$
return $(\ell', r')$
```

In-Place Implementation of Quick-Sort

```
quicksort($A, \ell, r$)
1 if $\ell \geq r$ return
2 $(\ell', r') \leftarrow$ patition$(A, \ell, r)$
3 quicksort$(A, \ell, \ell' - 1)$
4 quicksort$(A, r' + 1, r)$
```

- To sort an array A of size n, call quicksort$(A, 1, n)$.

Note: We pass the array A by reference, instead of by copying.

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- To merge two arrays, we need a third array with size equaling the total size of two arrays

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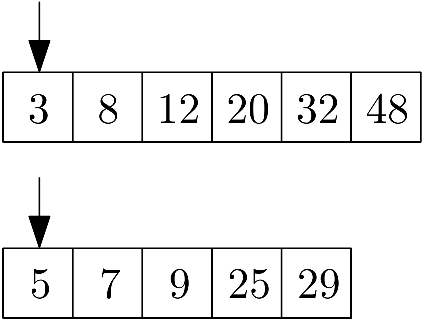

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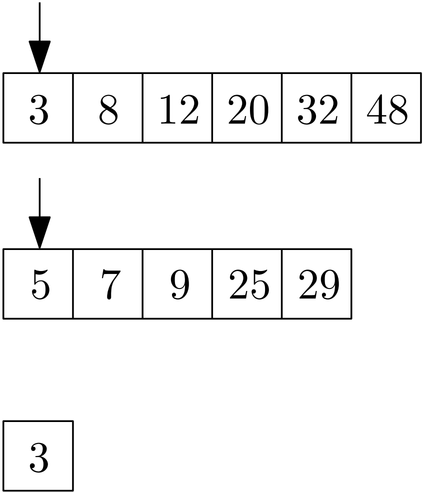

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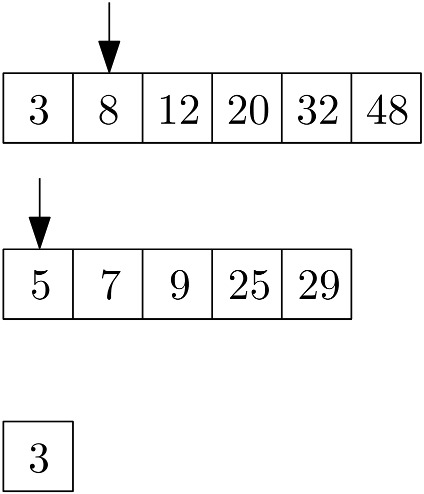

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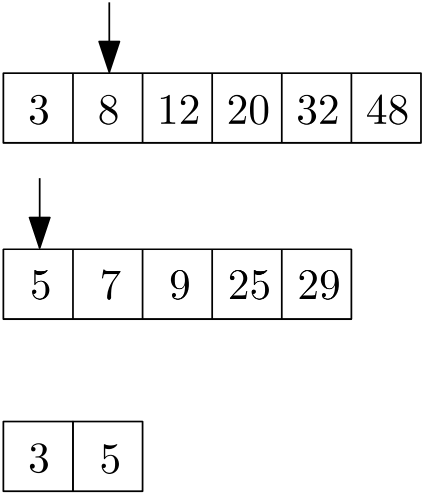

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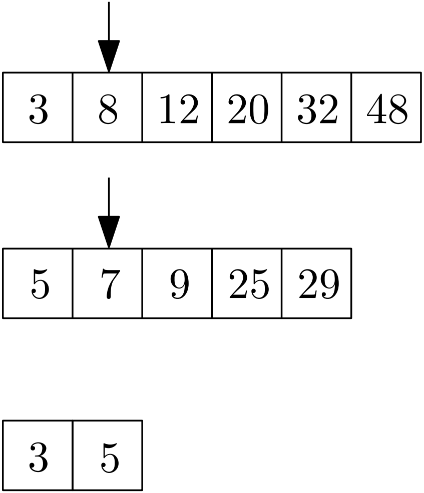

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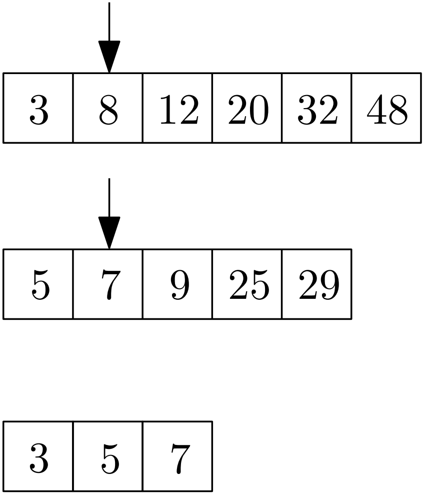

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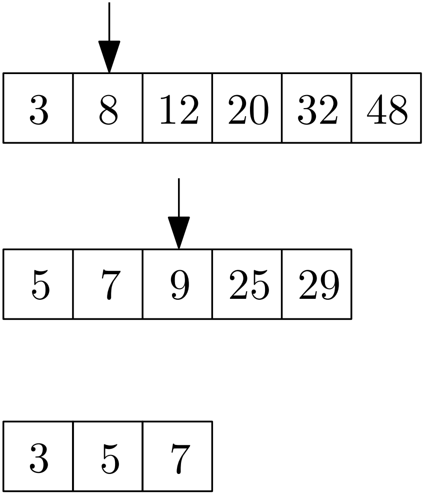

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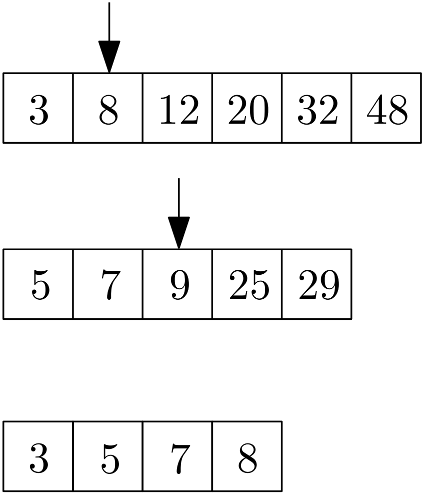

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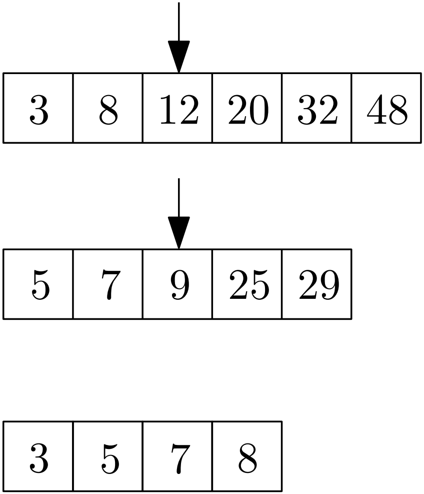

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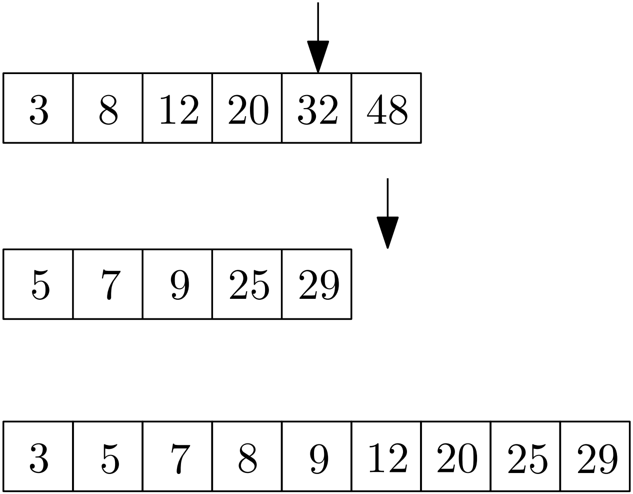
Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays



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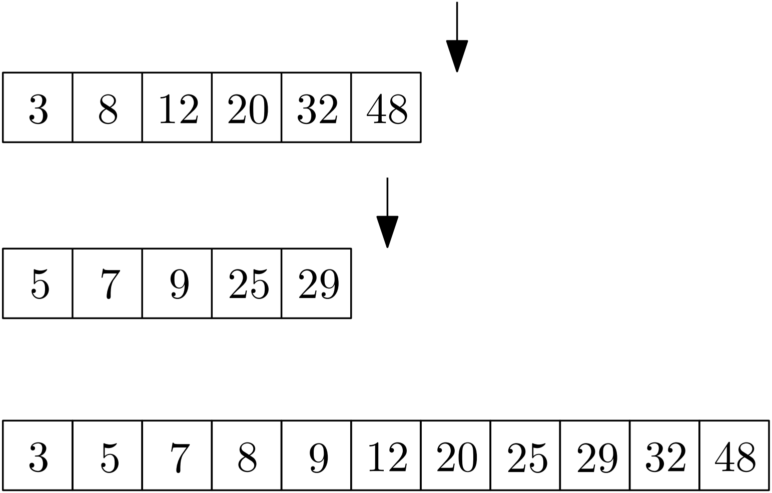

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1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
    - Quicksort
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Comparison-Based Sorting Algorithms

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Comparison-Based Sorting Algorithms

- To sort, we are only allowed to compare two elements
- We can not use "internal structures" of the elements

Lemma The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$.

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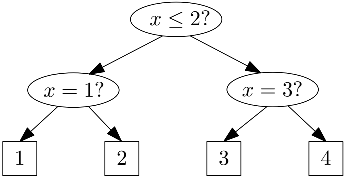

Comparison-Based Sorting Algorithms

Q: Can we do better than $O(n \log n)$ for sorting?

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- Bob has a permutation π over $\{1, 2, 3, \cdots, n\}$ in his hand.
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A: At least $\log_2 n! = \Theta(n \lg n)$

Outline

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Selection Problem

Input: a set A of n numbers, and $1 \le i \le n$

Output: the i-th smallest number in A

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- Sorting solves the problem in time $O(n \lg n)$.
- Our goal: $O(n)$ running time

Recall: Quicksort with Median Finder

```
quicksort($A, n$)
    if $n \leq 1$ then return $A$
    $x \leftarrow$ lower median of $A$
    $A_L \leftarrow$ elements in $A$ that are less than $x$          \\ Divide
    $A_R \leftarrow$ elements in $A$ that are greater than $x$       \\ Divide
    $B_L \leftarrow$ quicksort($A_L, A_L$.size)                      \\ Conquer
    $B_R \leftarrow$ quicksort($A_R, A_R$.size)                      \\ Conquer
    $t \leftarrow$ number of times $x$ appear $A$
    return the array obtained by concatenating $B_L$, the array
        containing $t$ copies of $x$, and $B_R$
```

Selection Algorithm with Median Finder

```
selection($(A, n, i)$)
if $n = 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ elements in $A$ that are less than $x$                  \\ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$               \\ Divide
if $i \leq A_L$.size then
    return selection($(A_L, A_L$.size$, i)$)                            \\ Conquer
elseif $i > n - A_R$.size then
    return selection($(A_R, A_R$.size$, i - (n - A_R$.size$))$)         \\ Conquer
else return $x$
```

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- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

Randomized Selection Algorithm

```
selection($(A, n, i)$)
if $n = 1$ then return $A$
$x \leftarrow$ random element of $A$ (called pivot)
$A_L \leftarrow$ elements in $A$ that are less than $x$        \\ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$     \\ Divide
if $i \leq A_L$.size then
    return selection($(A_L, A_L$.size$, i)$)                  \\ Conquer
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elseif $i > n - A_R$.size then
    return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ \\ Conquer
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- expected running time $= O(n)$

Outline

Polynomial Multiplication

Input: two polynomials of degree $n-1$

Output: product of two polynomials

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$$(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5)$$

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Example:

$$\begin{aligned}
& (3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\
= {} & 6x^6 - 9x^5 + 18x^4 - 15x^3 \\
& + 4x^5 - 6x^4 + 12x^3 - 10x^2 \\
& - 10x^4 + 15x^3 - 30x^2 + 25x \\
& + 8x^3 - 12x^2 + 24x - 20 \\
= {} & 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20
\end{aligned}$$

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&+8x^3-12x^2+24x-20\\
=\;&6x^6-5x^5+2x^4+20x^3-52x^2+49x-20
\end{aligned}$$

- **Input**: $(4,-5,2,3),(-5,6,-3,2)$
- **Output**: $(-20,49,-52,20,2,-5,6)$

Naïve Algorithm

```
polynomial-multiplication$(A, B, n)$
let $C[k] = 0$ for every $k = 0, 1, 2, \cdots, 2n - 2$
for $i \leftarrow 0$ to $n - 1$
    for $j \leftarrow 0$ to $n - 1$
        $C[i + j] \leftarrow C[i + j] + A[i] \times B[j]$
return $C$
```

Naïve Algorithm

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let $C[k] = 0$ for every $k = 0, 1, 2, \cdots, 2n-2$
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return $C$
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Running time: $O(n^2)$

Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x - 3)x^2 + (6x - 5)$$

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- $p(x)$: degree of $n-1$ (assume n is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

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$$pq = \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right)$$

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$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right)x^{n/2} + p_L q_L \end{aligned}$$

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$$\begin{aligned} \mathsf{multiply}(p, q) = {} & \mathsf{multiply}(p_H, q_H) \times x^n \\ & + \left(\mathsf{multiply}(p_H, q_L) + \mathsf{multiply}(p_L, q_H)\right) \times x^{n/2} \\ & + \mathsf{multiply}(p_L, q_L) \end{aligned}$$

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- Recurrence: $T(n) = 4T(n/2) + O(n)$

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- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

Divide-and-Conquer for Polynomial Multiplication

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$$r_H = \mathsf{multiply}(p_H, q_H)$$
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Divide-and-Conquer for Polynomial Multiplication

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$

Divide-and-Conquer for Polynomial Multiplication

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& + r_L
\end{aligned}
$$

- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

Assumption n is a power of 2. Arrays are 0-indexed.

```
multiply$(A, B, n)$
if $n = 1$ then return $(A[0]B[0])$
$A_L \leftarrow A[0 \mathinner{..} n/2 - 1], A_H \leftarrow A[n/2 \mathinner{..} n - 1]$
$B_L \leftarrow B[0 \mathinner{..} n/2 - 1], B_H \leftarrow B[n/2 \mathinner{..} n - 1]$
$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$
$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$
$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$
$C \leftarrow$ array of $(2n - 1)$ 0's
for $i \leftarrow 0$ to $n - 2$ do
    $C[i] \leftarrow C[i] + C_L[i]$
    $C[i + n] \leftarrow C[i + n] + C_H[i]$
    $C[i + n/2] \leftarrow C[i + n/2] + C_M[i] - C_L[i] - C_H[i]$
return $C$
```

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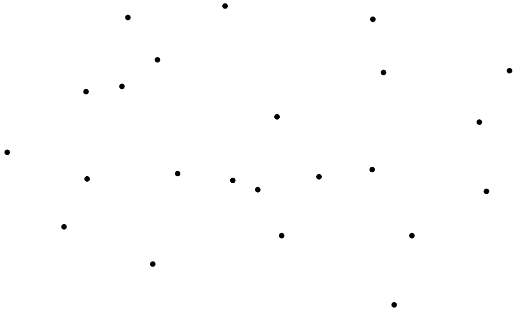
- Closest pair
- Convex hull
- Matrix multiplication
- FFT(Fast Fourier Transform): polynomial multiplication in $O(n \lg n)$ time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



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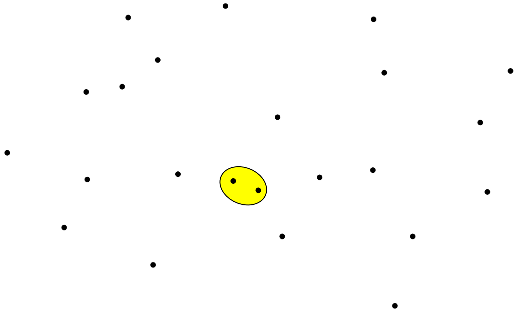

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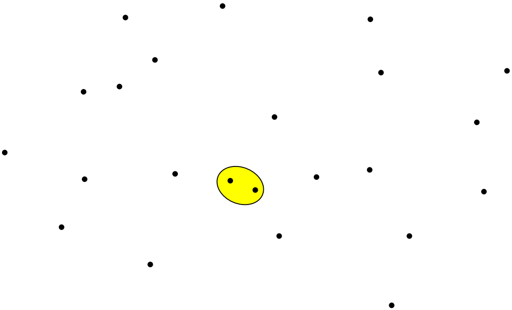
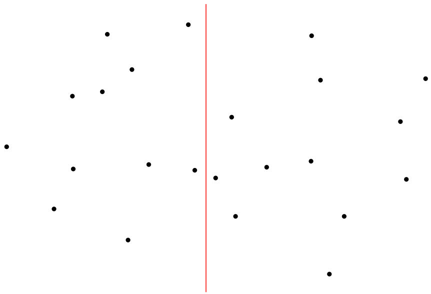


- Trivial algorithm: $O(n^2)$ running time

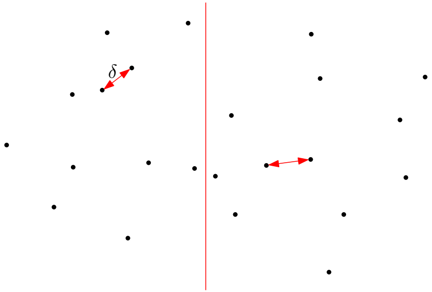
Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line



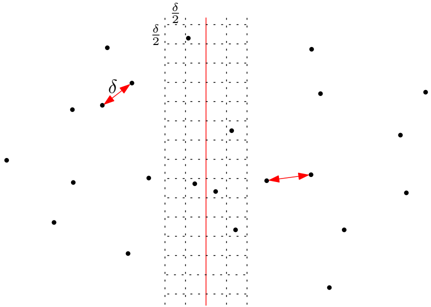
Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively

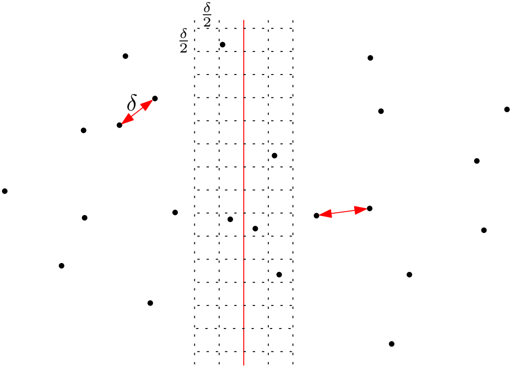


Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively
- **Combine**: Check if there is a closer pair between left-half and right-half



Divide-and-Conquer Algorithm for Closest Pair



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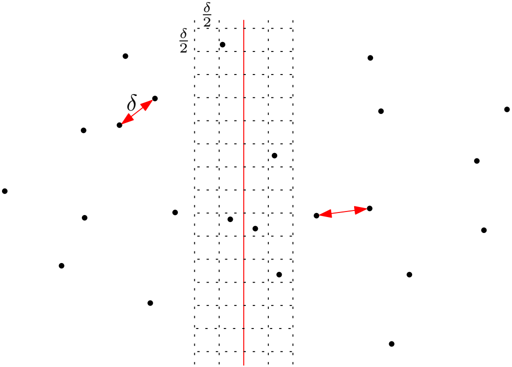

- Each box contains at most one pair

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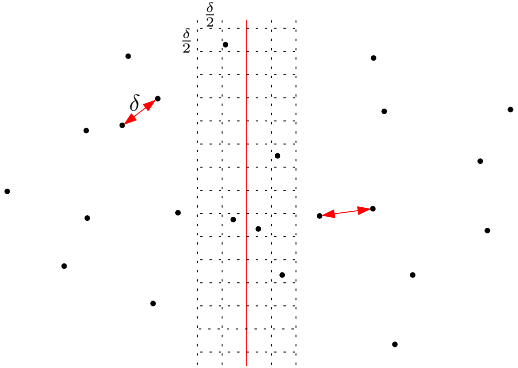

- Each box contains at most one pair
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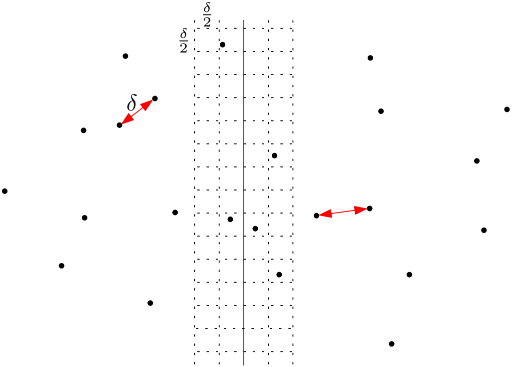

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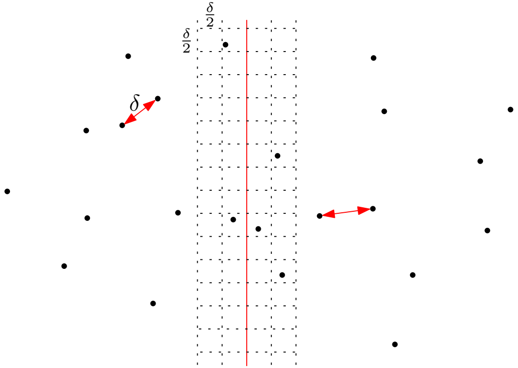

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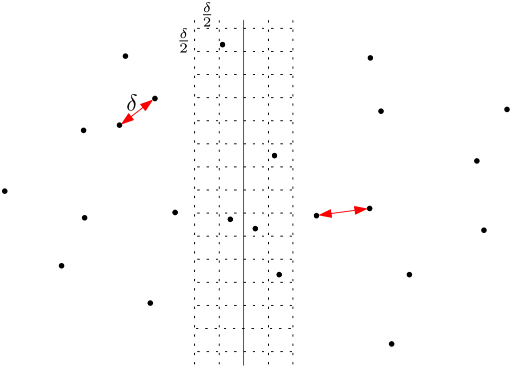

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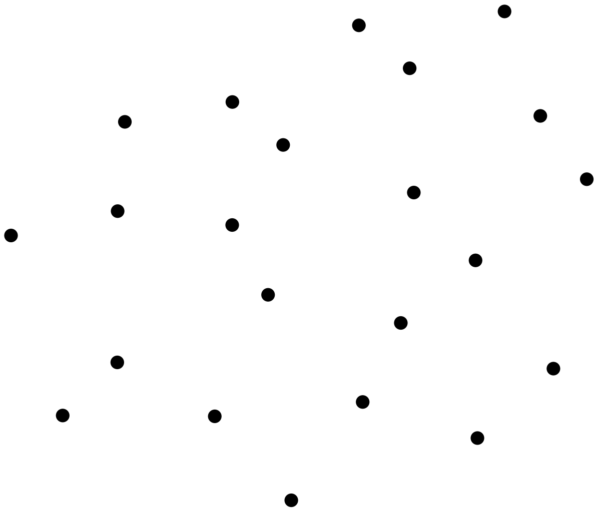
- Running time: $O(n \lg n)$

$O(n \lg n)$-Time Algorithm for Convex Hull



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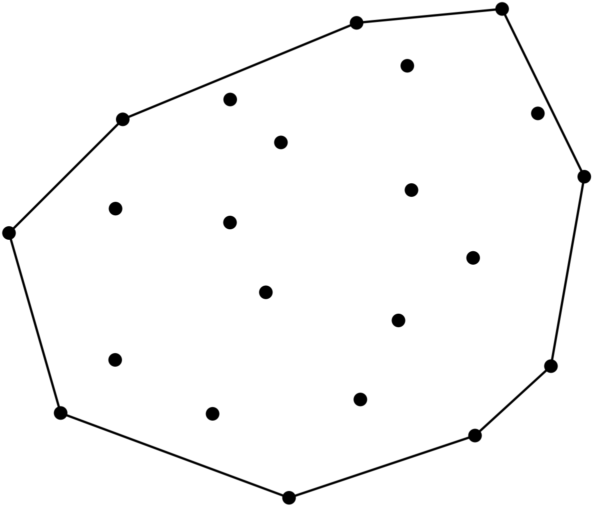

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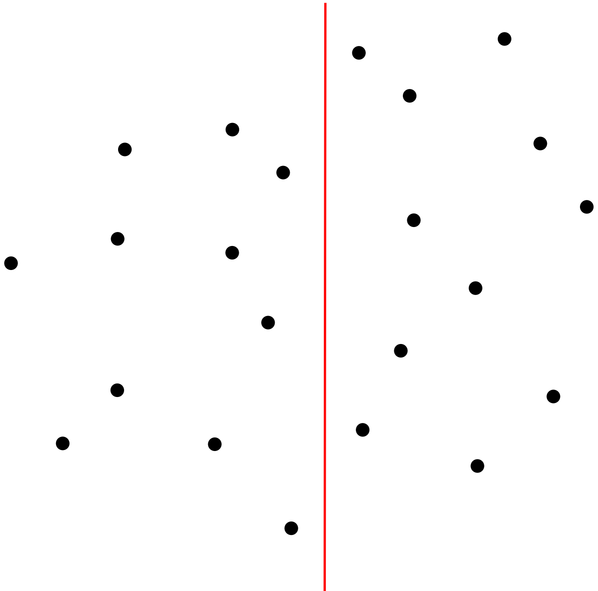

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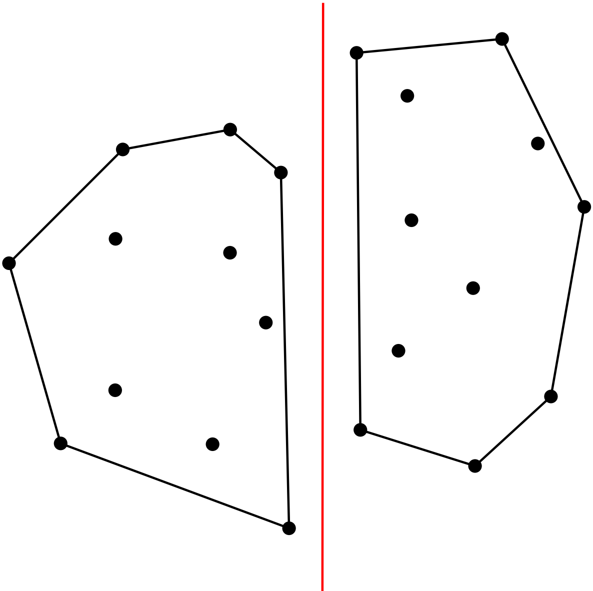

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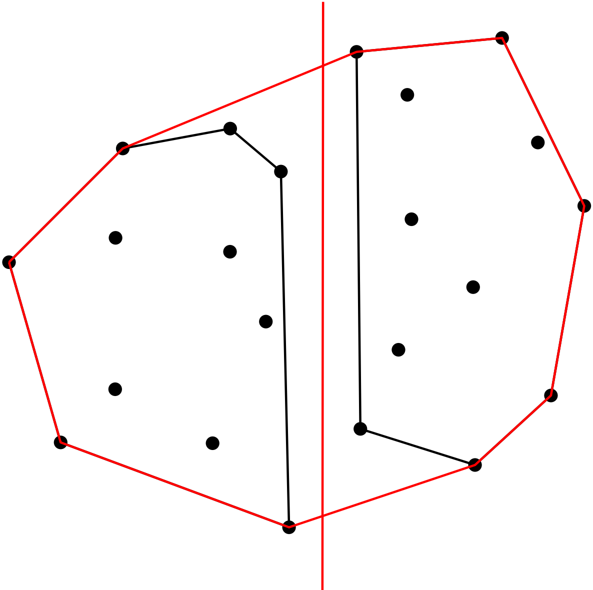

Strassen's Algorithm for Matrix Multiplication

Matrix Multiplication

Input: two $n \times n$ matrices A and B

Output: $C = AB$

Strassen's Algorithm for Matrix Multiplication

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Naive Algorithm: matrix-multiplication(A, B, n)

```
for $i \leftarrow 1$ to $n$
    for $j \leftarrow 1$ to $n$
        $C[i, j] \leftarrow 0$
        for $k \leftarrow 1$ to $n$
            $C[i, j] \leftarrow C[i, j] + A[i, k] \times B[k, j]$
return $C$
```

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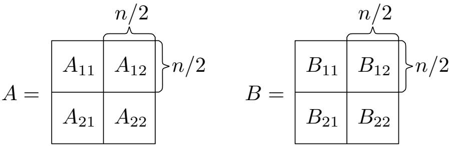
```

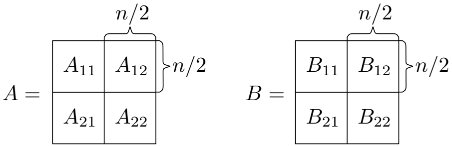
- running time $= O(n^3)$

Try to Use Divide-and-Conquer



- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- matrix_multiplication(A, B) recursively calls
  matrix_multiplication(A_{11}, B_{11}),
  matrix_multiplication(A_{12}, B_{21}),
  $\cdots$

Try to Use Divide-and-Conquer



$$A = \begin{array}{|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{n/2} & \\ \hline A_{11} & A_{12} & \} n/2 \\ \hline A_{21} & A_{22} & \\ \hline \end{array} \qquad B = \begin{array}{|c|c|l} \multicolumn{1}{c}{} & \multicolumn{1}{c}{n/2} & \\ \hline B_{11} & B_{12} & \} n/2 \\ \hline B_{21} & B_{22} & \\ \hline \end{array}$$

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- matrix_multiplication(A, B) recursively calls
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  matrix_multiplication(A_{12}, B_{21}),
  $\cdots$
- Recurrence for running time: $T(n) = 8T(n/2) + O(n^2)$
- $T(n) = O(n^3)$

Strassen's Algorithm

- $T(n) = 8T(n/2) + O(n^2)$
- Strassen's Algorithm: improve the number of multiplications from 8 to 7!
- New recurrence: $T(n) = \color{red}{7}T(n/2) + O(n^2)$

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- Strassen's Algorithm: improve the number of multiplications from 8 to 7!
- New recurrence: $T(n) = {\color{red}7}T(n/2) + O(n^2)$
- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

Outline

Methods for Solving Recurrences

- The recursion-tree method
- The master theorem

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$

Recursion-Tree Method

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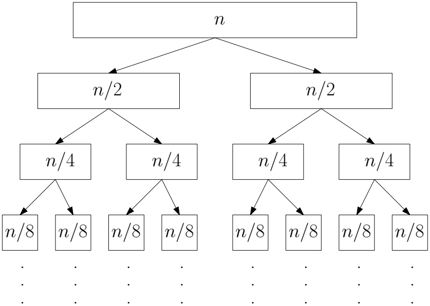
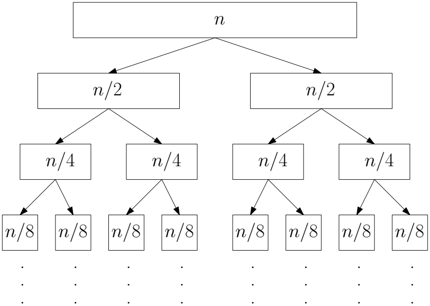


Recursion-Tree Method

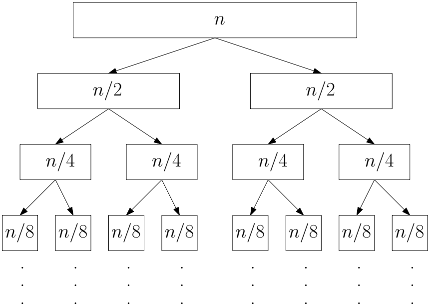
- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$

Recursion-Tree Method

- $T(n) = 2T(n/2) + O(n)$



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels

Recursion-Tree Method

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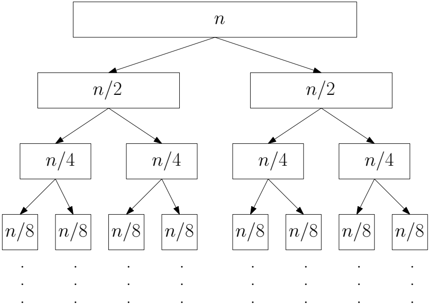

- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

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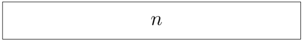

n

Recursion-Tree Method

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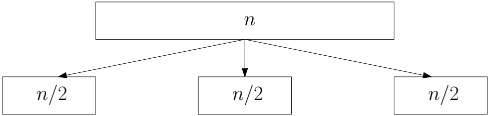

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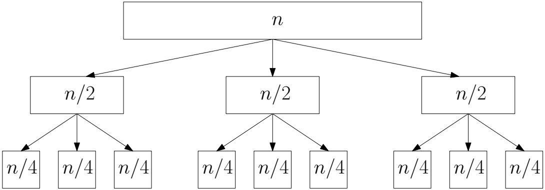

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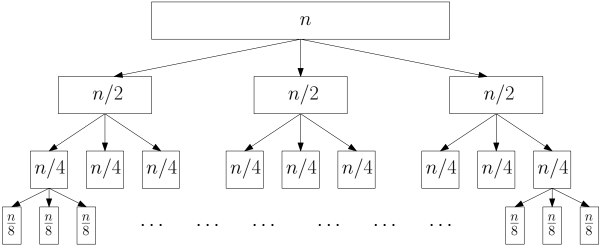

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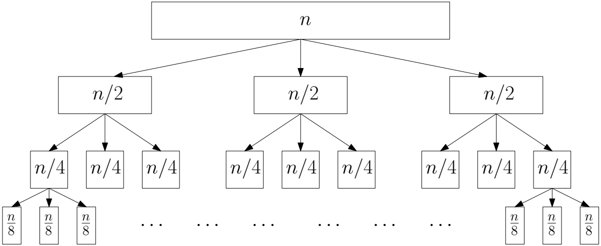

- Total running time at level i?

Recursion-Tree Method

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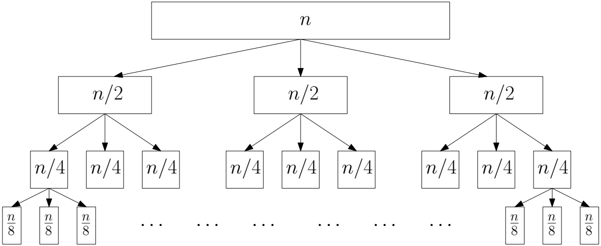

- Total running time at level i? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$

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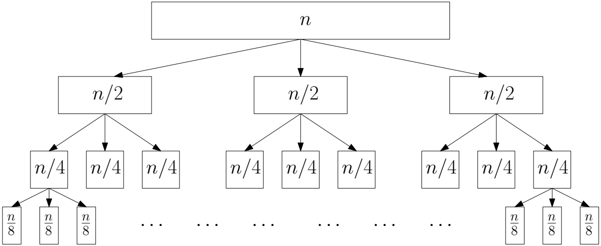

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Recursion-Tree Method

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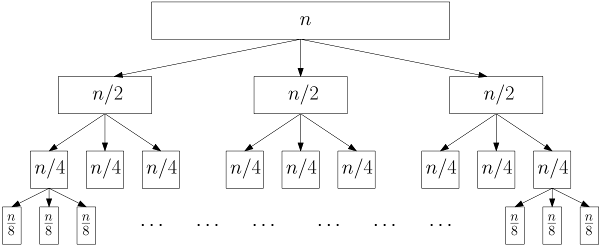

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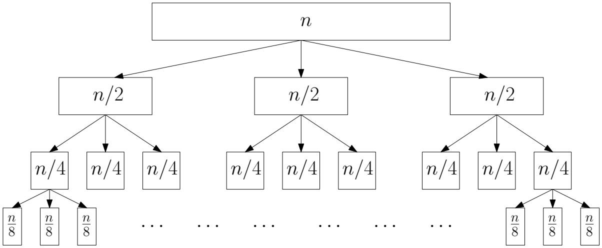

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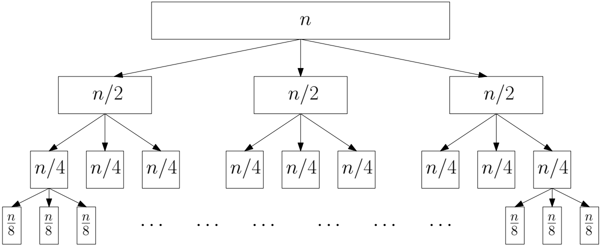

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$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{2}\right)^i n = O\left(n\left(\frac{3}{2}\right)^{\lg_2 n}\right) = O(3^{\lg_2 n}) = O(n^{\lg_2 3}).$$

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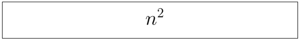

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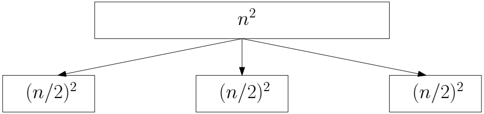

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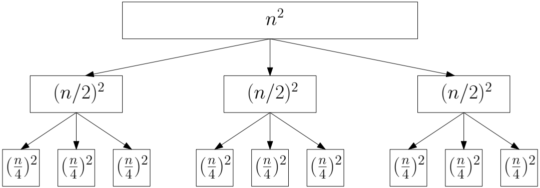

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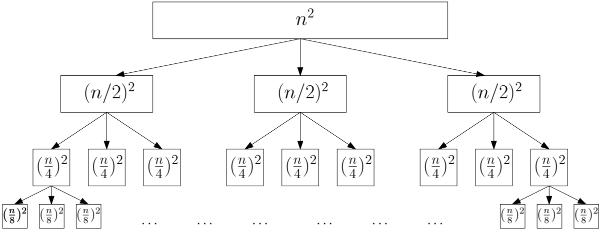

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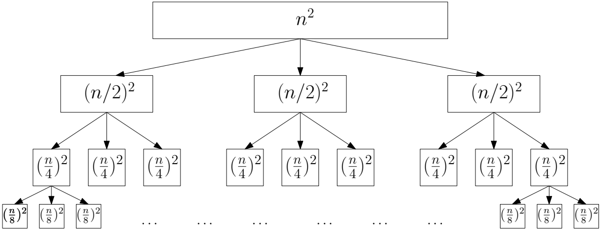

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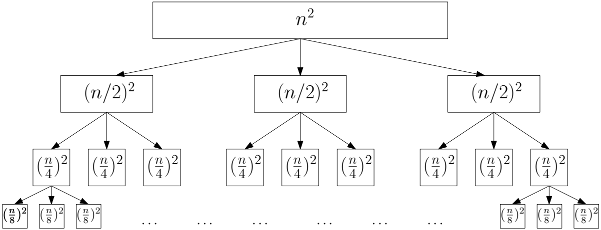

- Total running time at level i? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$

Recursion-Tree Method

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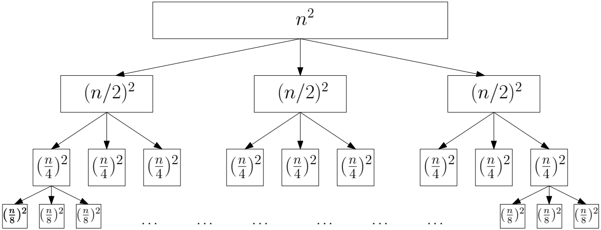

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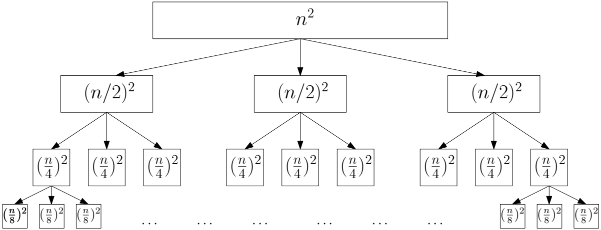

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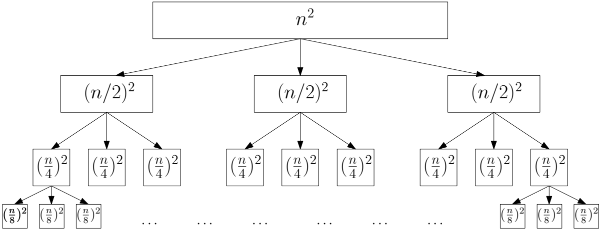

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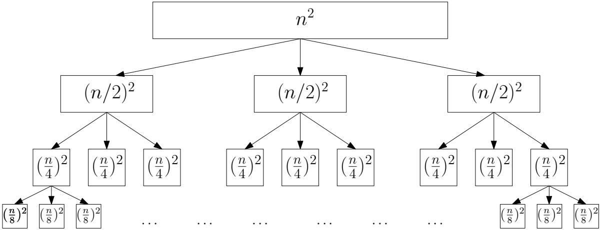

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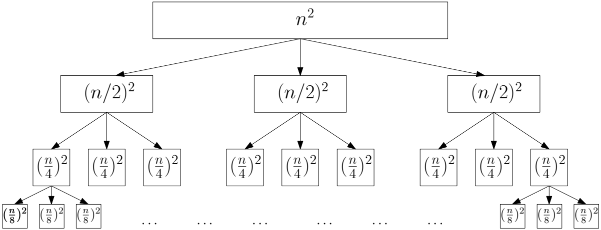

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$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$				$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\lg_2 3})$
$T(n) = 3T(n/2) + O(n^2)$				$O(n^2)$

Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$				$O(n^{\lg_2 3})$
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Master Theorem

Recurrences	a	b	c	time
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Theorem $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} & \text{if } c < \lg_b a \\ & \text{if } c = \lg_b a \\ & \text{if } c > \lg_b a \end{cases}$$

Master Theorem

Recurrences	a	b	c	time
$T(n) = 2T(n/2) + O(n)$	2	2	1	$O(n \lg n)$
$T(n) = 3T(n/2) + O(n)$	3	2	1	$O(n^{\lg_2 3})$
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- Ex: $T(n) = 2T(n/2) + O(n^2)$. Case 3.

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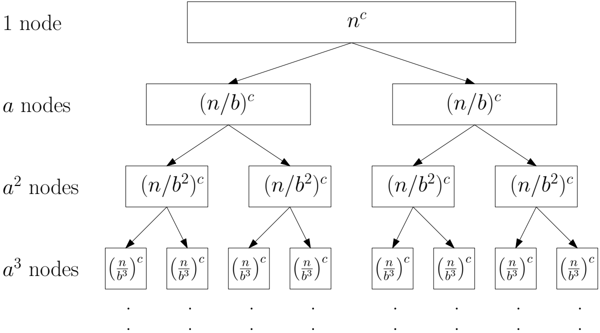
Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



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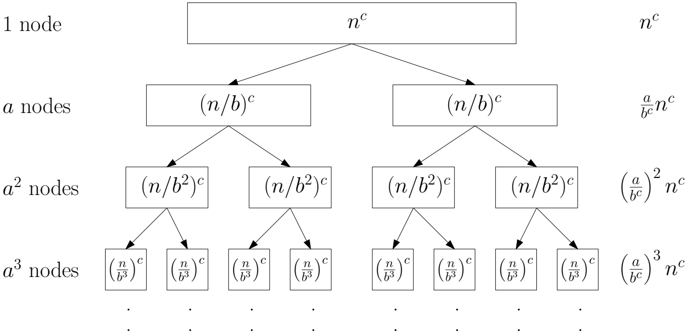

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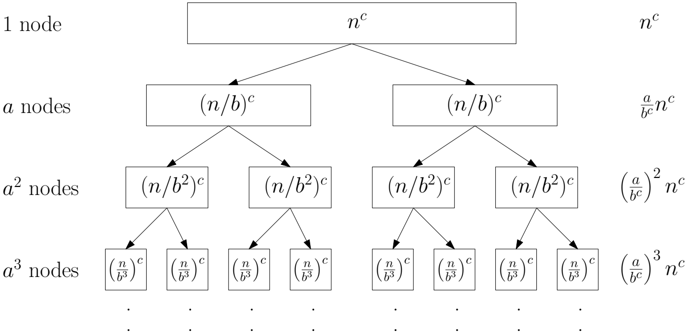

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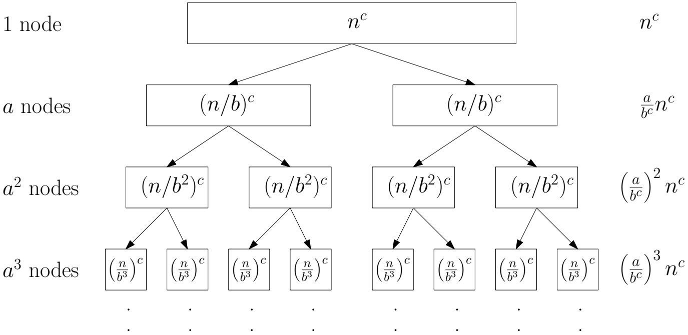

- $c < \lg_b a$: bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\lg_b n} n^c = n^{\lg_b a}$
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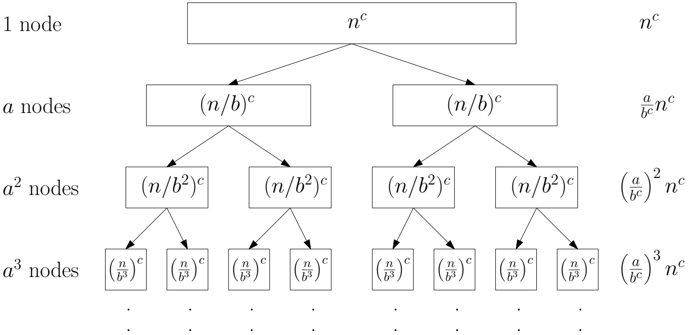

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- $c = \lg_b a$: all levels have same time: $n^c \lg_b n = O(n^c \lg n)$
- $c > \lg_b a$: top-level dominates: $O(n^c)$

Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
6. Solving Recurrences
7. **Computing n-th Fibonacci Number**

Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

n-th Fibonacci Number

Input: integer $n > 0$

Output: F_n

Computing F_n : Stupid Divide-and-Conquer Algorithm

Fib(n)

```
if $n = 0$ return 0
if $n = 1$ return 1
return Fib$(n-1)$ + Fib$(n-2)$
```

Q: Is the running time of the algorithm polynomial or exponential in n?

Computing F_n : Stupid Divide-and-Conquer Algorithm

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1 if $n = 0$ return 0
2 if $n = 1$ return 1
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Q: Is the running time of the algorithm polynomial or exponential in n?

A: Exponential

Computing F_n : Stupid Divide-and-Conquer Algorithm

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A: Exponential

- Running time is at least $\Omega(F_n)$

Computing F_n : Stupid Divide-and-Conquer Algorithm

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```
① if $n = 0$ return 0
② if $n = 1$ return 1
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Q: Is the running time of the algorithm polynomial or exponential in n?

A: Exponential

- Running time is at least $\Omega(F_n)$
- F_n is exponential in n

Computing F_n: Reasonable Algorithm

Fib(n)

```
$F[0] \leftarrow 0$
$F[1] \leftarrow 1$
for $i \leftarrow 2$ to $n$ do
    $F[i] \leftarrow F[i-1] + F[i-2]$
return $F[n]$
```

- Dynamic Programming

Computing F_n: Reasonable Algorithm

```
Fib($n$)
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- Dynamic Programming
- Running time $=$?

Computing F_n: Reasonable Algorithm

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- Dynamic Programming
- Running time = $O(n)$

Computing F_n: Even Better Algorithm

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\cdots$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

power(n)

```
if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$R \leftarrow$ power($\lfloor n/2 \rfloor$)
$R \leftarrow R \times R$
if $n$ is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
return $R$
```

Fib(n)

```
if $n = 0$ then return $0$
$M \leftarrow$ power($n - 1$)
return $M[1][1]$
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- Recurrence for running time? $T(n) = T(n/2) + O(1)$
- $T(n) = O(\lg n)$

Running time $= O(\lg n)$: We Cheated!

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- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time

Running time $= O(\lg n)$: We Cheated!

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- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time
- Even printing $F(n)$ requires time much larger than $O(\lg n)$

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- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time
- Even printing $F(n)$ requires time much larger than $O(\lg n)$

Fixing the Problem

To compute F_n, we need $O(\lg n)$ basic arithmetic operations on integers

Summary: Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

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- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, $\cdots$:
  $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \lg n)$

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- Matrix Multiplication:
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- Matrix Multiplication:
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- Usually, designing better algorithm for “combine” step is key to improve running time