CSE 431/531: Algorithm Analysis and Design (Spring 2018)
Greedy Algorithms

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo

Main Goal of Algorithm Design

- Design fast algorithms to solve problems

Main Goal of Algorithm Design

- Design fast algorithms to solve problems
- Design faster algorithms to solve problems

Main Goal of Algorithm Design

- Design fast algorithms to solve problems
- Design faster algorithms to solve problems

Def. The goal of an optimization problem is to find a valid solution with the minimum (or maximum) cost (or value).

Main Goal of Algorithm Design

- Design fast algorithms to solve problems
- Design faster algorithms to solve problems

Def. The goal of an optimization problem is to find a valid solution with the minimum (or maximum) cost (or value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

Main Goal of Algorithm Design

- Design polynomial-time algorithms to solve problems
- Design faster algorithms to solve problems

Def. The goal of an optimization problem is to find a valid solution with the minimum (or maximum) cost (or value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.

Main Goal of Algorithm Design

- Design polynomial-time algorithms to solve problems
- Design faster algorithms to solve problems

Def. The goal of an optimization problem is to find a valid solution with the minimum (or maximum) cost (or value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.

Main Goal of Algorithm Design

- Design polynomial-time algorithms to solve problems
- Design faster algorithms to solve problems

Def. The goal of an optimization problem is to find a valid solution with the minimum (or maximum) cost (or value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.
- $f(n)$ is polynomial if $f(n) = O(n^k)$ for some constant $k > 0$.

Main Goal of Algorithm Design

- Design efficient algorithms to solve problems
- Design faster algorithms to solve problems

Def. The goal of an optimization problem is to find a valid solution with the minimum (or maximum) cost (or value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.
- $f(n)$ is polynomial if $f(n) = O(n^k)$ for some constant $k > 0$.
- convention: polynomial time $=$ efficient

Main Goal of Algorithm Design

- Design efficient algorithms to solve problems
- Design more efficient algorithms to solve problems

Def. The goal of an optimization problem is to find a valid solution with the minimum (or maximum) cost (or value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.
- $f(n)$ is polynomial if $f(n) = O(n^k)$ for some constant $k > 0$.
- convention: polynomial time $=$ efficient

Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Outline

1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. Summary

Toy Problem 1: Bill Changing

Input: Integer $A \geq 0$

Currency denominations: $\$1, \$2, \$5, \$10, \$20$

Output: A way to pay A dollars using fewest number of bills

Toy Problem 1: Bill Changing

Input: Integer $A \geq 0$

Currency denominations: $\$1, \$2, \$5, \$10, \$20$

Output: A way to pay A dollars using fewest number of bills

Example:

- Input: 48
- Output: 5 bills, $\$48 = \$20 \times 2 + \$5 + \$2 + \$1$

Toy Problem 1: Bill Changing

Input: Integer $A \geq 0$

Currency denominations: $\$1, \$2, \$5, \$10, \$20$

Output: A way to pay A dollars using fewest number of bills

Example:

- Input: 48
- Output: 5 bills, $\$48 = \$20 \times 2 + \$5 + \$2 + \$1$

Cashier's Algorithm

```
1  while $A \geq 0$ do
2      $a \leftarrow \max\{t \in \{1, 2, 5, 10, 20\} : t \leq A\}$
3      pay a $\$a$ bill
4      $A \leftarrow A - a$
```

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a “reasonable” strategy

- strategy: choose the largest bill that does not exceed A

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

- strategy: choose the largest bill that does not exceed A
- the strategy is "reasonable": choosing a larger bill help us in minimizing the number of bills

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

- strategy: choose the largest bill that does not exceed A
- the strategy is "reasonable": choosing a larger bill help us in minimizing the number of bills
- The decision is irrevocable : once we choose a \$$a$ bill, we let $A \leftarrow A - a$ and proceed to the next

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid
- minimize $n_1 + n_2 + n_5 + n_{10} + n_{20}$ subject to $n_1 + 2n_2 + 5n_5 + 10n_{10} + 20n_{20} = A$

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid
- minimize $n_1 + n_2 + n_5 + n_{10} + n_{20}$ subject to $n_1 + 2n_2 + 5n_5 + 10n_{10} + 20n_{20} = A$

Obs.

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid
- minimize $n_1 + n_2 + n_5 + n_{10} + n_{20}$ subject to $n_1 + 2n_2 + 5n_5 + 10n_{10} + 20n_{20} = A$

Obs.

- $n_1 < 2$ $2 \leq A < 5$: pay a \$2 bill

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid
- minimize $n_1 + n_2 + n_5 + n_{10} + n_{20}$ subject to $n_1 + 2n_2 + 5n_5 + 10n_{10} + 20n_{20} = A$

Obs.

- $n_1 < 2$ $\qquad$ $2 \leq A < 5$: pay a \$2 bill
- $n_1 + 2n_2 < 5$ $\qquad$ $5 \leq A < 10$: pay a \$5 bill

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid
- minimize $n_1 + n_2 + n_5 + n_{10} + n_{20}$ subject to $n_1 + 2n_2 + 5n_5 + 10n_{10} + 20n_{20} = A$

Obs.

- $n_1 < 2$ — $2 \leq A < 5$: pay a \$2 bill
- $n_1 + 2n_2 < 5$ — $5 \leq A < 10$: pay a \$5 bill
- $n_1 + 2n_2 + 5n_5 < 10$ — $10 \leq A < 20$: pay a \$10 bill

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- $n_1, n_2, n_5, n_{10}, n_{20}$: number of \$1, \$2, \$5, \$10, \$20 bills paid
- minimize $n_1 + n_2 + n_5 + n_{10} + n_{20}$ subject to $n_1 + 2n_2 + 5n_5 + 10n_{10} + 20n_{20} = A$

Obs.

- $n_1 < 2$ $2 \leq A < 5$: pay a \$2 bill
- $n_1 + 2n_2 < 5$ $5 \leq A < 10$: pay a \$5 bill
- $n_1 + 2n_2 + 5n_5 < 10$ $10 \leq A < 20$: pay a \$10 bill
- $n_1 + 2n_2 + 5n_5 + 10n_{10} < 20$ $20 \leq A < \infty$: pay a \$20 bill

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- Trivial: in residual problem, we need to pay $A' = A - a$ dollars, using the fewest number of bills

Toy Example 2: Box Packing

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

Toy Example 2: Box Packing

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

Example:

- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: $45 \to 60, 20 \to 40, 19 \to 25$

Box Packing: Design a Safe Strategy

Q: Take box 1 (with capacity c_1). Which item should we put in box 1?

Box Packing: Design a Safe Strategy

Q: Take box 1 (with capacity c_1). Which item should we put in box 1?

A: The item of the largest size that can be put into the box.

Box Packing: Design a Safe Strategy

Q: Take box 1 (with capacity c_1). Which item should we put in box 1?

A: The item of the largest size that can be put into the box.

- putting the item gives us the easiest residual problem.

Box Packing: Design a Safe Strategy

Q: Take box 1 (with capacity c_1). Which item should we put in box 1?

A: The item of the largest size that can be put into the box.

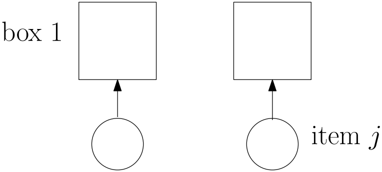
- putting the item gives us the easiest residual problem.
- formal proof via exchanging argument: $j =$ largest item that can be put into box 1.

Box Packing: Design a Safe Strategy

Q: Take box 1 (with capacity c_1). Which item should we put in box 1?

A: The item of the largest size that can be put into the box.

- putting the item gives us the easiest residual problem.
- formal proof via exchanging argument: $j =$ largest item that can be put into box 1.

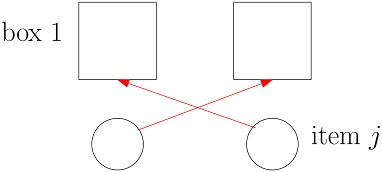


Box Packing: Design a Safe Strategy

Q: Take box 1 (with capacity c_1). Which item should we put in box 1?

A: The item of the largest size that can be put into the box.

- putting the item gives us the easiest residual problem.
- formal proof via exchanging argument: $j =$ largest item that can be put into box 1.



- Residual task: solve the instance obtained by removing box 1 and item j

Greedy Algorithm for Box Packing

```
$T \leftarrow \{1, 2, 3, \cdots, m\}$
for $i \leftarrow 1$ to $n$ do
    if some item in $T$ can be put into box $i$, then
        $j \leftarrow$ the largest item in $T$ that can be put into box $i$
        print("put item $j$ in box $i$")
        $T \leftarrow T \setminus \{j\}$
```

Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

Exchanging argument: let S be an arbitrary optimum solution. If S is consistent with the greedy choice, we are done. Otherwise, modify it to another optimum solution S' such that S' is consistent with the greedy choice.

Outline

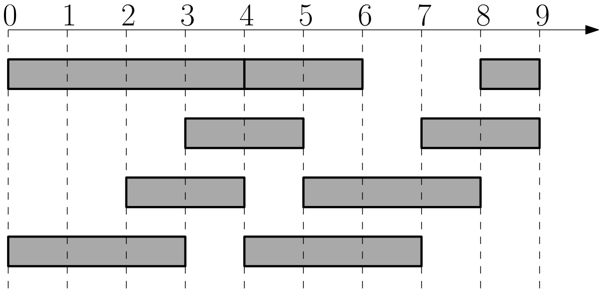
1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. Summary

Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs

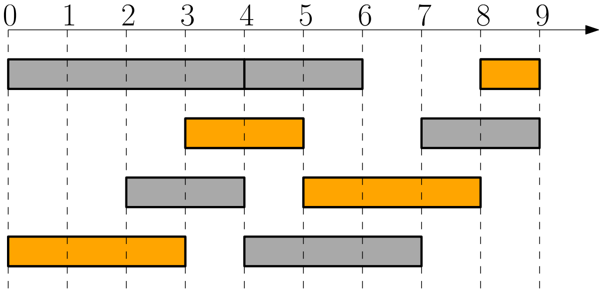


Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs



Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?

Greedy Algorithm for Interval Scheduling

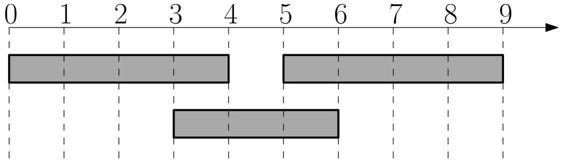
- Which of the following decisions are safe?
- Schedule the job with the smallest size?

Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!

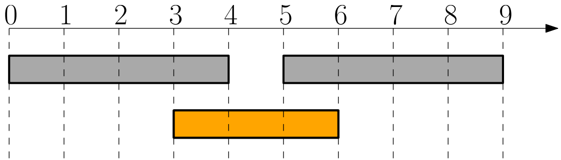
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!



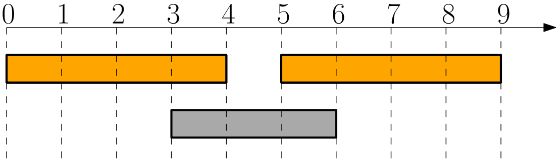
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!



Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!



Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!

Greedy Algorithm for Interval Scheduling

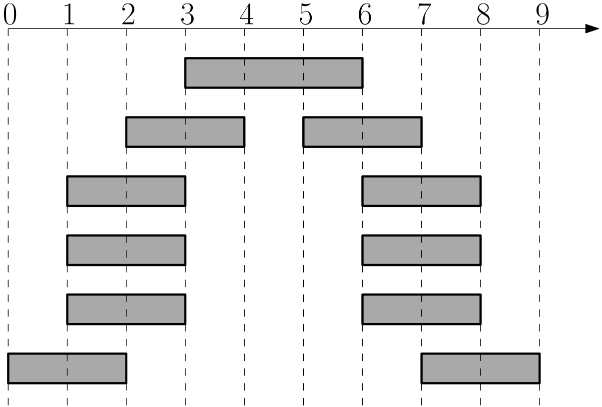
- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs?

Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!

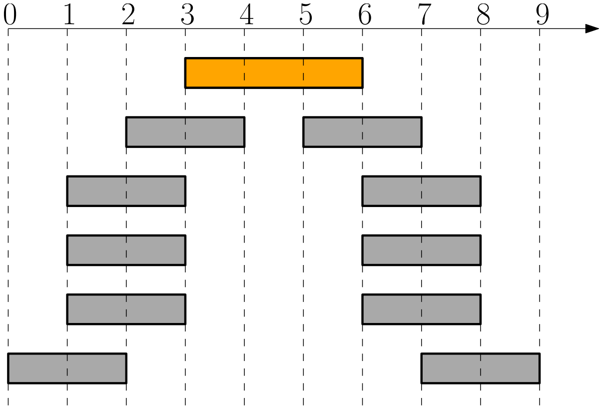
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!



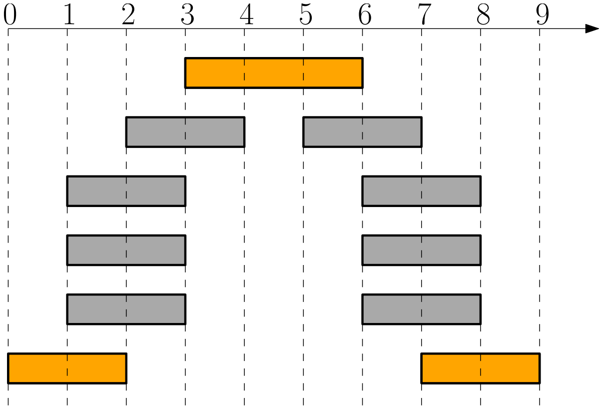
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!



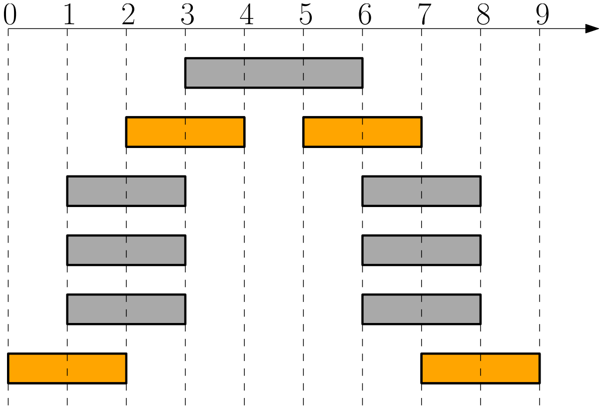
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!



Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!



Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!

Greedy Algorithm for Interval Scheduling

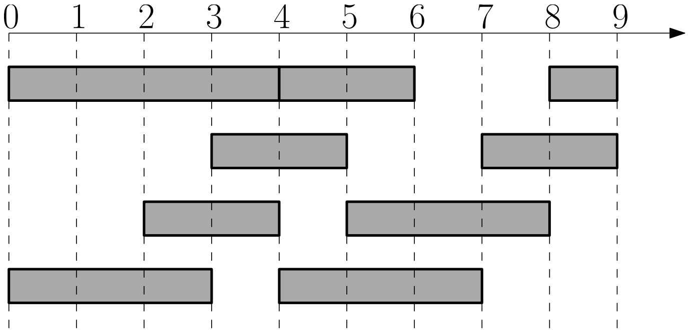
- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time?

Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time? Yes!

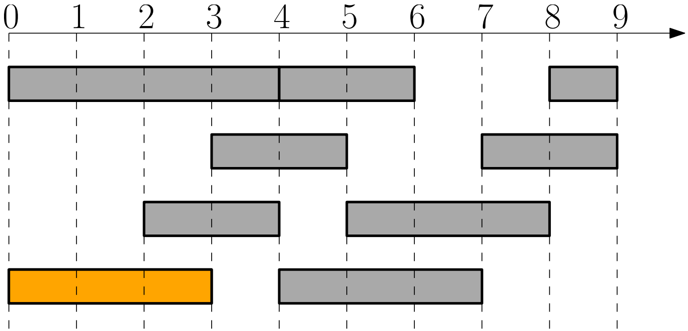
Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time? Yes!



Greedy Algorithm for Interval Scheduling

- Which of the following decisions are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time? Yes!



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

- Take an arbitrary optimum solution S

Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

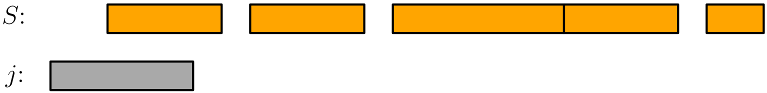
- Take an arbitrary optimum solution S
- If it contains j, done

Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

- Take an arbitrary optimum solution S
- If it contains j, done

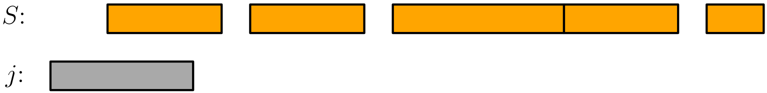


Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

- Take an arbitrary optimum solution S
- If it contains j, done
- Otherwise, replace the first job in S with j to obtain an new optimum schedule S'. ☐

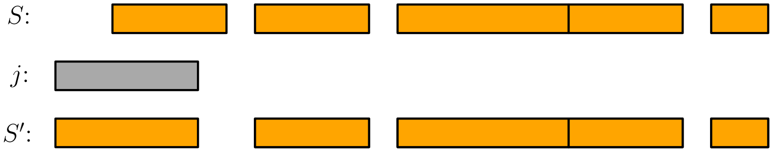


Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

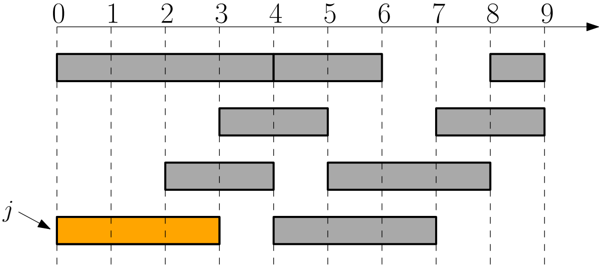
- Take an arbitrary optimum solution S
- If it contains j, done
- Otherwise, replace the first job in S with j to obtain an new optimum schedule S'. □



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

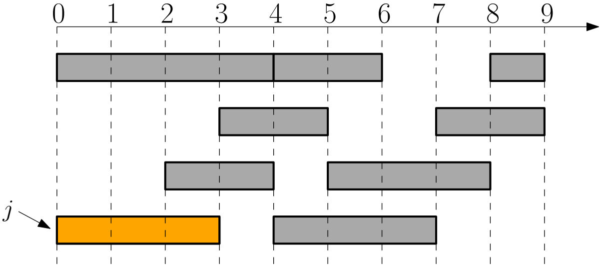
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem?



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

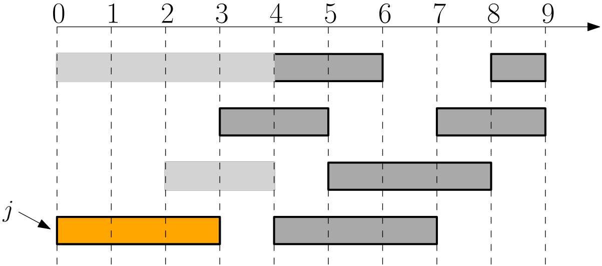
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem? Yes!



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

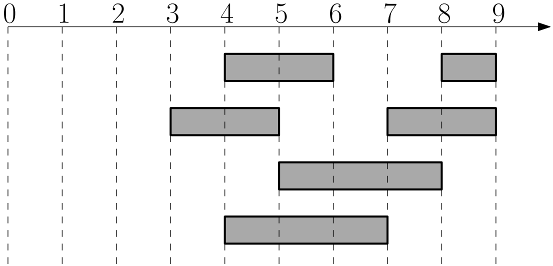
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem? Yes!



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem? Yes!



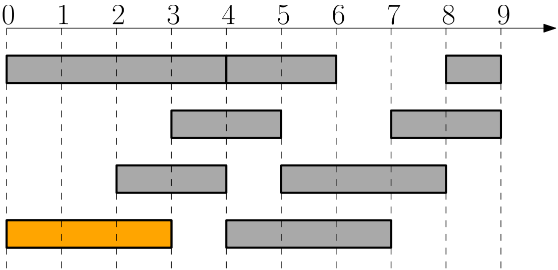
Greedy Algorithm for Interval Scheduling

```
Schedule($s, f, n$)
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```

Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

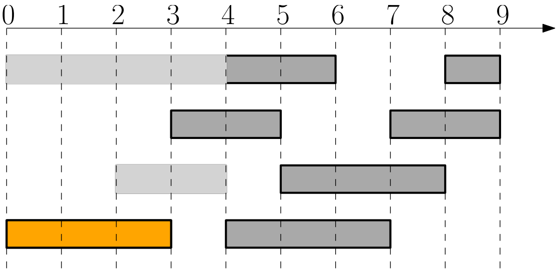
```
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```



Greedy Algorithm for Interval Scheduling

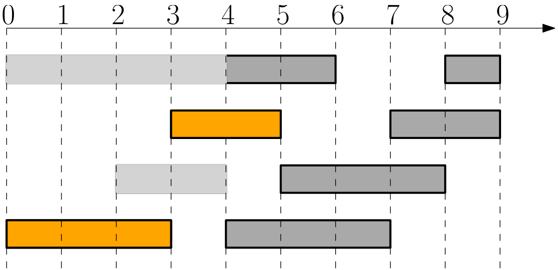
Schedule(s, f, n)

```
A ← {1, 2, ⋯, n}, S ← ∅
while A ≠ ∅
    j ← argmin_{j'∈A} f_{j'}
    S ← S ∪ {j}; A ← {j' ∈ A : s_{j'} ≥ f_j}
return S
```



Greedy Algorithm for Interval Scheduling

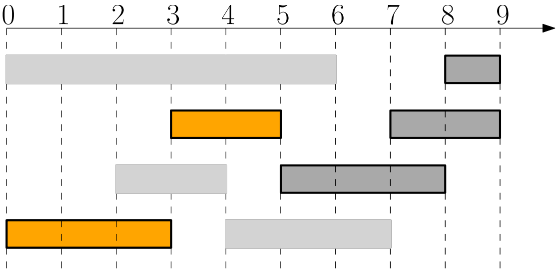
```
Schedule($s, f, n$)
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```



Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

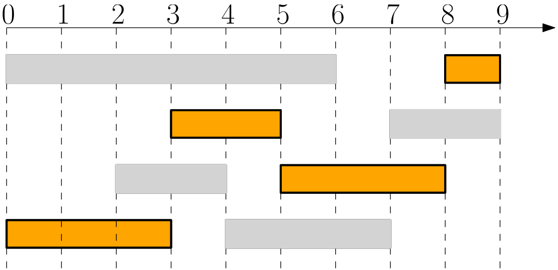
```
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```



Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

```
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```



Greedy Algorithm for Interval Scheduling

```
Schedule($s, f, n$)
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```

Running time of algorithm?

Greedy Algorithm for Interval Scheduling

```
Schedule($s, f, n$)
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```

Running time of algorithm?

- Naive implementation: $O(n^2)$ time

Greedy Algorithm for Interval Scheduling

```
Schedule($s, f, n$)
$A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
while $A \neq \emptyset$
    $j \leftarrow \arg\min_{j' \in A} f_{j'}$
    $S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
return $S$
```

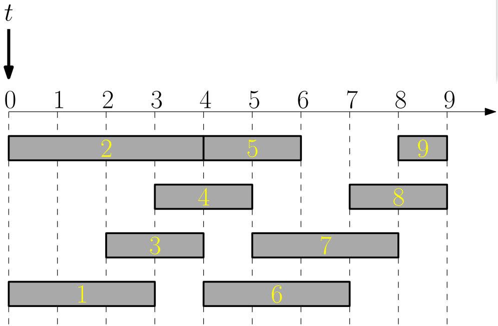
Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

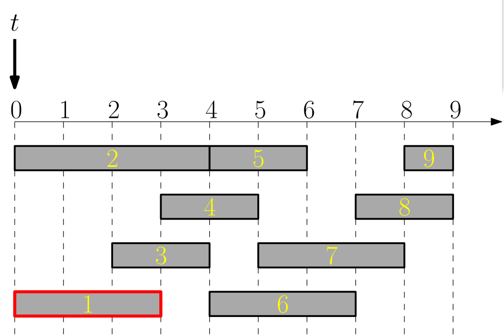
```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

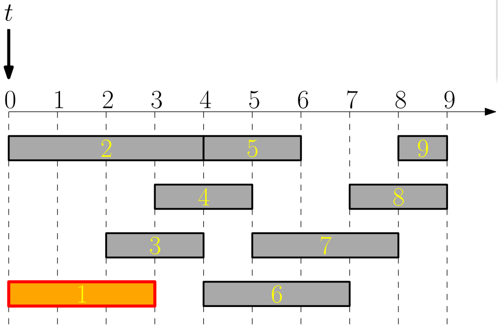
Schedule(s, f, n)

```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



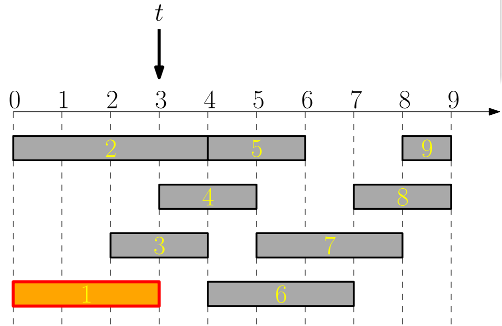
Clever Implementation of Greedy Algorithm

```
Schedule($s, f, n$)
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

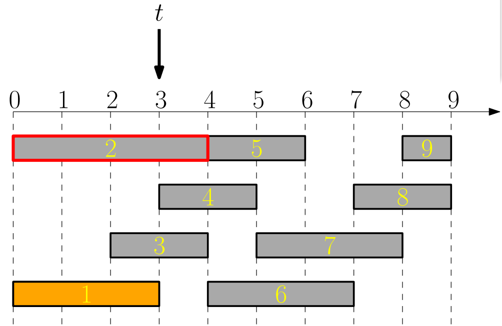
```
Schedule($s, f, n$)
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

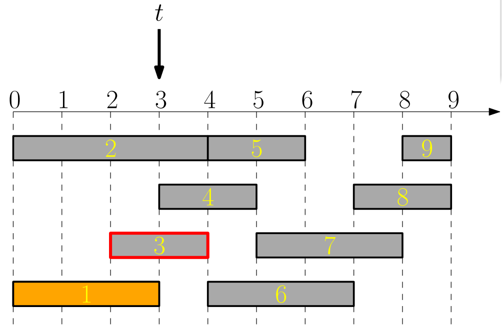
```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

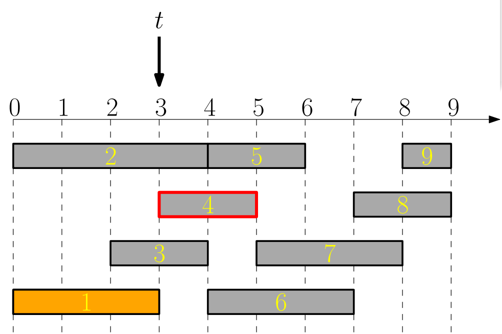
```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

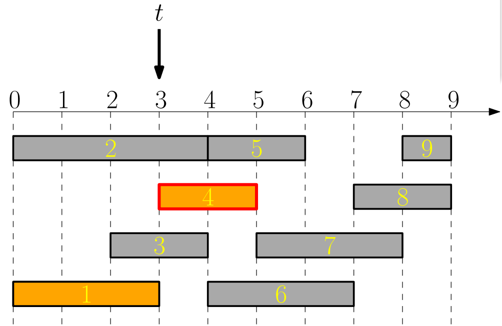
```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

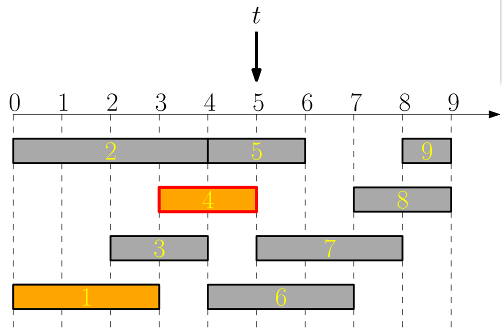
```
1 sort jobs according to $f$ values
2 $t \leftarrow 0$, $S \leftarrow \emptyset$
3 for every $j \in [n]$ according to non-decreasing order of $f_j$
4     if $s_j \geq t$ then
5         $S \leftarrow S \cup \{j\}$
6         $t \leftarrow f_j$
7 return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

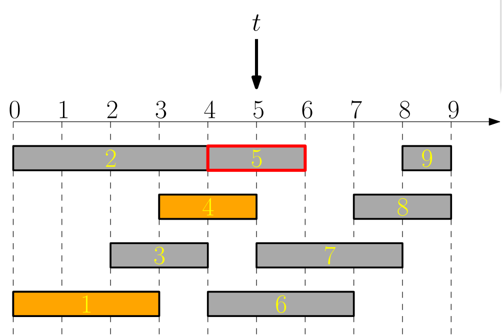
```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

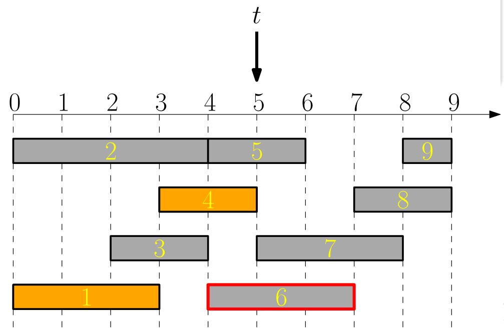
Schedule(s, f, n)

```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

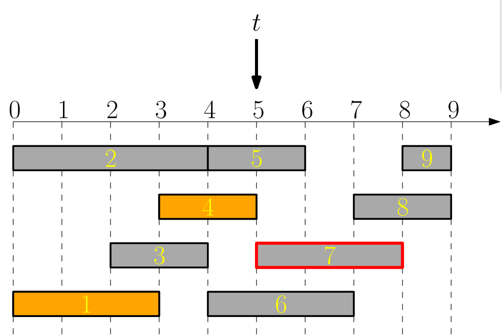
```
Schedule($s, f, n$)
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

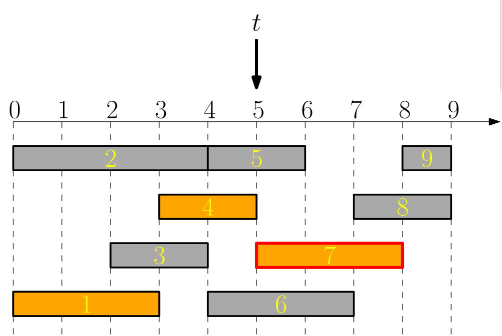
Schedule(s, f, n)

```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

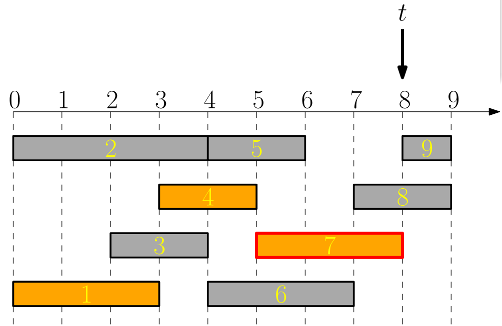
```
Schedule($s, f, n$)
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

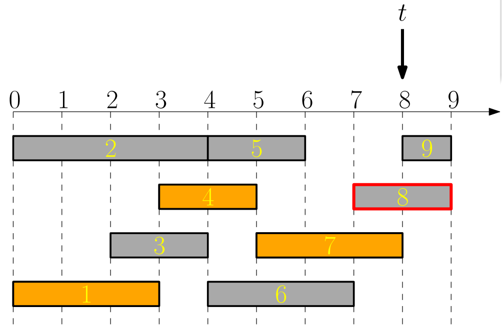
```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

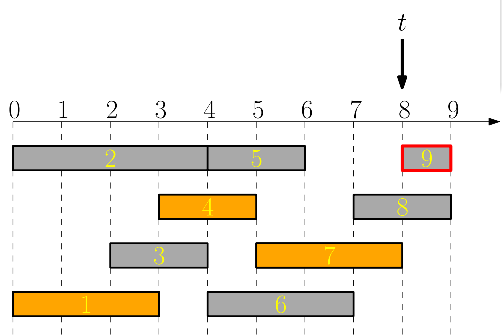
Schedule(s, f, n)

```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

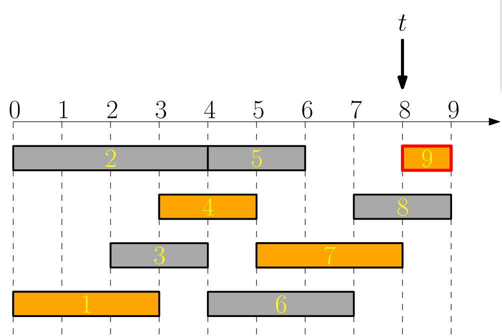
```
Schedule($s, f, n$)
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

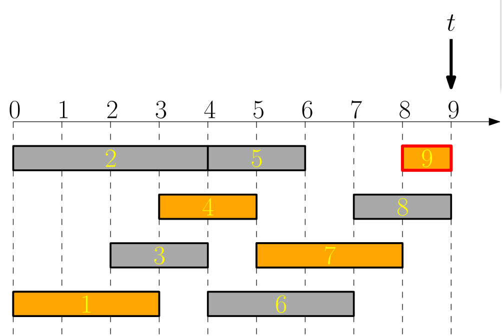
```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```



Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

```
sort jobs according to $f$ values
$t \leftarrow 0$, $S \leftarrow \emptyset$
for every $j \in [n]$ according to non-decreasing order of $f_j$
    if $s_j \geq t$ then
        $S \leftarrow S \cup \{j\}$
        $t \leftarrow f_j$
return $S$
```

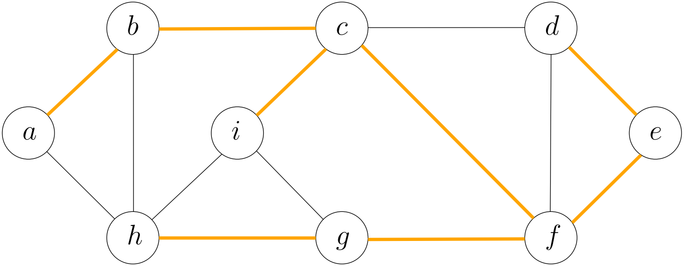


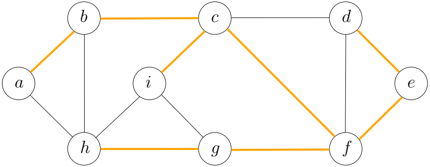
Outline

1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. Summary

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

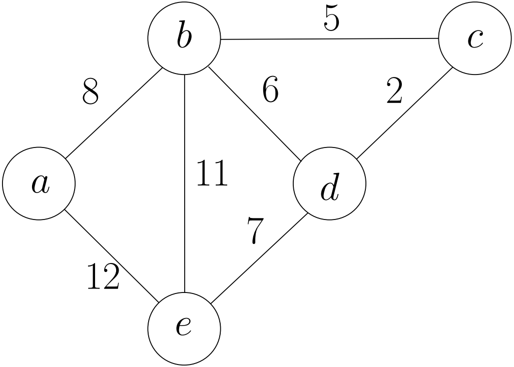
Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

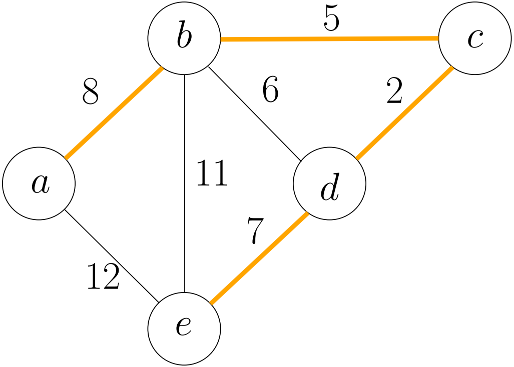
Output: the spanning tree T of G with the minimum total weight



Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight



Recall: Steps of Designing A Greedy Algorithm

- Design a "reasonable" strategy
- Prove that the reasonable strategy is "safe" (key, usually done by "exchanging argument")
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

Recall: Steps of Designing A Greedy Algorithm

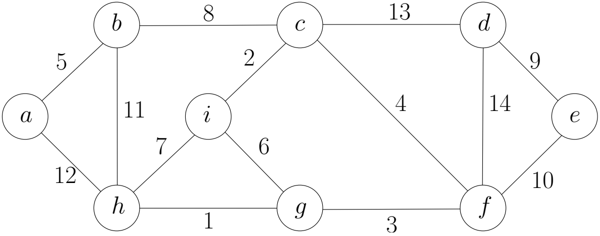
- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

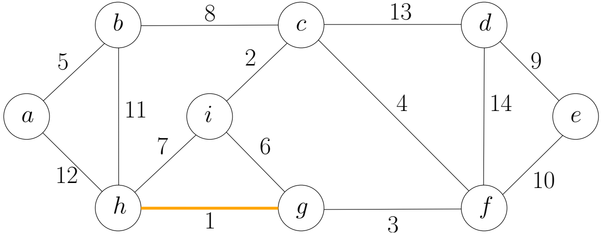
Two Classic Greedy Algorithms for MST

- Kruskal’s Algorithm
- Prim’s Algorithm

Outline



Q: Which edge can be safely included in the MST?



Q: Which edge can be safely included in the MST?

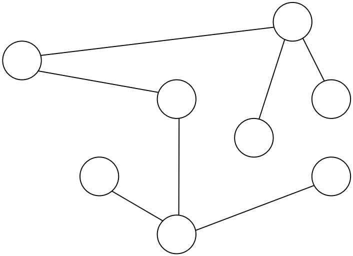
A: The edge with the smallest weight (lightest edge).

Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

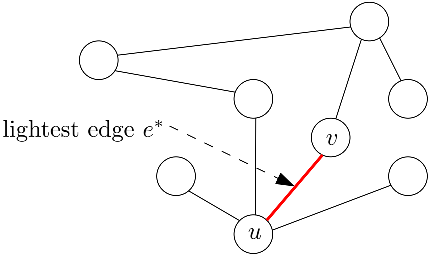
- Take a minimum spanning tree T



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

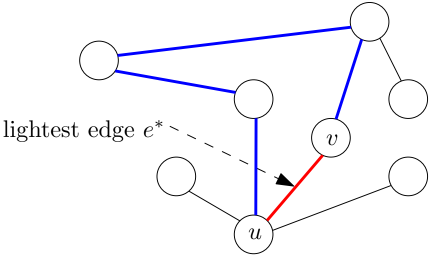
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

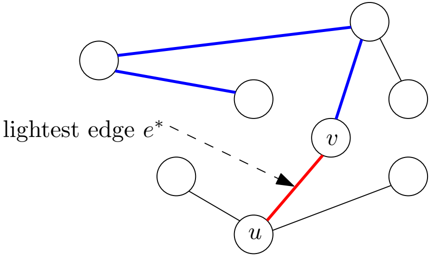
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

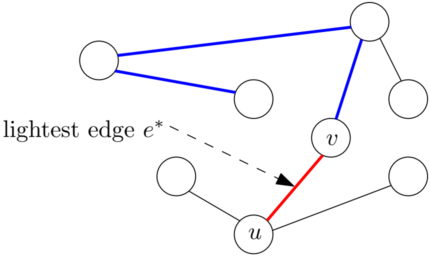
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v
- Remove any edge e in the path to obtain tree T'



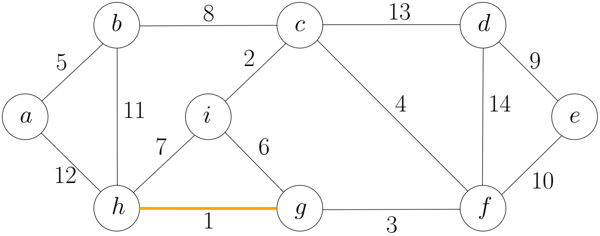
Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

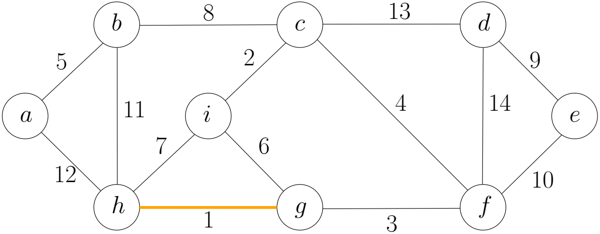
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v
- Remove any edge e in the path to obtain tree T'
- $w(e^*) \le w(e) \implies w(T') \le w(T)$: T' is also a MST ☐



Is the Residual Problem Still a MST Problem?

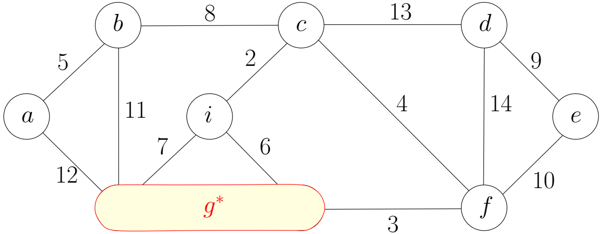


Is the Residual Problem Still a MST Problem?



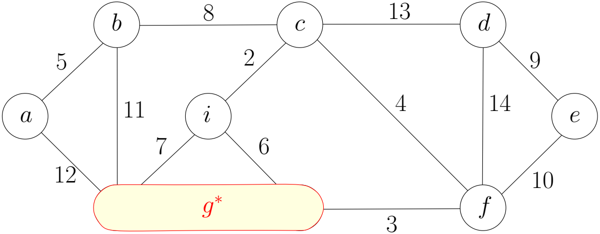
- Residual problem: find the minimum spanning tree that contains edge (g, h)

Is the Residual Problem Still a MST Problem?



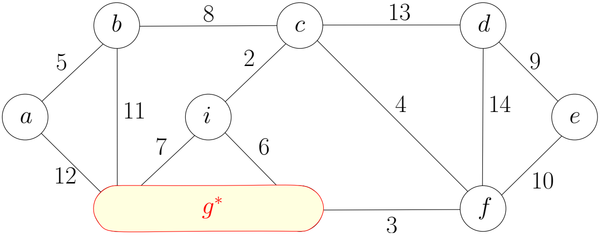
- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)

Is the Residual Problem Still a MST Problem?

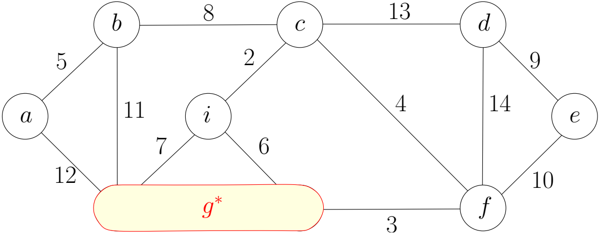


- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)

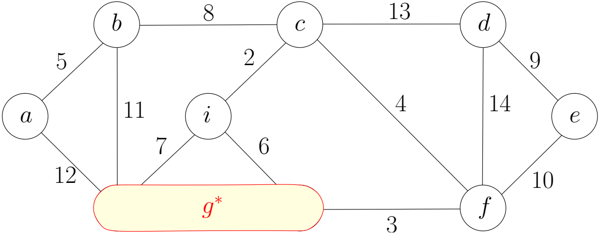


Contraction of an Edge (u, v)



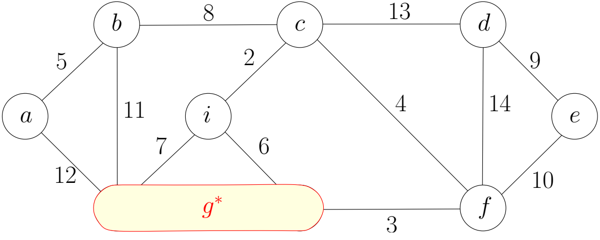
- Remove u and v from the graph, and add a new vertex u^*

Contraction of an Edge (u, v)



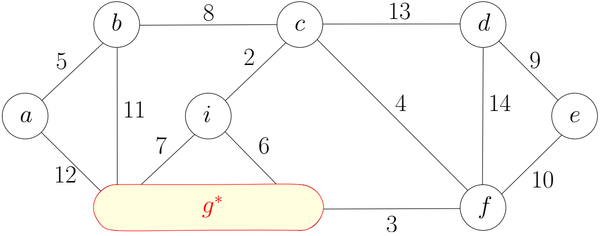
- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges parallel connecting u to v from E

Contraction of an Edge (u, v)



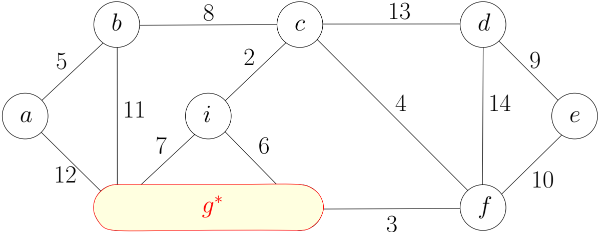
- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges parallel connecting u to v from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges parallel connecting u to v from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges parallel connecting u to v from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

Q: What edges are removed due to contractions?

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

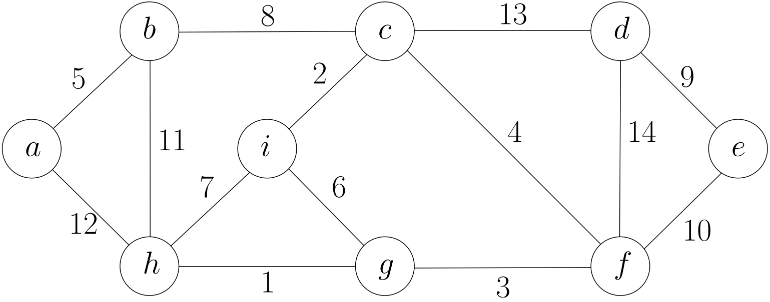
Q: What edges are removed due to contractions?

A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

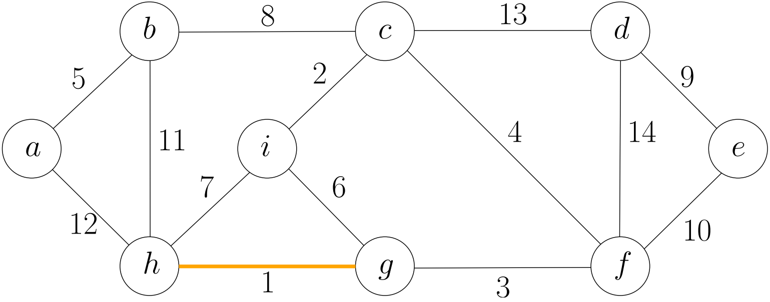
```
MST-Greedy($G, w$)
$F = \emptyset$
sort edges in $E$ in non-decreasing order of weights $w$
for each edge $(u, v)$ in the order
    if $u$ and $v$ are not connected by a path of edges in $F$
        $F = F \cup \{(u, v)\}$
return $(V, F)$
```

Kruskal's Algorithm: Example



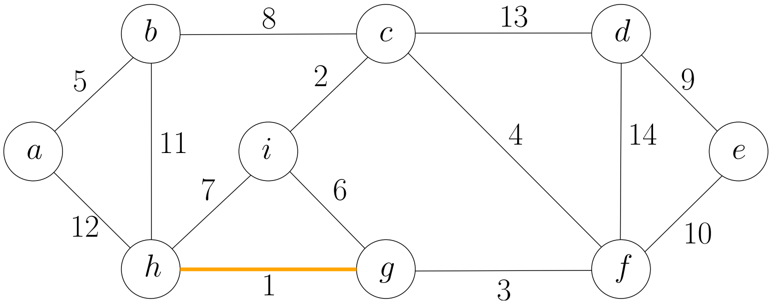
Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

Kruskal's Algorithm: Example



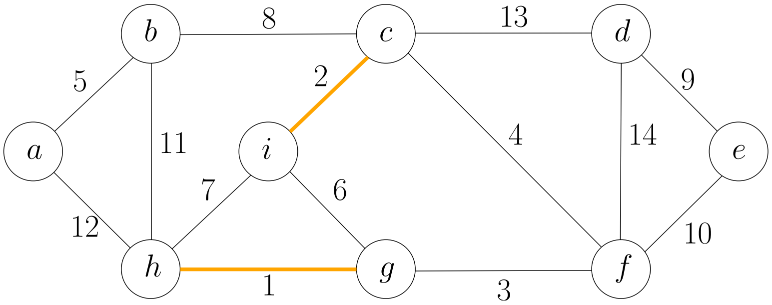
Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

Kruskal's Algorithm: Example



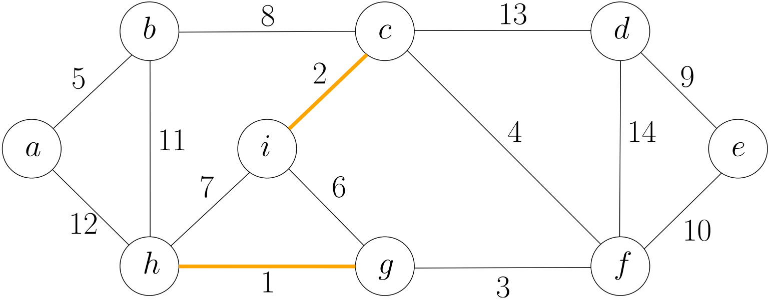
Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g, h\}, \{i\}$

Kruskal's Algorithm: Example



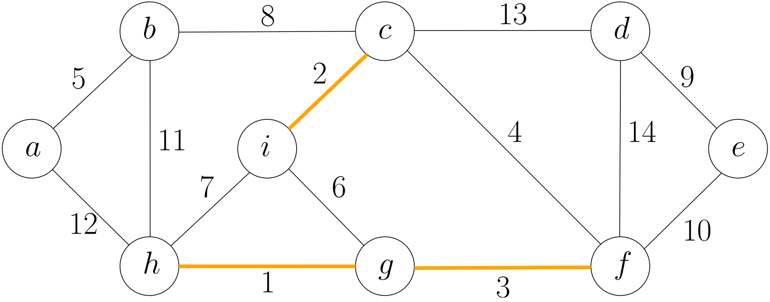
Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g, h\}, \{i\}$

Kruskal's Algorithm: Example



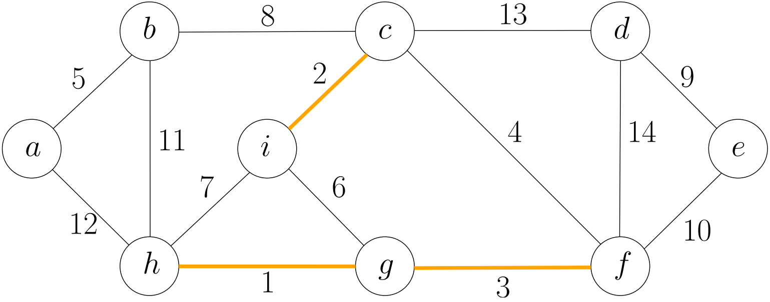
Sets: $\{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f\}, \{g, h\}$

Kruskal's Algorithm: Example



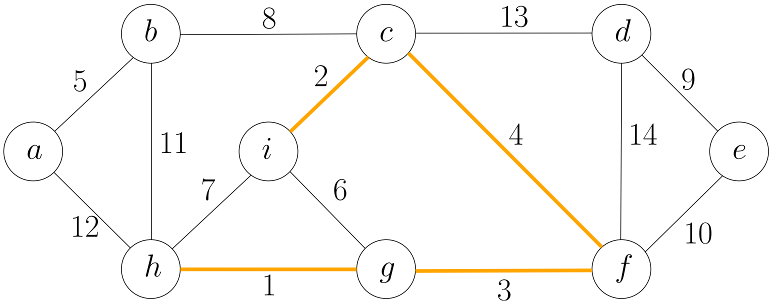
Sets: $\{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f\}, \{g, h\}$

Kruskal's Algorithm: Example



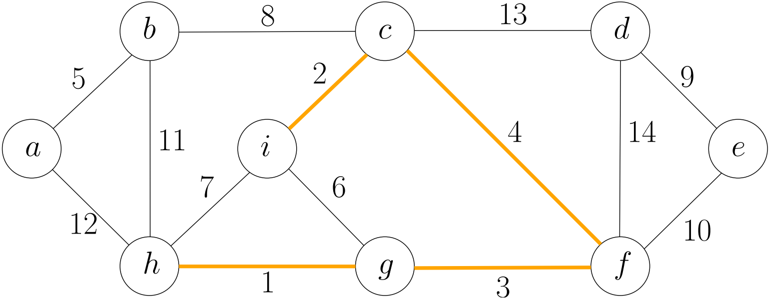
Sets: $\{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f, g, h\}$

Kruskal's Algorithm: Example



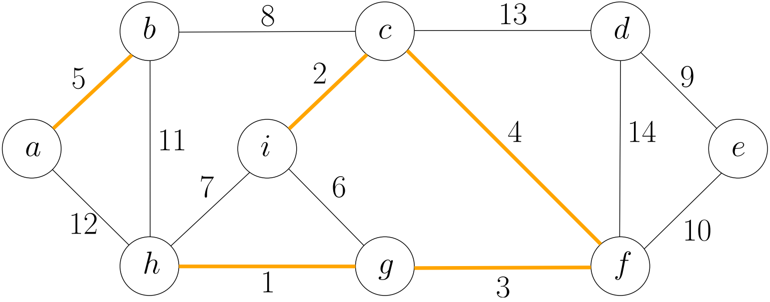
Sets: $\{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f, g, h\}$

Kruskal's Algorithm: Example



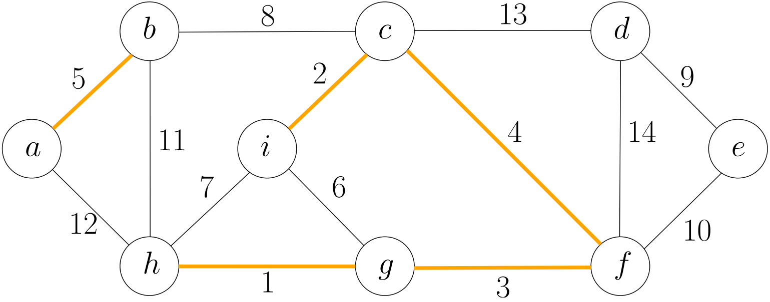
Sets: $\{a\}, \{b\}, \{c, i, f, g, h\}, \{d\}, \{e\}$

Kruskal's Algorithm: Example



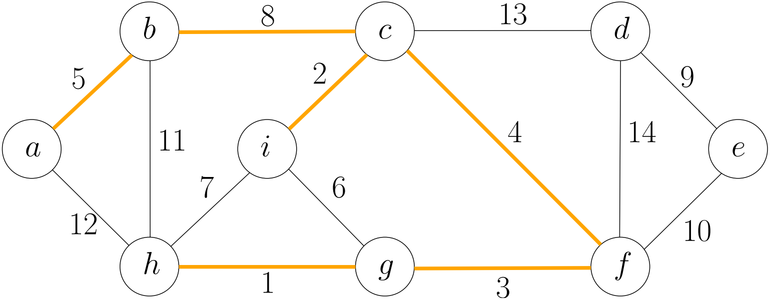
Sets: $\{a\}, \{b\}, \{c, i, f, g, h\}, \{d\}, \{e\}$

Kruskal's Algorithm: Example



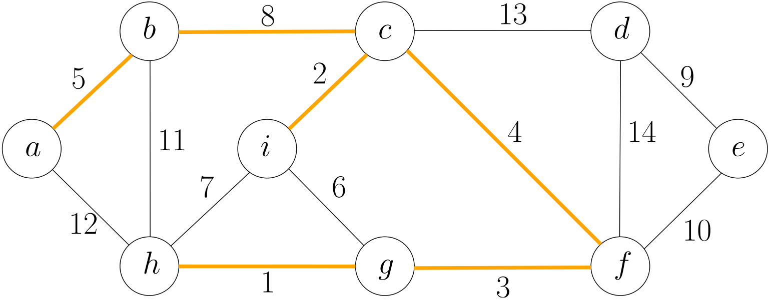
Sets: $\{a, b\}, \{c, i, f, g, h\}, \{d\}, \{e\}$

Kruskal's Algorithm: Example



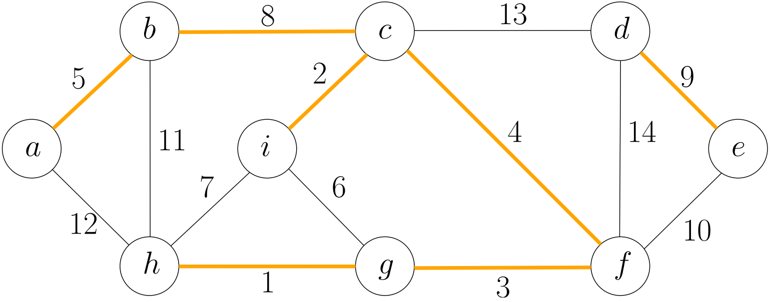
Sets: $\{a, b\}, \{c, i, f, g, h\}, \{d\}, \{e\}$

Kruskal's Algorithm: Example



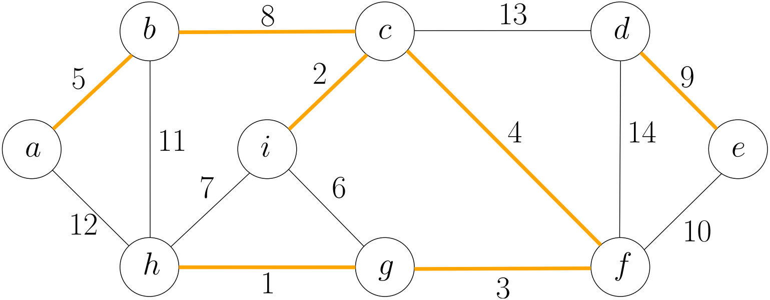
Sets: $\{a, b, c, i, f, g, h\}, \{d\}, \{e\}$

Kruskal's Algorithm: Example



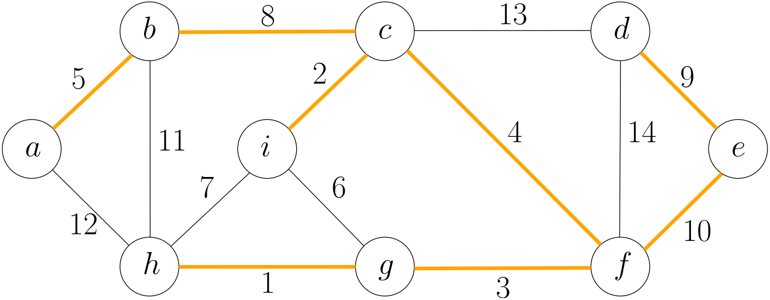
Sets: $\{a, b, c, i, f, g, h\}, \{d\}, \{e\}$

Kruskal's Algorithm: Example



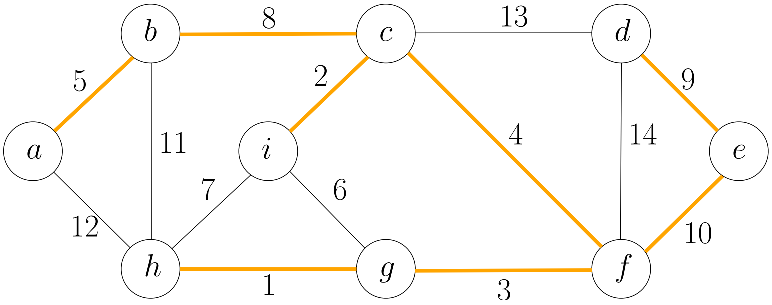
Sets: $\{a, b, c, i, f, g, h\}, \{d, e\}$

Kruskal's Algorithm: Example



Sets: $\{a, b, c, i, f, g, h\}, \{d, e\}$

Kruskal's Algorithm: Example



Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

```
$F \leftarrow \emptyset$
$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
sort the edges of $E$ in non-decreasing order of weights $w$
for each edge $(u, v) \in E$ in the order
    $S_u \leftarrow$ the set in $\mathcal{S}$ containing $u$
    $S_v \leftarrow$ the set in $\mathcal{S}$ containing $v$
    if $S_u \neq S_v$
        $F \leftarrow F \cup \{(u, v)\}$
        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
return $(V, F)$
```

Running Time of Kruskal's Algorithm

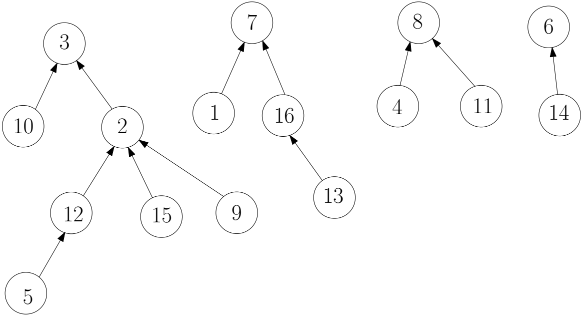
```
MST-Kruskal($G$, $w$)
❶  $F \leftarrow \emptyset$
❷  $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
❸  sort the edges of $E$ in non-decreasing order of weights $w$
❹  for each edge $(u, v) \in E$ in the order
❺      $S_u \leftarrow$ the set in $\mathcal{S}$ containing $u$
❻      $S_v \leftarrow$ the set in $\mathcal{S}$ containing $v$
❼      if $S_u \neq S_v$
❽          $F \leftarrow F \cup \{(u, v)\}$
❾          $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
❿  return $(V, F)$
```

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

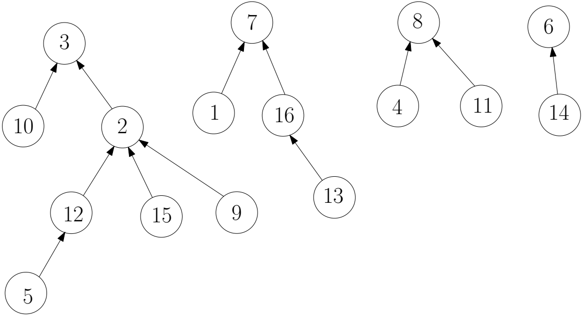
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition:
  $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$

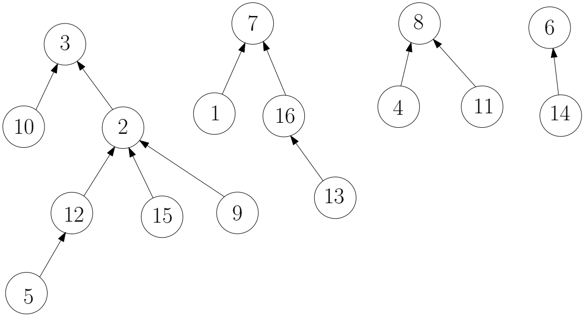


- $par[i]$: parent of i, ($par[i] =$ nil if i is a root).

Union-Find Data Structure

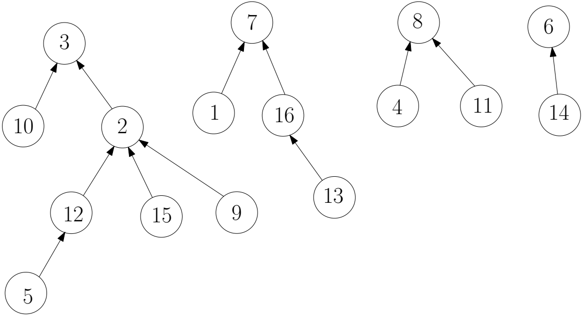


Union-Find Data Structure



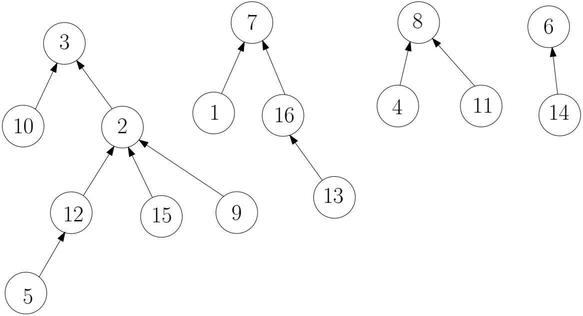
- Q: how can we check if u and v are in the same set?

Union-Find Data Structure



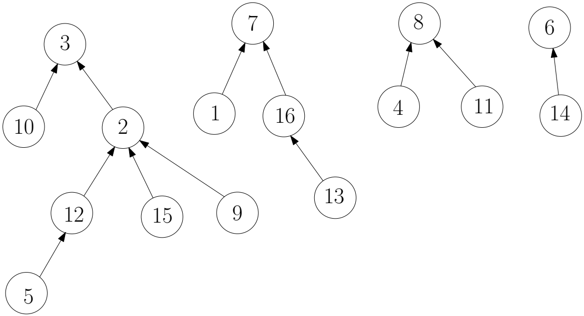
- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).

Union-Find Data Structure



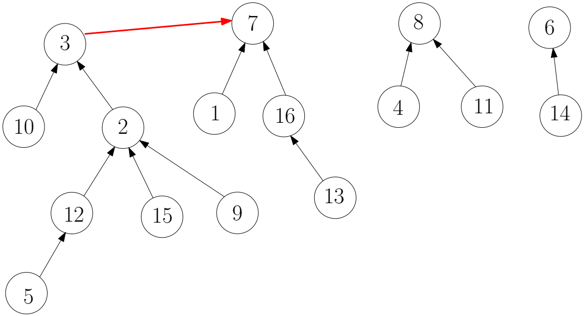
- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).
- root(u): the root of the tree containing u

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).
- root(u): the root of the tree containing u
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).
- root(u): the root of the tree containing u
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure

root(v)

```
if $par[v] = $ nil then
    return $v$
else
    return root($par[v]$)
```

Union-Find Data Structure

root(v)

```
if $par[v]$ = nil then
    return $v$
else
    return root($par[v]$)
```

- Problem: the tree might too deep; running time might be large

Union-Find Data Structure

root(v)

```
if $par[v] = $ nil then
    return $v$
else
    return root($par[v]$)
```

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.

Union-Find Data Structure

root(v)

```
if $par[v]$ = nil then
    return $v$
else
    return root($par[v]$)
```

root(v)

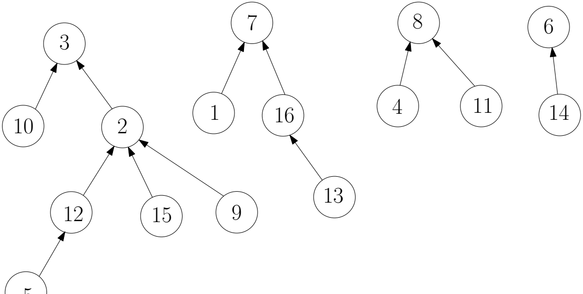
```
if par$[v]$ = nil then
    return $v$
else
    $par[v] \leftarrow$ root($par[v]$)
    return $par[v]$
```

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.

Union-Find Data Structure

root(v)

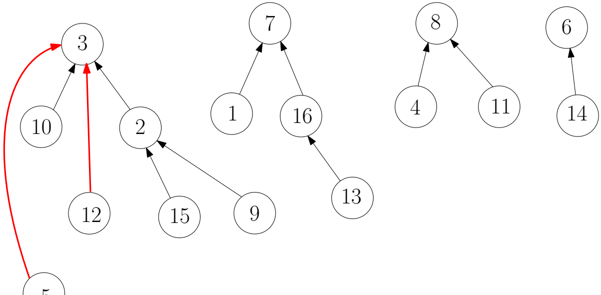
```
if $par[v]$ = nil then
    return $v$
else
    $par[v] \leftarrow$ root($par[v]$)
    return $par[v]$
```



Union-Find Data Structure

root(v)

```
if $par[v] = nil$ then
    return $v$
else
    $par[v] \leftarrow root(par[v])$
    return $par[v]$
```



MST-Kruskal(G, w)

```
$F \leftarrow \emptyset$
$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
sort the edges of $E$ in non-decreasing order of weights $w$
for each edge $(u, v) \in E$ in the order
    $S_u \leftarrow$ the set in $\mathcal{S}$ containing $u$
    $S_v \leftarrow$ the set in $\mathcal{S}$ containing $v$
    if $S_u \neq S_v$
        $F \leftarrow F \cup \{(u, v)\}$
        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
return $(V, F)$
```

MST-Kruskal(G, w)

```
1  $F \leftarrow \emptyset$
2  for every $v \in V$: let $par[v] \leftarrow$ nil
3  sort the edges of $E$ in non-decreasing order of weights $w$
4  for each edge $(u, v) \in E$ in the order
5      $u' \leftarrow$ root$(u)$
6      $v' \leftarrow$ root$(v)$
7      if $u' \neq v'$
8          $F \leftarrow F \cup \{(u, v)\}$
9          $par[u'] \leftarrow v'$
10 return $(V, F)$
```

- ②,⑤,⑥,⑦,⑨ takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

MST-Kruskal(G, w)

```
1  $F \leftarrow \emptyset$
2  for every $v \in V$: let $par[v] \leftarrow$ nil
3  sort the edges of $E$ in non-decreasing order of weights $w$
4  for each edge $(u, v) \in E$ in the order
5      $u' \leftarrow$ root$(u)$
6      $v' \leftarrow$ root$(v)$
7      if $u' \neq v'$
8          $F \leftarrow F \cup \{(u, v)\}$
9          $par[u'] \leftarrow v'$
10 return $(V, F)$
```

- ❷,❺,❻,❼,❾ takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

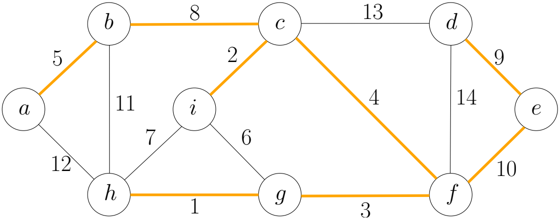
MST-Kruskal(G, w)

```
1  $F \leftarrow \emptyset$
2  for every $v \in V$: let $par[v] \leftarrow$ nil
3  sort the edges of $E$ in non-decreasing order of weights $w$
4  for each edge $(u, v) \in E$ in the order
5      $u' \leftarrow$ root$(u)$
6      $v' \leftarrow$ root$(v)$
7      if $u' \neq v'$
8          $F \leftarrow F \cup \{(u, v)\}$
9          $par[u'] \leftarrow v'$
10 return $(V, F)$
```

- Running time $=$ time for ③ $= O(m \lg n)$.

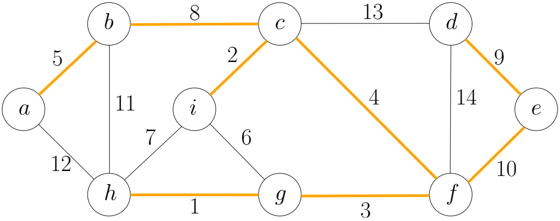
Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



Assumption Assume all edge weights are different.

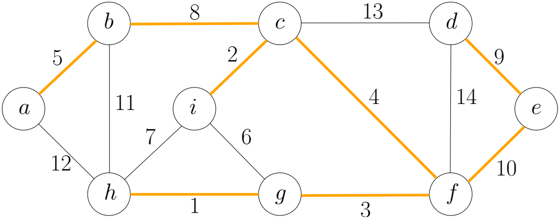
Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



- (i, g) is not in the MST because of cycle (i, c, f, g)

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. Summary

Two Methods to Build a MST

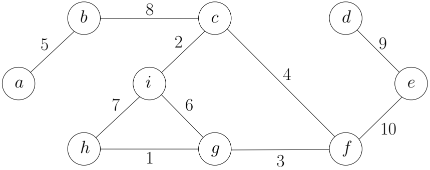
1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from F one by one until we obtain a spanning tree

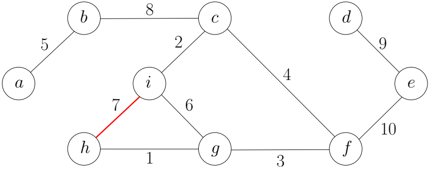
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from F one by one until we obtain a spanning tree



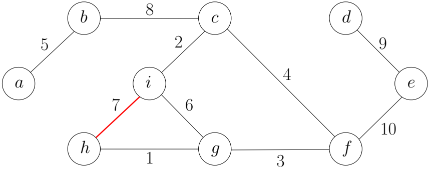
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from F one by one until we obtain a spanning tree



Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from F one by one until we obtain a spanning tree



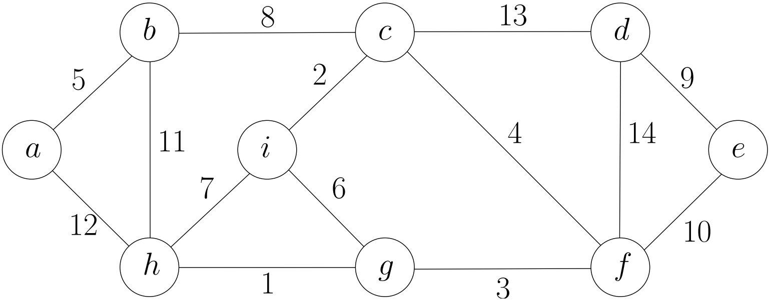
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

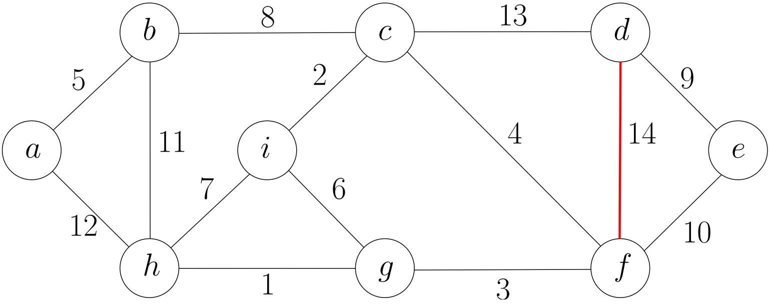
MST-Greedy(G, w)

```
$F \leftarrow E$
sort $E$ in non-increasing order of weights
for every $e$ in this order
    if $(V, F \setminus \{e\})$ is connected then
        $F \leftarrow F \setminus \{e\}$
return $(V, F)$
```

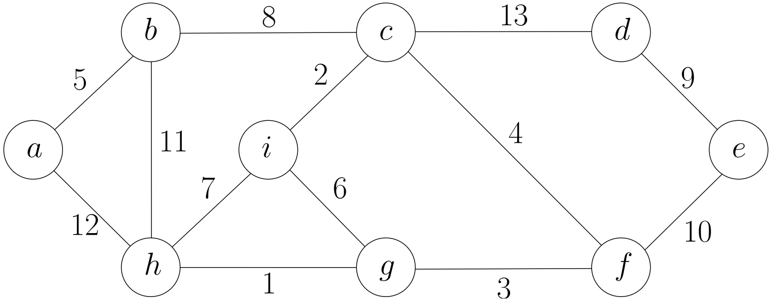
Reverse Kruskal's Algorithm: Example



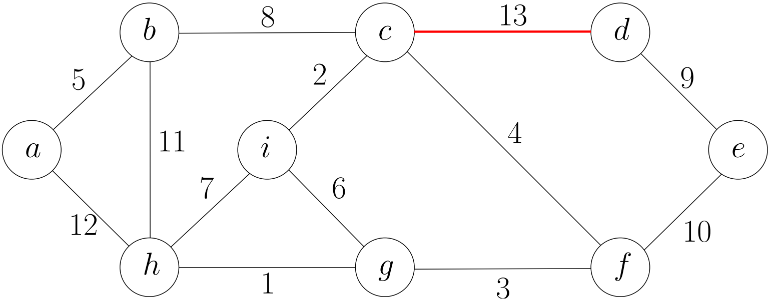
Reverse Kruskal's Algorithm: Example



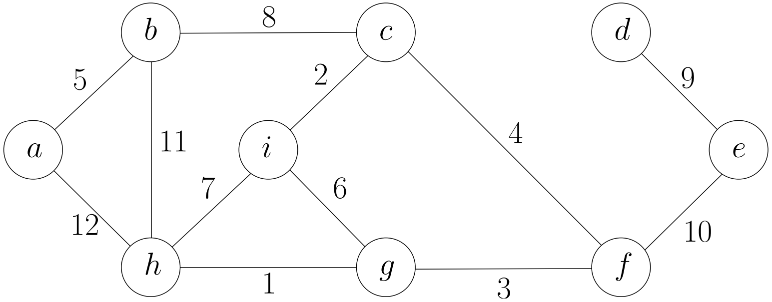
Reverse Kruskal's Algorithm: Example



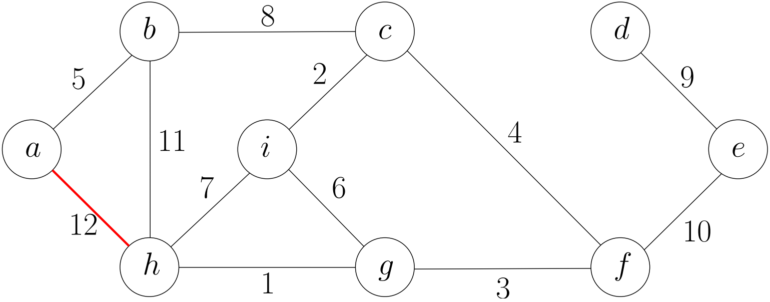
Reverse Kruskal's Algorithm: Example



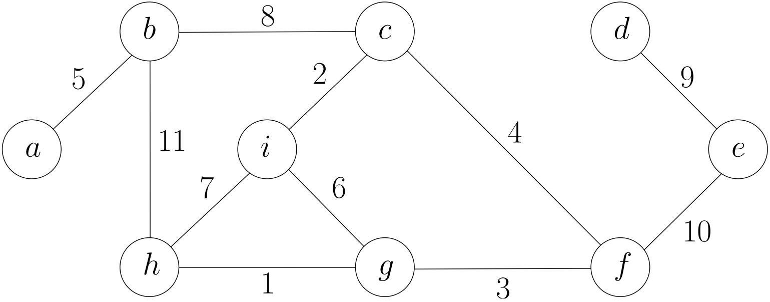
Reverse Kruskal's Algorithm: Example



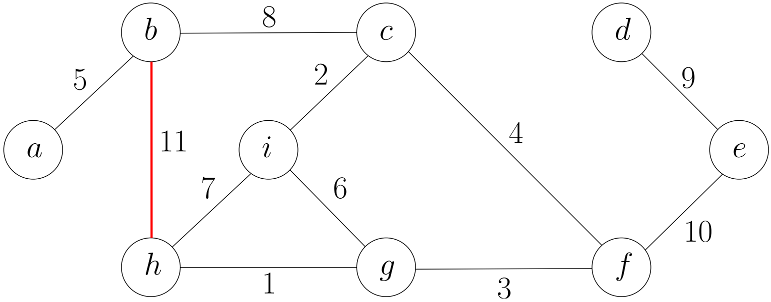
Reverse Kruskal's Algorithm: Example



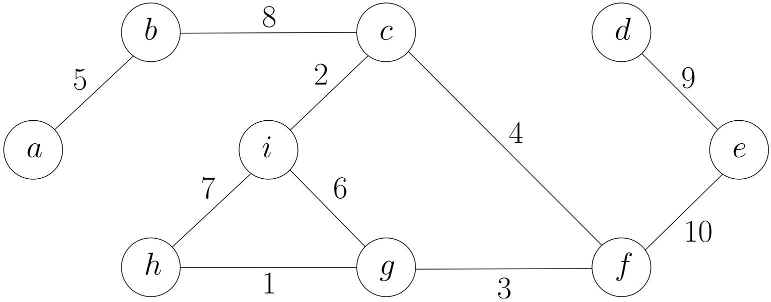
Reverse Kruskal's Algorithm: Example



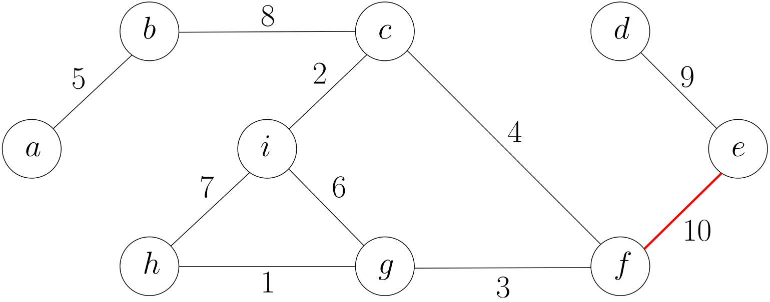
Reverse Kruskal's Algorithm: Example



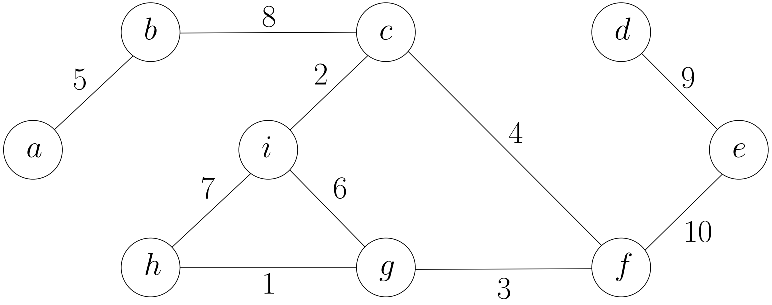
Reverse Kruskal's Algorithm: Example



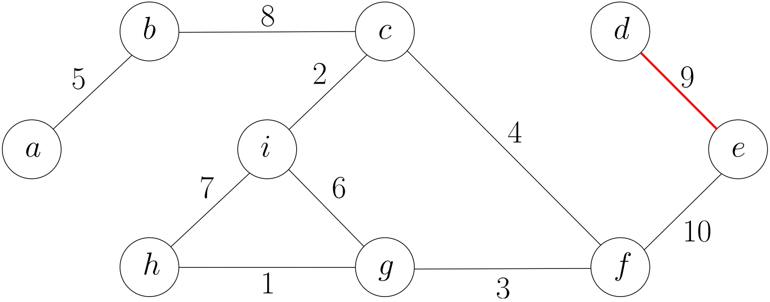
Reverse Kruskal's Algorithm: Example



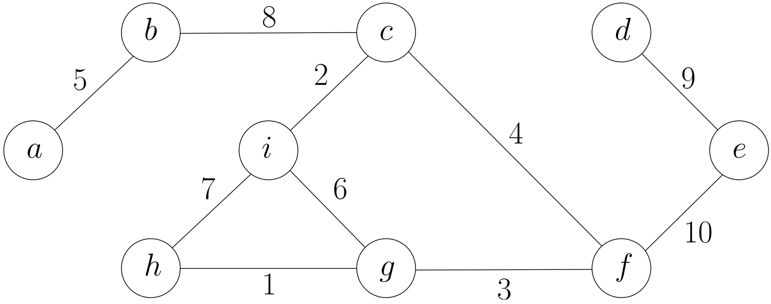
Reverse Kruskal's Algorithm: Example



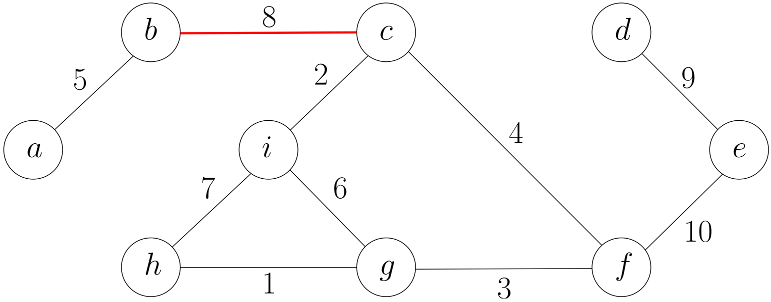
Reverse Kruskal's Algorithm: Example



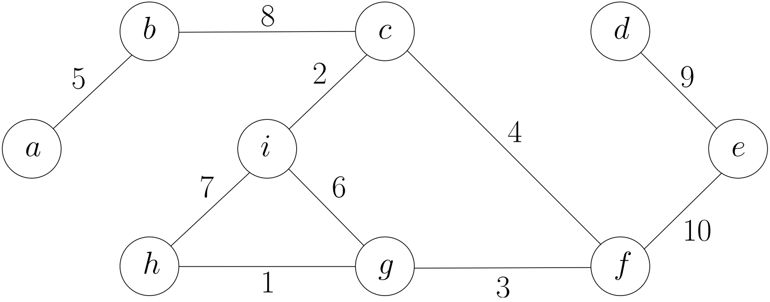
Reverse Kruskal's Algorithm: Example



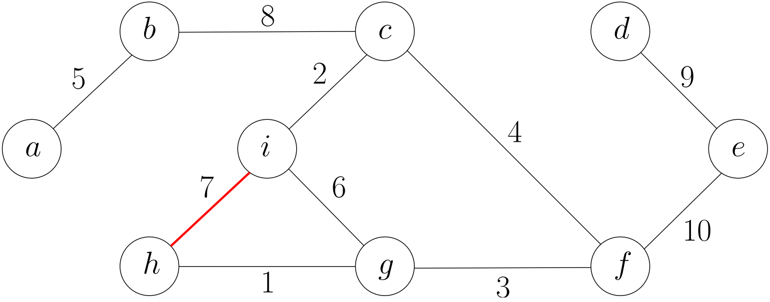
Reverse Kruskal's Algorithm: Example



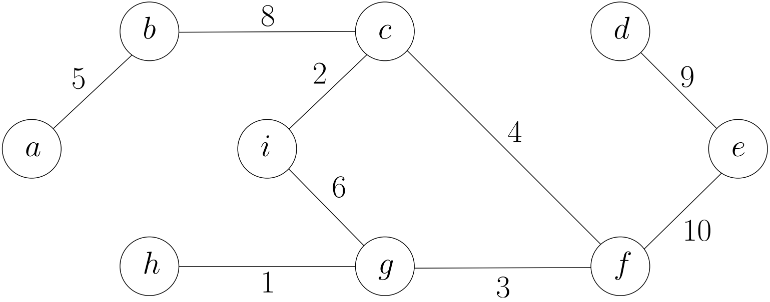
Reverse Kruskal's Algorithm: Example



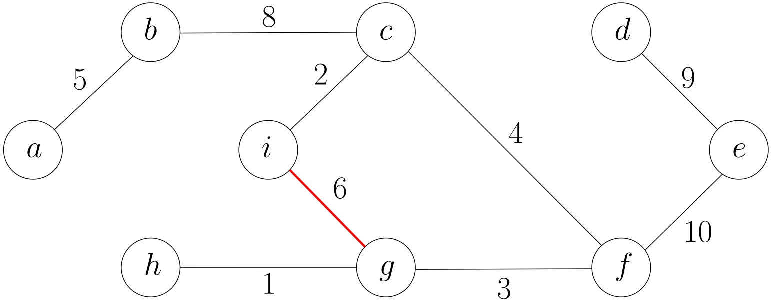
Reverse Kruskal's Algorithm: Example



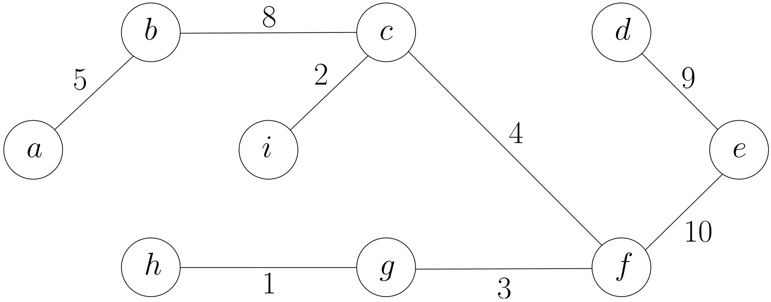
Reverse Kruskal's Algorithm: Example



Reverse Kruskal's Algorithm: Example



Reverse Kruskal's Algorithm: Example

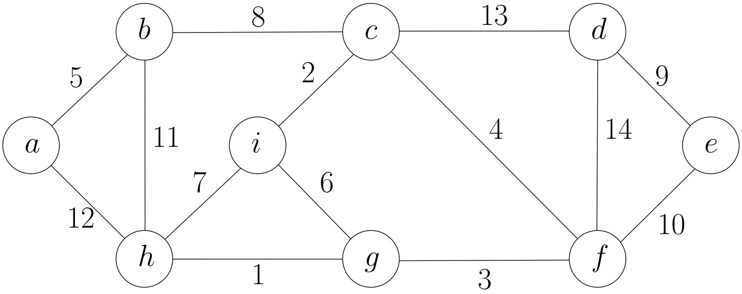


Outline

1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. Summary

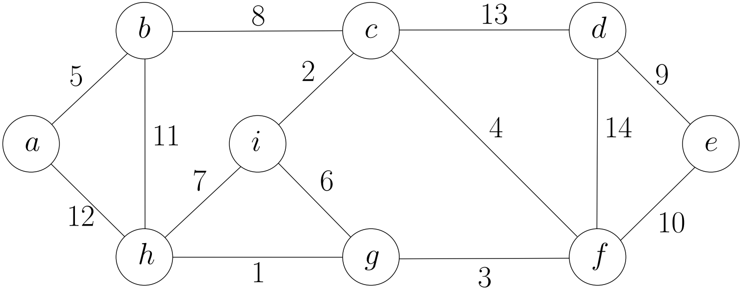
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



Design Greedy Strategy for MST

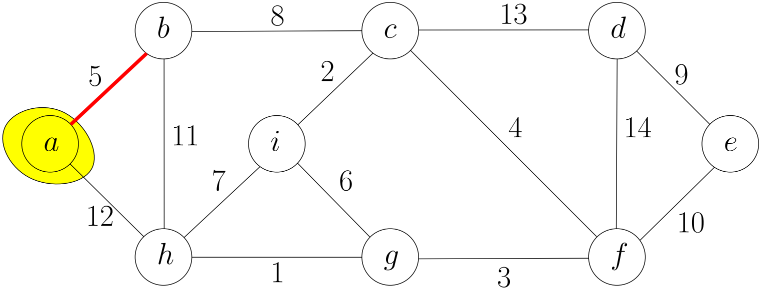
- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



- Greedy strategy for Prim's algorithm: choose the lightest edge incident to a.

Design Greedy Strategy for MST

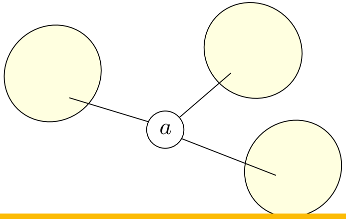
- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



- Greedy strategy for Prim's algorithm: choose the lightest edge incident to a.

Lemma It is safe to include the lightest edge incident to a.

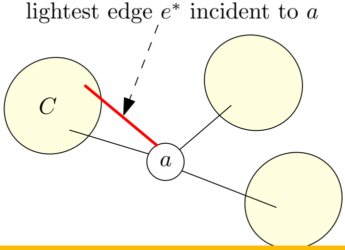
Lemma It is safe to include the lightest edge incident to a.



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

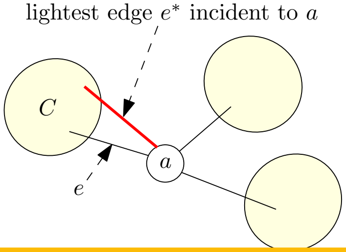
Lemma It is safe to include the lightest edge incident to a.



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C

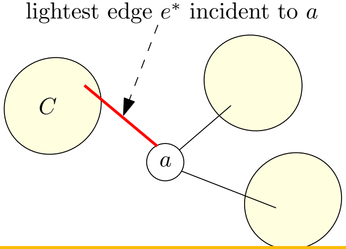
Lemma It is safe to include the lightest edge incident to a.



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C

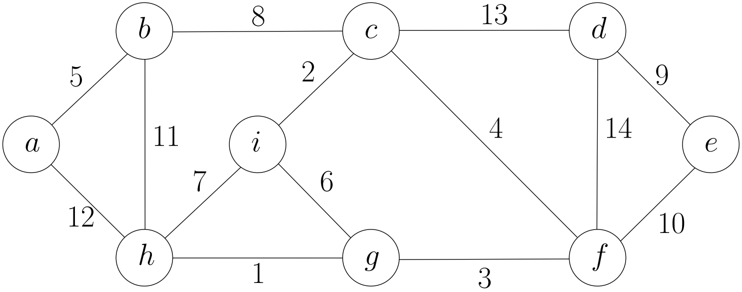
Lemma It is safe to include the lightest edge incident to a.



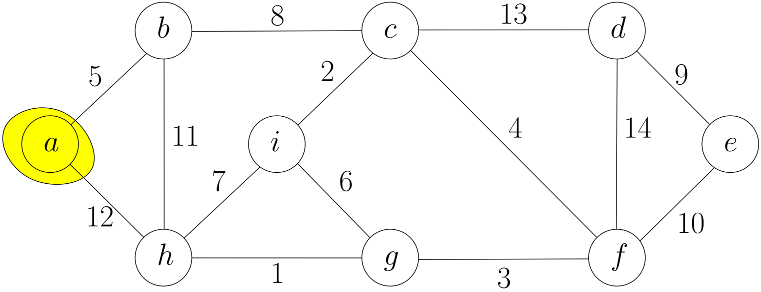
Proof.

- Let T be a MST
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus e \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ ☐

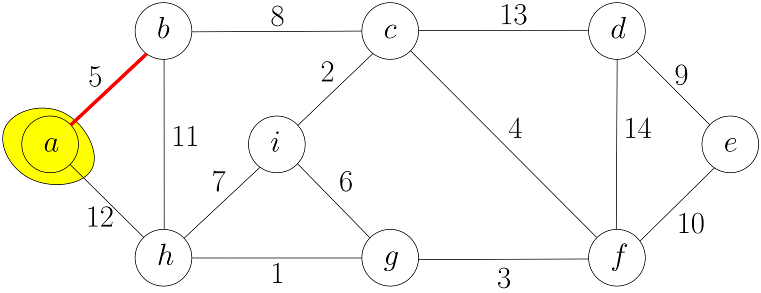
Prim's Algorithm: Example



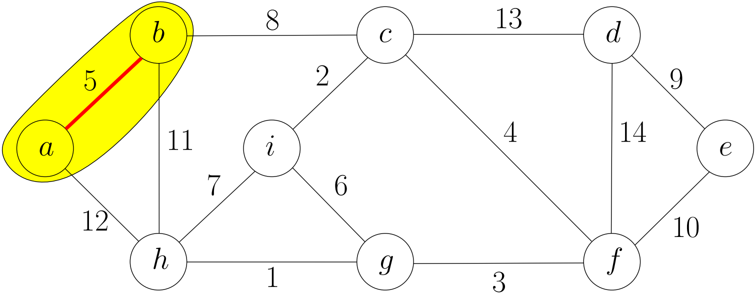
Prim's Algorithm: Example



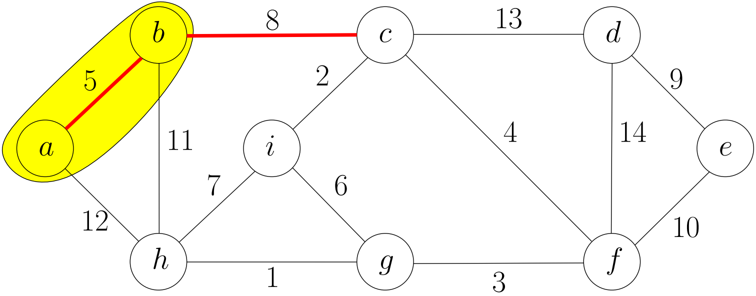
Prim's Algorithm: Example



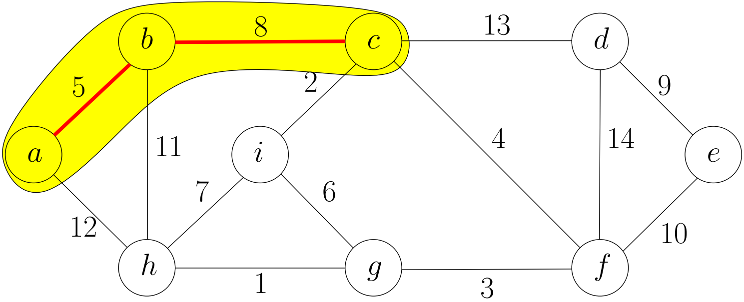
Prim's Algorithm: Example



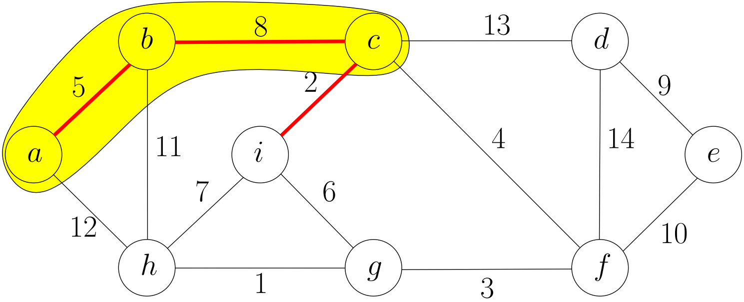
Prim's Algorithm: Example



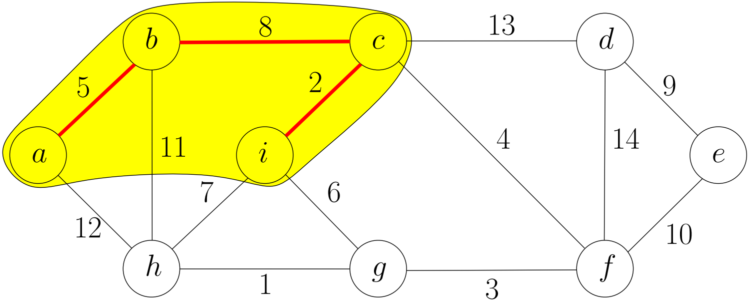
Prim's Algorithm: Example



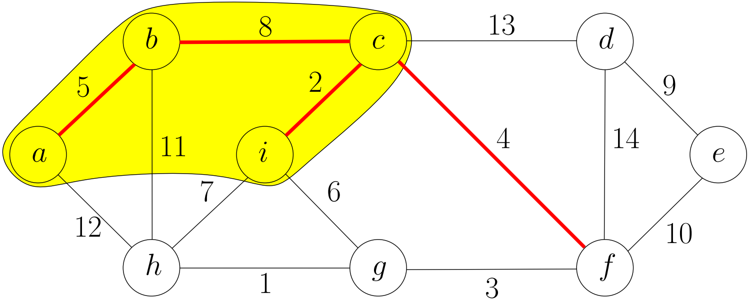
Prim's Algorithm: Example



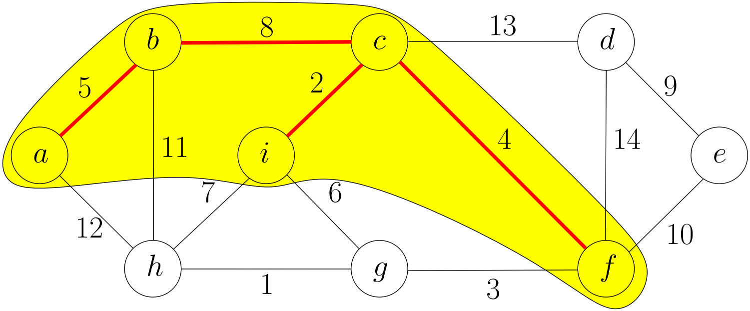
Prim's Algorithm: Example



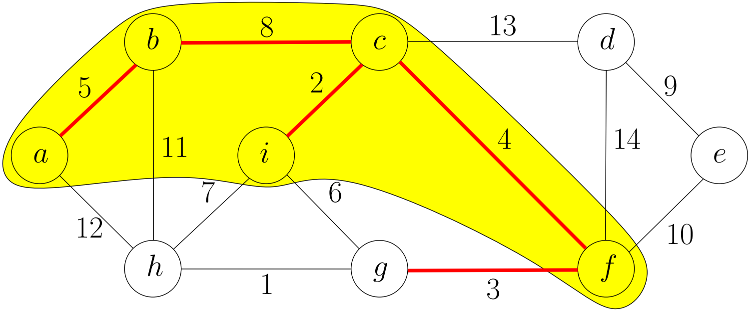
Prim's Algorithm: Example



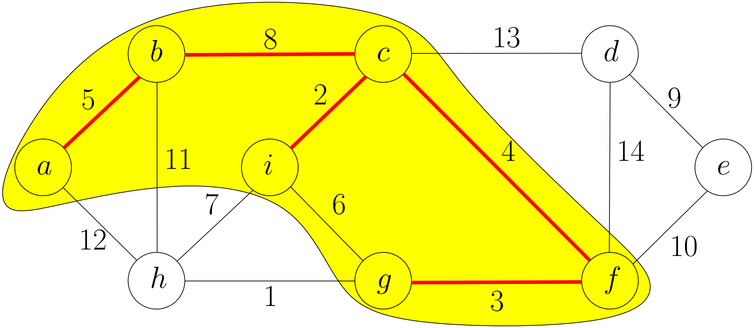
Prim's Algorithm: Example



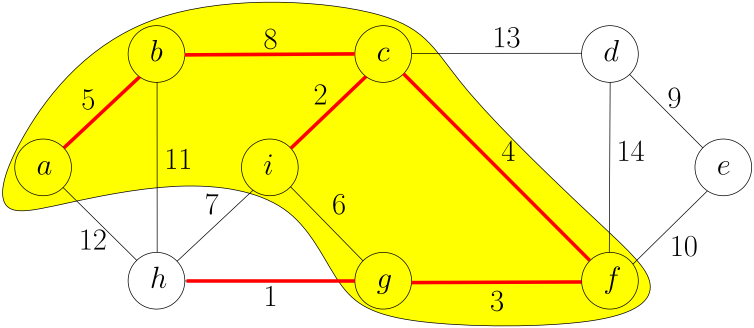
Prim's Algorithm: Example



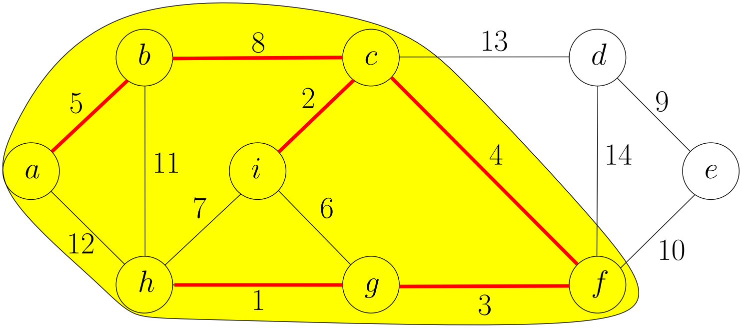
Prim's Algorithm: Example



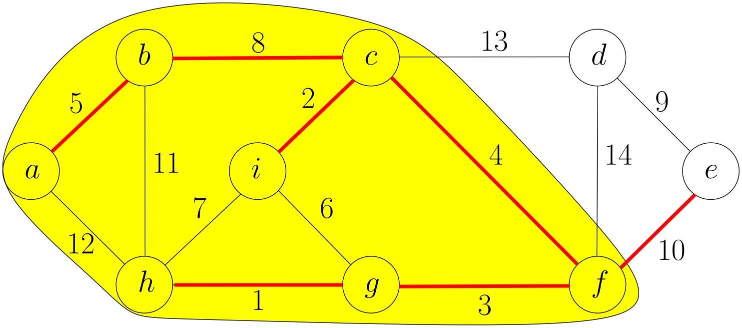
Prim's Algorithm: Example



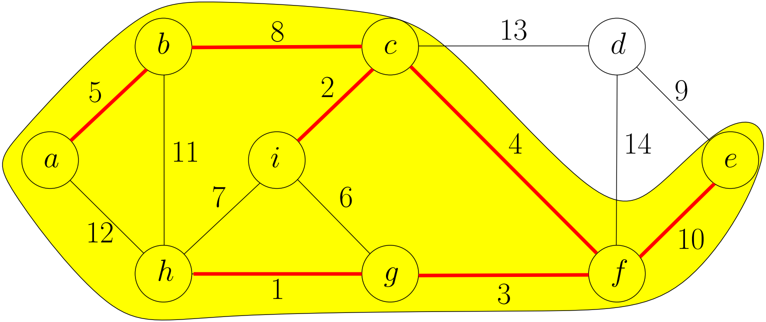
Prim's Algorithm: Example



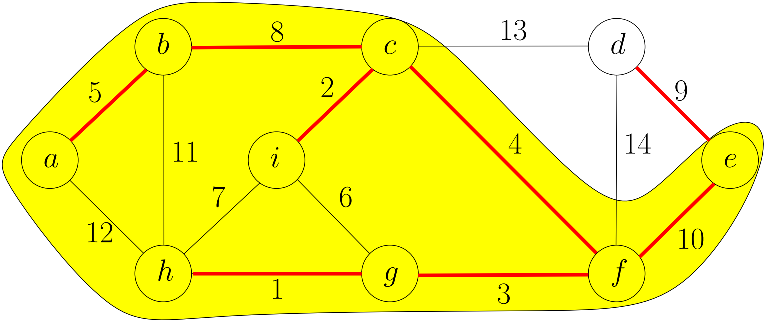
Prim's Algorithm: Example



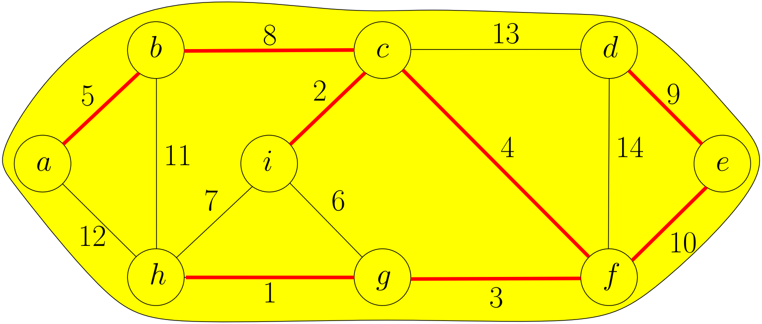
Prim's Algorithm: Example



Prim's Algorithm: Example



Prim's Algorithm: Example



Greedy Algorithm

```
MST-Greedy1($G, w$)
$S \leftarrow \{s\}$, where $s$ is arbitrary vertex in $V$
$F \leftarrow \emptyset$
while $S \neq V$
    $(u, v) \leftarrow$ lightest edge between $S$ and $V \setminus S$,
                                  where $u \in S$ and $v \in V \setminus S$
    $S \leftarrow S \cup \{v\}$
    $F \leftarrow F \cup \{(u, v)\}$
return $(V, F)$
```

Greedy Algorithm

MST-Greedy1(G, w)

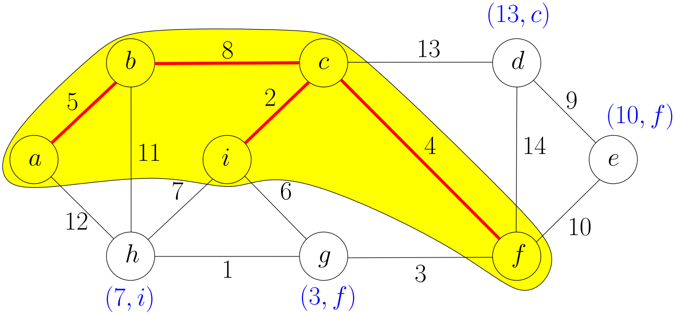
```
$S \leftarrow \{s\}$, where $s$ is arbitrary vertex in $V$
$F \leftarrow \emptyset$
while $S \neq V$
    $(u, v) \leftarrow$ lightest edge between $S$ and $V \setminus S$,
                          where $u \in S$ and $v \in V \setminus S$
    $S \leftarrow S \cup \{v\}$
    $F \leftarrow F \cup \{(u, v)\}$
return $(V, F)$
```

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S



Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

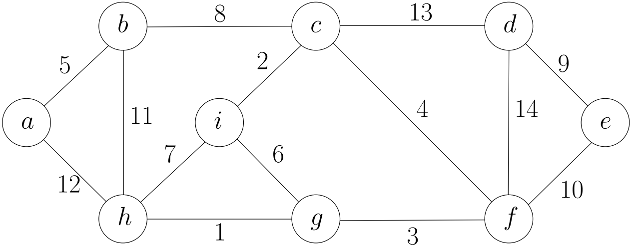
- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

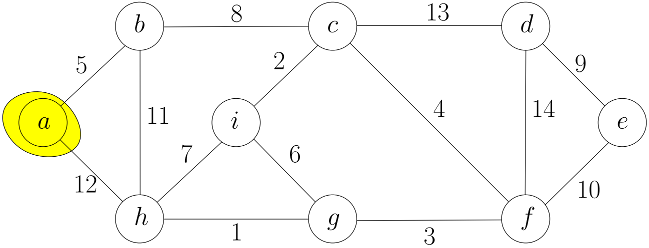
MST-Prim(G, w)

```
$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$, do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$
```

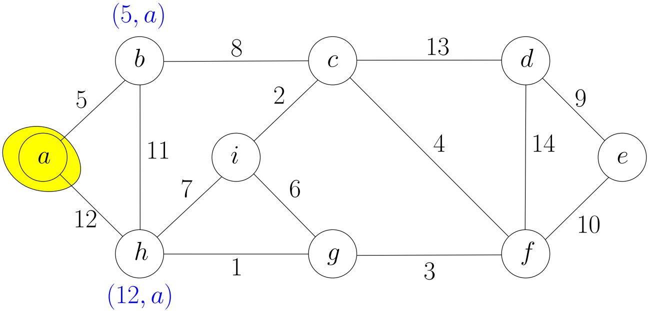
Example



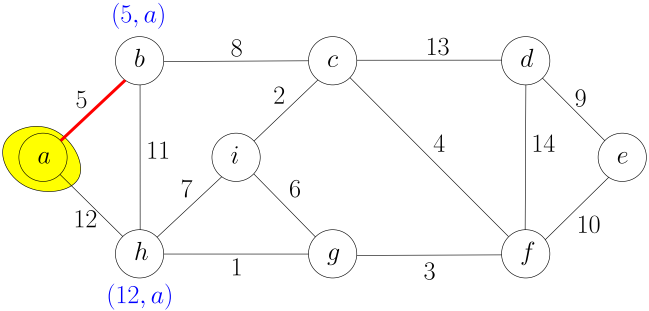
Example



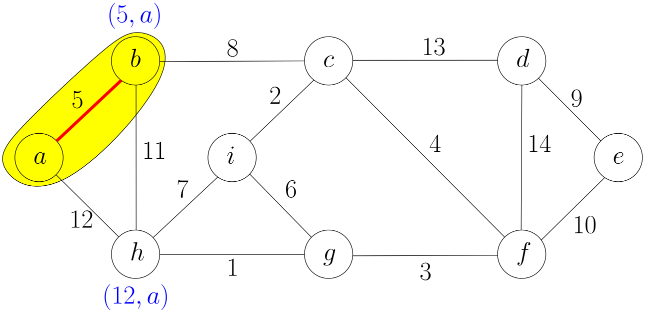
Example



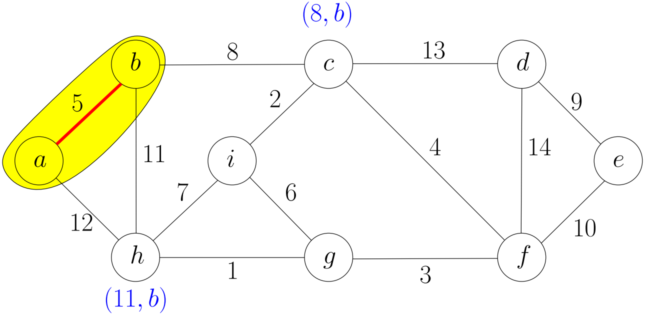
Example



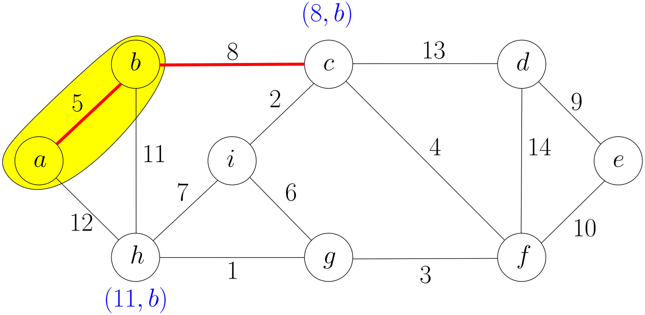
Example



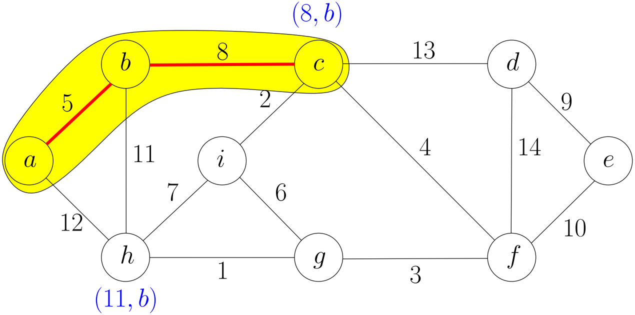
Example



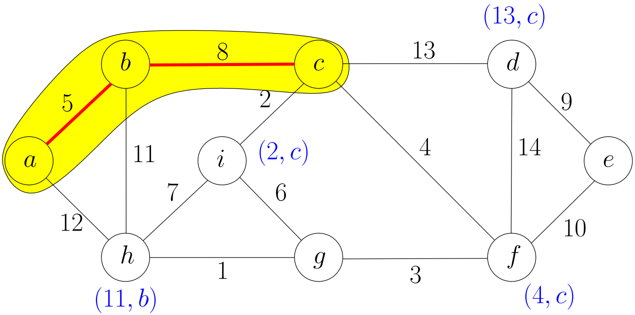
Example



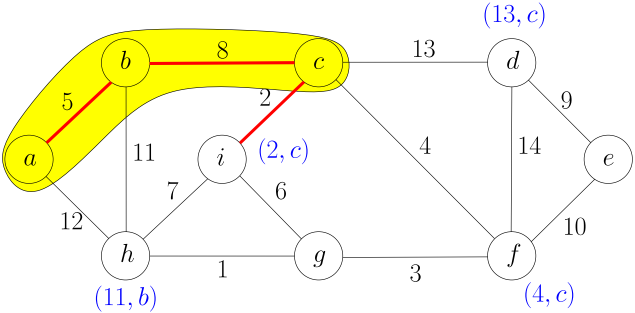
Example



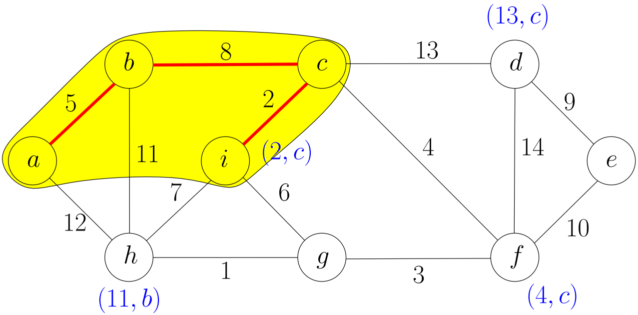
Example



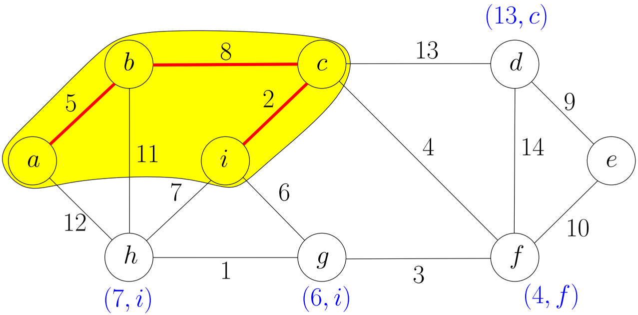
Example



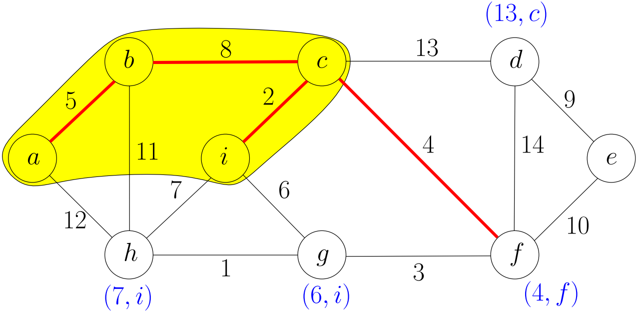
Example



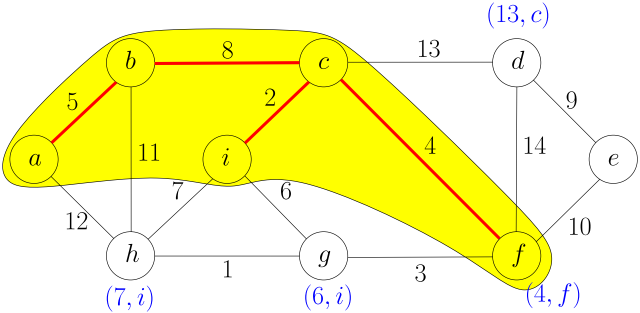
Example



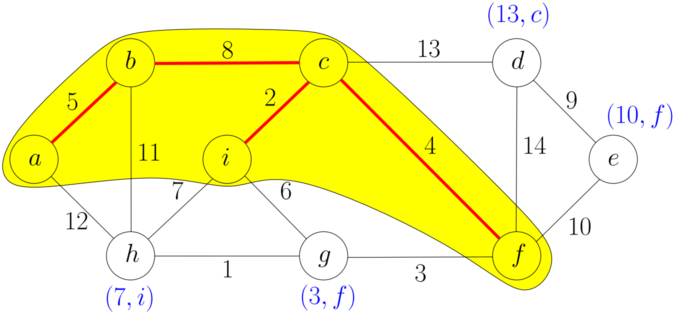
Example



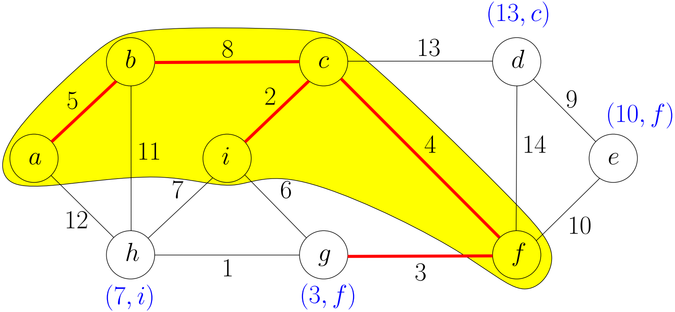
Example



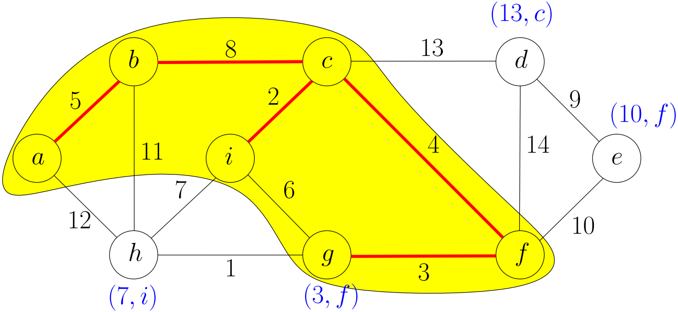
Example



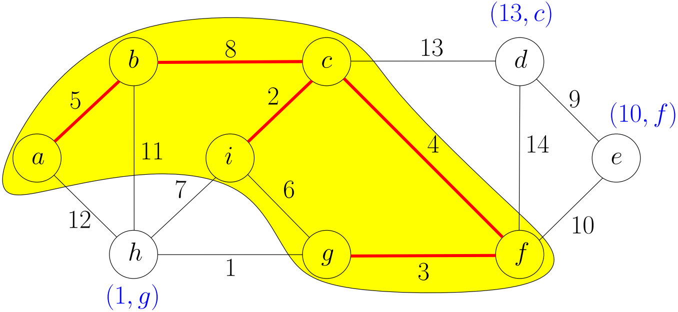
Example



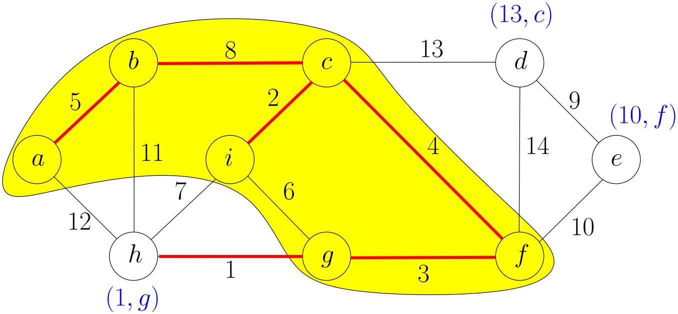
Example



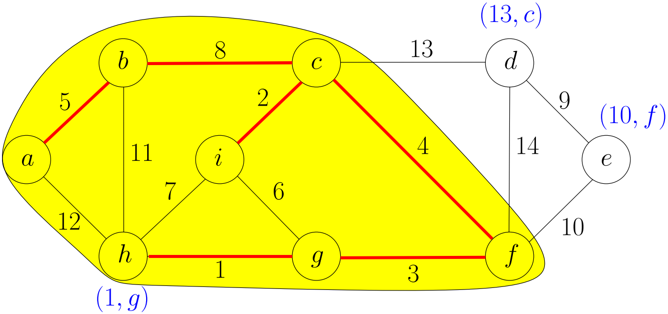
Example



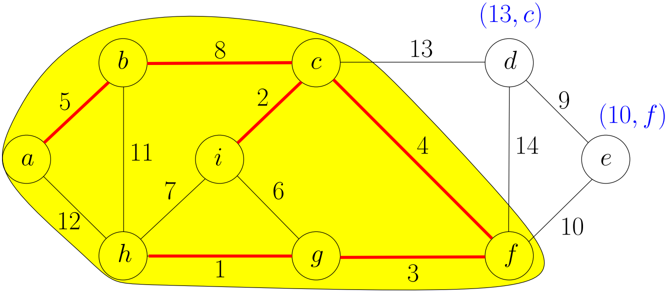
Example



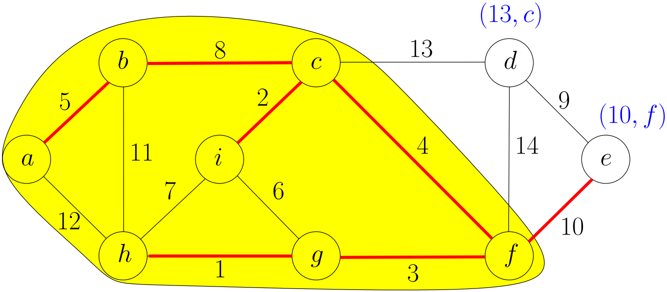
Example



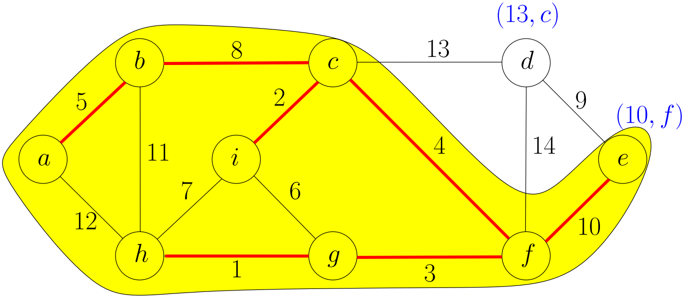
Example



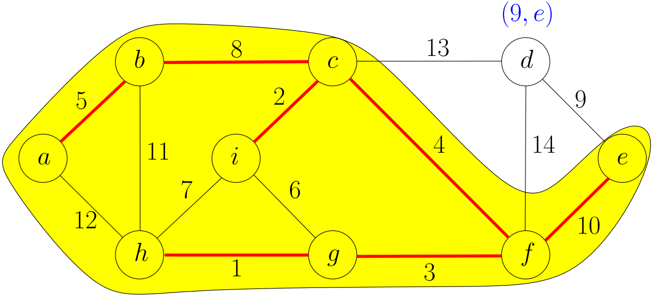
Example



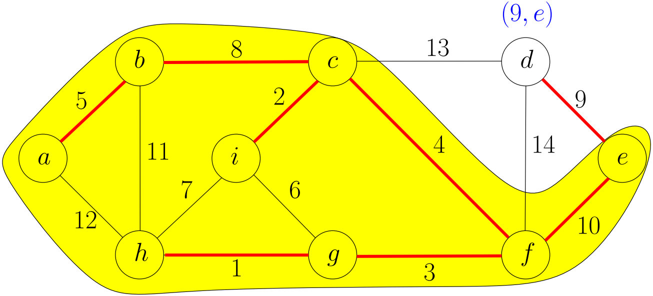
Example



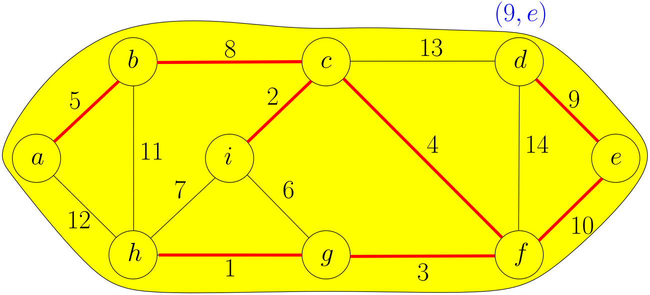
Example



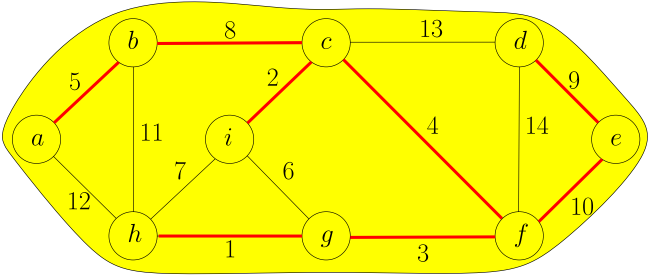
Example



Example



Example



Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u, v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u, v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v)\in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v)\in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value extract_min
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values. decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

Prim's Algorithm

MST-Prim(G, w)

```
$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$

while $S \neq V$, do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$
```

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

```
$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$, $Q$.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$
```

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+$ $O(m)\times$ (time for decrease_key)

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+\ O(m)\times$ (time for decrease_key)

concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+ O(m)\times$ (time for decrease_key)

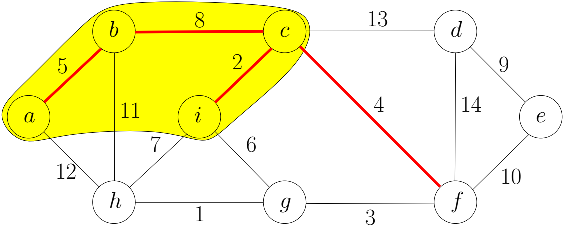
concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.

Assumption Assume all edge weights are different.

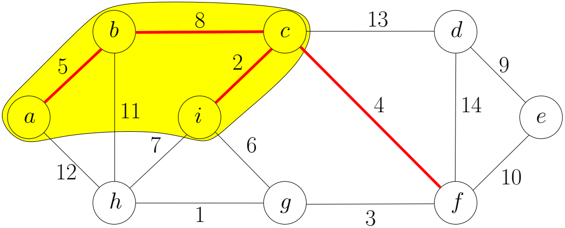
Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.



- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$

Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.



- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

“Evidence” for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

“Evidence” for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

Exactly one of the following is true:

- There is a cut in which e is the lightest edge
- There is a cycle in which e is the heaviest edge

"Evidence" for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

Exactly one of the following is true:

- There is a cut in which e is the lightest edge
- There is a cycle in which e is the heaviest edge

Thus, the minimum spanning tree is unique with assumption.

Outline

1. Toy Examples

2. Interval Scheduling

3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm

4. Single Source Shortest Paths
    - Dijkstra's Algorithm

5. Data Compression and Huffman Code

6. Summary

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

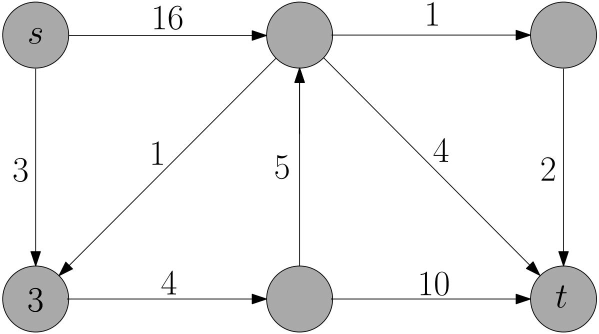
Output: shortest path from s to t

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

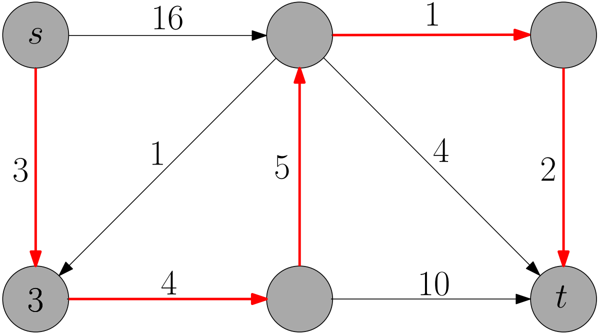
Output: shortest path from s to t



s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$
$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest path from s to t



Single Source Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Single Source Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

Single Source Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s \in V$
$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

Single Source Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s \in V$
$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

- Shortest path from s to v may contain $\Omega(n)$ edges

- Shortest path from s to v may contain $\Omega(n)$ edges
- There are $\Omega(n)$ different vertices v

- Shortest path from s to v may contain $\Omega(n)$ edges
- There are $\Omega(n)$ different vertices v
- Thus, printing out all shortest paths may take time $\Omega(n^2)$

- Shortest path from s to v may contain $\Omega(n)$ edges
- There are $\Omega(n)$ different vertices v
- Thus, printing out all shortest paths may take time $\Omega(n^2)$
- Not acceptable if graph is sparse

Shortest Path Tree

Shortest Path Tree

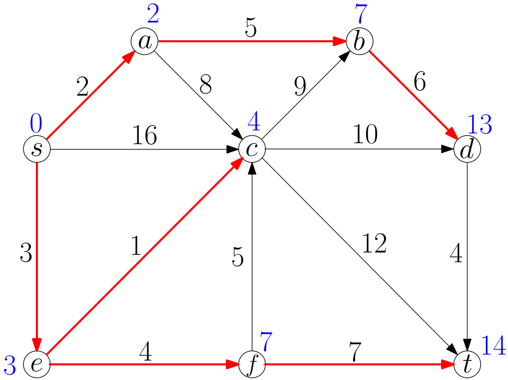
- $O(n)$-size data structure to represent all shortest paths

Shortest Path Tree

- $O(n)$-size data structure to represent all shortest paths
- For every vertex v, we only need to remember the parent of v: second-to-last vertex in the shortest path from s to v (why?)

Shortest Path Tree

- $O(n)$-size data structure to represent all shortest paths
- For every vertex v, we only need to remember the parent of v: second-to-last vertex in the shortest path from s to v (why?)



Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi(v), v \in V \setminus s$: the parent of v

$d(v), v \in V \setminus s$: the length of shortest path from s to v

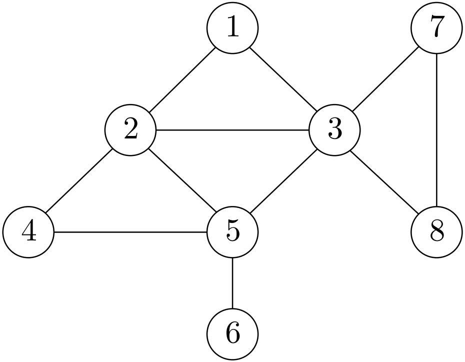
Q: How to compute shortest paths from s when all edges have weight 1?

Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s

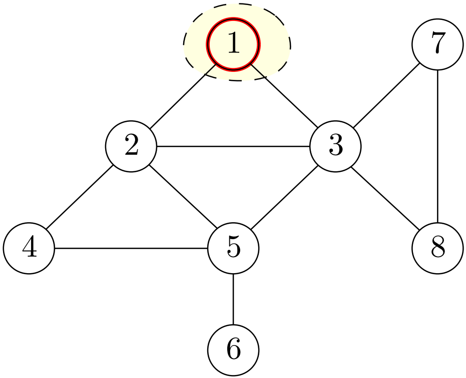
Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s



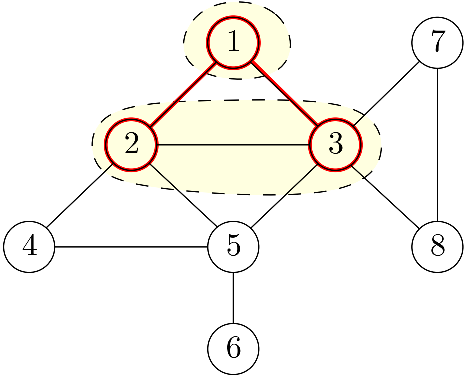
Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s



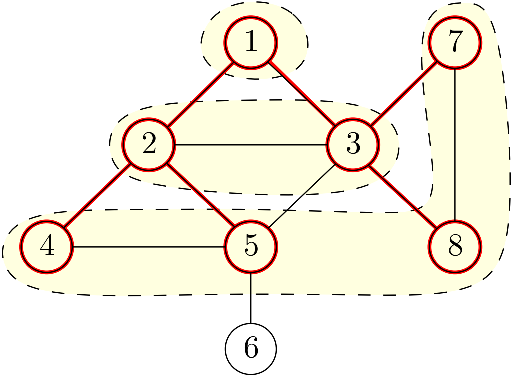
Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s



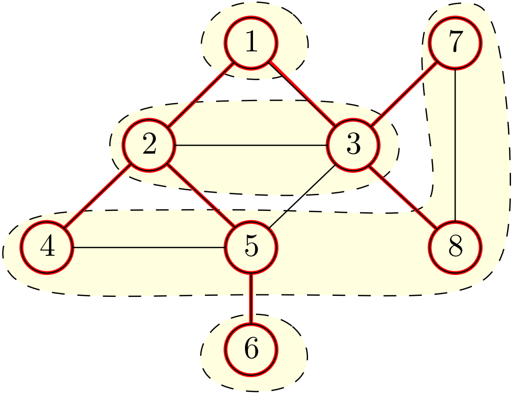
Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s



Q: How to compute shortest paths from s when all edges have weight 1?

A: Breadth first search (BFS) from source s



Assumption Weights $w(u, v)$ are integers (w.l.o.g).

Assumption Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a pah of $w(u, v)$ unit-weight edges

Assumption Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a pah of $w(u, v)$ unit-weight edges

Shortest Path Algorithm by Running BFS

1. replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2. run BFS
3. $\pi(v) =$ vertex from which v is visited
4. $d(v) =$ index of the level containing v

Assumption Weights $w(u,v)$ are integers (w.l.o.g).

- An edge of weight $w(u,v)$ is equivalent to a pah of $w(u,v)$ unit-weight edges

Shortest Path Algorithm by Running BFS

1. replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2. run BFS
3. $\pi(v) =$ vertex from which v is visited
4. $d(v) =$ index of the level containing v

- Problem: $w(u,v)$ may be too large!

Assumption Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a pah of $w(u, v)$ unit-weight edges

Shortest Path Algorithm by Running BFS

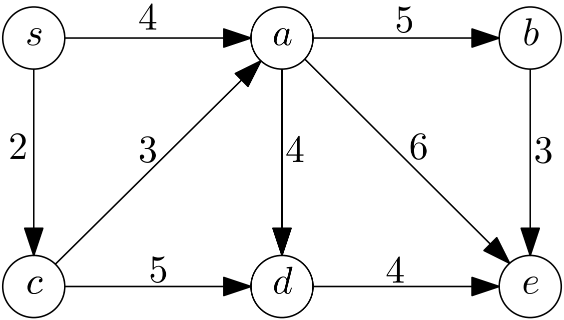
1. replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2. run BFS virtually
3. $\pi(v) =$ vertex from which v is visited
4. $d(v) =$ index of the level containing v

- Problem: $w(u, v)$ may be too large!

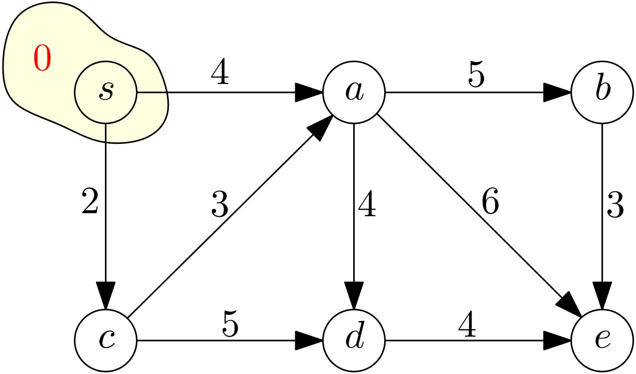
Shortest Path Algorithm by Running BFS Virtually

```
$S \leftarrow \{s\}, d(s) \leftarrow 0$
while $|S| \leq n$
    find a $v \notin S$ that minimizes $\min_{u \in S: (u,v) \in E} \{d(u) + w(u,v)\}$
    $S \leftarrow S \cup \{v\}$
    $d(v) \leftarrow \min_{u \in S: (u,v) \in E} \{d(u) + w(u,v)\}$
```

Virtual BFS: Example

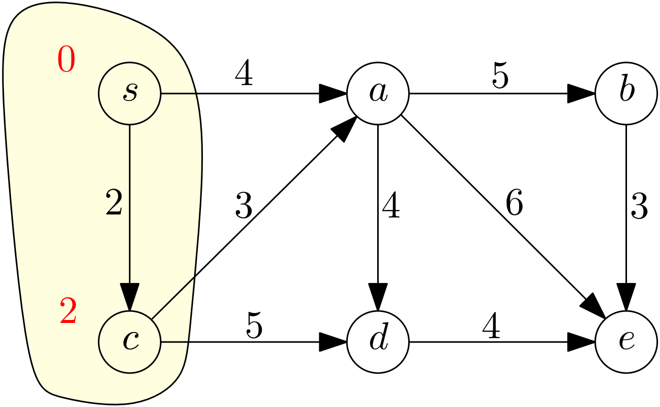


Virtual BFS: Example



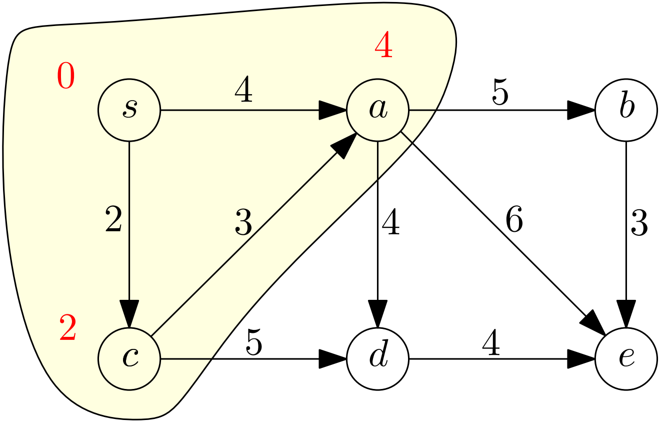
Time 0

Virtual BFS: Example



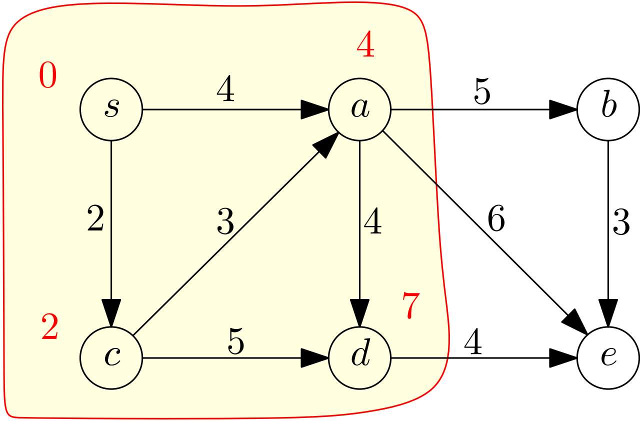
Time 2

Virtual BFS: Example



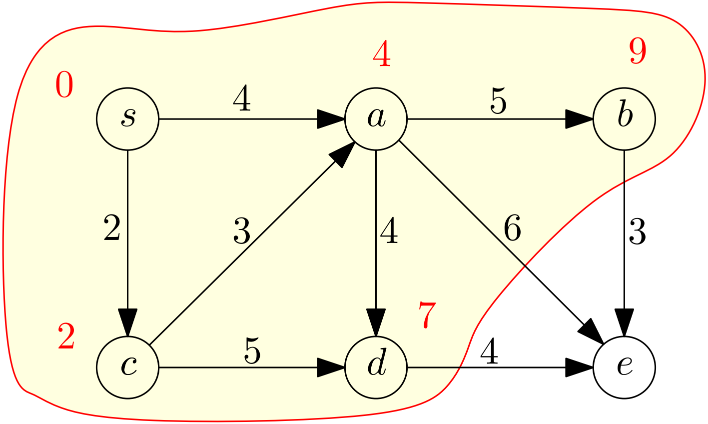
Time 4

Virtual BFS: Example



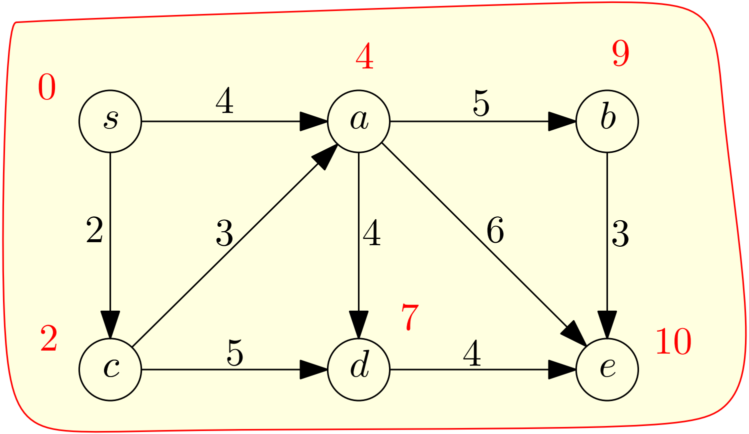
Time 7

Virtual BFS: Example



Time 9

Virtual BFS: Example



Time 10

Outline

1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. Summary

Dijkstra's Algorithm

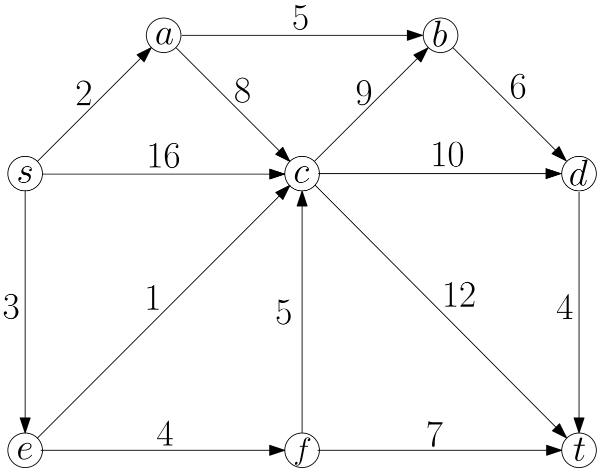
```
Dijkstra($G, w, s$)
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    add $u$ to $S$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $d(u) + w(u, v) < d(v)$ then
            $d(v) \leftarrow d(u) + w(u, v)$
            $\pi(v) \leftarrow u$
return $(d, \pi)$
```

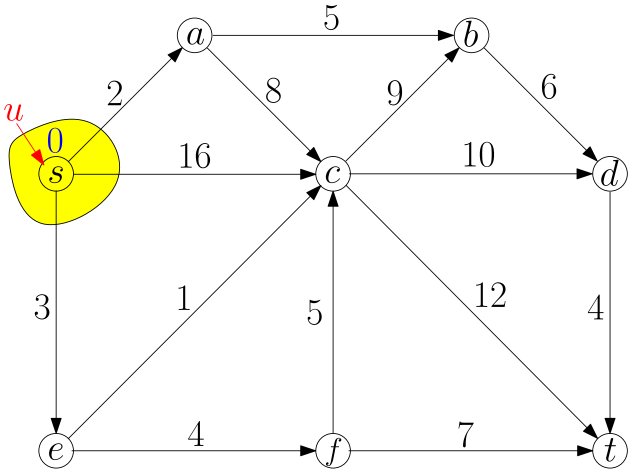
Dijkstra's Algorithm

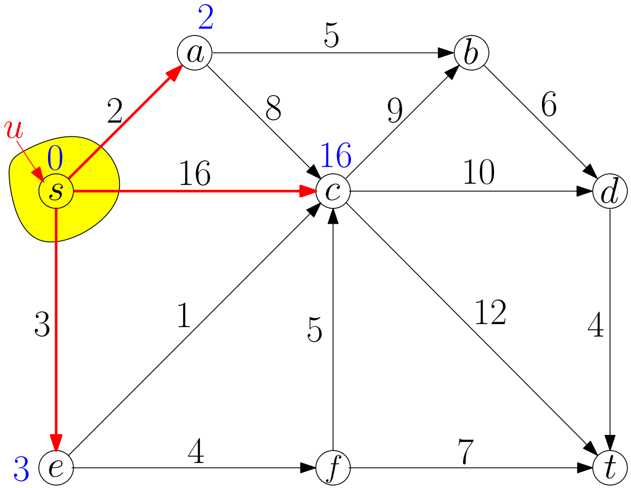
Dijkstra(G, w, s)

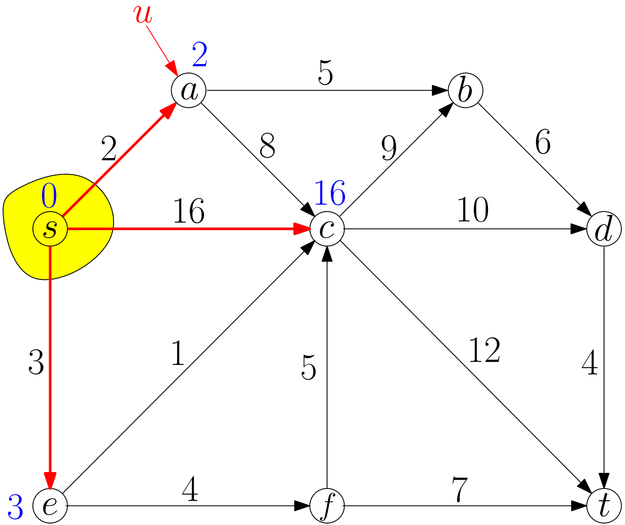
```
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    add $u$ to $S$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $d(u) + w(u, v) < d(v)$ then
            $d(v) \leftarrow d(u) + w(u, v)$
            $\pi(v) \leftarrow u$
return $(d, \pi)$
```

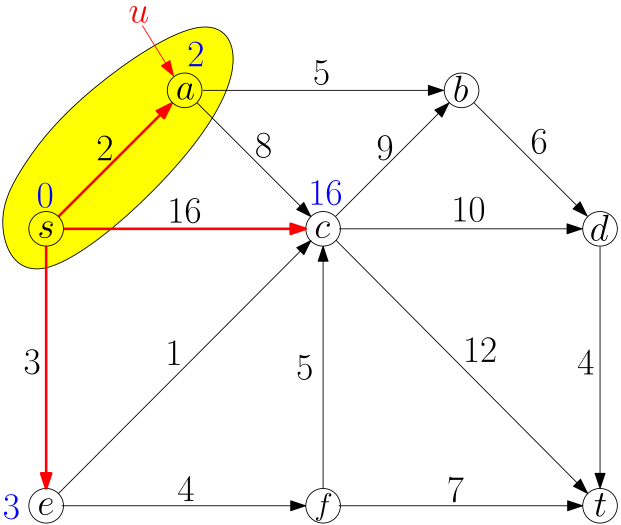
- Running time $= O(n^2)$


a
b
s
c
d
e
f
t
5
2
8
9
6
16
10
3
1
5
12
4
4
7

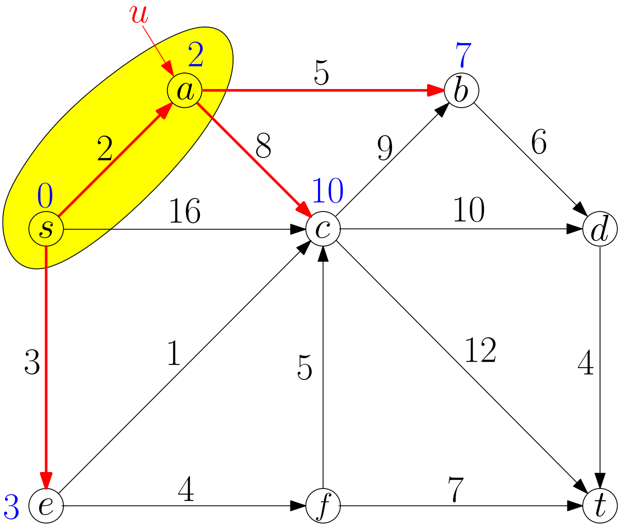

a
b
c
d
s
e
f
t
u
0
5
2
8
9
6
16
10
3
1
5
12
4
4
7

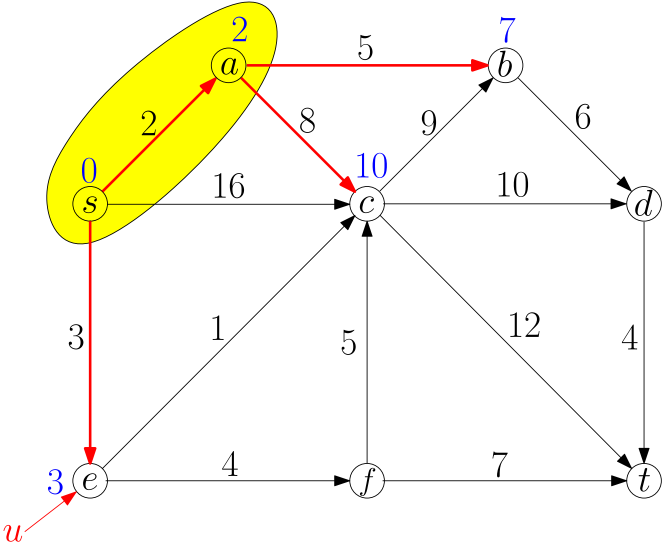

2
a
5
b
8
9
6
u
0
16
s
16
c
10
d
1
5
12
4
3
3
e
4
f
7
t

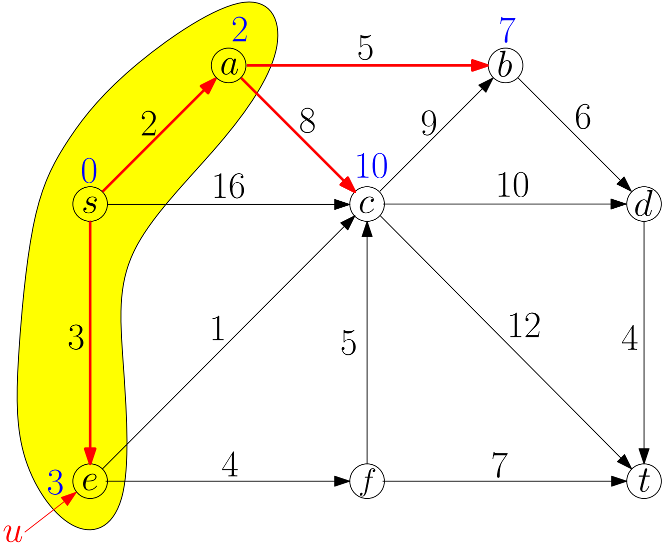

u
2
a
5
b
2
8
9
6
0
s
16
16
c
10
d
3
1
5
12
4
3
e
4
f
7
t

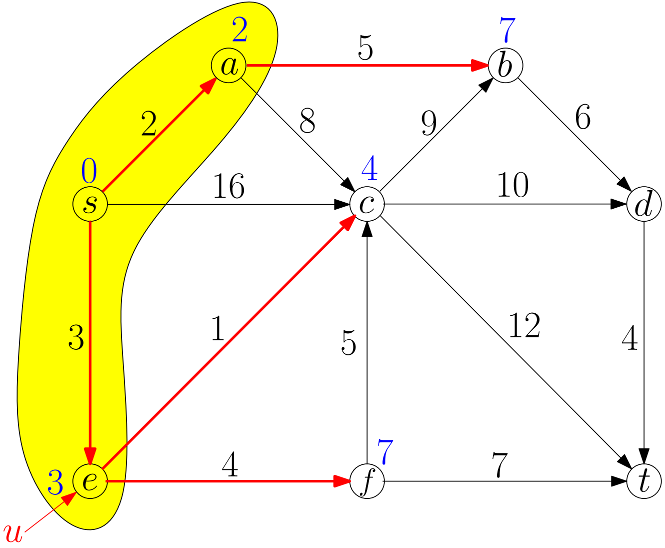


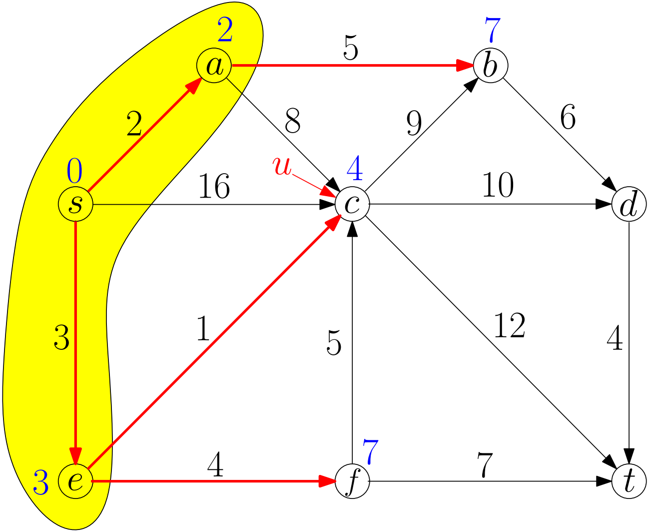
u
2
a
5
b
2
8
9
6
0
s
16
16
c
10
d
3
1
5
12
4
3
e
4
f
7
t


u
2
a
5
7
b
2
8
9
6
0
s
16
10
c
10
d
3
1
5
12
4
4
7
3
e
f
t

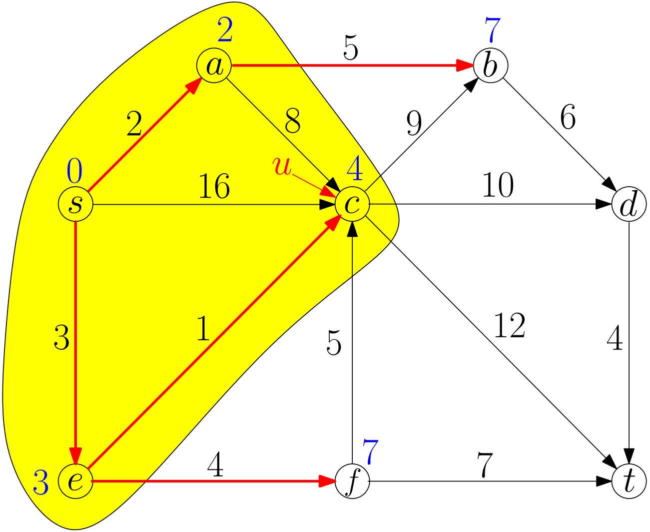

2
5
7
a
b
2
8
9
6
0
10
16
10
s
c
d
3
1
5
12
4
4
7
3
e
f
t
u

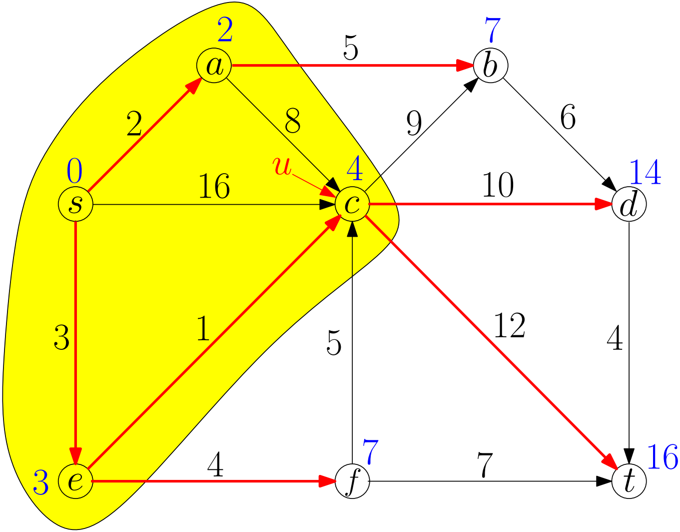

2
a
7
b
5
8
9
6
0
s
16
10
c
10
d
3
1
5
12
4
3
e
4
f
7
t
u

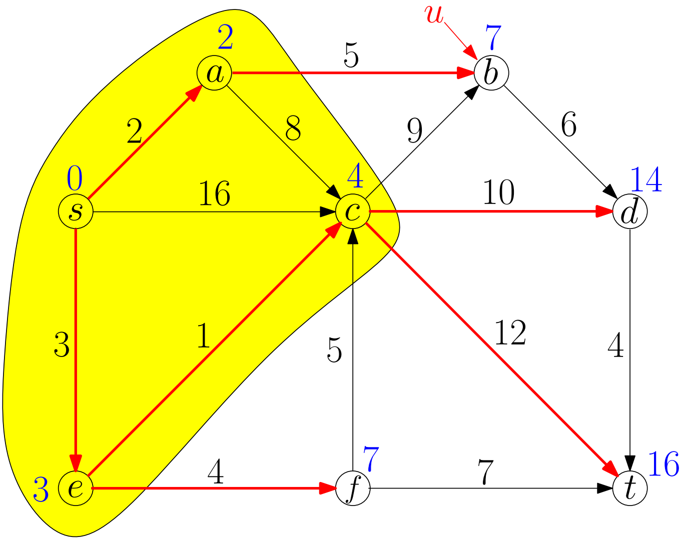

2
a
5
7
b
2
8
9
6
0
s
16
4
c
10
d
3
1
5
12
4
4
7
7
3
e
f
t
u

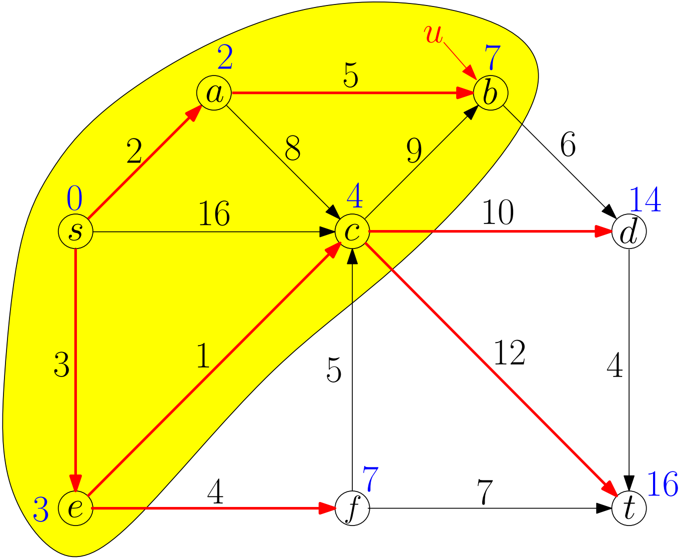


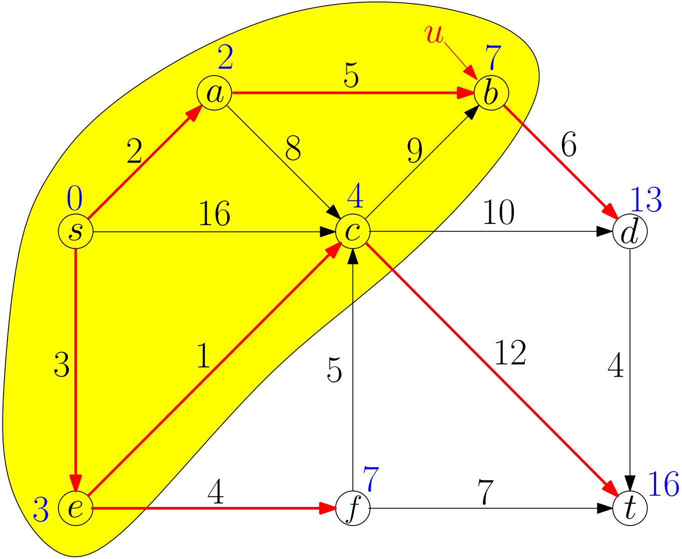
2
a
5
7
b
2
8
9
6
u
4
0
s
16
c
10
d
3
1
5
12
4
7
3
e
4
f
7
t

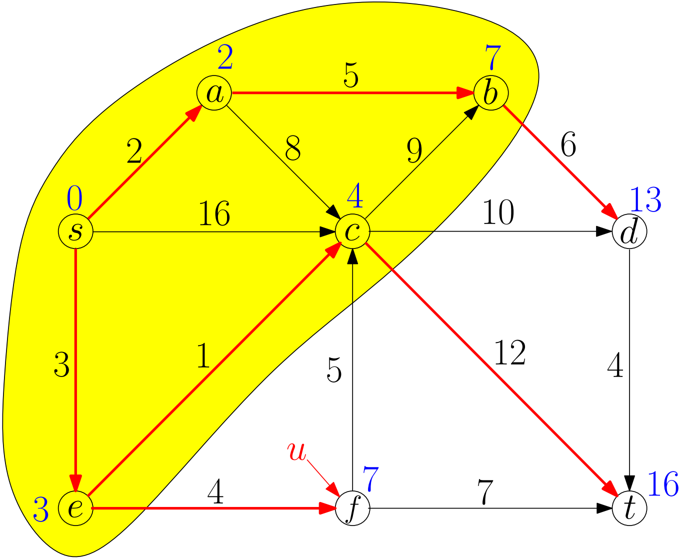

2
a
5
7
b
2
8
9
6
0
u
4
s
16
c
10
d
3
1
5
12
4
7
7
3
e
4
f
t

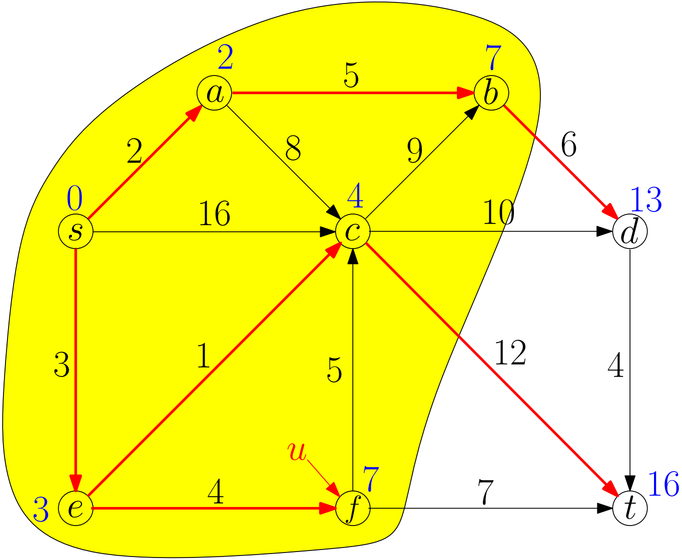

2
a
5
7
b
2
8
9
6
0
s
16
u
4
c
10
14
d
3
1
5
12
4
3
e
4
7
f
7
16
t

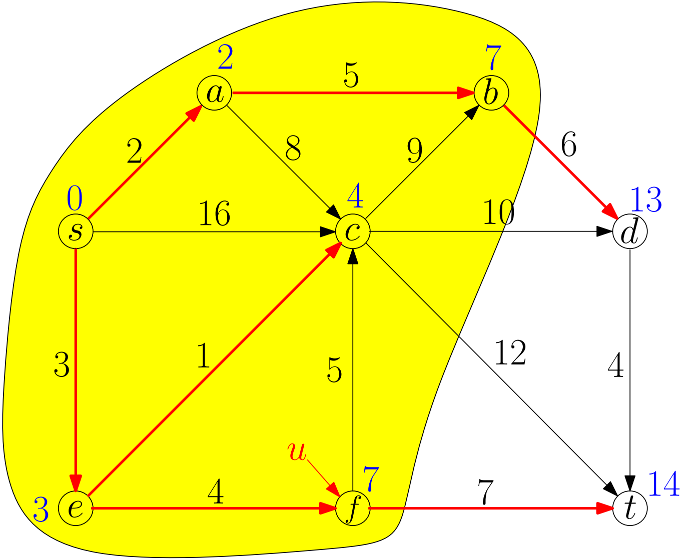

u
7
2
5
a
b
2
8
9
6
0
4
14
s
16
c
10
d
3
1
5
12
4
3
e
4
7
f
7
t
16

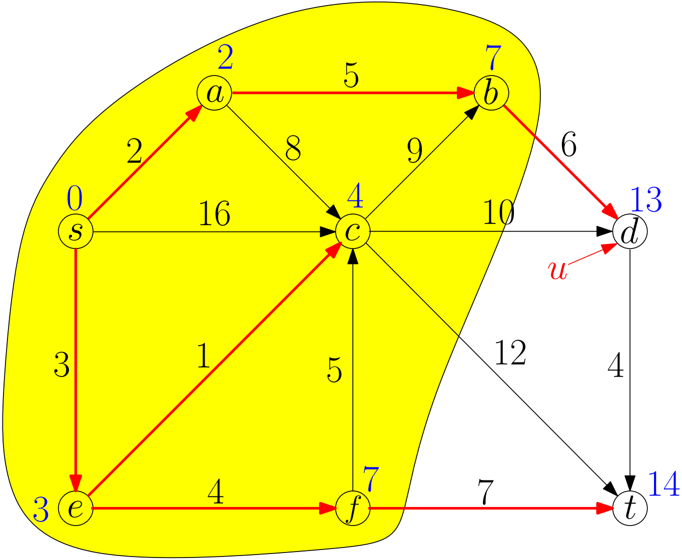

u
2
7
a
5
b
2
8
9
6
0
4
14
s
16
c
10
d
3
1
5
12
4
7
16
3
e
4
f
7
t

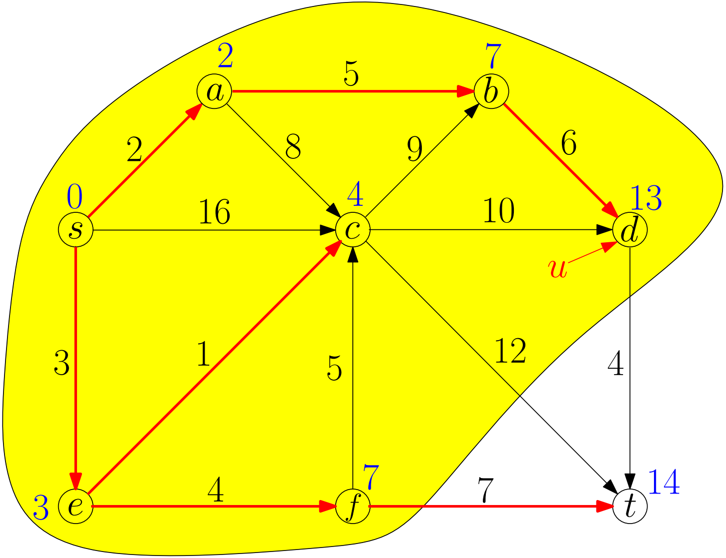

u
s
a
b
c
d
e
f
t
0
2
7
4
13
3
7
16
5
2
8
9
6
16
10
3
1
5
12
4
4
7

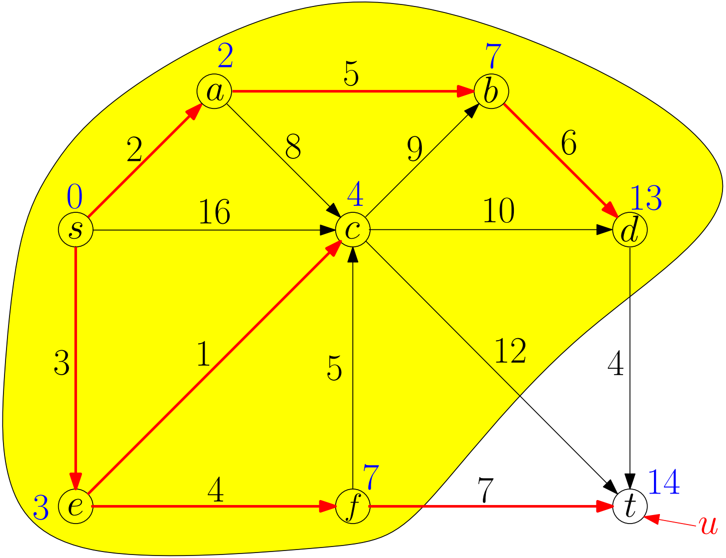

2
7
5
a
b
2
8
9
6
0
4
13
s
16
c
10
d
3
1
12
5
4
u
7
16
3
e
4
f
7
t

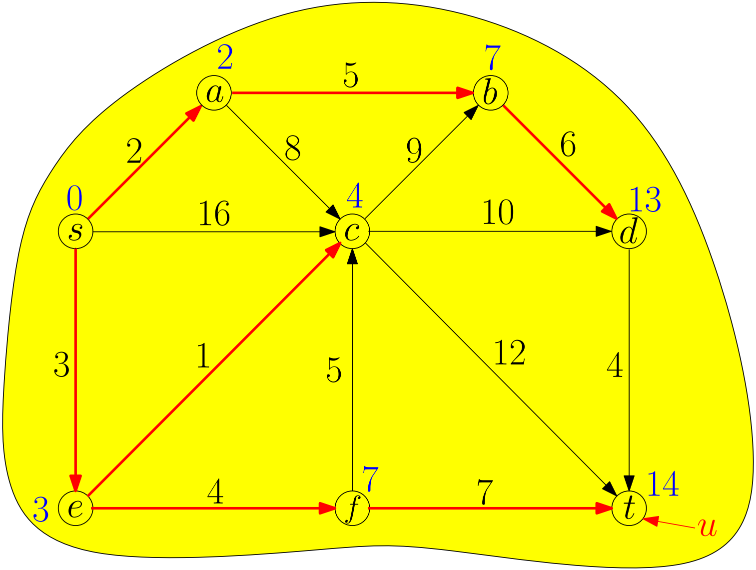

0
2
7
4
13
3
7
16
s
a
b
c
d
e
f
t
u
5
2
8
9
6
16
10
3
1
5
12
4
4
7


2
7
a
5
b
2
8
9
6
0
4
13
s
16
c
10
d
3
1
5
12
4
u
7
14
3
e
4
f
7
t


2
7
5
a
b
2
8
9
6
0
4
13
16
10
s
c
d
u
3
1
5
12
4
3
e
4
7
f
7
t
14


2
7
5
a
b
2
8
9
6
0
4
13
s
16
c
10
d
u
3
1
5
12
4
3
e
4
7
f
7
t
14


2
a
5
7
b
8
9
6
2
0
s
16
4
c
10
13
d
3
1
5
12
4
3
e
4
7
f
7
t
14
u


2
a
7
b
5
2
8
9
6
0
s
16
4
c
10
13
d
3
1
5
12
4
4
7
7
14
3
e
f
t
u

Improved Running Time using Priority Queue

Dijkstra(G, w, s)

```

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $d(u) + w(u, v) < d(v)$ then
            $d(v) \leftarrow d(u) + w(u, v)$, $Q$.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $(\pi, d)$
```

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

```
$s \leftarrow$ arbitrary vertex in $G$
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$, $Q$.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$
```

Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. Summary

Encoding Symbols Using Bits

- assume: 8 symbols a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per symbol

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \rightarrow 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per symbol

Q: What if some symbols appear more frequently than the others in expectation?

Q: If some symbols appear more frequently than the others in expectation, can we have a better encoding scheme?

A: Maybe. Using variable-length encoding scheme.

Idea

- using fewer bits for symbols that are more frequently used, and more bits for symbols that are less frequently used.

Need to use prefix codes to guarantee a unique decoding.

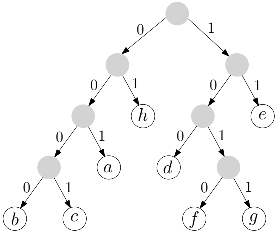
Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \rightarrow \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

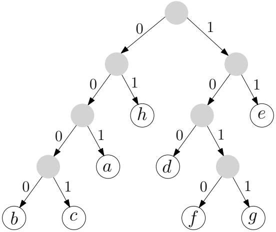
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

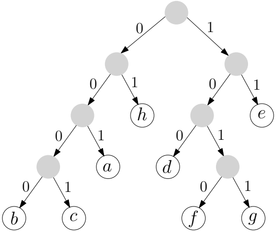


- 0001001100000001011110100001001

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

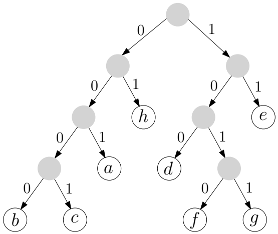


- 0001/001100000001011110100001001
- c

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

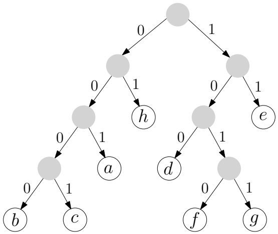


- 0001/001/100000001011110100001001
- ca

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

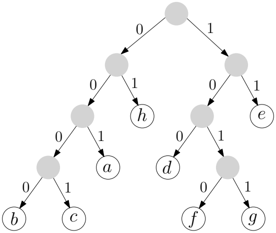


- 0001/001/100/0000101111010000 1001
- cad

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



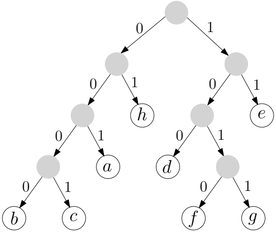
- 0001/001/100/0000/01011110100001001
- cadb

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100

e	f	g	h
11	1010	1011	01

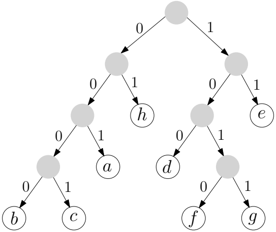


- 0001/001/100/0000/01/011110100001001
- cadbh

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

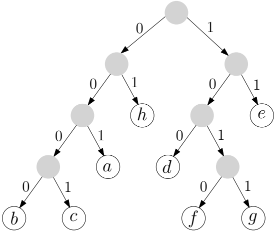


- 0001/001/100/0000/01/01/1110100001001
- cadbhh

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

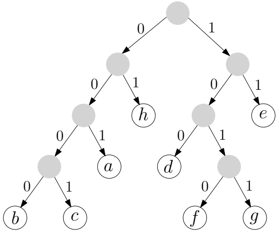


- 0001/001/100/0000/01/01/11/10100001001
- cadbhhe

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

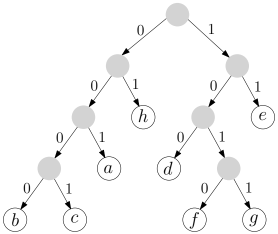


- 0001/001/100/0000/01/01/11/1010/0001001
- cadbhhef

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01

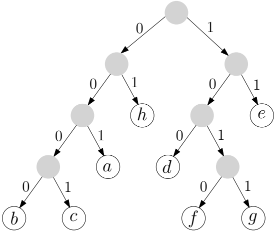


- 0001/001/100/0000/01/01/11/1010/0001/001
- cadbhhefc

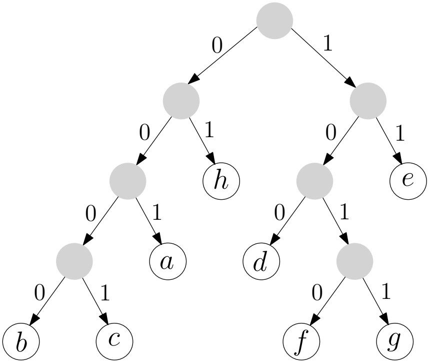
Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

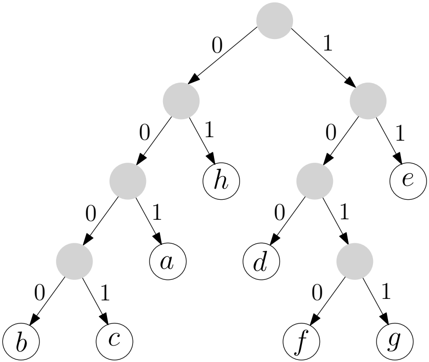
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



- 0001/001/100/0000/01/01/11/1010/0001/001/
- cadbhhefca

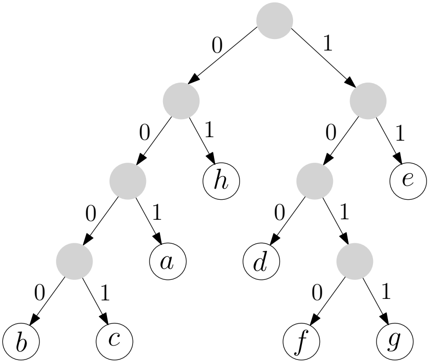


Properties of Encoding Tree



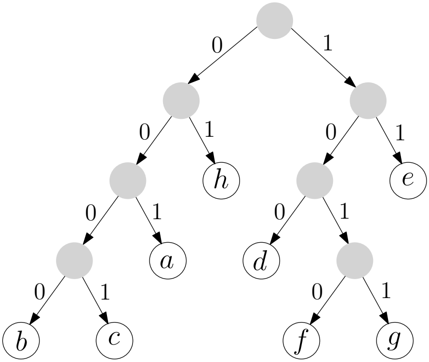
Properties of Encoding Tree

- Rooted binary tree



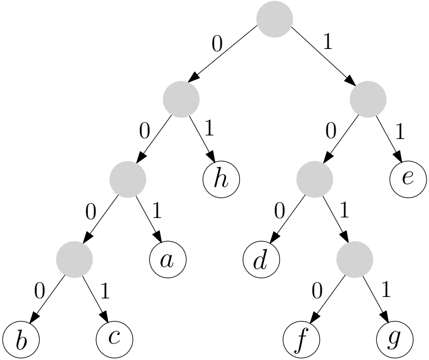
Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1



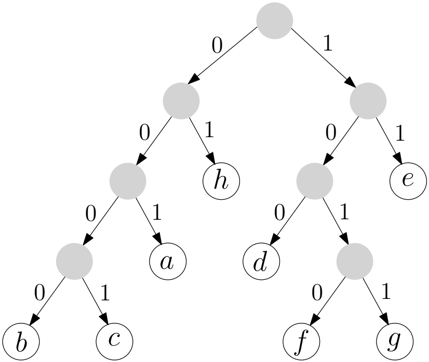
Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some symbol



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some symbol
- If coding scheme is not wasteful: a non-leaf has exactly two children



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some symbol
- If coding scheme is not wasteful: a non-leaf has exactly two children

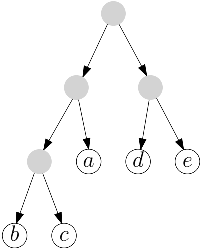
Best Prefix Codes

Input: frequencies of letters in a message

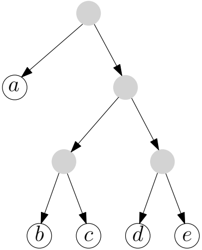
Output: prefix coding scheme giving the shortest encoding for the message

example

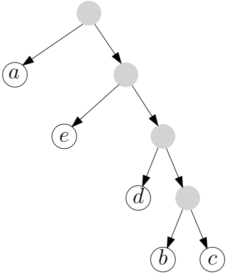
symbols	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



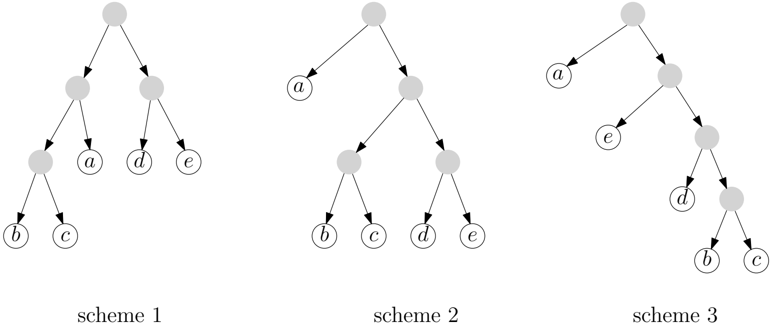
scheme 2



scheme 3

example

symbols	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total $= 89$
scheme 2 length	1	3	3	3	3	total $= 87$
scheme 3 length	1	4	4	3	2	total $= 84$



scheme 1 scheme 2 scheme 3

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

- the code for some letter?

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

- the code for some letter?
- hard to design a strategy; residual problem is complicated.

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

- the code for some letter?
- hard to design a strategy; residual problem is complicated.

- a partition of letters into left and right sub-trees?

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

- the code for some letter?
- hard to design a strategy; residual problem is complicated.

- a partition of letters into left and right sub-trees?
- not clear how to design the greedy algorithm

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

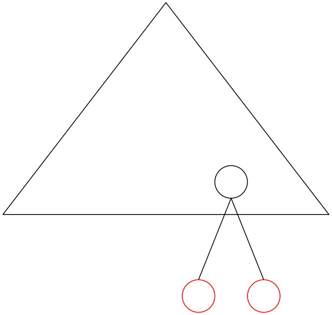
- the code for some letter?
- hard to design a strategy; residual problem is complicated.

- a partition of letters into left and right sub-trees?
- not clear how to design the greedy algorithm

A: Choose two letters and make them brothers in the tree.

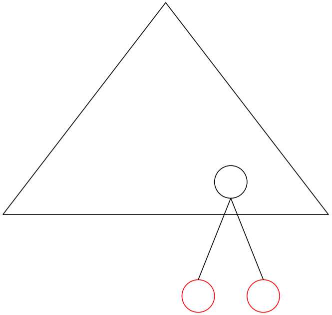
Which Two symbols Can Be Safely Put Together As Brothers?

- Focus a tree structure, without leaf labeling



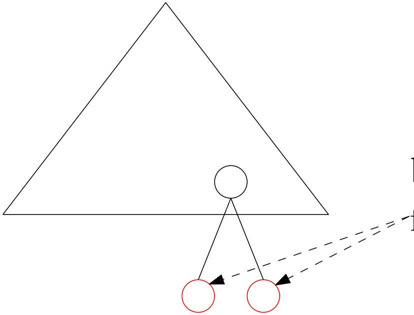
Which Two symbols Can Be Safely Put Together As Brothers?

- Focus a tree structure, without leaf labeling
- There are two deepest leaves that are brothers



Which Two symbols Can Be Safely Put Together As Brothers?

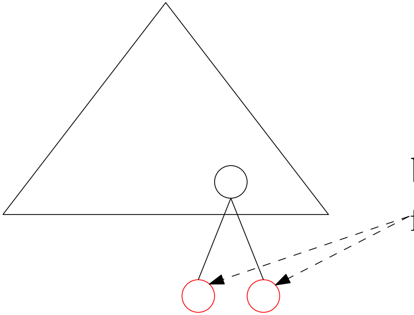
- Focus a tree structure, without leaf labeling
- There are two deepest leaves that are brothers



best to put the two least frenquent symbols here!

Which Two symbols Can Be Safely Put Together As Brothers?

- Focus a tree structure, without leaf labeling
- There are two deepest leaves that are brothers
- It is safe to make the two least frequent symbols brothers!



best to put the two least frenquent symbols here!

- It is safe to make the two least frequent symbols brothers!

- It is safe to make the two least frequent symbols brothers!

Lemma There is an optimum encoding tree, where the two least frequent symbols are brothers.

- It is safe to make the two least frequent symbols brothers!

Lemma There is an optimum encoding tree, where the two least frequent symbols are brothers.

- So we can make the two least frequent symbols brothers; the decision is irrevocable.

- It is safe to make the two least frequent symbols brothers!

Lemma There is an optimum encoding tree, where the two least frequent symbols are brothers.

- So we can make the two least frequent symbols brothers; the decision is irrevocable.

Q: Is the residual problem an instance of the best prefix codes problem?

- It is safe to make the two least frequent symbols brothers!

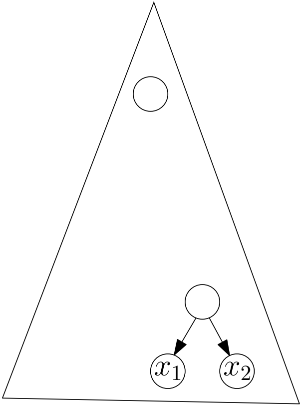
Lemma There is an optimum encoding tree, where the two least frequent symbols are brothers.

- So we can make the two least frequent symbols brothers; the decision is irrevocable.

Q: Is the residual problem an instance of the best prefix codes problem?

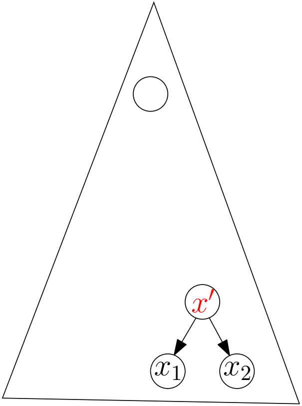
A: Yes, although the answer is not immediate.

- f_x: the frequency of the symbol x in the support.
- x_1 and x_2: the two symbols we decided to put together.
- d_x the depth of symbol x in our output encoding tree.



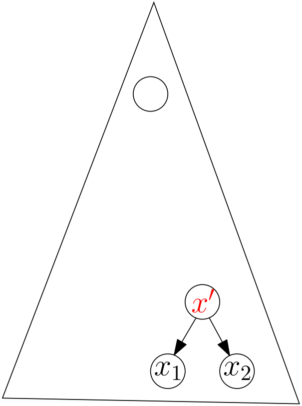
$$\begin{aligned} &\sum_{x \in S} f_x d_x \\ =& \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\ =& \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \end{aligned}$$

- f_x: the frequency of the symbol x in the support.
- x_1 and x_2: the two symbols we decided to put together.
- d_x the depth of symbol x in our output encoding tree.



$$\begin{aligned}
&\sum_{x \in S} f_x d_x \\
&= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
&= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
\end{aligned}$$

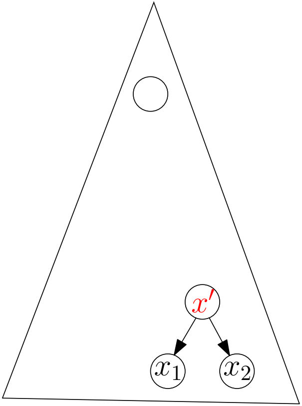
- f_x: the frequency of the symbol x in the support.
- x_1 and x_2: the two symbols we decided to put together.
- d_x the depth of symbol x in our output encoding tree.



$$\begin{aligned}
&\sum_{x \in S} f_x d_x \\
= &\sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
= &\sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
\end{aligned}$$

Def: $f_{x'} = f_{x_1} + f_{x_2}$

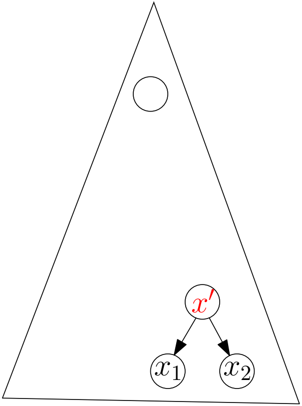
- f_x: the frequency of the symbol x in the support.
- x_1 and x_2: the two symbols we decided to put together.
- d_x the depth of symbol x in our output encoding tree.



$$\begin{aligned}
&\sum_{x\in S} f_x d_x \\
&= \sum_{x\in S\setminus\{x_1,x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
&= \sum_{x\in S\setminus\{x_1,x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \\
&= \sum_{x\in S\setminus\{x_1,x_2\}} f_x d_x + f_{x'}(d_{x'} + 1)
\end{aligned}$$

Def: $f_{x'} = f_{x_1} + f_{x_2}$

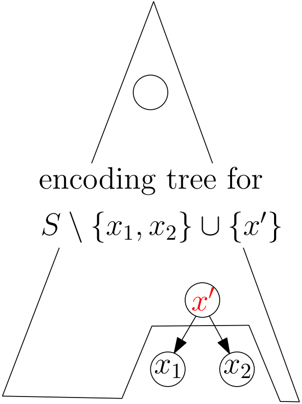
- f_x: the frequency of the symbol x in the support.
- x_1 and x_2: the two symbols we decided to put together.
- d_x the depth of symbol x in our output encoding tree.



$$
\begin{aligned}
& \sum_{x \in S} f_x d_x \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x'} (d_{x'} + 1) \\
= & \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x + f_{x'}
\end{aligned}
$$

Def: $f_{x'} = f_{x_1} + f_{x_2}$

- f_x: the frequency of the symbol x in the support.
- x_1 and x_2: the two symbols we decided to put together.
- d_x the depth of symbol x in our output encoding tree.



$$\begin{aligned}
&\sum_{x \in S} f_x d_x \\
&= \sum_{x \in S\setminus\{x_1,x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
&= \sum_{x \in S\setminus\{x_1,x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \\
&= \sum_{x \in S\setminus\{x_1,x_2\}} f_x d_x + f_{x'}(d_{x'} + 1) \\
&= \sum_{x \in S\setminus\{x_1,x_2\}\cup\{x'\}} f_x d_x + f_{x'}
\end{aligned}$$

Def: $f_{x'} = f_{x_1} + f_{x_2}$

In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with symbols $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

Huffman codes: Recursive Algorithm

Huffman(S, f)

```
if $|S| > 1$ then
    let $x_1, x_2$ be the two symbols with the smallest $f$ values
    introduce a new symbol $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
    $S' \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
    call Huffman$(S', f|_{S'})$ to build an encoding tree $T'$
    let $T$ be obtained from $T'$ by adding $x_1, x_2$ as two children of $x'$
    return $T$
else
    let $x$ be the symbol in $S$
    return a tree with a single node labeled $x$
```

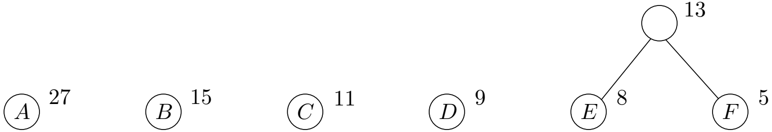
Huffman codes: Iterative Algorithm

Huffman(S, f)

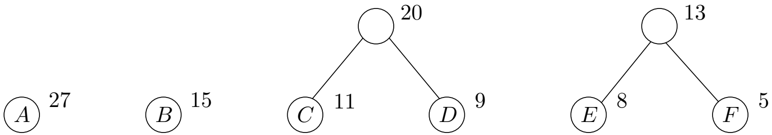
```
while $|S| > 1$ do
    let $x_1, x_2$ be the two symbols with the smallest $f$ values
    introduce a new symbol $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
    let $x_1$ and $x_2$ be the two children of $x'$
    $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
return the tree constructed
```

Example

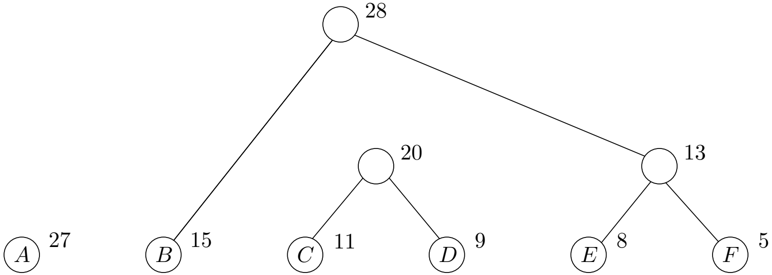
Example



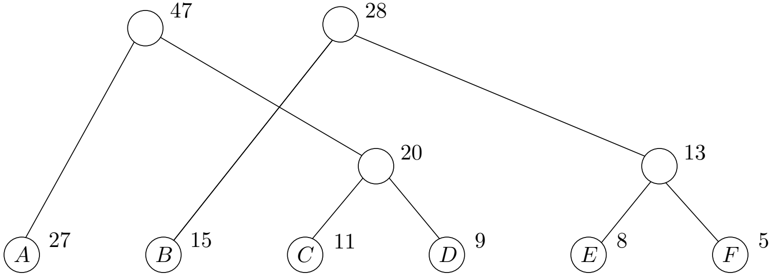
Example



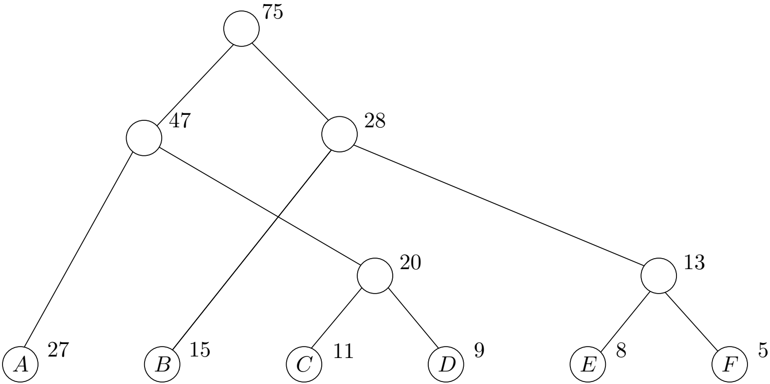
Example



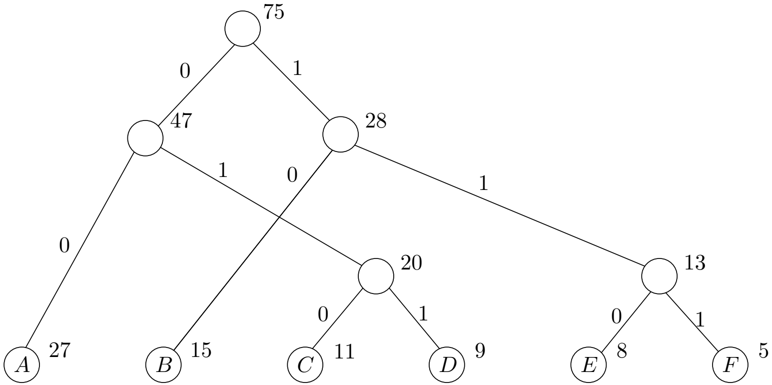
Example



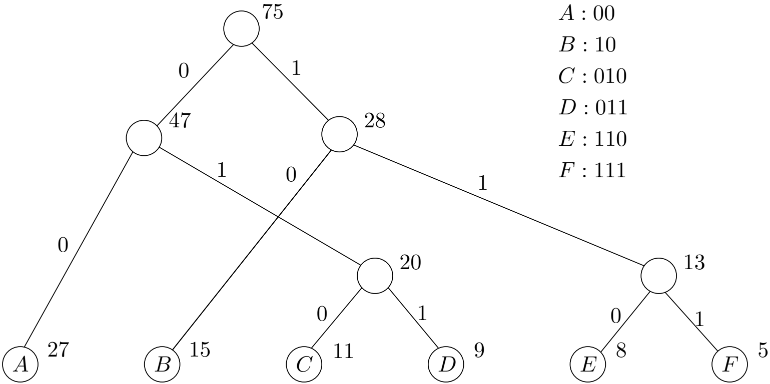
Example



Example



Example



$A : 00$

$B : 10$

$C : 010$

$D : 011$

$E : 110$

$F : 111$

Algorithm using Priority Queue

Huffman(S, f)

```
$Q \leftarrow$ build-priority-queue$(S)$
while $Q.\mathrm{size} > 1$ do
    $x_1 \leftarrow Q.\mathrm{extract\text{-}min}()$
    $x_2 \leftarrow Q.\mathrm{extract\text{-}min}()$
    introduce a new symbol $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
    let $x_1$ and $x_2$ be the two children of $x'$
    $Q.\mathrm{insert}(x')$
return the tree constructed
```

Outline

1. Toy Examples
2. Interval Scheduling
3. Minimum Spanning Tree
    - Kruskal's Algorithm
    - Reverse-Kruskal's Algorithm
    - Prim's Algorithm
4. Single Source Shortest Paths
    - Dijkstra's Algorithm
5. Data Compression and Huffman Code
6. **Summary**

Summary for Greedy Algorithms

1. Design a “reasonable” strategy

Summary for Greedy Algorithms

1. Design a “reasonable” strategy
    - Interval scheduling problem: schedule the job j^* with the earliest deadline

Summary for Greedy Algorithms

1. Design a “reasonable” strategy
    - Interval scheduling problem: schedule the job j^* with the earliest deadline
    - Kruskal's algorithm for MST: select lightest edge e^*

Summary for Greedy Algorithms

1. Design a “reasonable” strategy
    - Interval scheduling problem: schedule the job j^* with the earliest deadline
    - Kruskal’s algorithm for MST: select lightest edge e^*
    - Inverse Kruskal’s algorithm for MST: drop the heaviest non-bridge edge e^*

Summary for Greedy Algorithms

1. Design a "reasonable" strategy
    - Interval scheduling problem: schedule the job j^* with the earliest deadline
    - Kruskal's algorithm for MST: select lightest edge e^*
    - Inverse Kruskal's algorithm for MST: drop the heaviest non-bridge edge e^*
    - Prim's algorithm for MST: select the lightest edge e^* incident to a specified vertex s

Summary for Greedy Algorithms

1. Design a “reasonable” strategy
    - Interval scheduling problem: schedule the job j^* with the earliest deadline
    - Kruskal’s algorithm for MST: select lightest edge e^*
    - Inverse Kruskal’s algorithm for MST: drop the heaviest non-bridge edge e^*
    - Prim’s algorithm for MST: select the lightest edge e^* incident to a specified vertex s
    - Huffman codes: make the two least frequent symbols brothers

Summary for Greedy Algorithms

1. Design “reasonable” strategy
2. Prove that the reasonable strategy is “safe”

Summary for Greedy Algorithms

1. Design “reasonable” strategy
2. Prove that the reasonable strategy is “safe”

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

- Usually done by "exchange argument"

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

- Usually done by "exchange argument"
- Interval scheduling problem: exchange j^* with the first job in an optimal solution

Summary for Greedy Algorithms

1. Design “reasonable” strategy
2. Prove that the reasonable strategy is “safe”

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

- Usually done by “exchange argument”
- Interval scheduling problem: exchange j^* with the first job in an optimal solution
- Kruskal’s algorithm: exchange e^* with some edge e in the cycle in $T \cup \{e^*\}$

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

- Usually done by "exchange argument"
- Interval scheduling problem: exchange j^* with the first job in an optimal solution
- Kruskal's algorithm: exchange e^* with some edge e in the cycle in $T \cup \{e^*\}$
- Prim's algorithm: exchange e^* with some other edge e incident to s

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"
3. Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"
3. Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
    - Interval scheduling problem: remove j^* and the jobs it conflicts with

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"
3. Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
    - Interval scheduling problem: remove j^* and the jobs it conflicts with
    - Kruskal and Prim's algorithms: contracting e^*

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"
3. Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
    - Interval scheduling problem: remove j^* and the jobs it conflicts with
    - Kruskal and Prim's algorithms: contracting e^*
    - Inverse Kruskal's algorithm: remove e^*

Summary for Greedy Algorithms

1. Design "reasonable" strategy
2. Prove that the reasonable strategy is "safe"
3. Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
    - Interval scheduling problem: remove j^* and the jobs it conflicts with
    - Kruskal and Prim's algorithms: contracting e^*
    - Inverse Kruskal's algorithm: remove e^*
    - Huffman codes: merge two symbols into one

Summary for Greedy Algorithms

- Dijkstra's algorithm does not quite fit in the framework.

Summary for Greedy Algorithms

- Dijkstra's algorithm does not quite fit in the framework.
- It combines "greedy algorithm" and "dynamic programming"

Summary for Greedy Algorithms

- Dijkstra's algorithm does not quite fit in the framework.
- It combines "greedy algorithm" and "dynamic programming"
- Greedy algorithm: each time select the vertex in $V \setminus S$ with the smallest d value and add it to S

Summary for Greedy Algorithms

- Dijkstra's algorithm does not quite fit in the framework.
- It combines "greedy algorithm" and "dynamic programming"
- Greedy algorithm: each time select the vertex in $V \setminus S$ with the smallest d value and add it to S
- Dynamic programming: remember the d values of vertices in S for future use

Summary for Greedy Algorithms

- Dijkstra's algorithm does not quite fit in the framework.
- It combines "greedy algorithm" and "dynamic programming"
- Greedy algorithm: each time select the vertex in $V \setminus S$ with the smallest d value and add it to S
- Dynamic programming: remember the d values of vertices in S for future use
- Dijkstra's algorithm is very similar to Prim's algorithm for MST