

CSE 431/531: Algorithm Analysis and Design (Spring 2018)

Graph Basics

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*Department of Computer Science and Engineering
University at Buffalo*

- 1 Graphs
- 2 Connectivity and Graph Traversal
 - Testing Bipartiteness
- 3 Topological Ordering

Examples of Graphs



Figure: Road Networks



Figure: Internet



Figure: Social Networks

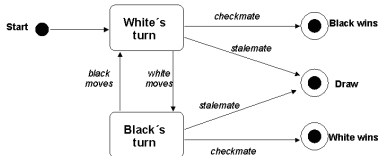
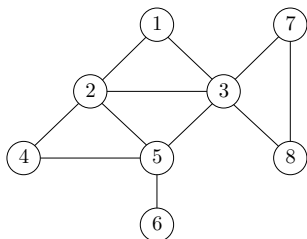


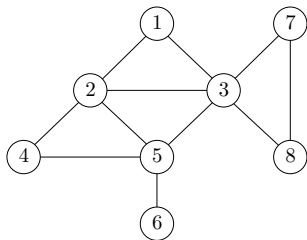
Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



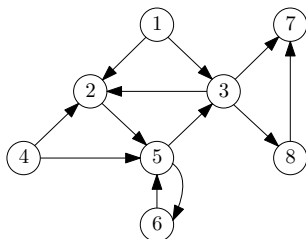
- V : set of vertices (nodes);
- E : pairwise relationships among V ;
 - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

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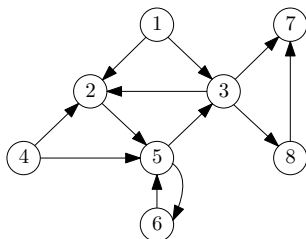
- V : set of vertices (nodes);
 - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- E : pairwise relationships among V ;
 - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
 - $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$

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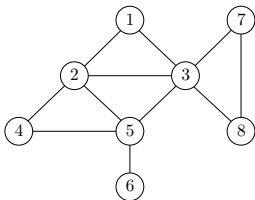
Directed Graph $G = (V, E)$



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Abuse of Notations

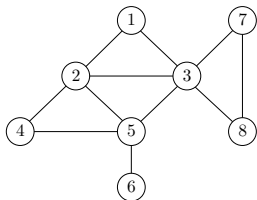
- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



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- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

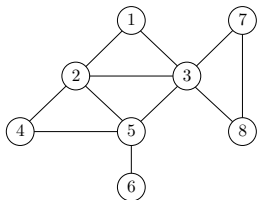
Representation of Graphs



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Adjacency matrix
 - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
 - A is symmetric if graph is undirected

Representation of Graphs



1: [2] → [3]

6: [5]

2: [1] → [3] → [4] → [5]

7: [3] → [8]

3: [1] → [2] → [5] → [7] → [8]

4: [2] → [5]

8: [3] → [7]

5: [2] → [3] → [4] → [6]

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- Linked lists
 - For every vertex v , there is a linked list containing all **neighbours** of v .

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n : number of vertices
- m : number of edges, assuming $n - 1 \leq m \leq n(n - 1)/2$
- d_v : number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u, v) \in E$		
time to list all neighbours of v		

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memory usage	$O(n^2)$	$O(m)$
time to check $(u, v) \in E$	$O(1)$	
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time to list all neighbours of v	$O(n)$	

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Outline

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Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)

two vertices $s, t \in V$

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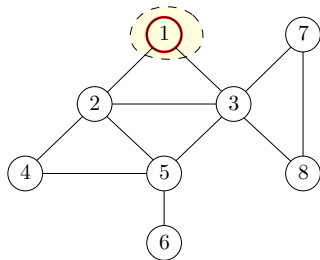
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 - Breadth-First Search (BFS)
 - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \dots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \dots \cup L_j$ and have an edge to a vertex in L_j

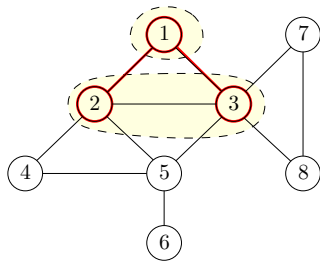
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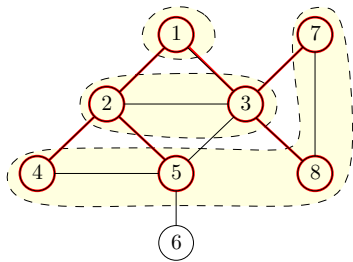
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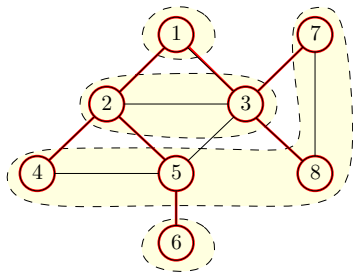
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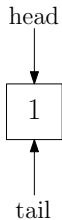
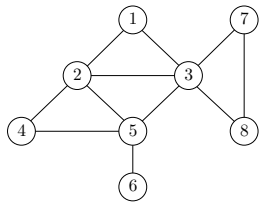
Implementing BFS using a Queue

BFS(s)

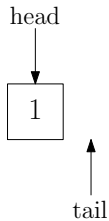
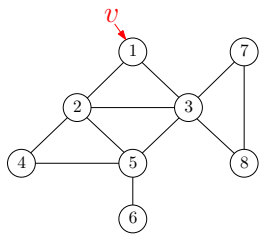
- 1 $head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
- 2 mark s as “visited” and all other vertices as “unvisited”
- 3 while $head \geq tail$
- 4 $v \leftarrow queue[tail], tail \leftarrow tail + 1$
- 5 for all neighbours u of v
- 6 if u is “unvisited” then
- 7 $head \leftarrow head + 1, queue[head] = u$
- 8 mark u as “visited”

- Running time: $O(n + m)$.

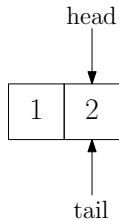
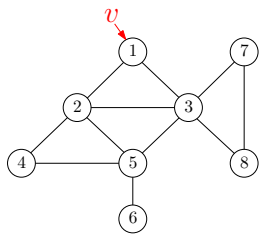
Example of BFS via Queue



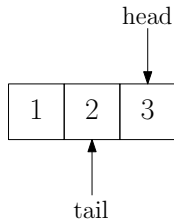
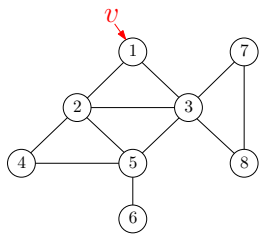
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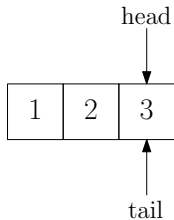
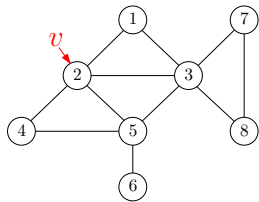
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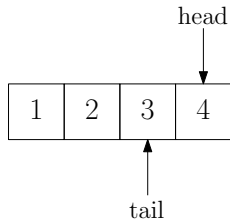
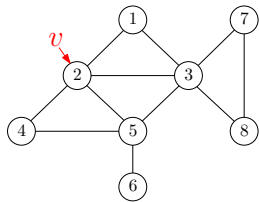
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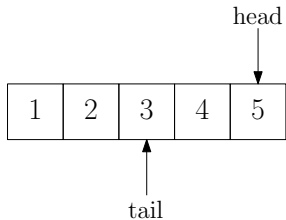
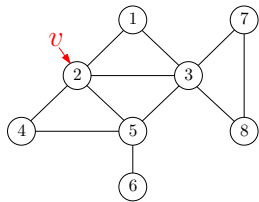
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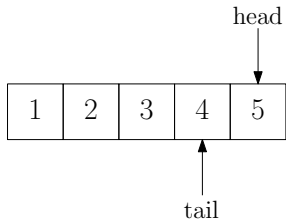
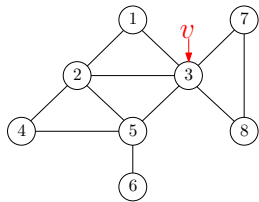
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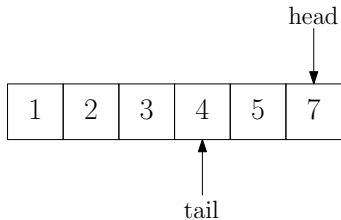
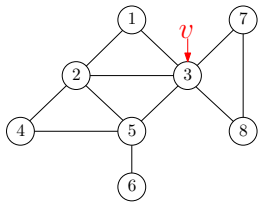
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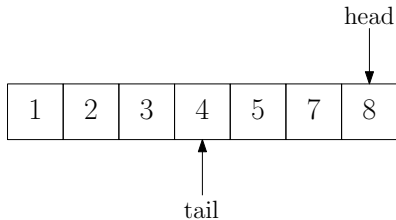
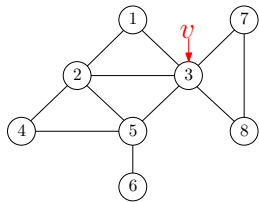
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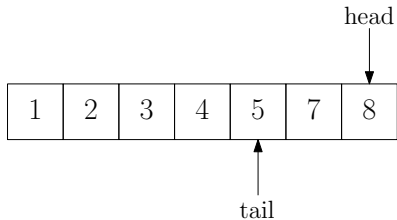
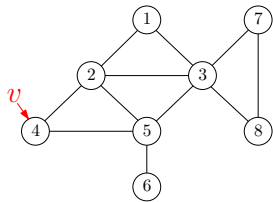
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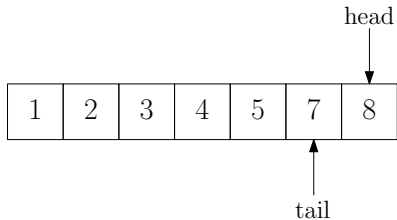
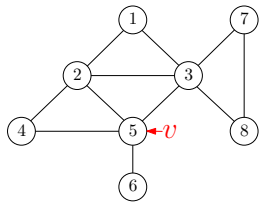
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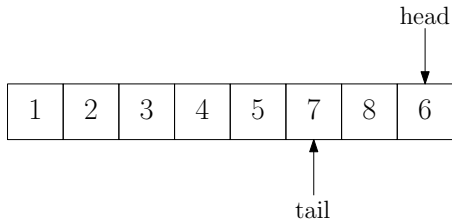
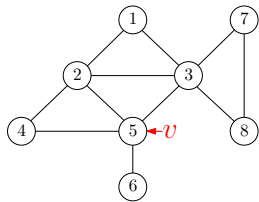
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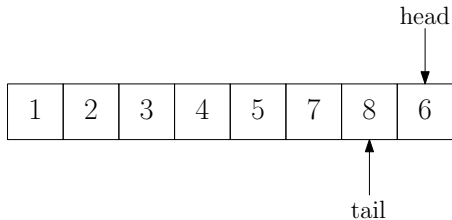
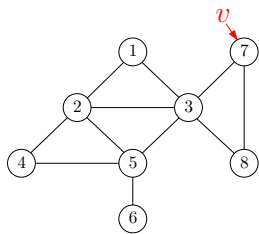
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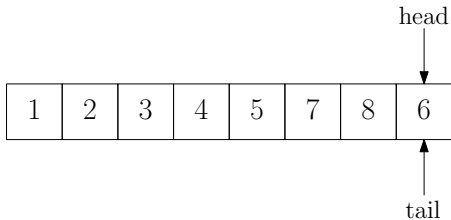
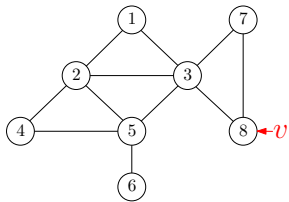
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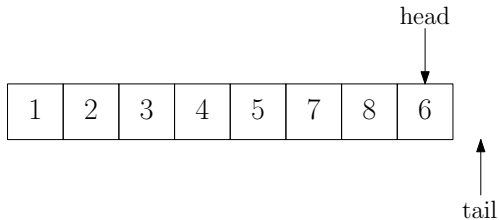
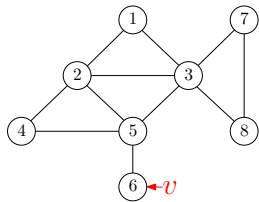
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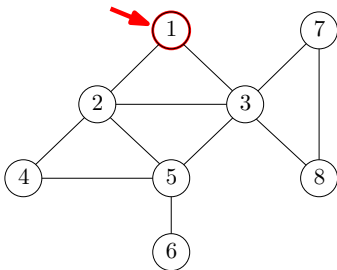


Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
- When reach an already-visited vertex (“dead-end”), go back
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- If tried all edges leading out of the current vertex, go back

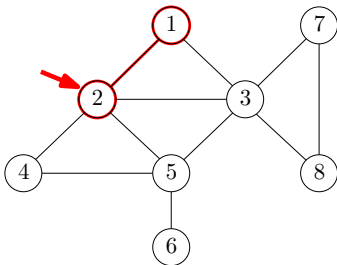
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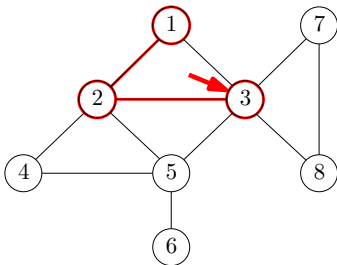
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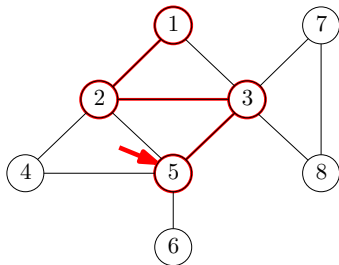
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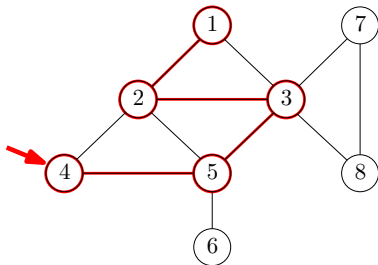
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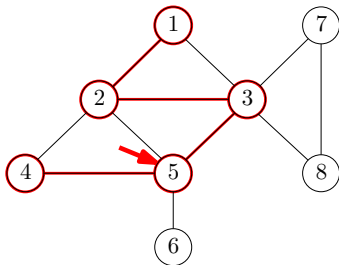
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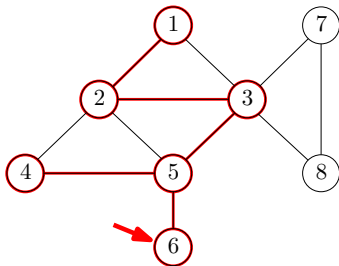
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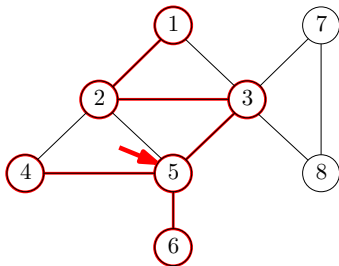
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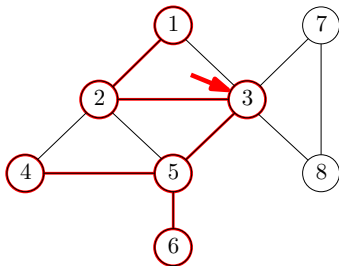
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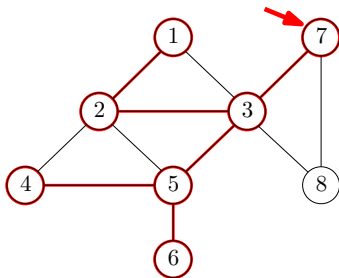
Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
- When reach an already-visited vertex (“dead-end”), go back
- Travel through the next edge
- If tried all edges leading out of the current vertex, go back



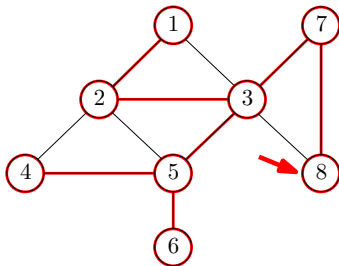
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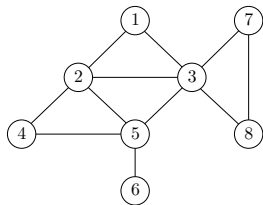
Implementing DFS using a Stack

DFS(s)

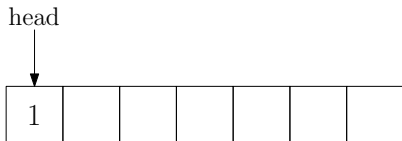
- 1 $head \leftarrow 1, stack[1] \leftarrow s$
- 2 mark all vertices as “unexplored”
- 3 while $head \geq 1$
- 4 $v \leftarrow stack[head], head \leftarrow head - 1$
- 5 if v is unexplored then
- 6 mark v as “explored”
- 7 for all neighbours u of v
- 8 if u is not explored then
- 9 $head \leftarrow head + 1, stack[head] = u$

- Running time: $O(n + m)$.

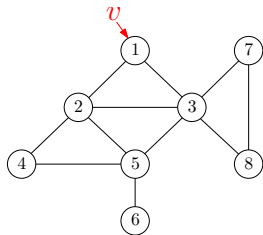
Example of DFS using Stack



explored vertices:

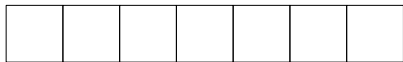


Example of DFS using Stack

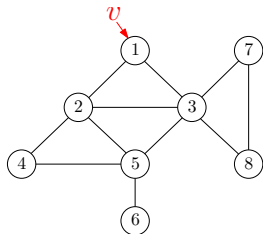


explored vertices:

head

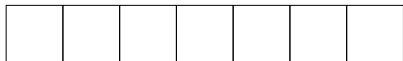


Example of DFS using Stack

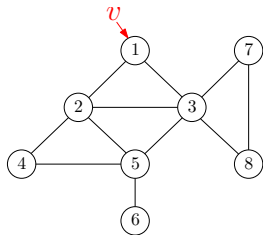


explored vertices: 1

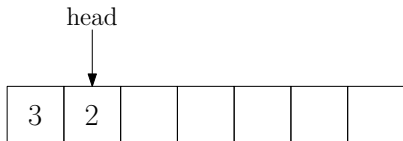
head



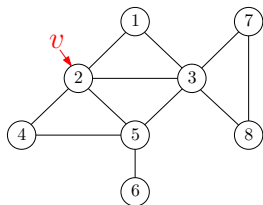
Example of DFS using Stack



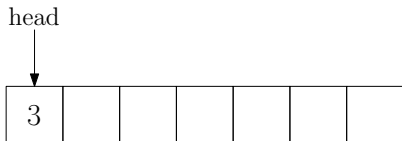
explored vertices: 1



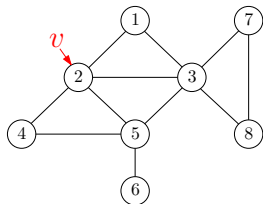
Example of DFS using Stack



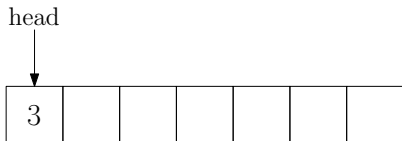
explored vertices: 1



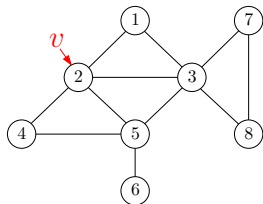
Example of DFS using Stack



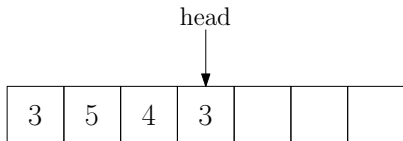
explored vertices: 1 2



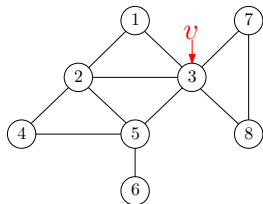
Example of DFS using Stack



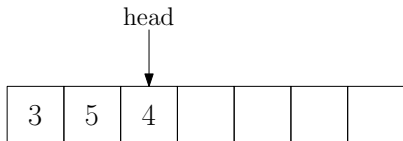
explored vertices: 1 2



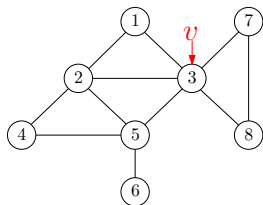
Example of DFS using Stack



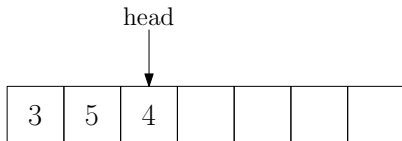
explored vertices: 1 2



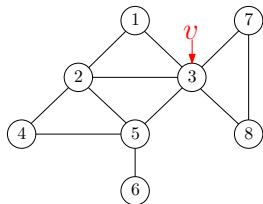
Example of DFS using Stack



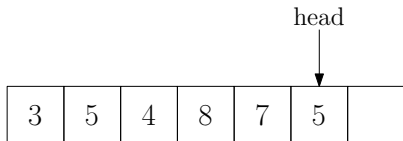
explored vertices: 1 2 3



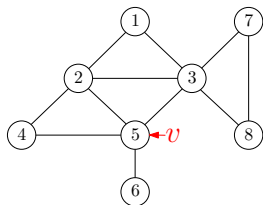
Example of DFS using Stack



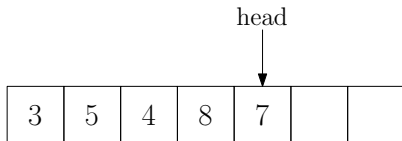
explored vertices: 1 2 3



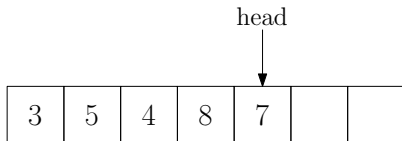
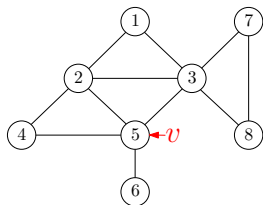
Example of DFS using Stack



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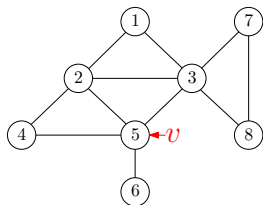


Example of DFS using Stack



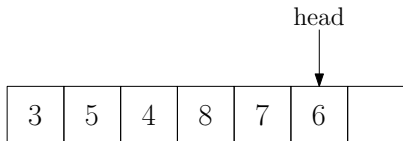
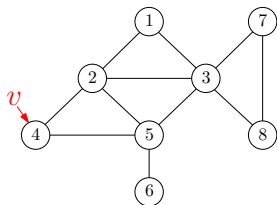
explored vertices: 1 2 3 5

Example of DFS using Stack



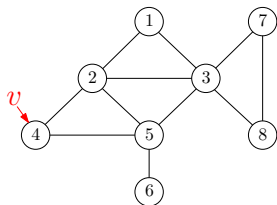
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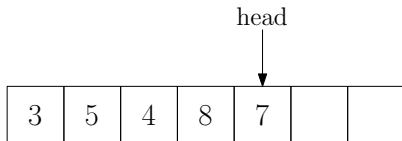
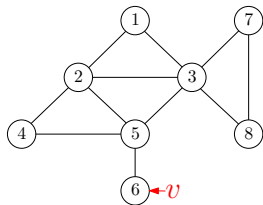
explored vertices: 1 2 3 5

Example of DFS using Stack



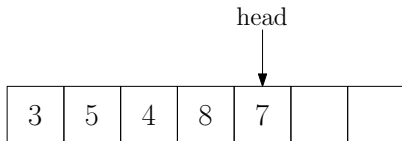
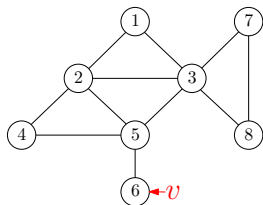
explored vertices: 1 2 3 5 4

Example of DFS using Stack



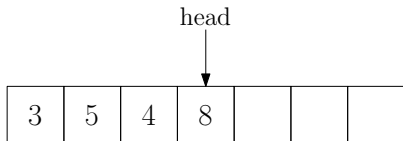
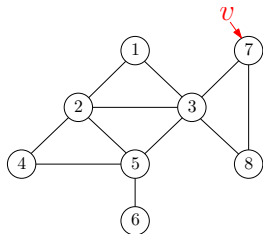
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Example of DFS using Stack



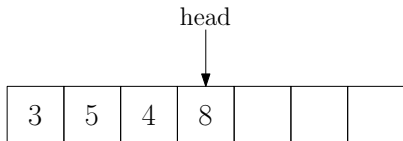
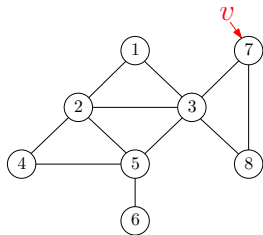
explored vertices: 1 2 3 5 4 6

Example of DFS using Stack



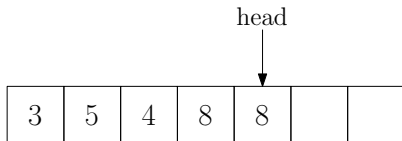
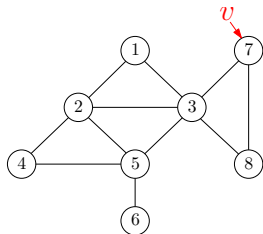
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Example of DFS using Stack



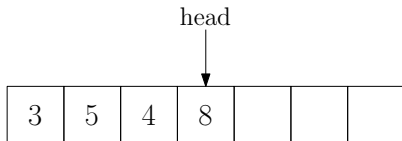
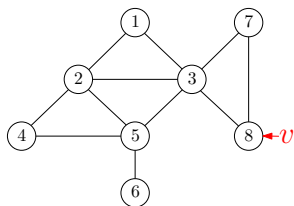
explored vertices: 1 2 3 5 4 6 7

Example of DFS using Stack



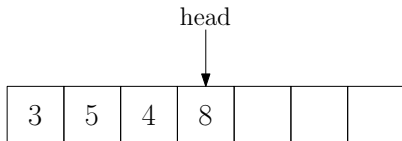
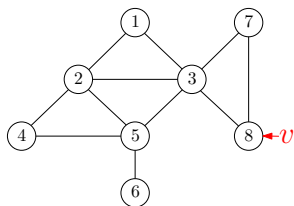
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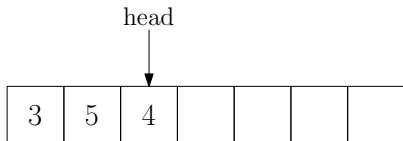
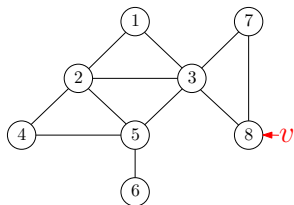
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Example of DFS using Stack



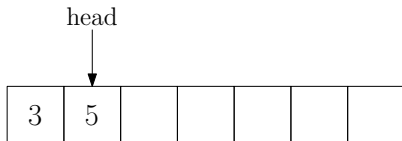
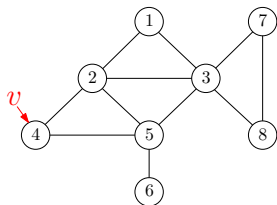
explored vertices: 1 2 3 5 4 6 7 8

Example of DFS using Stack



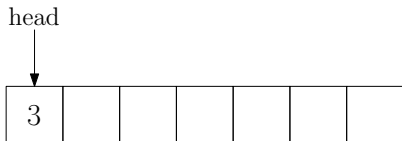
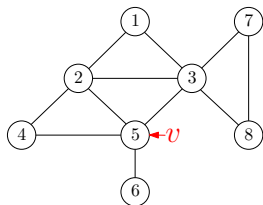
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Example of DFS using Stack



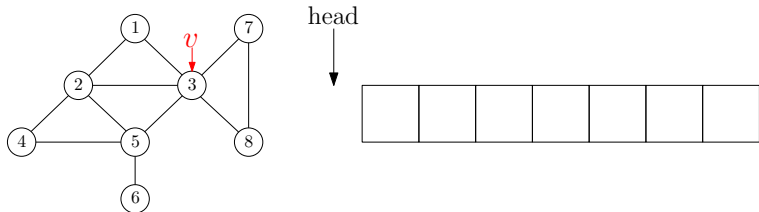
explored vertices: 1 2 3 5 4 6 7 8

Example of DFS using Stack



explored vertices: 1 2 3 5 4 6 7 8

Example of DFS using Stack



explored vertices: 1 2 3 5 4 6 7 8

Implementing DFS using Recursion

DFS(s)

- 1 mark all vertices as “unexplored”
- 2 recursive-DFS(s)

recursive-DFS(v)

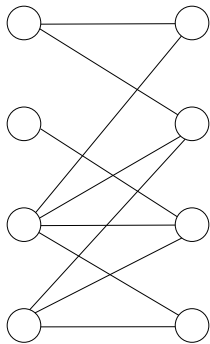
- 1 if v is explored then return
- 2 mark v as “explored”
- 3 for all neighbours u of v
- 4 recursive-DFS(u)

Outline

- 1 Graphs
- 2 Connectivity and Graph Traversal
 - Testing Bipartiteness
- 3 Topological Ordering

Testing Bipartiteness: Applications of BFS

Def. A graph $G = (V, E)$ is a **bipartite graph** if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, we have either $u \in L, v \in R$ or $v \in L, u \in R$.



Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$

Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g

Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
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- Neighbors of s must be in R

Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L

Testing Bipartiteness

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- ...

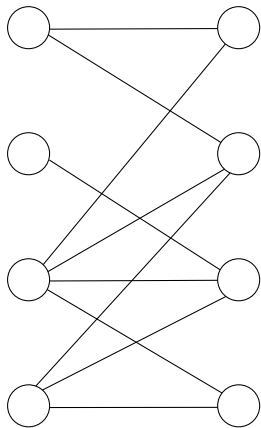
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
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- ...
- Report “not a bipartite graph” if contradiction was found

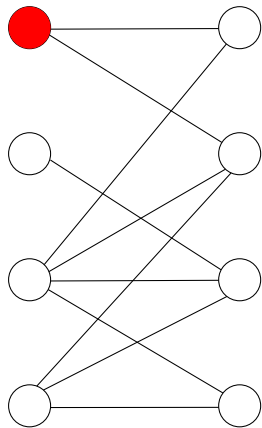
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L
- ...
- Report “not a bipartite graph” if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

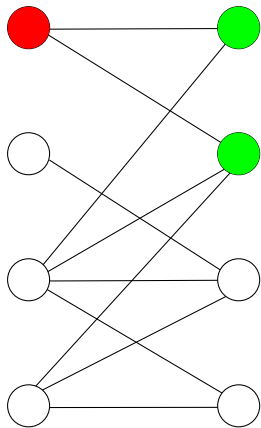
Test Bipartiteness



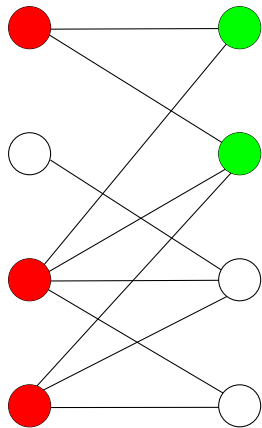
Test Bipartiteness



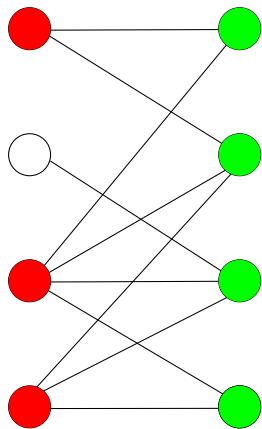
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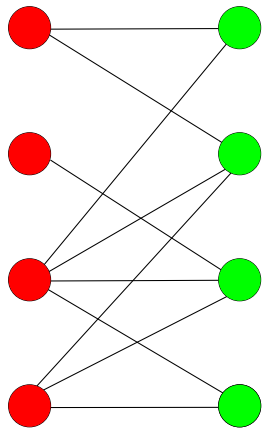
Test Bipartiteness



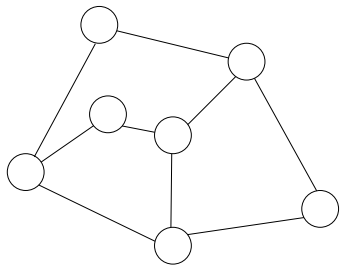
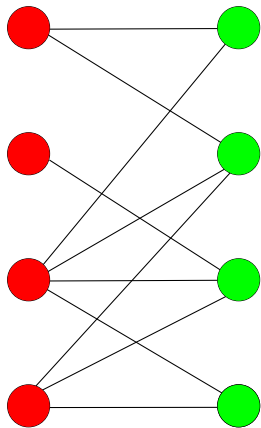
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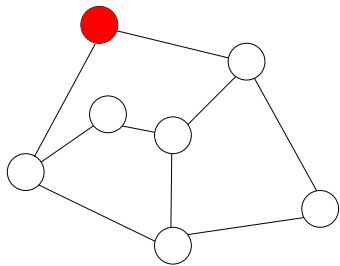
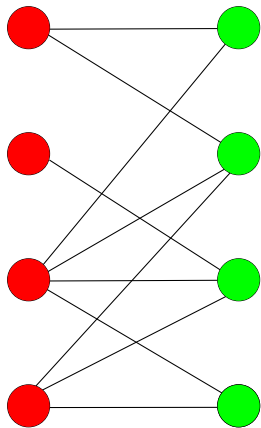
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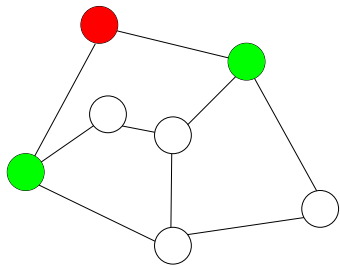
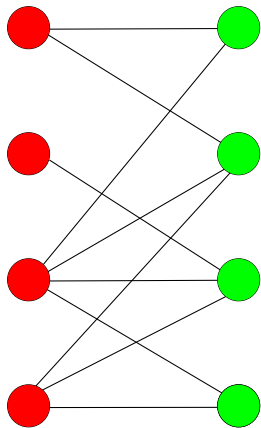
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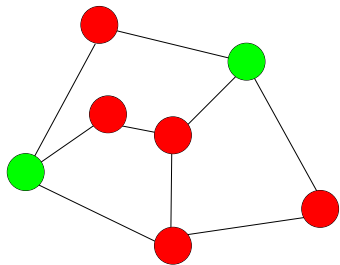
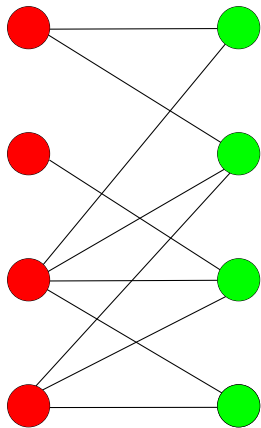
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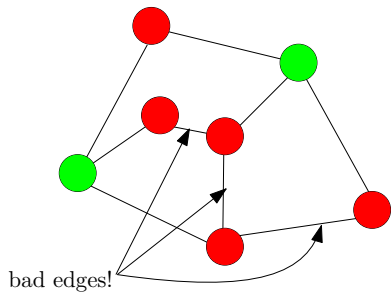
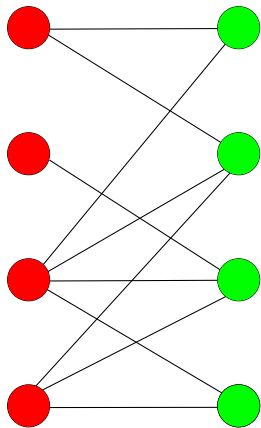
Test Bipartiteness



Test Bipartiteness



Test Bipartiteness



Testing Bipartiteness using BFS

BFS(s)

- 1 $head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
- 2 mark s as “visited” and all other vertices as “unvisited”
- 3 while $head \geq tail$
- 4 $v \leftarrow queue[tail], tail \leftarrow tail + 1$
- 5 for all neighbours u of v
- 6 if u is “unvisited” then
- 7 $head \leftarrow head + 1, queue[head] = u$
- 8 mark u as “visited”

Testing Bipartiteness using BFS

test-bipartiteness(s)

- 1 $head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
- 2 mark s as “visited” and all other vertices as “unvisited”
- 3 $color[s] \leftarrow 0$
- 4 while $head \geq tail$
- 5 $v \leftarrow queue[tail], tail \leftarrow tail + 1$
- 6 for all neighbours u of v
- 7 if u is “unvisited” then
- 8 $head \leftarrow head + 1, queue[head] = u$
- 9 mark u as “visited”
- 10 $color[u] \leftarrow 1 - color[v]$
- 11 elseif $color[u] = color[v]$ then
- 12 print(“ G is not bipartite”) and exit

Testing Bipartiteness using BFS

- 1 mark all vertices as “unvisited”
- 2 for each vertex $v \in V$
- 3 if v is “unvisited” then
- 4 test-bipartiteness(v)
- 5 print(“ G is bipartite”)

Testing Bipartiteness using BFS

- 1 mark all vertices as “unvisited”
- 2 for each vertex $v \in V$
- 3 if v is “unvisited” then
- 4 test-bipartiteness(v)
- 5 print(“ G is bipartite”)

Obs. Running time of algorithm = $O(n + m)$

Testing Bipartiteness using BFS

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- 2 for each vertex $v \in V$
- 3 if v is “unvisited” then
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- 5 print(“ G is bipartite”)

Obs. Running time of algorithm = $O(n + m)$

Homework problem: using DFS to implement test-bipartiteness.

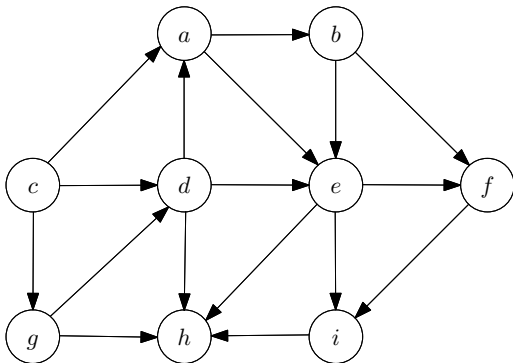
- 1 Graphs
- 2 Connectivity and Graph Traversal
 - Testing Bipartiteness
- 3 Topological Ordering

Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

Output: 1-to-1 function $\pi : V \rightarrow \{1, 2, 3 \dots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$

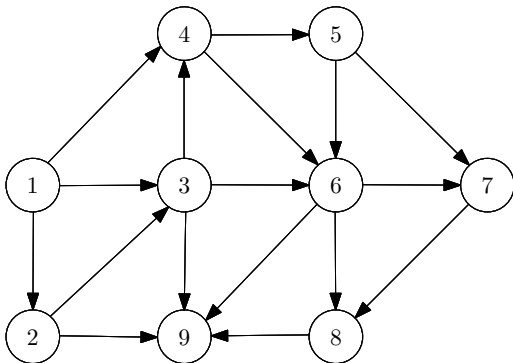


Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

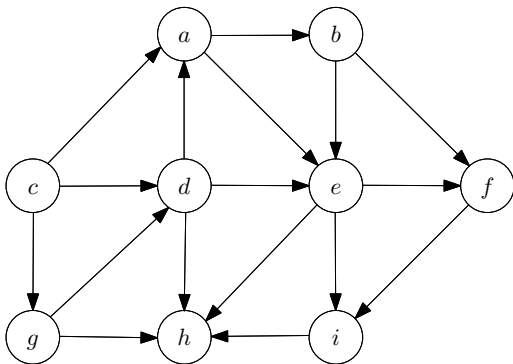
Output: 1-to-1 function $\pi : V \rightarrow \{1, 2, 3 \dots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



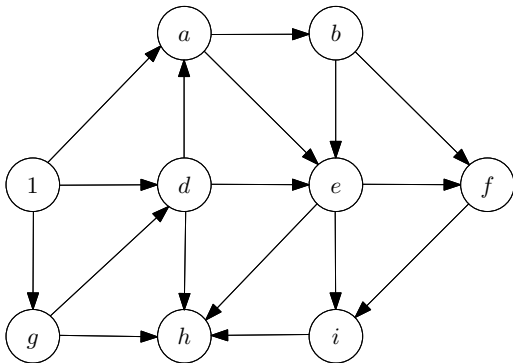
Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



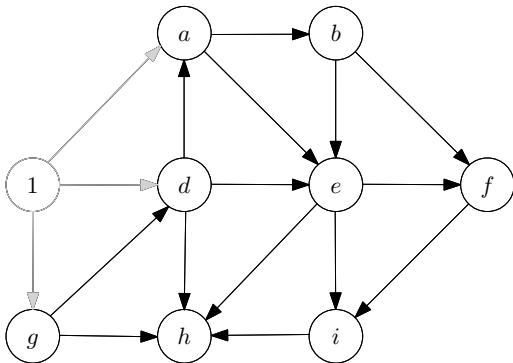
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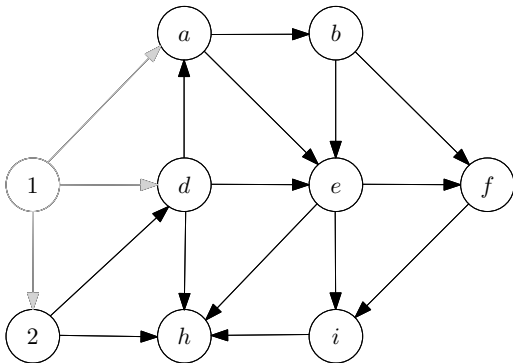
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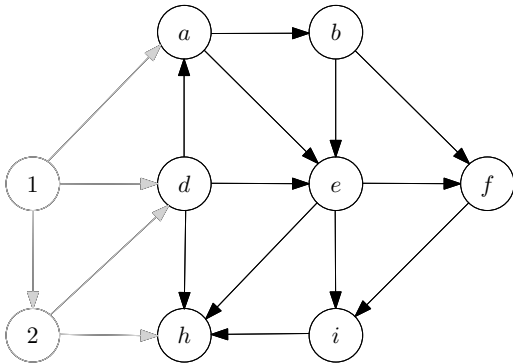
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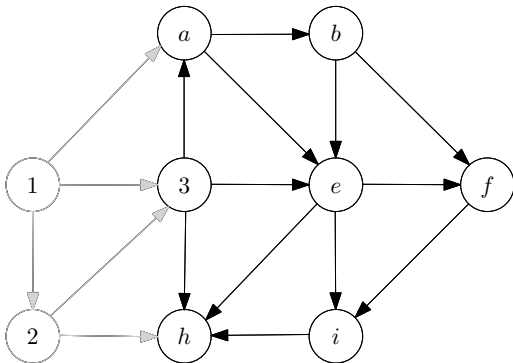
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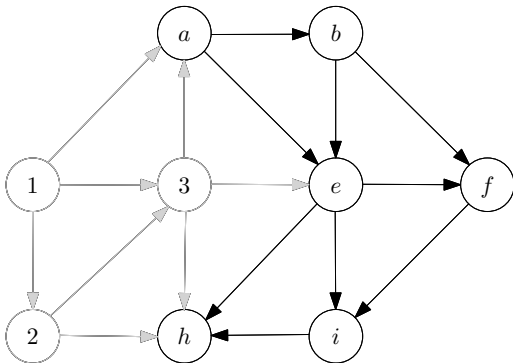
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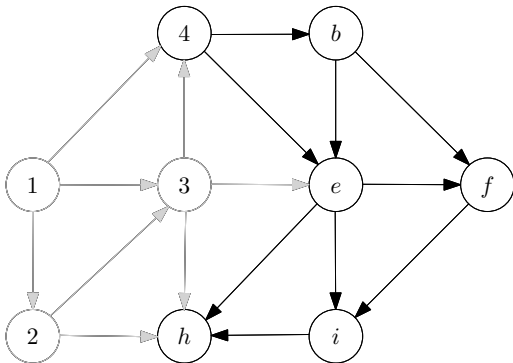
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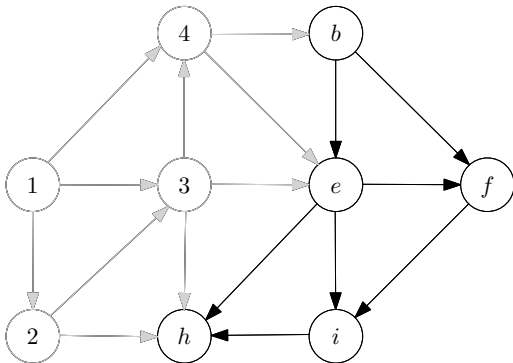
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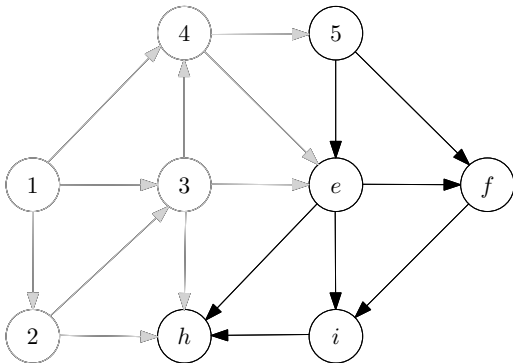
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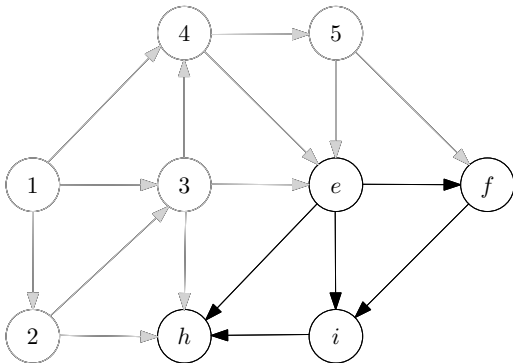
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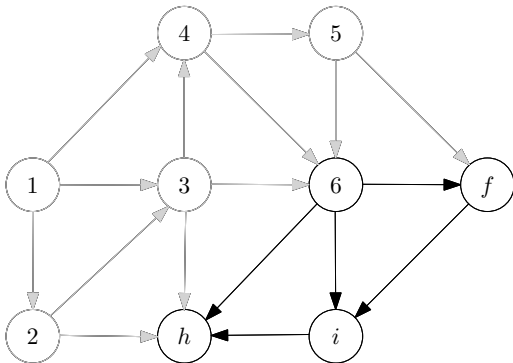
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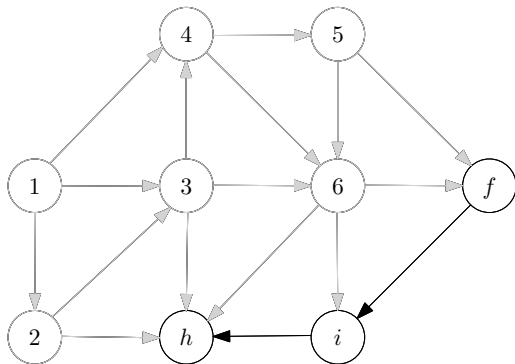
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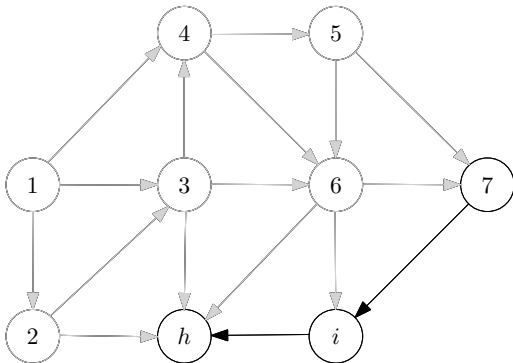
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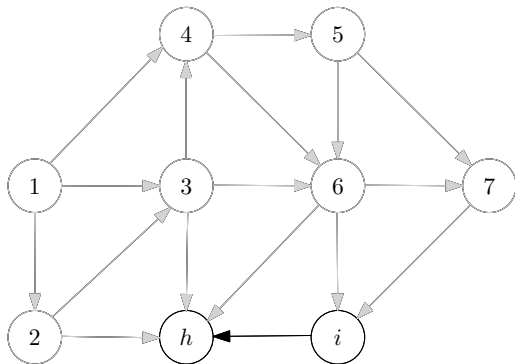
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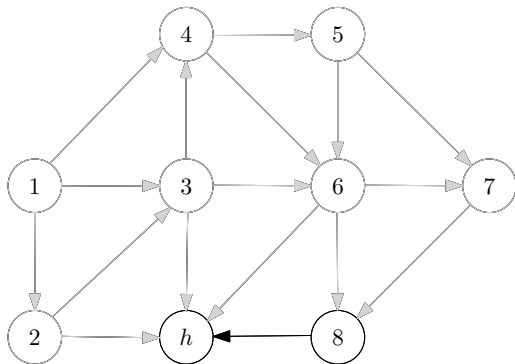
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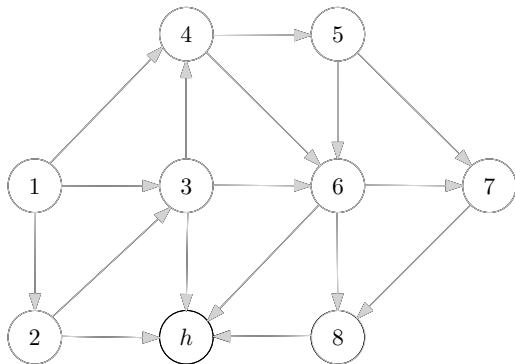
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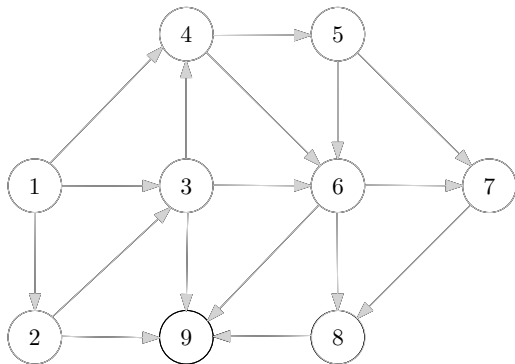
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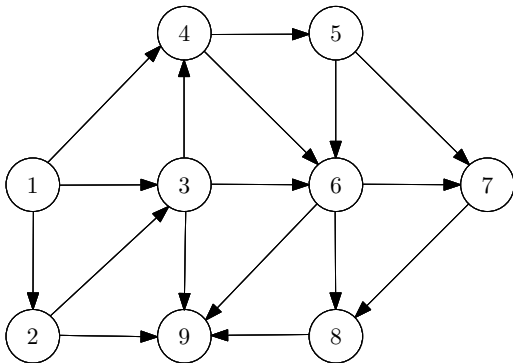
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Q: How to make the algorithm as efficient as possible?

Topological Ordering

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Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

- 1 let $d_v \leftarrow 0$ for every $v \in V$
- 2 for every $v \in V$
- 3 for every u such that $(v, u) \in E$
- 4 $d_u \leftarrow d_u + 1$
- 5 $S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
- 6 while $S \neq \emptyset$
- 7 $v \leftarrow$ arbitrary vertex in $S, S \leftarrow S \setminus \{v\}$
- 8 $i \leftarrow i + 1, \pi(v) \leftarrow i$
- 9 for every u such that $(v, u) \in E$
- 10 $d_u \leftarrow d_u - 1$
- 11 if $d_u = 0$ then add u to S
- 12 if $i < n$ then output "not a DAG"

- S can be represented using a queue or a stack
- Running time = $O(n + m)$