

CSE 431/531: Algorithm Analysis and Design (Spring 2018)

Dynamic Programming

Lecturer: Shi Li

*Department of Computer Science and Engineering
University at Buffalo*

Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many **independent** sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many **overlapping** sub-problems
- Build solutions for larger and larger sub-problems
- Use a **table** to store solutions for sub-problems for reuse

Recall: Computing the n -th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots

Fib(n)

- 1 $F[0] \leftarrow 0$
- 2 $F[1] \leftarrow 1$
- 3 for $i \leftarrow 2$ to n do
- 4 $F[i] \leftarrow F[i - 1] + F[i - 2]$
- 5 return $F[n]$

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- Store each $F[i]$ for future use.

Outline

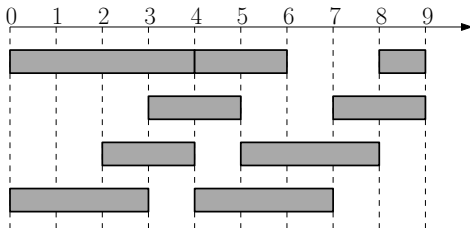
- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem
- 3 Knapsack Problem
- 4 Longest Common Subsequence
 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Graphs with Negative Weights
 - Shortest Paths in Directed Acyclic Graphs
 - Bellman-Ford Algorithm
- 6 All-Pair Shortest Paths and Floyd-Warshall
- 7 Matrix Chain Multiplication
- 8 Summary

Recall: Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs

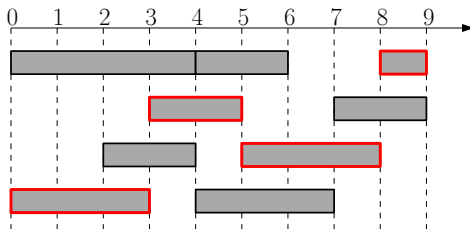


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Weighted Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-weight subset of mutually compatible jobs

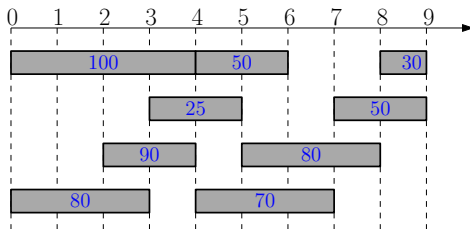
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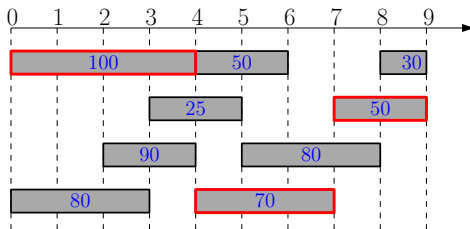
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Hard to Design a Greedy Algorithm

Q: Which job is safe to schedule?

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- Job with the earliest finish time?

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- Job with the largest weight?

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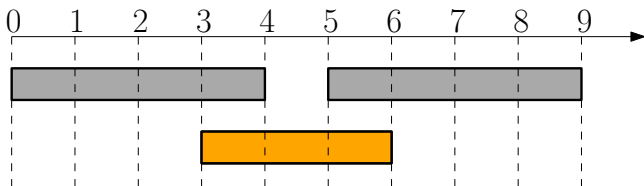
No, when weights are equal, this is the shortest job

Hard to Design a Greedy Algorithm

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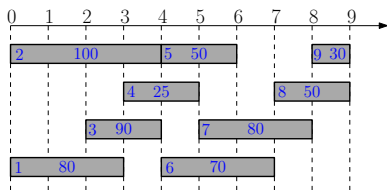
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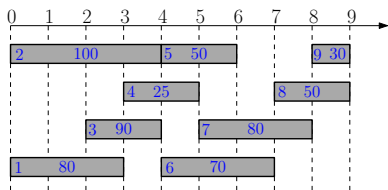
Designing a Dynamic Programming Algorithm

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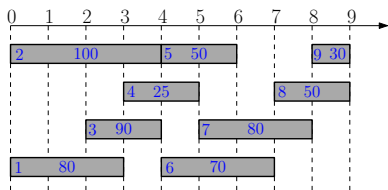
- Sort jobs according to non-decreasing order of finish times

Designing a Dynamic Programming Algorithm



- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \dots, i\}$

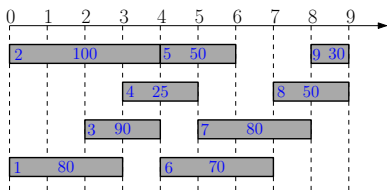
Designing a Dynamic Programming Algorithm



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i	$opt[i]$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

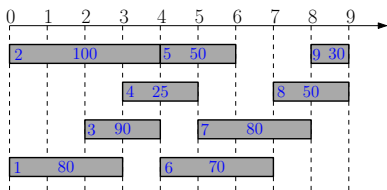
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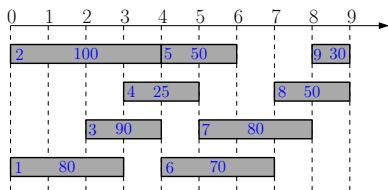
Designing a Dynamic Programming Algorithm



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i	$opt[i]$
0	0
1	80
2	
3	
4	
5	
6	
7	
8	
9	

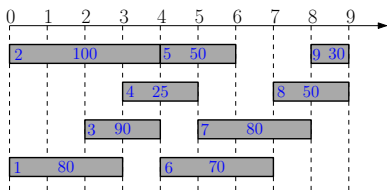
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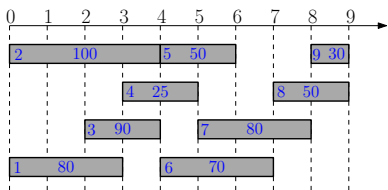
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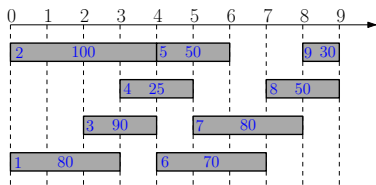
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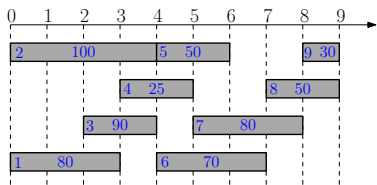
i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

Designing a Dynamic Programming Algorithm



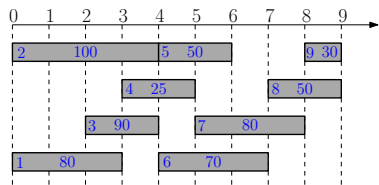
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- $opt[i]$: optimal value for the instance

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \dots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \dots, opt[i - 1]$

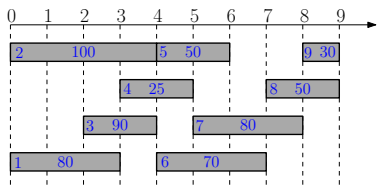
Designing a Dynamic Programming Algorithm



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Q: The value of optimal solution that **does not contain** i ?

Designing a Dynamic Programming Algorithm

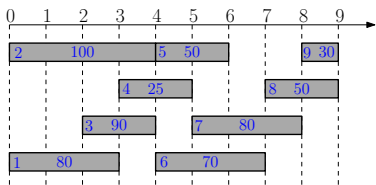


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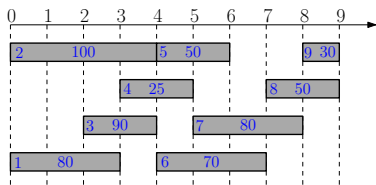
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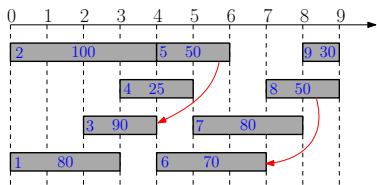
Q: The value of optimal solution that **does not contain** i ?

A: $opt[i-1]$

Q: The value of optimal solution that **contains** job i ?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

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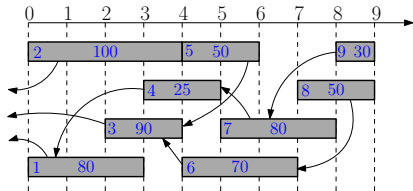
Recursion for $opt[i]$:

$$opt[i] = \max \{opt[i - 1], v_i + opt[p_i]\}$$

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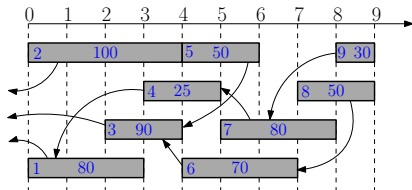


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] =$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

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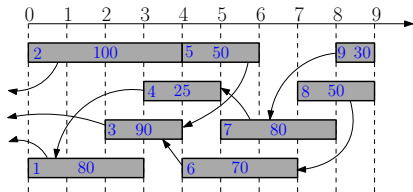


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Designing a Dynamic Programming Algorithm

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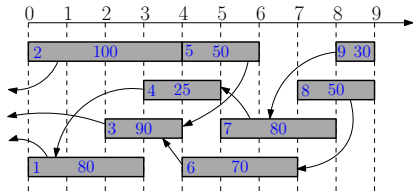


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Designing a Dynamic Programming Algorithm

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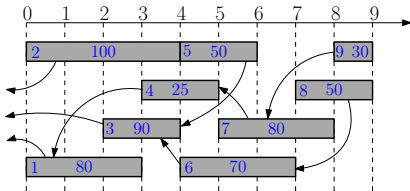


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Designing a Dynamic Programming Algorithm

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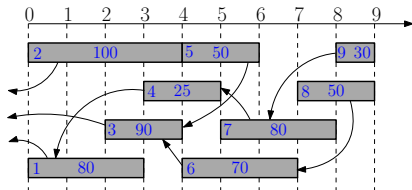


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- $opt[3] = \max\{opt[2], 90 + opt[0]\}$
- $opt[4] =$
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Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

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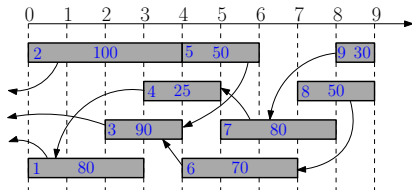


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Designing a Dynamic Programming Algorithm

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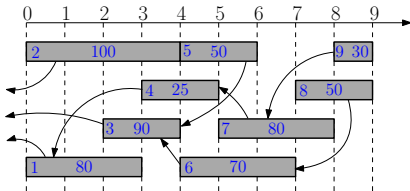


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- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\}$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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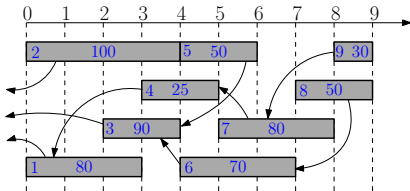


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Designing a Dynamic Programming Algorithm

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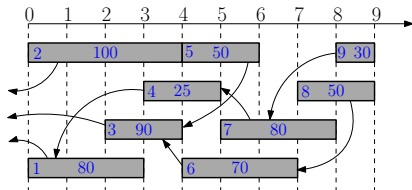


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- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\}$

Designing a Dynamic Programming Algorithm

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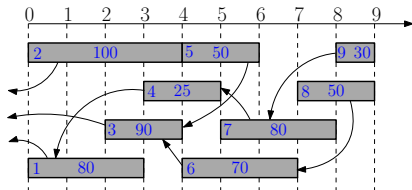


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Designing a Dynamic Programming Algorithm

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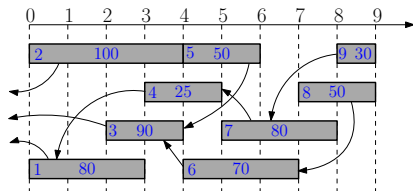


- $opt[0] = 0, opt[1] = 80, opt[2] = 100$
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Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max \{opt[i - 1], v_i + opt[p_i]\}$$



- $opt[0] = 0$, $opt[1] = 80$, $opt[2] = 100$
- $opt[3] = 100$, $opt[4] = 105$, $opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Recursive Algorithm to Compute $opt[n]$

- 1 sort jobs by non-decreasing order of finishing times
- 2 compute p_1, p_2, \dots, p_n
- 3 return $compute-opt(n)$

$compute-opt(i)$

- 1 if $i = 0$ then
- 2 return 0
- 3 else
- 4 return $\max\{compute-opt(i - 1), v_i + compute-opt(p_i)\}$

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- Running time can be exponential in n

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- Running time can be exponential in n
- Reason: we are computed each $opt[i]$ many times

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- 3 else
- 4 return $\max\{\text{compute-opt}(i - 1), v_i + \text{compute-opt}(p_i)\}$

- Running time can be exponential in n
- Reason: we are computed each $opt[i]$ many times
- Solution: store the value of $opt[i]$, so it's computed only once

Memoized Recursive Algorithm

- 1 sort jobs by non-decreasing order of finishing times
- 2 compute p_1, p_2, \dots, p_n
- 3 $opt[0] \leftarrow 0$ and $opt[i] \leftarrow \perp$ for every $i = 1, 2, 3, \dots, n$
- 4 return compute-opt(n)

compute-opt(i)

- 1 if $opt[i] = \perp$ then
- 2 $opt[i] \leftarrow \max\{\text{compute-opt}(i - 1), v_i + \text{compute-opt}(p_i)\}$
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- Running time sorting: $O(n \lg n)$

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Dynamic Programming

- 1 sort jobs by non-decreasing order of finishing times
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- 3 $opt[0] \leftarrow 0$
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Dynamic Programming

- 1 sort jobs by non-decreasing order of finishing times
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How Can We Recover the Optimum Schedule?

- 1 sort jobs by non-decreasing order of finishing times
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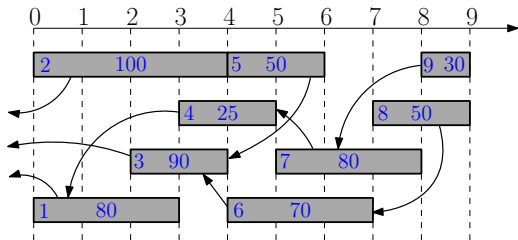
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- 1 $i \leftarrow n, S \leftarrow \emptyset$
- 2 while $i \neq 0$
- 3 if $b[i] = N$
- 4 $i \leftarrow i - 1$
- 5 else
- 6 $S \leftarrow S \cup \{i\}$
- 7 $i \leftarrow p_i$
- 8 return S

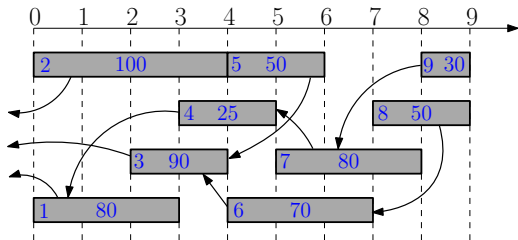
Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	\perp
1	80	
2	100	
3	100	
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8	220	
9	220	



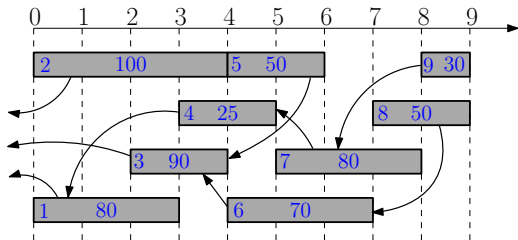
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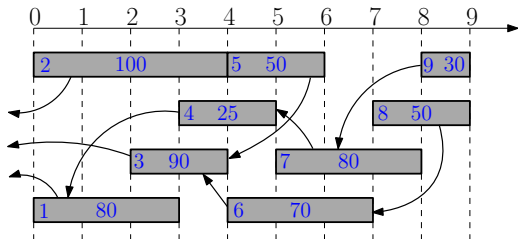
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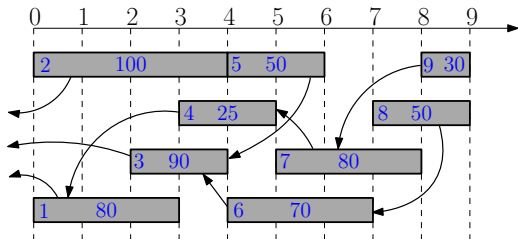
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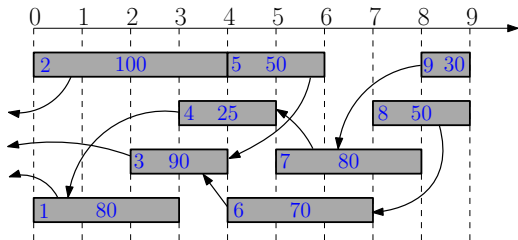
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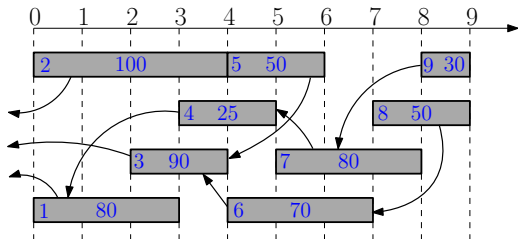
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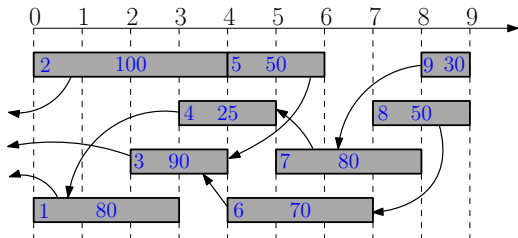
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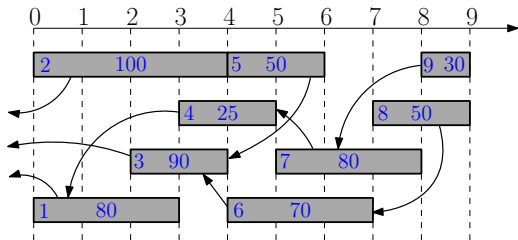
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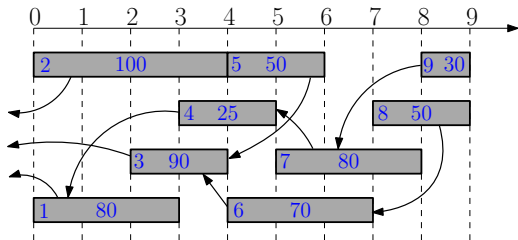
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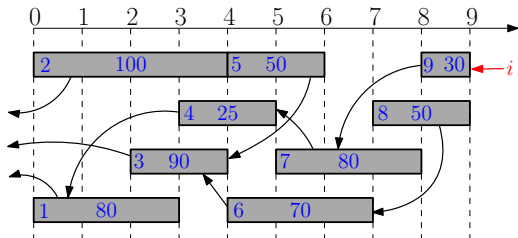
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8	220	Y
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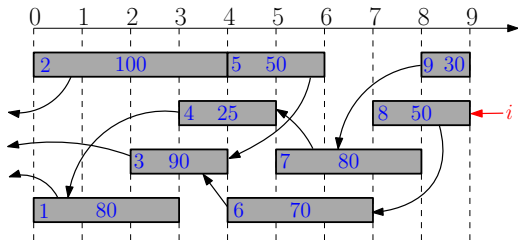
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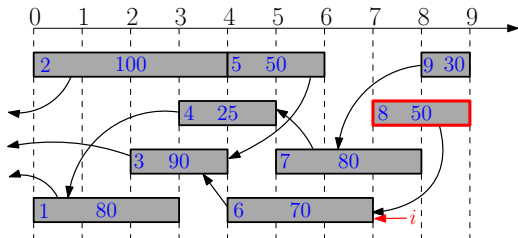
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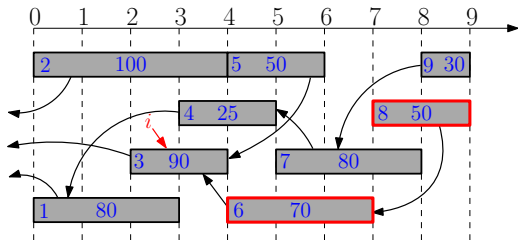
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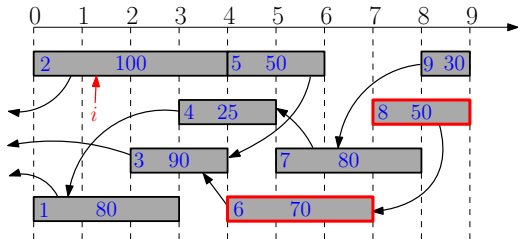
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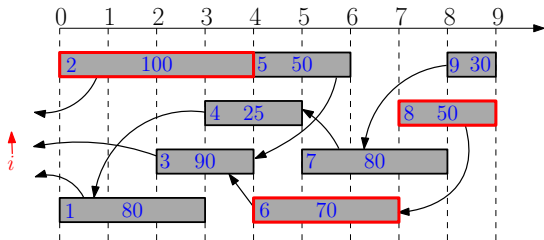
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7	185	Y
8	220	Y
9	220	N



Dynamic Programming

- Break up a problem into many **overlapping** sub-problems
- Build solutions for larger and larger sub-problems
- Use a **table** to store solutions for sub-problems for reuse

Outline

- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem**
- 3 Knapsack Problem
- 4 Longest Common Subsequence
 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Graphs with Negative Weights
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Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \quad \text{s.t. } \sum_{i \in S} w_i \leq W.$$

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- Motivation: you have budget W , and want to buy a subset of items, so as to spend as much money as possible.

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Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$

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- Motivation: you have budget W , and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Greedy Algorithms for Subset Sum

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A: No. $W = 100, n = 3, w = (51, 50, 50)$.

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Greedy Algorithms for Subset Sum

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Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \dots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that **does not contain** i ?

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Q: The value of the optimum solution that **does not contain** i ?

A: $opt[i - 1, W']$

Q: The value of the optimum solution that **contains** i ?

A: $opt[i - 1, W' - w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \dots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} i = 0 \\ i > 0, w_i > W' \\ i > 0, w_i \leq W' \end{cases}$$

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Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \dots, w_i)$;
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$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i - 1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{l} opt[i - 1, W'] \\ opt[i - 1, W' - w_i] + w_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

Dynamic Programming

- 1 for $W' \leftarrow 0$ to W
- 2 $opt[0, W'] \leftarrow 0$
- 3 for $i \leftarrow 1$ to n
- 4 for $W' \leftarrow 0$ to W
- 5 $opt[i, W'] \leftarrow opt[i - 1, W']$
- 6 if $w_i \geq W'$ and $opt[i - 1, W' - w_i] + w_i \geq opt[i, W']$
 then
- 7 $opt[i, W'] \leftarrow opt[i - 1, W' - w_i] + w_i$
- 8 return $opt[n, W]$

Recover the Optimum Set

- 1 for $W' \leftarrow 0$ to W
- 2 $opt[0, W'] \leftarrow 0$
- 3 for $i \leftarrow 1$ to n
- 4 for $W' \leftarrow 0$ to W
- 5 $opt[i, W'] \leftarrow opt[i - 1, W']$
- 6 $b[i, W'] \leftarrow \mathbf{N}$
- 7 if $w_i \leq W'$ and $opt[i - 1, W' - w_i] + w_i \geq opt[i, W']$
then
- 8 $opt[i, W'] \leftarrow opt[i - 1, W' - w_i] + w_i$
- 9 $b[i, W'] \leftarrow \mathbf{Y}$
- 10 return $opt[n, W]$

Recover the Optimum Set

- 1 $i \leftarrow n, W' \leftarrow W, S \leftarrow \emptyset$
- 2 while $i > 0$
- 3 if $b[i, W'] = Y$ then
- 4 $W' \leftarrow W' - w_i$
- 5 $S \leftarrow S \cup \{i\}$
- 6 $i \leftarrow i - 1$
- 7 return S

Running Time of Algorithm

- 1 for $W' \leftarrow 0$ to W
- 2 $opt[0, W'] \leftarrow 0$
- 3 for $i \leftarrow 1$ to n
- 4 for $W' \leftarrow 0$ to W
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- 6 if $w_i \leq W'$ and $opt[i - 1, W' - w_i] + w_i \geq opt[i, W']$
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- 8 return $opt[n, W]$

Running Time of Algorithm

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1 for  $W' \leftarrow 0$  to  $W$ 
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3 for  $i \leftarrow 1$  to  $n$ 
4   for  $W' \leftarrow 0$  to  $W$ 
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6     if  $w_i \leq W'$  and  $opt[i - 1, W' - w_i] + w_i \geq opt[i, W']$ 
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8          $opt[i, W'] \leftarrow opt[i - 1, W' - w_i] + w_i$ 
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- Running time is $O(nW)$

Running Time of Algorithm

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- Running time is $O(nW)$
- Running time is **pseudo-polynomial** because it depends on value of the input integers.

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Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \quad \text{s.t.} \quad \sum_{i \in S} w_i \leq W.$$

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a value $v_i > 0$ for each item i

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- Motivation: you have budget W , and want to buy a subset of items of maximum total value

Greedy Algorithm

- 1 sort items according to non-increasing order of v_i/w_i
- 2 for each item in the ordering
- 3 take the item if we have enough budget

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- 1 sort items according to non-increasing order of v_i/w_i
 - 2 for each item in the ordering
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- Bad example: $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.

Greedy Algorithm

- 1 sort items according to non-increasing order of v_i/w_i
 - 2 for each item in the ordering
 - 3 take the item if we have enough budget
- Bad example: $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.
 - Optimum takes item 2 and greedy takes item 1.

Fractional Knapsack Problem

Input: integer bound $W > 0$,

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a vector $(\alpha_1, \alpha_2, \dots, \alpha_n) \in [0, 1]^n$ that

$$\text{maximizes } \sum_{i=1}^n \alpha_i v_i \quad \text{s.t. } \sum_{i=1}^n \alpha_i w_i \leq W.$$

Fractional Knapsack Problem

Input: integer bound $W > 0$,
a set of n items, each with an integer weight $w_i > 0$
a value $v_i > 0$ for each item i

Output: a vector $(\alpha_1, \alpha_2, \dots, \alpha_n) \in [0, 1]^n$ that

$$\text{maximizes } \sum_{i=1}^n \alpha_i v_i \quad \text{s.t. } \sum_{i=1}^n \alpha_i w_i \leq W.$$

Greedy Algorithm for Fractional Knapsack

- 1 sort items according to non-increasing order of v_i/w_i ,
- 2 for each item according to the ordering, take as much fraction of the item as possible.

Greedy is Optimum for Fractional Knapsack

Greedy Algorithm for Fractional Knapsack

- 1 sort items according to non-increasing order of v_i/w_i ,
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Greedy Algorithm for Fractional Knapsack

- 1 sort items according to non-increasing order of v_i/w_i ,
- 2 for each item according to the ordering, take as much fraction of the item as possible.

- $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.

Greedy is Optimum for Fractional Knapsack

Greedy Algorithm for Fractional Knapsack

- 1 sort items according to non-increasing order of v_i/w_i ,
- 2 for each item according to the ordering, take as much fraction of the item as possible.

- $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.
- $\alpha_1 = 1, \alpha_2 = 0.99, \text{value} = 1.1 + 99 = 100.1$.

Greedy is Optimum for Fractional Knapsack

Greedy Algorithm for Fractional Knapsack

- 1 sort items according to non-increasing order of v_i/w_i ,
 - 2 for each item according to the ordering, take as much fraction of the item as possible.
- $W = 100, n = 2, w = (1, 100), v = (1.1, 100)$.
 - $\alpha_1 = 1, \alpha_2 = 0.99, \text{value} = 1.1 + 99 = 100.1$.
 - Idea of proof: exchanging argument. (Left as homework exercise).

DP for ($\{0, 1\}$ -)Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \dots, i\}$.
- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \dots, W$.

$$opt[i, W'] = \begin{cases} & i = 0 \\ & i > 0, w_i > W' \\ & i > 0, w_i \leq W' \end{cases}$$

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- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \dots, W$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i - 1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{l} opt[i - 1, W'] \\ opt[i - 1, W' - w_i] + v_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

Avoiding Unnecessary Computation and Memory Using Memoized Algorithm and Hash Map

compute-opt(i, W')

- 1 if $opt[i, W'] \neq \perp$ return $opt[i, W']$
- 2 if $i = 0$ then $r \leftarrow 0$
- 3 else
- 4 $r \leftarrow \text{compute-opt}(i - 1, W')$
- 5 if $w_i \leq W'$ then
- 6 $r' \leftarrow \text{compute-opt}(i - 1, W' - w_i) + v_i$
- 7 if $r' > r$ then $r \leftarrow r'$
- 8 $opt[i, W'] \leftarrow r$
- 9 return r

- Use hash map for opt

Exercise: Items with 3 Parameters

Input: integer bounds $W > 0$, $Z > 0$,
a set of n items, each with an integer weight $w_i > 0$
a size $z_i > 0$ for each item i
a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\begin{aligned} & \text{maximizes } \sum_{i \in S} v_i && \text{s.t.} \\ & \sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} z_i \leq Z \end{aligned}$$

Outline

- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem
- 3 Knapsack Problem
- 4 Longest Common Subsequence**
 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Graphs with Negative Weights
 - Shortest Paths in Directed Acyclic Graphs
 - Bellman-Ford Algorithm
- 6 All-Pair Shortest Paths and Floyd-Warshall
- 7 Matrix Chain Multiplication
- 8 Summary

Subsequence

- $A = bacdca$
- $C = adca$

Subsequence

- $A = bacdca$
- $C = adca$
- C is a subsequence of A

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- $A = bacdca$
- $C = adca$
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Def. Given two sequences $A[1 .. n]$ and $C[1 .. t]$ of letters, C is called a **subsequence** of A if there exists integers

$1 \leq i_1 < i_2 < i_3 < \dots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \dots, t$.

Subsequence

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$1 \leq i_1 < i_2 < i_3 < \dots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \dots, t$.

- Exercise: how to check if sequence C is a subsequence of A ?

Longest Common Subsequence

Input: $A[1 .. n]$ and $B[1 .. m]$

Output: the longest common subsequence of A and B

Example:

- $A = \text{'bacdca'}$
- $B = \text{'adbcdca'}$

Longest Common Subsequence

Input: $A[1 .. n]$ and $B[1 .. m]$

Output: the longest common subsequence of A and B

Example:

- $A = 'bacdca'$
- $B = 'adbcdca'$
- $LCS(A, B) = 'adca'$

Longest Common Subsequence

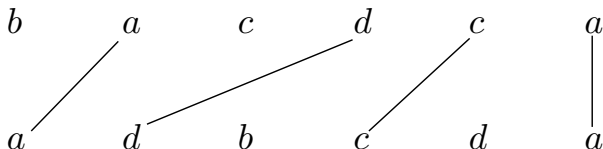
Input: $A[1 .. n]$ and $B[1 .. m]$

Output: the longest common subsequence of A and B

Example:

- $A = 'bacdca'$
 - $B = 'adbcdca'$
 - $LCS(A, B) = 'adca'$
-
- Applications: edit distance (diff), similarity of DNAs

Matching View of LCS



- Goal of LCS: find a maximum-size non-crossing matching between letters in A and letters in B .

Reduce to Subproblems

- $A = \text{'bacdca'}$
- $B = \text{'adbcdca'}$

Reduce to Subproblems

- $A = \text{'bacdca'}$
- $B = \text{'adbcdca'}$

Reduce to Subproblems

- $A = \text{'bacdc'}$
- $B = \text{'adbcd'}$

Reduce to Subproblems

- $A = \text{'bacdc'}$
- $B = \text{'adbcd'}$
- either the last letter of A is not matched:
- or the last letter of B is not matched:

Reduce to Subproblems

- $A = \text{'bacdc'}$
- $B = \text{'adbcd'}$
- either the last letter of A is not matched:
 - need to compute $\text{LCS}(\text{'bacdc'}, \text{'adbcd'})$
- or the last letter of B is not matched:

Reduce to Subproblems

- $A = \text{'bacdc'}$
- $B = \text{'adbcd'}$
- either the last letter of A is not matched:
 - need to compute $\text{LCS}(\text{'bacdc'}, \text{'adbcd'})$
- or the last letter of B is not matched:
 - need to compute $\text{LCS}(\text{'bacd'}, \text{'adbcd'})$

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} & \text{if } A[i] = B[j] \\ & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} opt[i - 1, j - 1] + 1 & \text{if } A[i] = B[j] \\ & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} opt[i - 1, j - 1] + 1 & \text{if } A[i] = B[j] \\ \max \begin{cases} opt[i - 1, j] \\ opt[i, j - 1] \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

```
1 for  $j \leftarrow 0$  to  $m$  do
2    $opt[0, j] \leftarrow 0$ 
3 for  $i \leftarrow 1$  to  $n$ 
4    $opt[i, 0] \leftarrow 0$ 
5   for  $j \leftarrow 1$  to  $m$ 
6     if  $A[i] = B[j]$  then
7        $opt[i, j] \leftarrow opt[i - 1, j - 1] + 1$ 
8     elseif  $opt[i, j - 1] \geq opt[i - 1, j]$  then
9        $opt[i, j] \leftarrow opt[i, j - 1]$ 
10    else
11       $opt[i, j] \leftarrow opt[i - 1, j]$ 
```

Dynamic Programming for LCS

- 1 for $j \leftarrow 0$ to m do
- 2 $opt[0, j] \leftarrow 0$
- 3 for $i \leftarrow 1$ to n
- 4 $opt[i, 0] \leftarrow 0$
- 5 for $j \leftarrow 1$ to m
- 6 if $A[i] = B[j]$ then
- 7 $opt[i, j] \leftarrow opt[i - 1, j - 1] + 1, \pi[i, j] \leftarrow \text{"↖"}$
- 8 elseif $opt[i, j - 1] \geq opt[i - 1, j]$ then
- 9 $opt[i, j] \leftarrow opt[i, j - 1], \pi[i, j] \leftarrow \text{"←"}$
- 10 else
- 11 $opt[i, j] \leftarrow opt[i - 1, j], \pi[i, j] \leftarrow \text{"↑"}$

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥						
2	0 ⊥						
3	0 ⊥						
4	0 ⊥						
5	0 ⊥						
6	0 ⊥						

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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←					
2	0 ⊥						
3	0 ⊥						
4	0 ⊥						
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2	0 ⊥						
3	0 ⊥						
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1	0 ⊥	0 ←	0 ←	1 ↖			
2	0 ⊥						
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←		
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0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	
2	0 ⊥						
3	0 ⊥						
4	0 ⊥						
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥						
3	0 ⊥						
4	0 ⊥						
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1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
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0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖					
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0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←				
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4	0 ⊥						
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
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0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥						
4	0 ⊥						
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥						
4	0 ⊥						
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
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4	0 ⊥						
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6	0 ⊥						

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<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←				
4	0 ⊥						
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
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<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←			
4	0 ⊥						
5	0 ⊥						
6	0 ⊥						

Example

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<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	
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<i>A</i>	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥						
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥						
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑					
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖				
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←			
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←		
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥						
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑					
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑				
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←			
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖		
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥						

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖					

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑				

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←			

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑		

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	

Example

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
6	0 ⊥	1 ↖	2 ↑	2 ←	3 ↑	3 ←	4 ↖

Example: Find Common Subsequence

	1	2	3	4	5	6
<i>A</i>	b	a	c	d	c	a
<i>B</i>	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥	0 ⊥
1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
2	0 ⊥	1 ↖	1 ←	1 ←	1 ←	1 ←	2 ↖
3	0 ⊥	1 ↑	1 ←	1 ←	2 ↖	2 ←	2 ←
4	0 ⊥	1 ↑	2 ↖	2 ←	2 ←	3 ↖	3 ←
5	0 ⊥	1 ↑	2 ↑	2 ←	3 ↖	3 ←	3 ←
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1	0 ⊥	0 ←	0 ←	1 ↖	1 ←	1 ←	1 ←
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Find Common Subsequence

- 1 $i \leftarrow n, j \leftarrow m, S \leftarrow ""$
- 2 while $i > 0$ and $j > 0$
- 3 if $\pi[i, j] = "\searrow"$ then
- 4 $S \leftarrow A[i] \bowtie S, i \leftarrow i - 1, j \leftarrow j - 1$
- 5 else if $\pi[i, j] = "\uparrow"$
- 6 $i \leftarrow i - 1$
- 7 else
- 8 $j \leftarrow j - 1$
- 9 return S

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A

each time we can delete a letter from A or insert a letter to A

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Obs. $\#OPs = \text{length}(A) + \text{length}(B) - 2 \cdot \text{length}(\text{LCS}(A, B))$

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- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem
- 3 Knapsack Problem
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 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Graphs with Negative Weights
 - Shortest Paths in Directed Acyclic Graphs
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Computing the Length of LCS

```
1 for  $j \leftarrow 0$  to  $m$  do
2    $opt[0, j] \leftarrow 0$ 
3 for  $i \leftarrow 1$  to  $n$ 
4    $opt[i, 0] \leftarrow 0$ 
5   for  $j \leftarrow 1$  to  $m$ 
6     if  $A[i] = B[j]$ 
7        $opt[i, j] \leftarrow opt[i - 1, j - 1] + 1$ 
8     elseif  $opt[i, j - 1] \geq opt[i - 1, j]$ 
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10    else
11       $opt[i, j] \leftarrow opt[i - 1, j]$ 
```

Obs. The i -th row of table only depends on $(i - 1)$ -th row.

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Q: How to use this observation to reduce space?

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A: We only keep two rows: the $(i - 1)$ -th row and the i -th row.

Linear Space Algorithm to Compute Length of LCS

```
1 for  $j \leftarrow 0$  to  $m$  do
2    $opt[0, j] \leftarrow 0$ 
3 for  $i \leftarrow 1$  to  $n$ 
4    $opt[i \bmod 2, 0] \leftarrow 0$ 
5   for  $j \leftarrow 1$  to  $m$ 
6     if  $A[i] = B[j]$ 
7        $opt[i \bmod 2, j] \leftarrow opt[i - 1 \bmod 2, j - 1] + 1$ 
8     elseif  $opt[i \bmod 2, j - 1] \geq opt[i - 1 \bmod 2, j]$ 
9        $opt[i \bmod 2, j] \leftarrow opt[i \bmod 2, j - 1]$ 
10    else
11       $opt[i \bmod 2, j] \leftarrow opt[i - 1 \bmod 2, j]$ 
12 return  $opt[n \bmod 2, m]$ 
```

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Recall: Single Source Shortest Path Problem

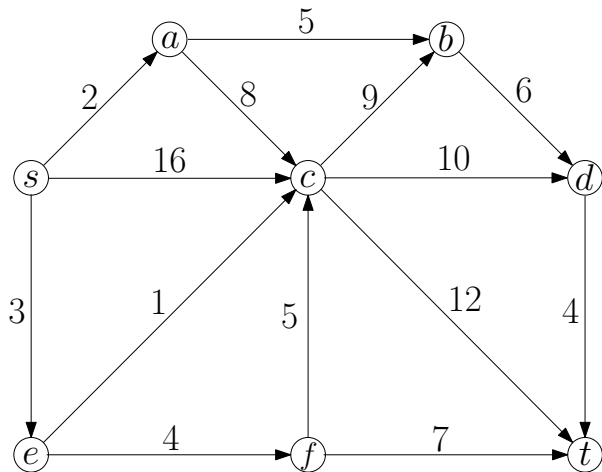
Single Source Shortest Paths

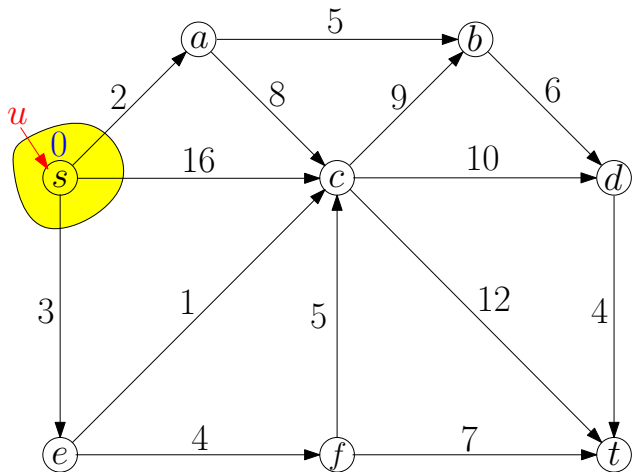
Input: directed graph $G = (V, E)$, $s \in V$

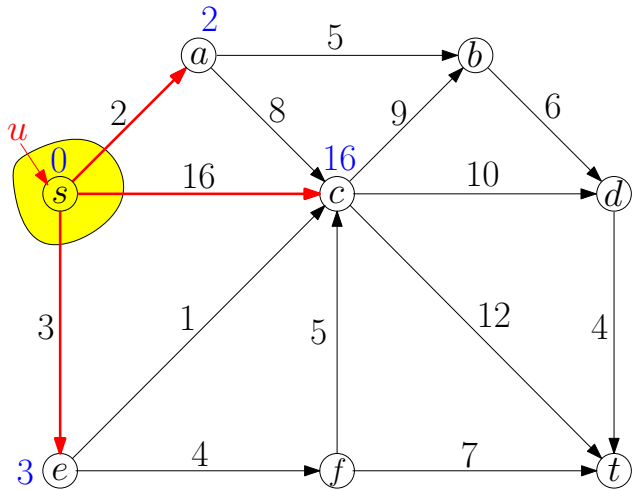
$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

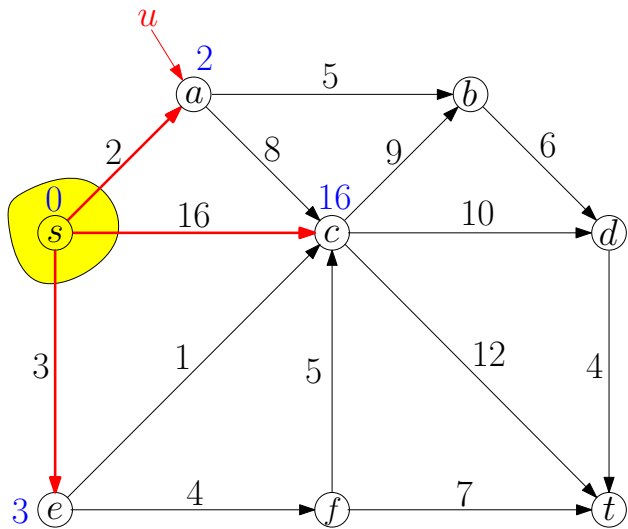
Output: shortest paths from s to all other vertices $v \in V$

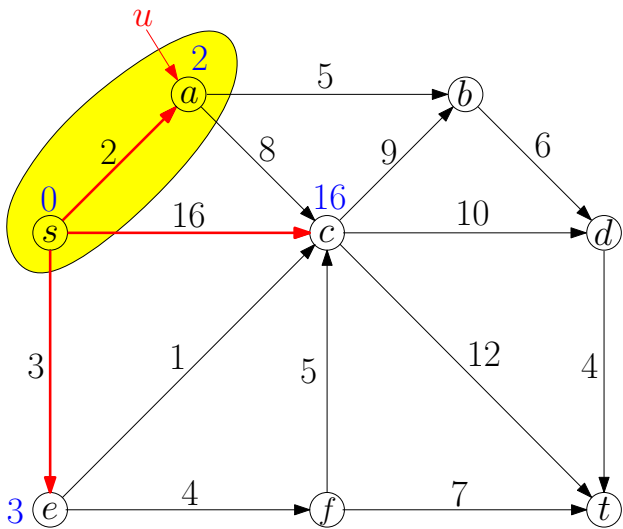
- Algorithm for the problem: Dijkstra's algorithm

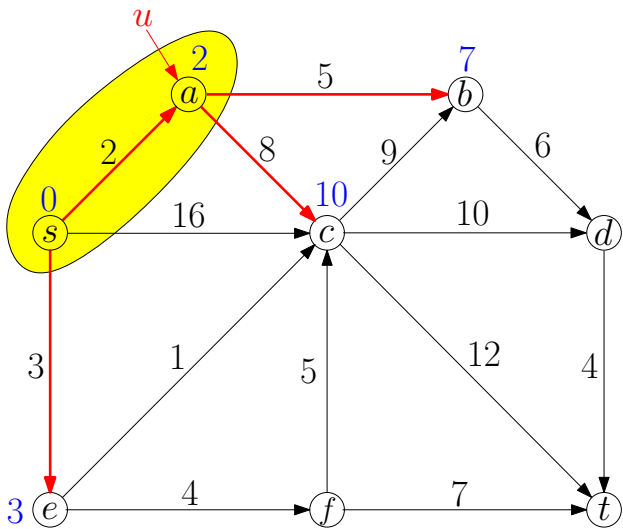


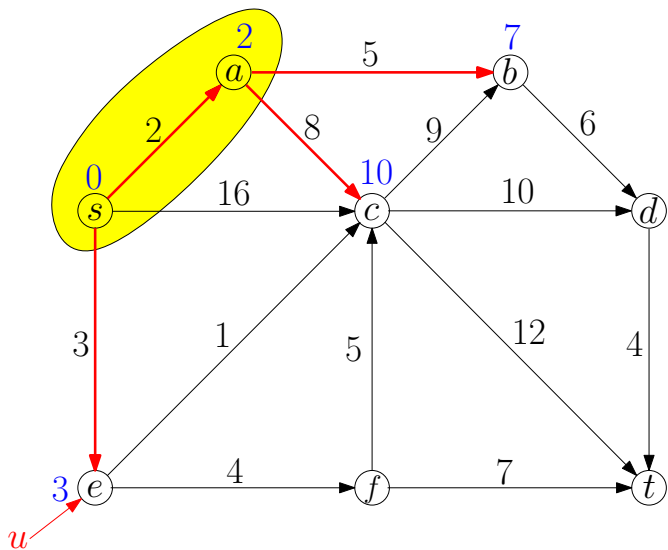


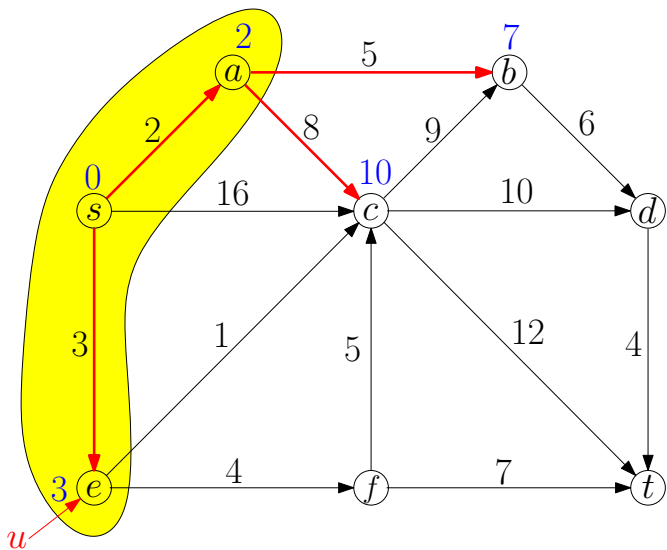


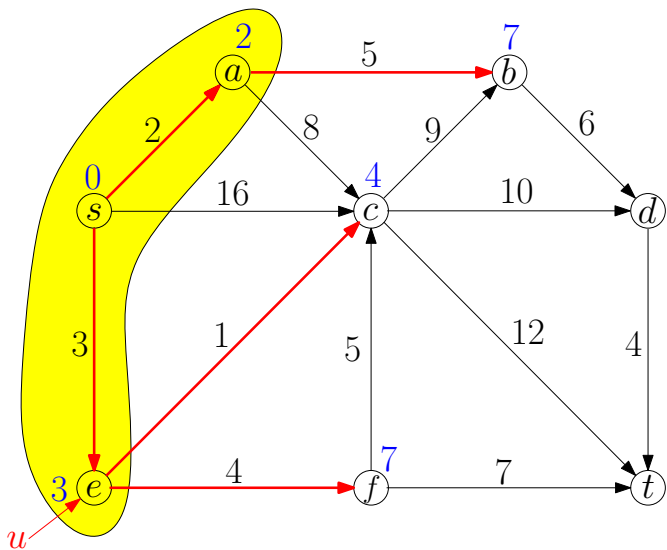


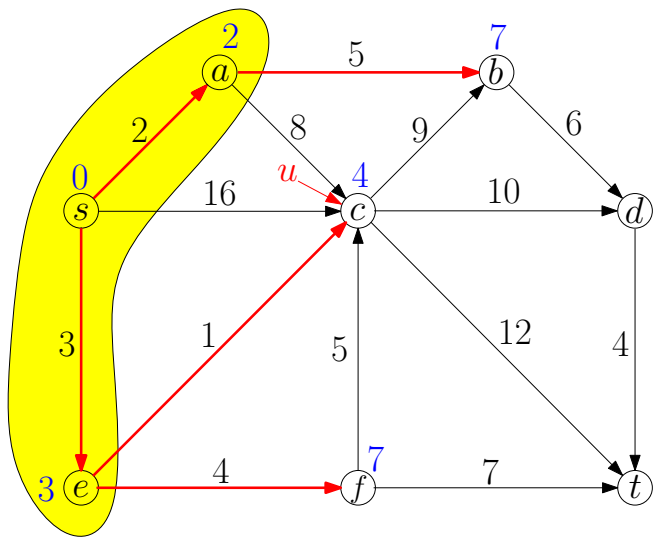


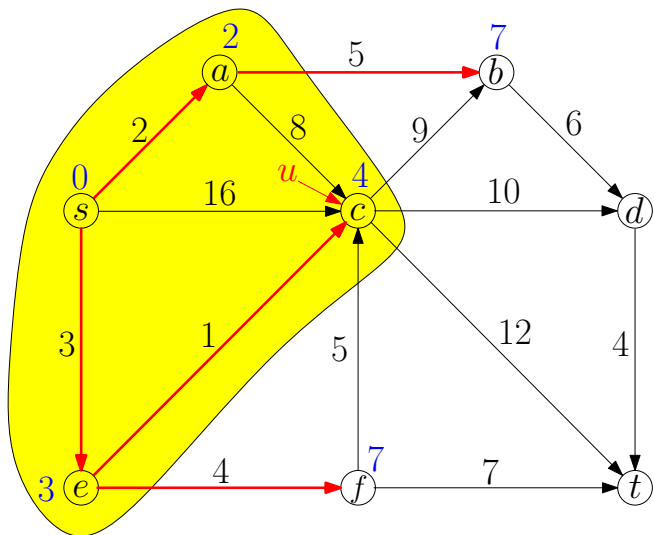


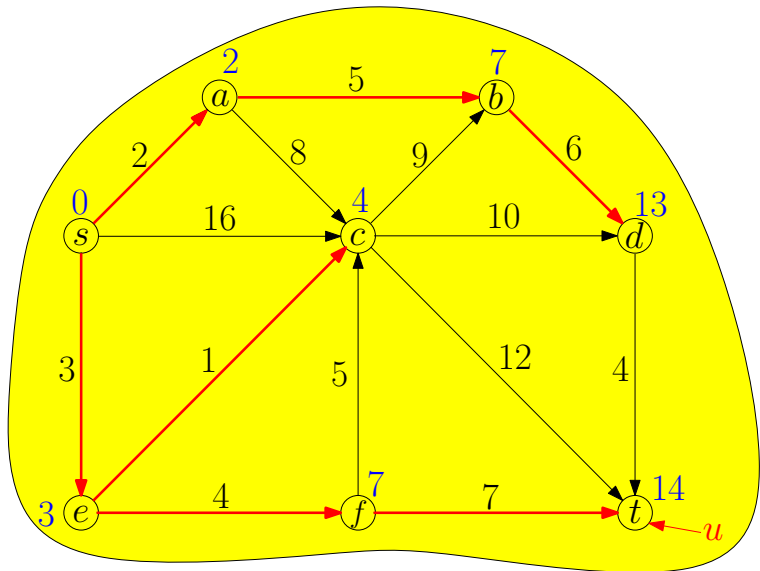












Dijkstra's Algorithm Using Priority Queue

Dijkstra(G, w, s)

- 1 $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
- 2 $Q \leftarrow$ empty queue, for each $v \in V$: $Q.insert(v, d(v))$
- 3 while $S \neq V$, do
- 4 $u \leftarrow Q.extract_min()$
- 5 $S \leftarrow S \cup \{u\}$
- 6 for each $v \in V \setminus S$ such that $(u, v) \in E$
- 7 if $d(u) + w(u, v) < d(v)$ then
- 8 $d(v) \leftarrow d(u) + w(u, v)$, $Q.decrease_key(v, d(v))$
- 9 $\pi(v) \leftarrow u$
- 10 return (π, d)

- Running time = $O(m + n \lg n)$.

Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \rightarrow \mathbb{R}$

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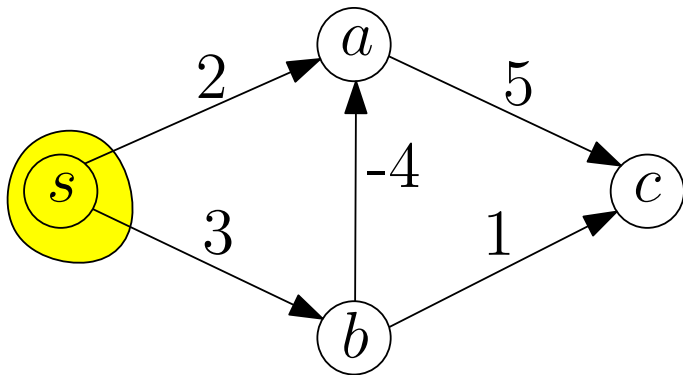
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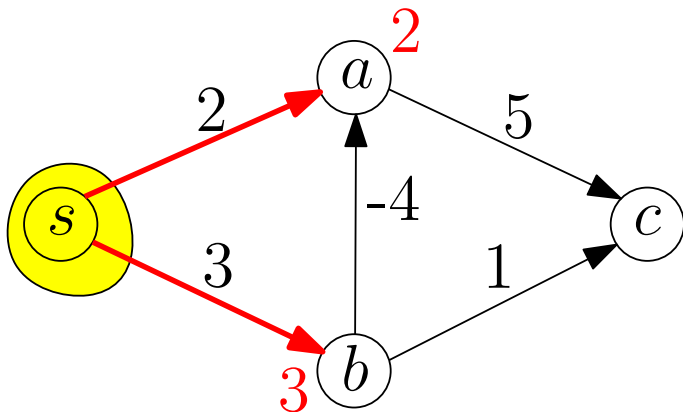
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- **Dijkstra's algorithm does not work any more!**

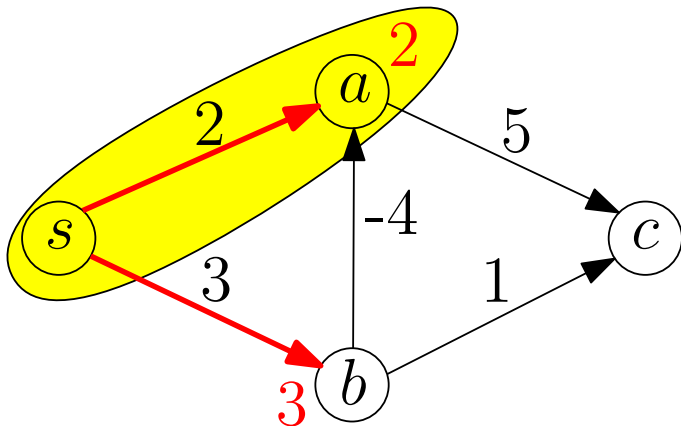
Dijkstra's Algorithm Fails if We Have Negative Weights



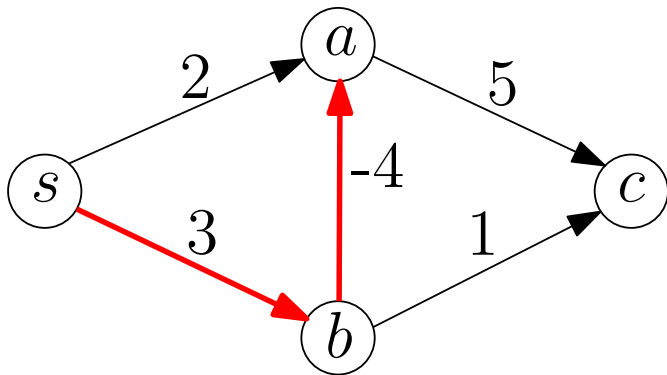
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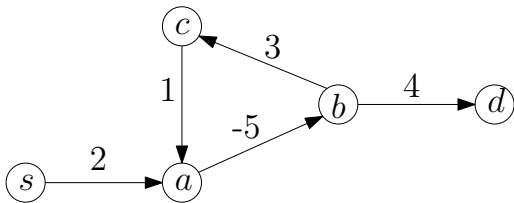


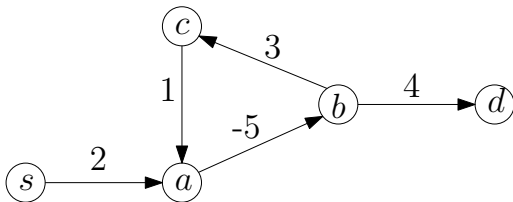
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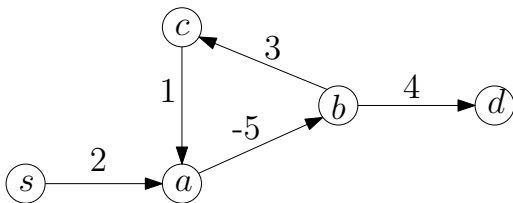
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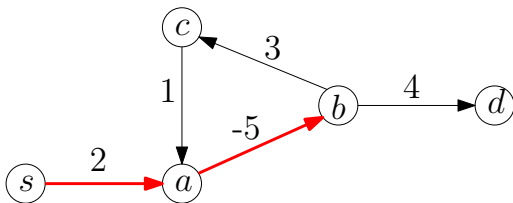


Q: What is the length of the shortest path from s to d ?



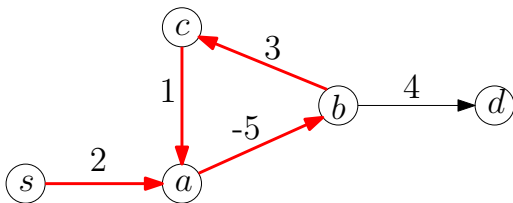
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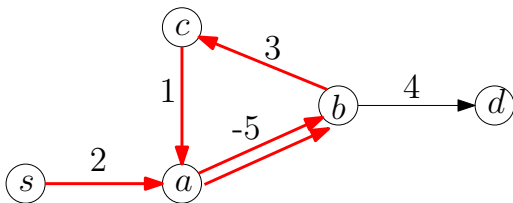
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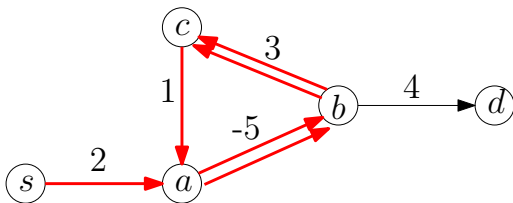
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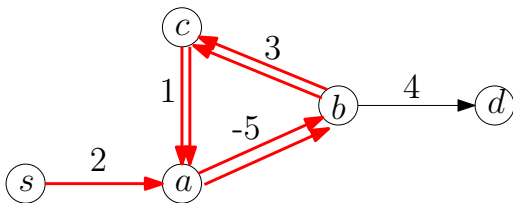
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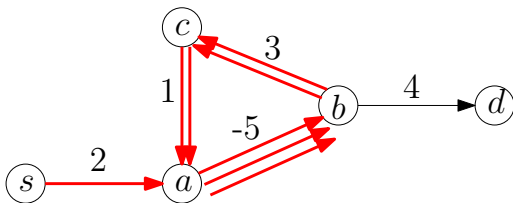
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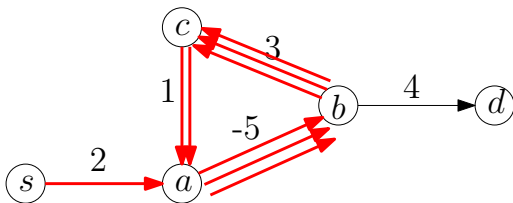
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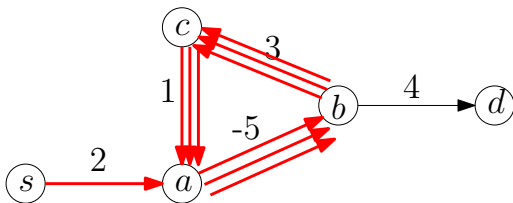
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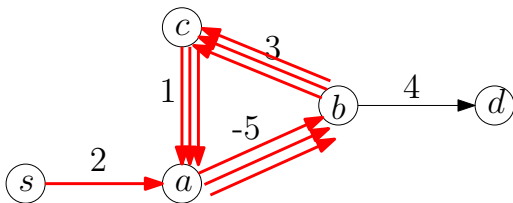
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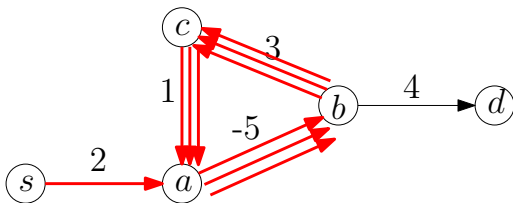
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Def. A negative cycle is a cycle in which the total weight of edges is negative.

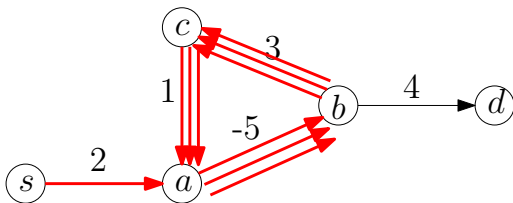


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Dealing with Negative Cycles



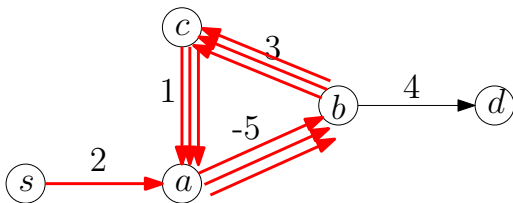
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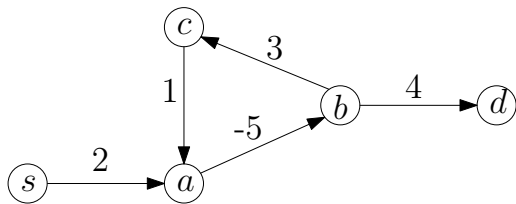
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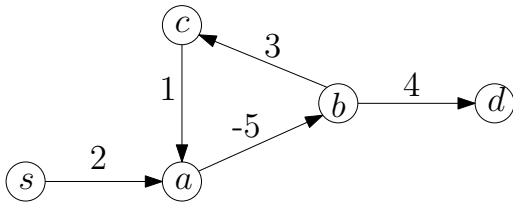
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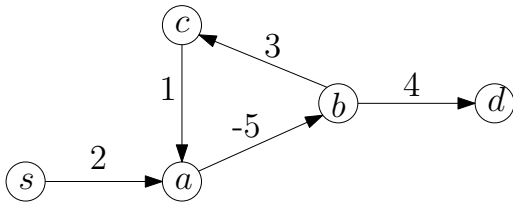
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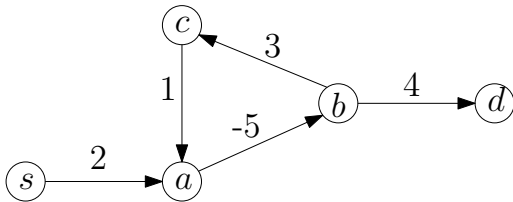


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Q: What is the length of the shortest **simple** path from s to d ?

A: 1

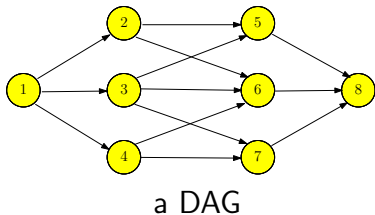
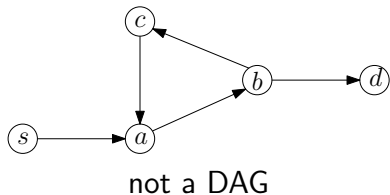
- Unfortunately, computing the shortest simple between two vertices is an **NP-hard** problem.

Outline

- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem
- 3 Knapsack Problem
- 4 Longest Common Subsequence
 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Graphs with Negative Weights**
 - Shortest Paths in Directed Acyclic Graphs
 - Bellman-Ford Algorithm
- 6 All-Pair Shortest Paths and Floyd-Warshall
- 7 Matrix Chain Multiplication
- 8 Summary

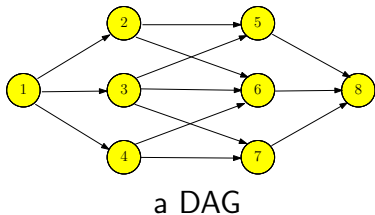
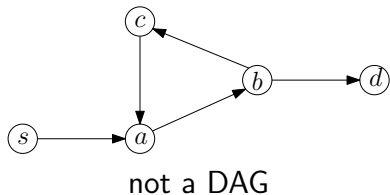
Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



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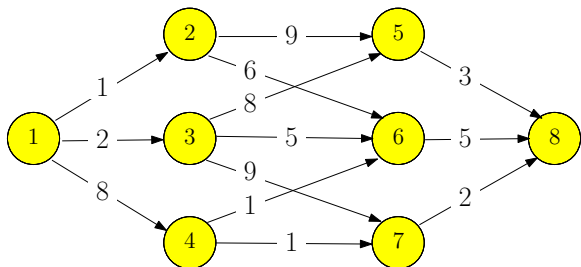
Lemma A directed graph is a DAG if and only if its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \rightarrow \mathbb{R}$.

Assume $V = \{1, 2, 3, \dots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i , for every $i \in V$

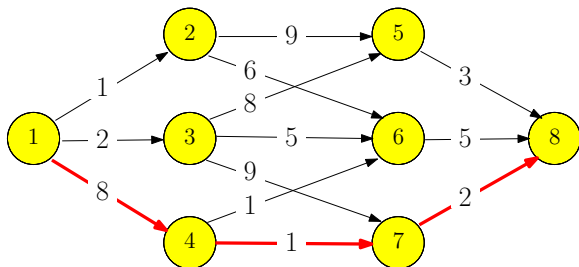


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$$f[i] = \begin{cases} & i = 1 \\ & i = 2, 3, \dots, n \end{cases}$$

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$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i) \in E} \{f(j) + w(j,i)\} & i = 2, 3, \dots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

Shortest Paths in DAG

- 1 $f[1] \leftarrow 0$
- 2 for $i \leftarrow 2$ to n do
- 3 $f[i] \leftarrow \infty$
- 4 for each incoming edge $(j, i) \in E$ of i
- 5 if $f[j] + w(j, i) < f[i]$
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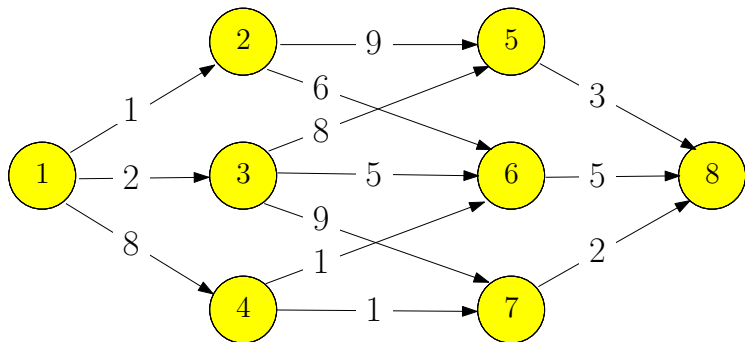
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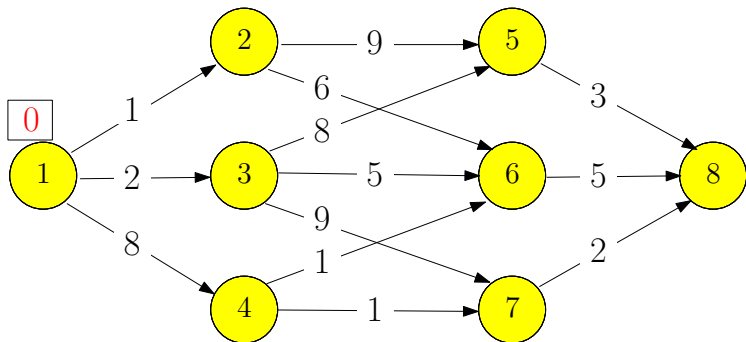
print-path(t)

- 1 if $t = 1$ then
- 2 print(1)
- 3 return
- 4 print-path($\pi(t)$)
- 5 print(", ", t)

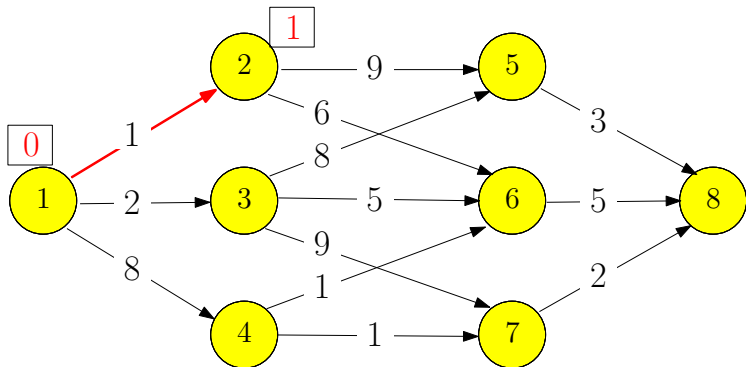
Example



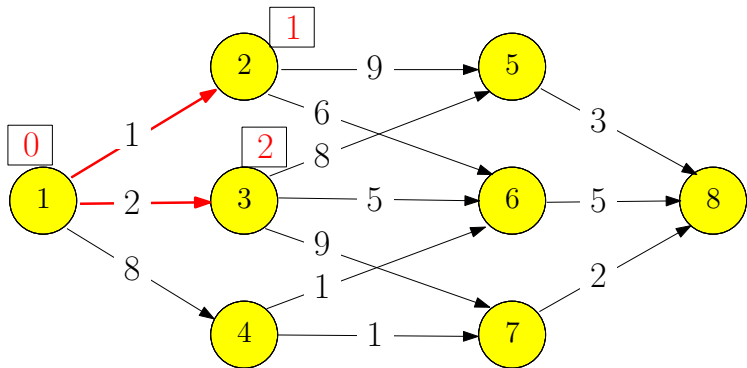
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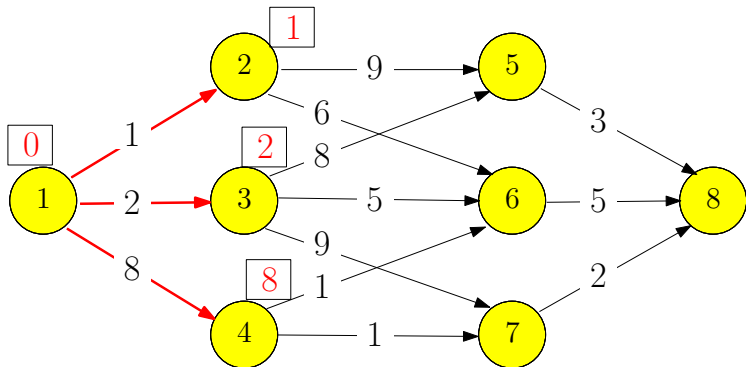
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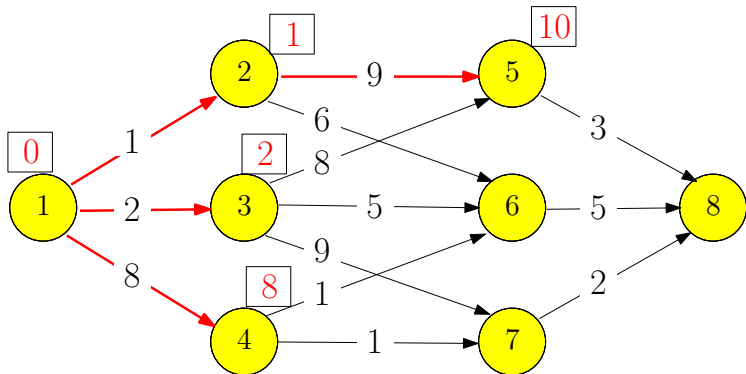
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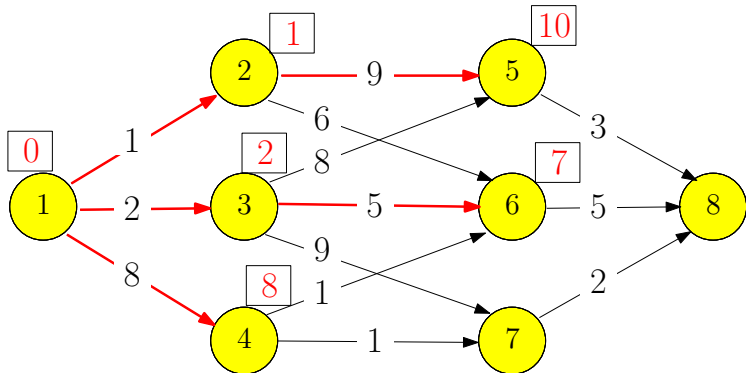
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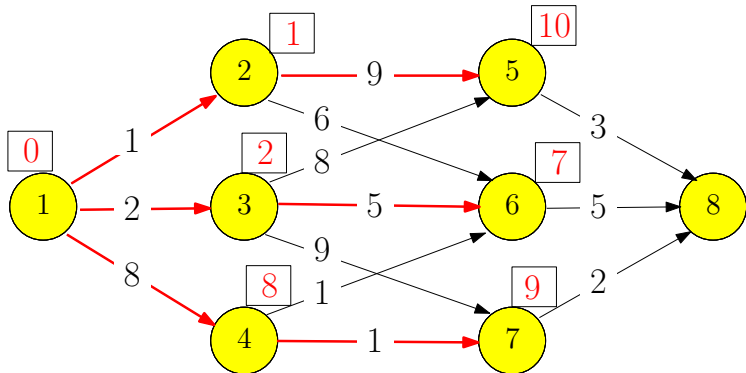
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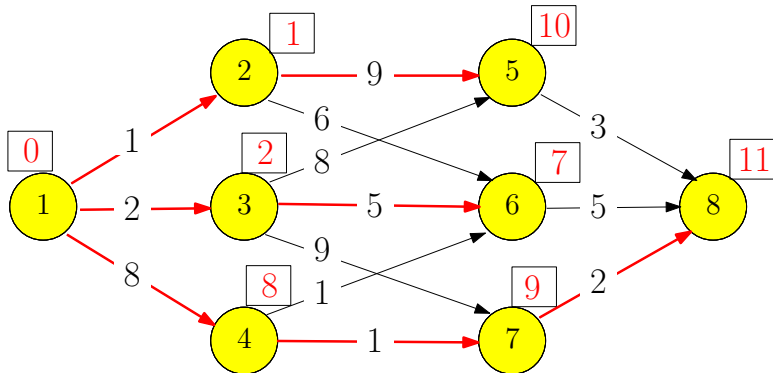
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Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \rightarrow \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

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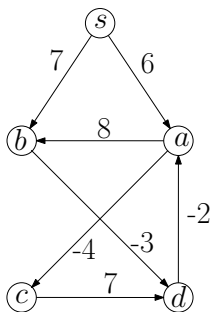
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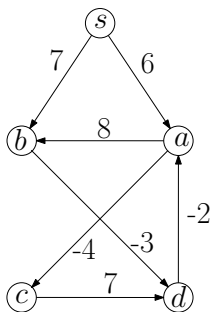
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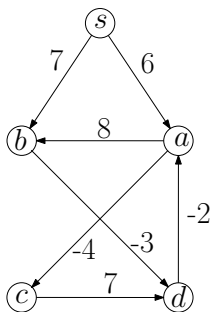
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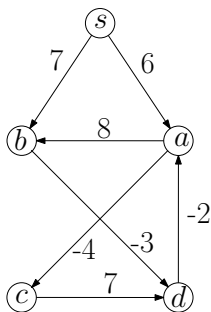
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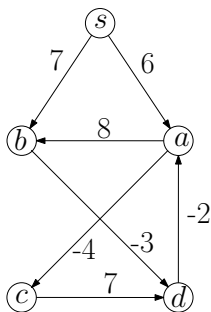
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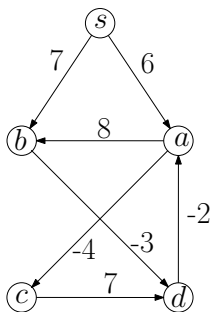
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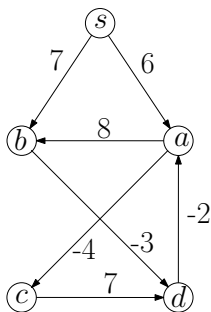
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$$f^\ell[v] = \left\{ \begin{array}{l} \end{array} \right.$$

$$\ell = 0, v = s$$

$$\ell = 0, v \neq s$$

$$\ell > 0$$



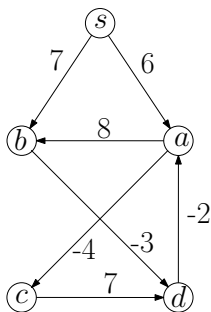
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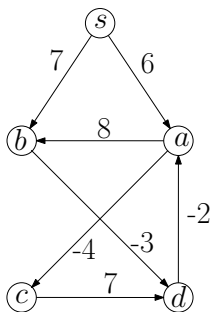
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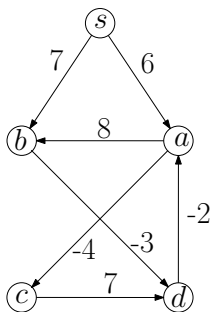
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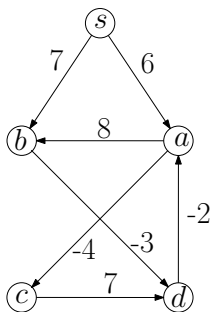
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$$f^{\ell-1}[v]$$

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$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \left\{ \begin{array}{l} f^{\ell-1}[v] \\ \min_{u:(u,v) \in E} (f^{\ell-1}[u] + w(u, v)) \end{array} \right. & \ell > 0 \end{cases}$$

dynamic-programming(G, w, s)

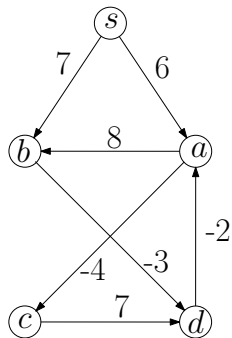
- 1 $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
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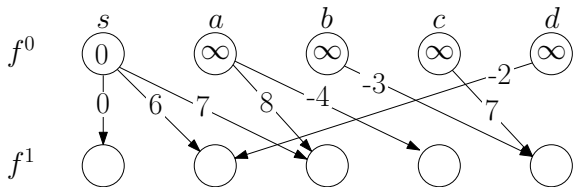
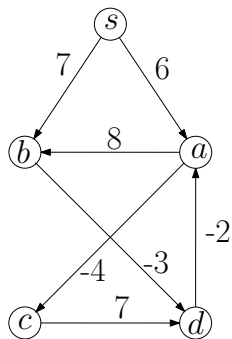
Obs. Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges

Dynamic Programming: Example

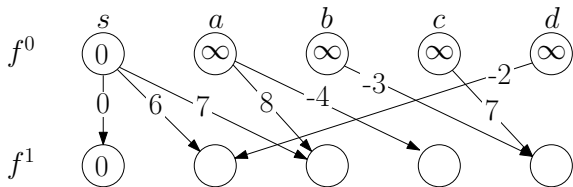
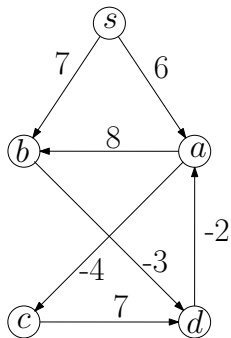


$$f^0 \quad \begin{matrix} s \\ \textcircled{0} \end{matrix} \quad \begin{matrix} a \\ \textcircled{\infty} \end{matrix} \quad \begin{matrix} b \\ \textcircled{\infty} \end{matrix} \quad \begin{matrix} c \\ \textcircled{\infty} \end{matrix} \quad \begin{matrix} d \\ \textcircled{\infty} \end{matrix}$$

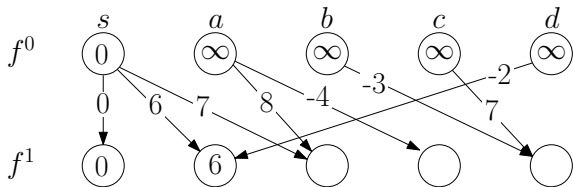
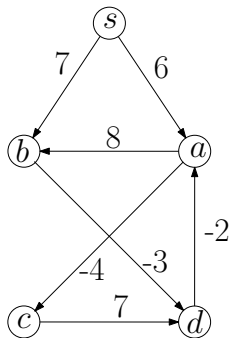
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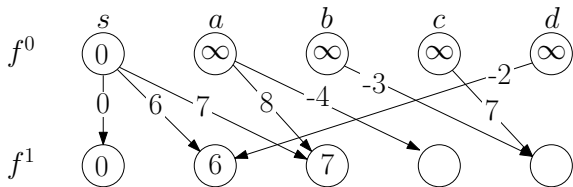
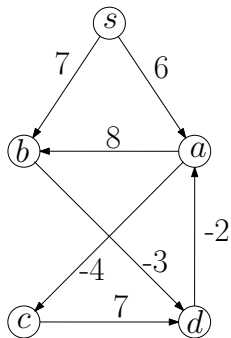
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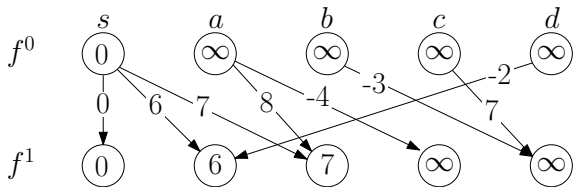
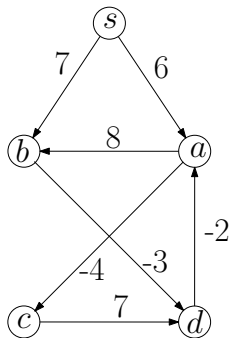
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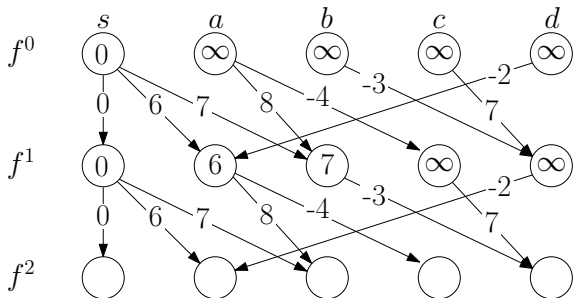
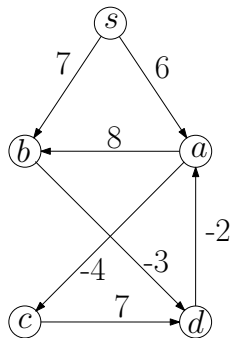
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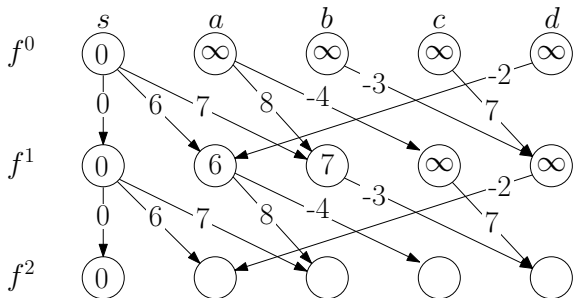
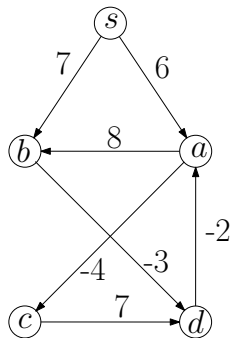
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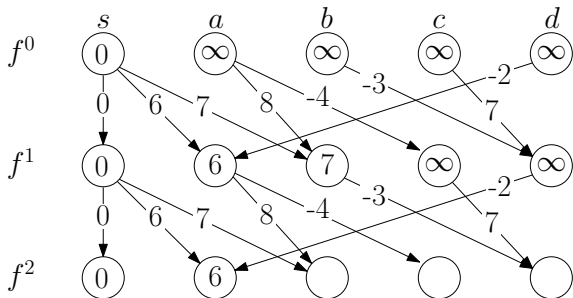
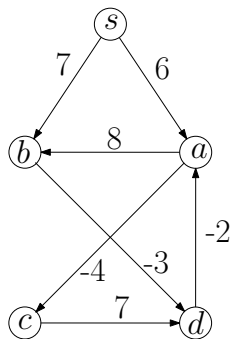
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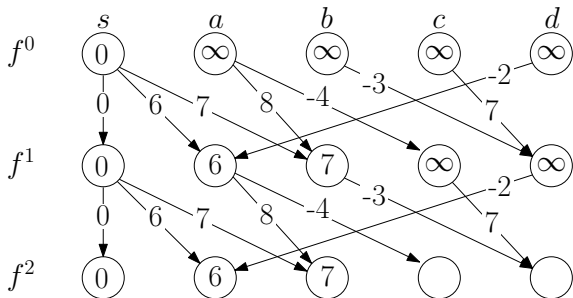
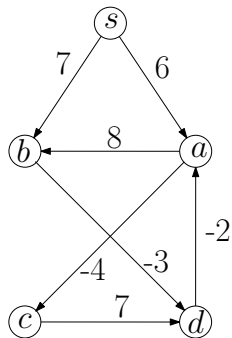
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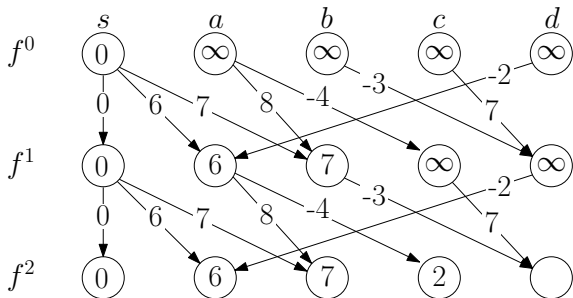
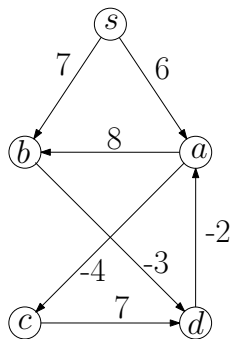
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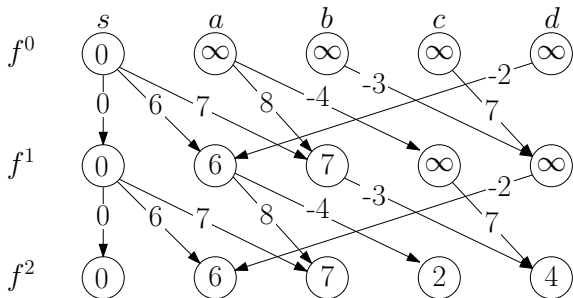
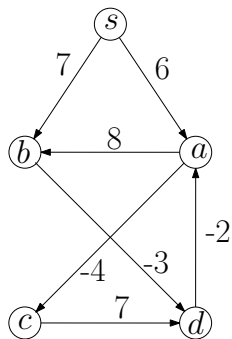
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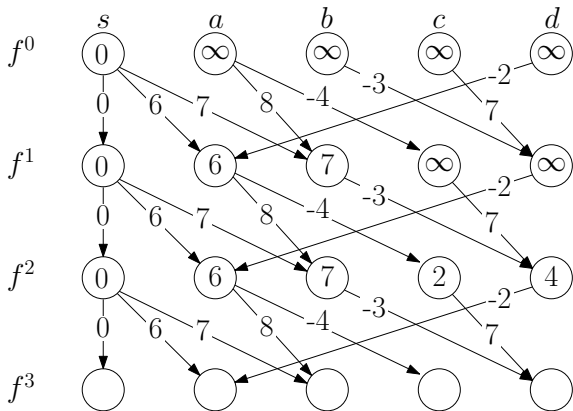
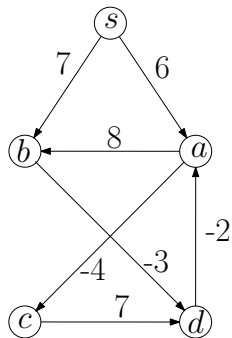
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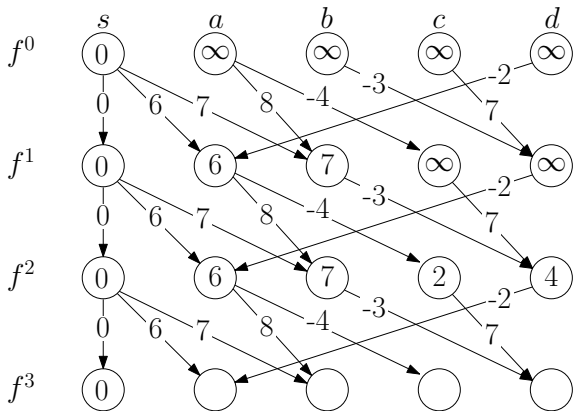
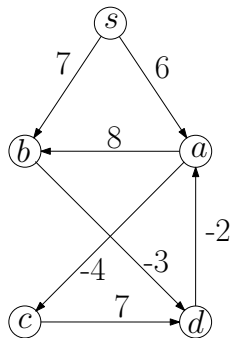
Dynamic Programming: Example



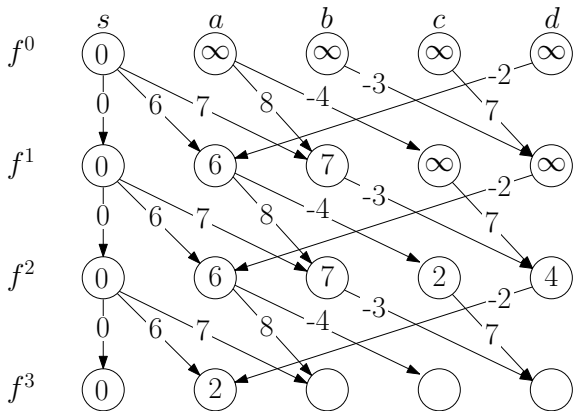
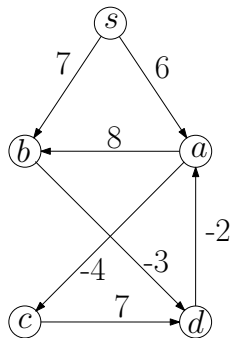
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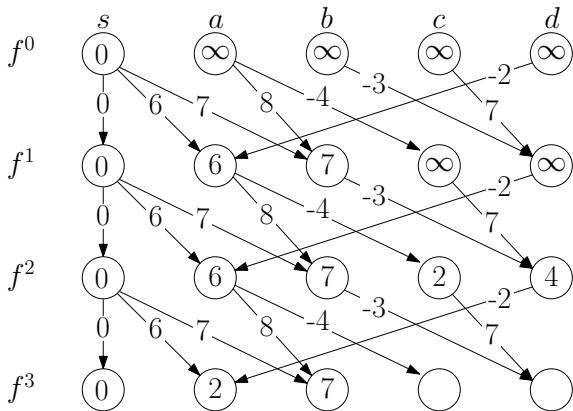
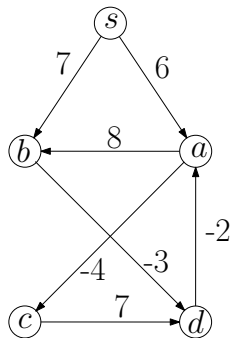
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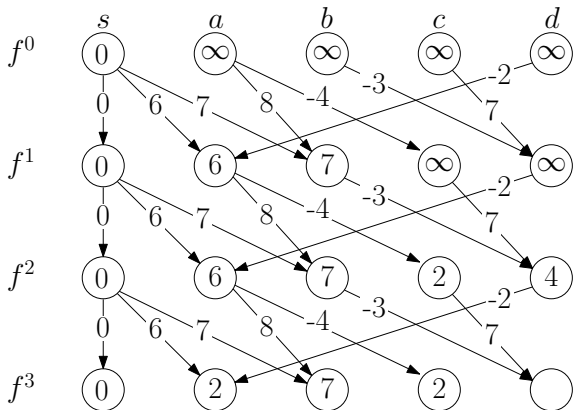
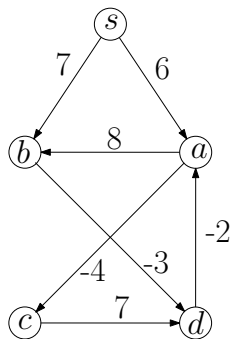
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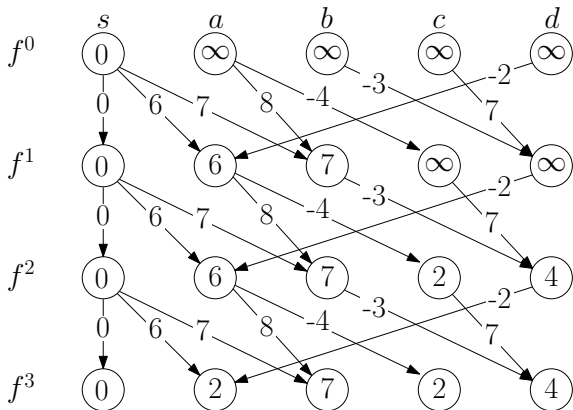
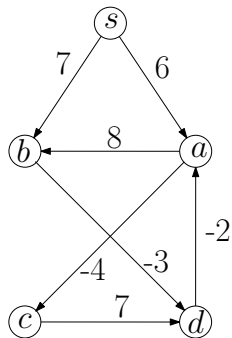
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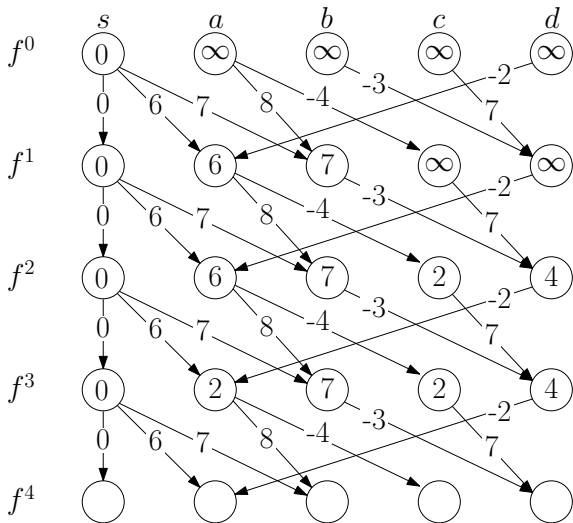
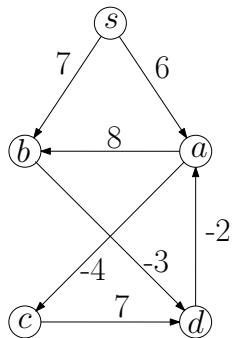
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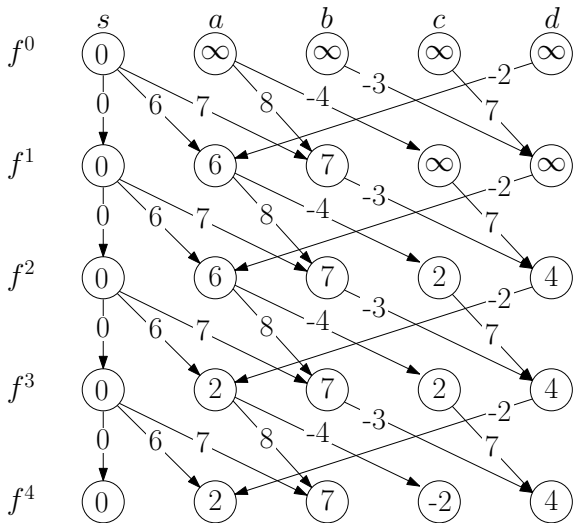
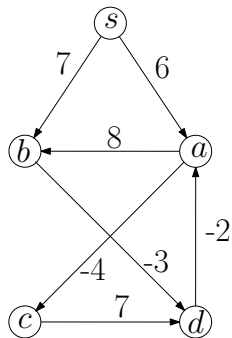
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Dynamic Programming: Example



dynamic-programming(G, w, s)

- 1 $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
- 2 for $\ell \leftarrow 1$ to $n - 1$ do
- 3 copy $f^{\ell-1} \rightarrow f^\ell$
- 4 for each $(u, v) \in E$
- 5 if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$
- 6 $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
- 7 return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges

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Q: What if there are negative cycles?

Dynamic Programming With Negative Cycle Detection

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- 7 for each $(u, v) \in E$
- 8 if $f^{n-1}[u] + w(u, v) < f^{n-1}[v]$
- 9 report “negative cycle exists” and exit
- 10 return $(f^{n-1}[v])_{v \in V}$

Dynamic Programming with Better Space Usage

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- f^ℓ only depends on $f^{\ell-1}$: only need to vectors

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 - After iteration ℓ , $f[v]$ is **at most** the length of the shortest path from s to v that uses at most ℓ edges
 - $f[v]$ is always the length of some path from s to v

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- After iteration ℓ , $f[v]$ is **at most** the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v
- **Assuming there are no negative cycles, after iteration $n - 1$, $f[v] =$ length of shortest path from s to v**

Bellman-Ford Algorithm

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- 2 for $\ell \leftarrow 1$ to n do
- 3 $updated \leftarrow false$
- 4 for each $(u, v) \in E$
- 5 if $f[u] + w(u, v) < f[v]$
- 6 $f[v] \leftarrow f[u] + w(u, v)$
- 7 $updated \leftarrow true$
- 8 if not $updated$, then return f
- 9 output "negative cycle exists"

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- $\pi[v]$: the parent of v in the shortest path tree
- Running time = $O(nm)$

Outline

- 1 Weighted Interval Scheduling
- 2 Subset Sum Problem
- 3 Knapsack Problem
- 4 Longest Common Subsequence
 - Longest Common Subsequence in Linear Space
- 5 Shortest Paths in Graphs with Negative Weights
 - Shortest Paths in Directed Acyclic Graphs
 - Bellman-Ford Algorithm
- 6 All-Pair Shortest Paths and Floyd-Warshall**
- 7 Matrix Chain Multiplication
- 8 Summary

All-Pair Shortest Paths

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Input: directed graph $G = (V, E)$,
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- For simplicity, extend the w values to non-edges:

$$w(i, j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\ \infty & i \neq j, (i, j) \notin E \end{cases}$$

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$$f^k[i, j] = \begin{cases} w(i, j) & k = 0 \\ \min \left\{ \begin{array}{l} f^{k-1}[i, j] \\ \min_{i < k < j} \{ f^{k-1}[i, k] + f^{k-1}[k, j] \} \end{array} \right. & k = 1, 2, \dots, n \end{cases}$$

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Floyd-Warshall(G, w)

- 1 $f^0 \leftarrow w$
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- Running time = $O(n^3)$.

Recovering Shortest Paths

Floyd-Warshall(G, w)

- 1 $f \leftarrow w, \pi[i, j] \leftarrow \perp$ for every $i, j \in V$
- 2 for $k \leftarrow 1$ to n do
- 3 for $i \leftarrow 1$ to n do
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- 5 if $f[i, k] + f[k, j] < f[i, j]$ then
- 6 $f[i, j] \leftarrow f[i, k] + f[k, j], \pi[i, j] \leftarrow k$

Recovering Shortest Paths

Floyd-Warshall(G, w)

- 1 $f \leftarrow w, \pi[i, j] \leftarrow \perp$ for every $i, j \in V$
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print-path(i, j)

- 1 if $\pi[i, j] = \perp$ then
- 2 if $i \neq j$ then print($i, ","$)
- 3 else
- 4 print-path($i, \pi[i, j]$), print-path($\pi[i, j], j$)

Detecting Negative Cycles

Floyd-Warshall(G, w)

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- 8 for $i \leftarrow 1$ to n do
- 9 for $j \leftarrow 1$ to n do
- 10 if $f[i, k] + f[k, j] < f[i, j]$ then
- 11 report “negative cycle exists” and exit

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Matrix Chain Multiplication

Matrix Chain Multiplication

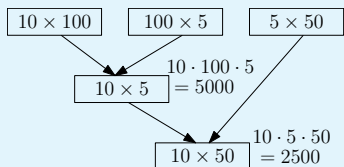
Input: n matrices A_1, A_2, \dots, A_n of sizes $r_1 \times c_1, r_2 \times c_2, \dots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \dots, n - 1$.

Output: the order of computing $A_1 A_2 \dots A_n$ with the minimum number of multiplications

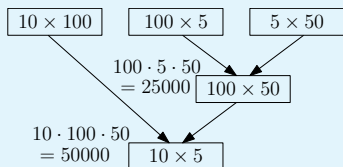
Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100$, $A_2 : 100 \times 5$, $A_3 : 5 \times 50$



$$\text{cost} = 5000 + 2500 = 7500$$

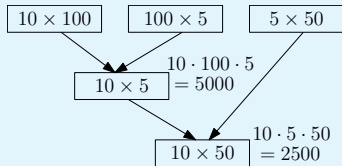


$$\text{cost} = 25000 + 50000 = 75000$$

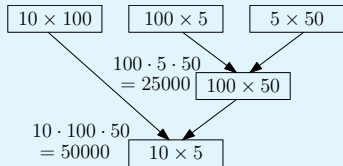
- $(A_1A_2)A_3: 10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1(A_2A_3): 100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

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- $opt[i, j]$: the minimum cost of computing $A_iA_{i+1} \cdots A_j$

$$opt[i, j] = \begin{cases} 0 & i = j \\ \min_{k:i \leq k < j} (opt[i, k] + opt[k + 1, j] + r_i c_k c_j) & i < j \end{cases}$$

matrix-chain-multiplication($n, r[1..n], c[1..n]$)

- 1 let $opt[i, i] \leftarrow 0$ for every $i = 1, 2, \dots, n$
- 2 for $\ell \leftarrow 2$ to n
- 3 for $i \leftarrow 1$ to $n - \ell + 1$
- 4 $j \leftarrow i + \ell - 1$
- 5 $opt[i, j] \leftarrow \infty$
- 6 for $k \leftarrow i$ to $j - 1$
- 7 if $opt[i, k] + opt[k + 1, j] + r_i c_k c_j < opt[i, j]$
- 8 $opt[i, j] \leftarrow opt[i, k] + opt[k + 1, j] + r_i c_k c_j$
- 9 return $opt[1, n]$

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Dynamic Programming

- Break up a problem into many **overlapping** sub-problems
- Build solutions for larger and larger sub-problems
- Use a **table** to store solutions for sub-problems for reuse

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i] =$ value of instance defined by jobs $\{1, 2, \dots, i\}$
- Subset sum, knapsack: $opt[i, W'] =$ value of instance with items $\{1, 2, \dots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j] =$ value of instance defined by $A[1..i]$ and $B[1..j]$
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- Matrix chain multiplication: $opt[i, j]$ = value of instances defined by matrices i to j
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Bellman-Ford: $f^\ell[v]$ = length of shortest path from s to v that uses at most ℓ edges
- Floyd-Warshall: $f^k[i, j]$ = length of shortest path from i to j that only uses $\{1, 2, \dots, k\}$ as intermediate vertices

Exercise: Counting Number of Domino Coverings

Input: n

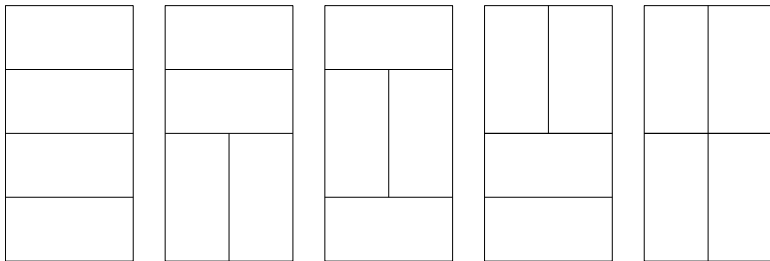
Output: number of ways to cover a $n \times 2$ grid using domino tiles

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- **Example:** 5 different ways if $n = 4$:

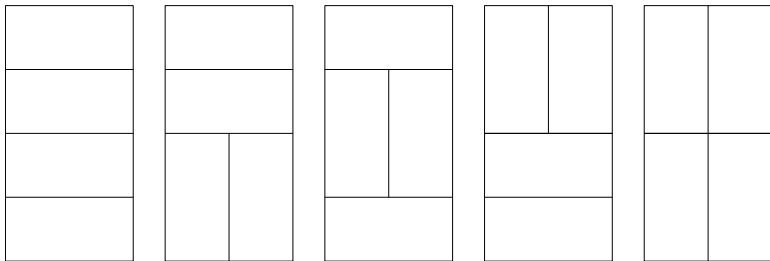


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- **Example:** 5 different ways if $n = 4$:



- How about number of ways to cover a $n \times 3$ grid?

Exercise: Maximum-Weight Subset with Gaps

Input: n , integers $w_1, w_2, \dots, w_n \geq 0$.

Output: a set $S \subseteq \{1, 2, 3, \dots, n\}$ that

$$\text{maximizes } \sum_{i \in S} w_i \quad \text{s.t.}$$

$$\forall i, j \in S, i \neq j, \text{ we have } |i - j| \geq 2.$$

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- Example: $n = 7, w = (10, 80, 100, 90, 30, 50, 70)$
- Choose items 2, 4, 7: value = $80 + 90 + 70 = 240$

Def. Given a sequence $A = (a_1, a_2, \dots, a_n)$ of n numbers, an increasing subsequence of A is a subsequence $(A_{i_1}, A_{i_2}, A_{i_3}, \dots, A_{i_t})$ such that $1 \leq i_1 < i_2 < i_3 < \dots < i_t \leq n$ and $a_{i_1} < a_{i_2} < a_{i_3} < \dots < a_{i_t}$.

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Input: $A = (a_1, a_2, \dots, a_n)$ of n numbers

Output: The length of the longest increasing sub-sequence of A

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Exercise: Longest Oscillating Subsequence

Input: A sequence A of n numbers

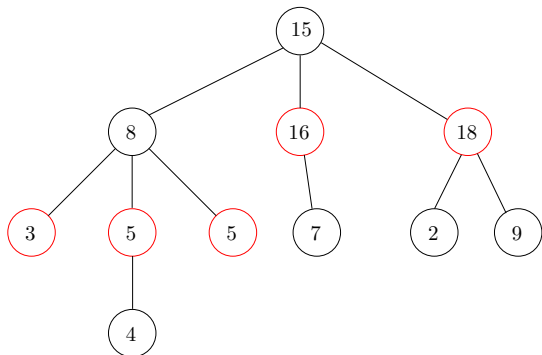
Output: The length of the longest oscillating subsequence of A

- Recall: an independent set of $G = (V, E)$ is a set $U \subseteq V$ such that there are no edges between vertices in U .

Maximum Weighted Independent Set in A Tree

Input: a tree with node weights

Output: the independent set of the tree with the maximum total weight



maximum-weight independent set has weight 47.