CSE 431/531: Algorithm Analysis and Design (Spring 2018)

Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo

Outline

- Course Webpage (contains schedule, policies, homeworks and slides):
  http://www.cse.buffalo.edu/~shil/courses/CSE531/
- Please sign up the course on Piazza from course webpage polls, asking/answering questions.

- Time and locatiion:
  - MoWeFr, 9:00-9:50am
  - Talbert 107
- Lecturer:
  - Shi Li, shil@buffalo.edu
  - Office hours: TBD
- TAs
  - TBD

You should know:

You should know:

- Mathematical Tools
  - Mathematical inductions
  - Probabilities and random variables

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  - Probabilities and random variables
- Data Structures
  - Stacks, queues, linked lists
- Some Programming Experience
  - E.g., C, C++, Java or Python

You Will Learn

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    - Sorting
    - Shortest paths
    - Minimum spanning tree

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  - Greedy algorithms
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  - Dynamic programming
  - Linear Programming

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- NP-completeness

Tentative Schedule

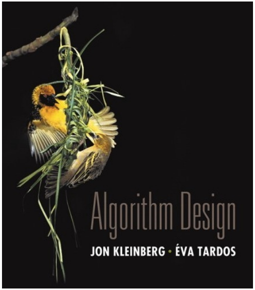
- Introduction, 3 lectures
- Basic Graph Algorithms, 3 lectures
- Greedy Algorihtms, 6 lectures (include recitation)
- Divide and Conquer, 6 lectures (include recitation)
- In-Class Exam #1, Mar 12, 2018
- Dynamic Programming, 6 lectures (include recitation)
- Linear Programming, 6 lectures (include recitation)
- In-Class Exam #2, Apr 18, 2018
- NP-Completeness 6 lectures (include recitation)
- Final Review, 1 lecture

- Exercise problems will be posted before each recitation class

Textbook

Textbook (Highly Recommended):

- Algorithm Design, 1st Edition, by *Jon Kleinberg* and *Eva Tardos*



Other Reference Books

- Introduction to Algorithms, Third Edition, *Thomas Cormen, Charles Leiserson, Rondald Rivest, Clifford Stein*

Reading Before Classes

- Highly recommended: read the correspondent sections from the textbook (or reference book) before classes
- Slides will be posted online before class

Grading

- 40% for homeworks
  - 6 homeworks, 5 of which contain programming problems

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  - 6 homeworks, 5 of which contain programming problems
- 60% for two in-class exams + final exam

$$
\begin{aligned}
&\max\{E1 \times 5\% + F \times 25\%, E1 \times 15\% + F \times 15\%\} \\
&+ \max\{E2 \times 5\% + F \times 25\%, E2 \times 15\% + F \times 15\%\} \\
&E1, E2, F \in [0, 100]
\end{aligned}
$$

For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussing
  - Must write down solutions on your own, in your own words
  - Write down names of students you collaborated with

For Homeworks, You Are Not Allowed to

- Use external resources
  - Can't Google or ask questions online for solutions
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- Need to implement the algorithms by your self
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Late policy

- You have one late credit
- turn in a homework late for three days using the late credit
- no other late submissions will be accepted

Exams

- Closed-book
- Can bring one A4 handwritten sheet

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Questions?

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What is an Algorithm?

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- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
- Computational problem: specifies the input/output relationship.
- An algorithm solves a computational problem if it produces the correct output for any given input.

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

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- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$
- $(270, 210) \to (210, 60) \to (60, 30) \to (30, 0)$

Examples

Sorting

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \le a'_2 \le \cdots \le a'_n$

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- Algorithms: insertion sort, merge sort, quicksort, ...

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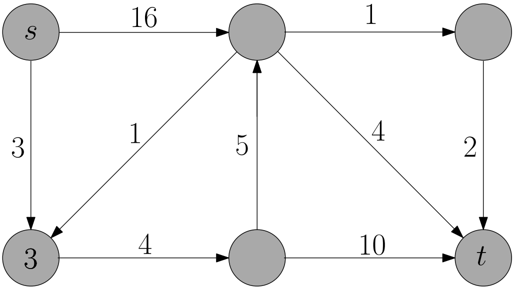

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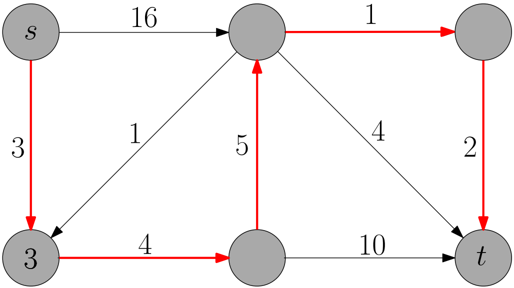

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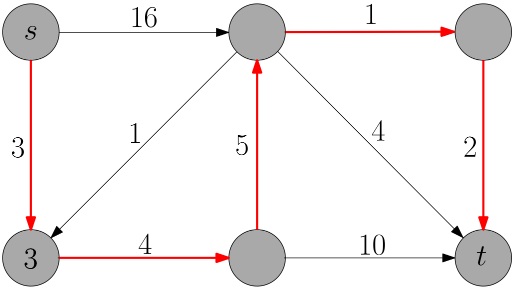

- Algorithm: Dijkstra's algorithm

Algorithm = Computer Program?

- Algorithm: "abstract", can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: "concrete", implementation of algorithm, associated with a particular programming language

Pseudo-Code

Pseudo-Code:

Euclidean(a, b)

1. while $b > 0$
2. $\quad (a, b) \leftarrow (b, a \bmod b)$
3. return a

C++ program:

```
int Euclidean(int a, int b){
    int c;
    while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```

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3. fundamental
4. it is fun!

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Insertion-Sort

- At the end of j-th iteration, make the first j numbers sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
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insertion-sort(A, n)

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for j ← 2 to n
    key ← A[j]
    i ← j − 1
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Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : {\color{red}53}, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : {\color{red}12, 53}, 35, 21, 59, 15$$

$$\text{after } j = 3 : {\color{red}12, 35, 53}, 21, 59, 15$$

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$$\text{after } j = 6 : {\color{red}12, 15, 21, 35, 53, 59}$$

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- Q: How fast is the computer?
- Q: Programming language?
- A: Important idea: asymptotic analysis
  - Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

- Ignoring lower order terms
- Ignoring leading constant

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- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
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O-notation allows us to

- ignore architecture of computer
- ignore programming language

Asymptotic Analysis of Insertion Sort

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- Worst-case running time for iteration j in the outer loop?
  Answer: $O(j)$
- Total running time $= \sum_{j=2}^{n} O(j) = O(n^2)$ (informal)

Computation Model

- Random-Access Machine (RAM) model: read $A[j]$ takes $O(1)$ time.
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- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort, heap sort, ...

- Remember to sign up for Piazza.

Questions?

Outline

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

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- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \le cg(n), \forall n \ge n_0\}.$$

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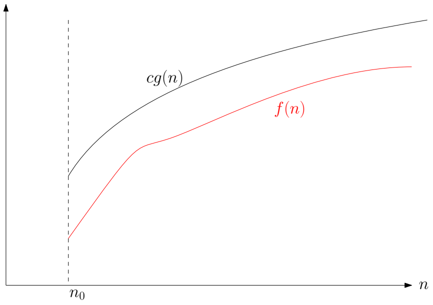

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Proof.

Let $c = 4$ and $n_0 = 10$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$
$$= -n^2 + 40n \le 0.$$
$$3n^2 + 2n \le c(n^2 - 10n)$$

□

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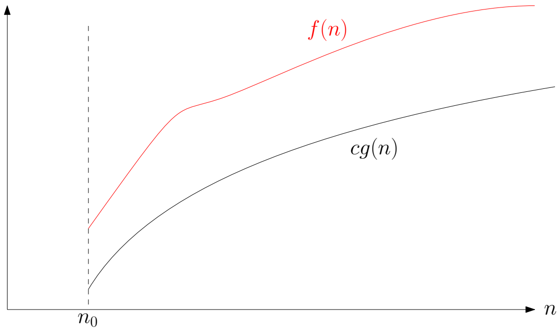

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Theorem $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$.

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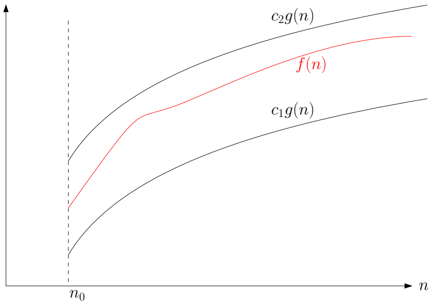

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Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Exercise

For each pair of functions f, g in the following table, indicate whether f is O, Ω or Θ of g.

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$3n - 50$	$n^2 - 7n$			
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$$f(n) = n^2$$

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases}$$

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- $3n^2 - 10n - 5 = O(n^2)$ is simpler.

o and ω-Notations

o-Notation For a function $g(n)$,

$$o(g(n)) = \{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

ω-Notation For a function $g(n)$,

$$\omega(g(n)) = \{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0\}.$$

Example:

- $3n^2 + 5n + 10 = o(n^2 \lg n)$.
- $3n^2 + 5n + 10 = \omega(n^2 / \lg n)$.

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Questions?

Outline

$O(n)$ (Linear) Running Time

Computing the sum of n numbers

sum(A, n)

1. $S \leftarrow 0$
2. for $i \leftarrow 1$ to n
3. $\quad S \leftarrow S + A[i]$
4. return S

$O(n)$ (Linear) Running Time

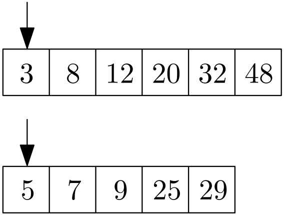
- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

$O(n)$ (Linear) Running Time

- Merge two sorted arrays



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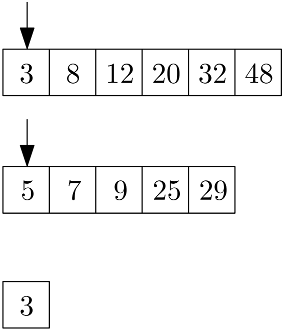

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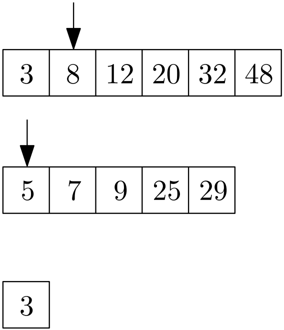


$O(n)$ (Linear) Running Time

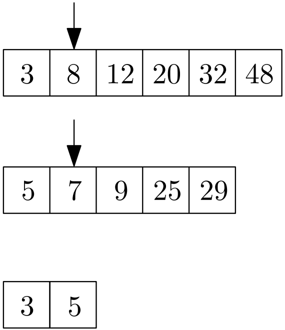
- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

3	5

- Merge two sorted arrays



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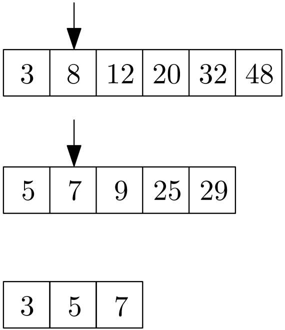

$O(n)$ (Linear) Running Time

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3	5	7

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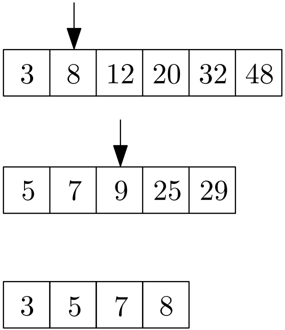

$O(n)$ (Linear) Running Time

- Merge two sorted arrays

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3	5	7	8

$O(n)$ (Linear) Running Time

- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

3	5	7	8	9	12	20	25	29

$O(n)$ (Linear) Running Time

- Merge two sorted arrays

3	8	12	20	32	48

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3	5	7	8	9	12	20	25	29	32	48

$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\ B and C are sorted, with length n_1 and n_2

$A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \le n_1$ and $j \le n_2$
    if $(B[i] \le C[j])$ then
        append $B[i]$ to A; $i \leftarrow i + 1$
    else
        append $C[j]$ to A; $j \leftarrow j + 1$
if $i \le n_1$ then append $B[i..n_1]$ to A
if $j \le n_2$ then append $C[j..n_2]$ to A
return A

$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\\\ B and C are sorted, with length n_1 and n_2

```
A ← []; i ← 1; j ← 1
while i ≤ n1 and j ≤ n2
    if (B[i] ≤ C[j]) then
        append B[i] to A; i ← i + 1
    else
        append C[j] to A; j ← j + 1
if i ≤ n1 then append B[i..n1] to A
if j ≤ n2 then append C[j..n2] to A
return A
```

Running time $= O(n)$ where $n = n_1 + n_2$.

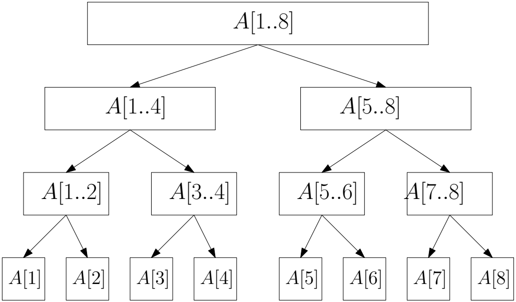
$O(n \lg n)$ Running Time

merge-sort(A, n)

```
if n = 1 then
    return A
else
    B ← merge-sort(A[1..⌊n/2⌋], ⌊n/2⌋)
    C ← merge-sort(A[⌊n/2⌋ + 1..n], n − ⌊n/2⌋)
return merge(B, C, ⌊n/2⌋, n − ⌊n/2⌋)
```

$O(n \lg n)$ Running Time

- Merge-Sort



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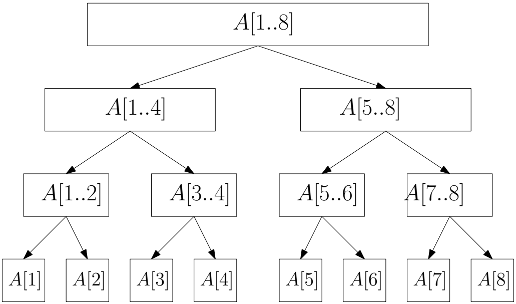
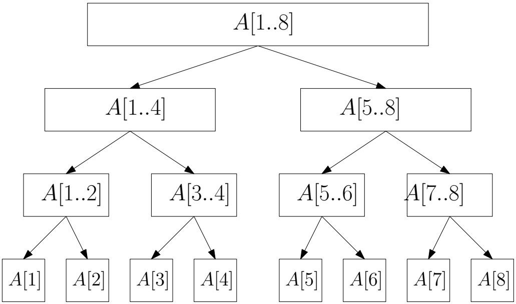


- Each level takes running time $O(n)$

$O(n \lg n)$ Running Time

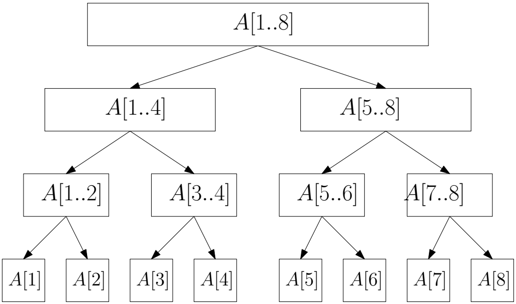
- Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels

$O(n \lg n)$ Running Time

- Merge-Sort



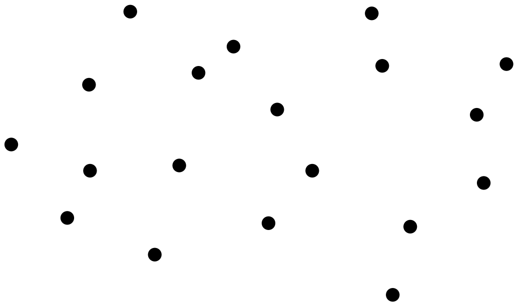
- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



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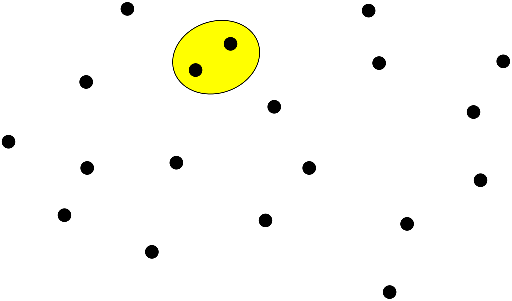

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair(x, y, n)

1. $bestd \leftarrow \infty$
2. for $i \leftarrow 1$ to $n - 1$
3. for $j \leftarrow i + 1$ to n
4. $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
5. if $d < bestd$ then
6. $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
7. return $(besti, bestj)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

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closest-pair(x, y, n)

```
bestd ← ∞
for i ← 1 to n − 1
    for j ← i + 1 to n
        d ← sqrt((x[i] − x[j])^2 + (y[i] − y[j])^2)
        if d < bestd then
            besti ← i, bestj ← j, bestd ← d
return (besti, bestj)
```

Closest pair can be solved in $O(n \lg n)$ time!

$O(n^3)$ (Cubic) Running Time

Multiply two matrices of size $n \times n$

matrix-multiplication(A, B, n)

$C \leftarrow$ matrix of size $n \times n$, with all entries being 0
for $i \leftarrow 1$ to n
    for $j \leftarrow 1$ to n
        for $k \leftarrow 1$ to n
            $C[i,k] \leftarrow C[i,k] + A[i,j] \times B[j,k]$
return C

$O(n^k)$ Running Time for Integer $k \geq 4$

Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

$O(n^k)$ Running Time for Integer $k \geq 4$

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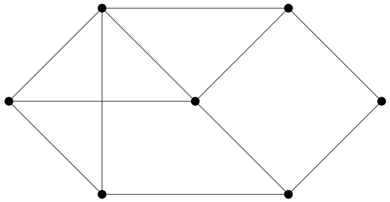

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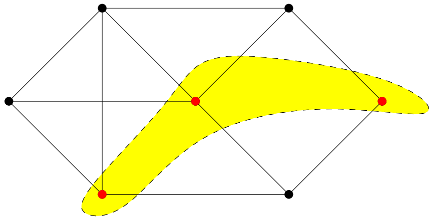

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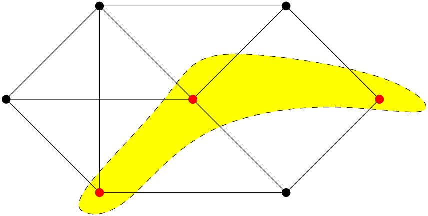


Independent set of size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

$O(n^k)$ Running Time for Integer $k \geq 4$

Independent Set of Size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

independent-set$(G = (V, E))$

1. for every set $S \subseteq V$ of size k
2. $\quad b \leftarrow$ true
3. $\quad$ for every $u, v \in S$
4. $\quad\quad$ if $(u, v) \in E$ then $b \leftarrow$ false
5. $\quad$ if b return true
6. return false

Running time $= O(\frac{n^k}{k!} \times k^2) = O(n^k)$ (assume k is a constant)

Beyond Polynomial Time: $O(2^n)$

Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the maximum independent set of G

max-independent-set($G = (V, E)$)

$R \leftarrow \emptyset$
for every set $S \subseteq V$
    $b \leftarrow$ true
    for every $u, v \in S$
        if $(u, v) \in E$ then $b \leftarrow$ false
    if b and $|S| > |R|$ then $R \leftarrow S$
return R

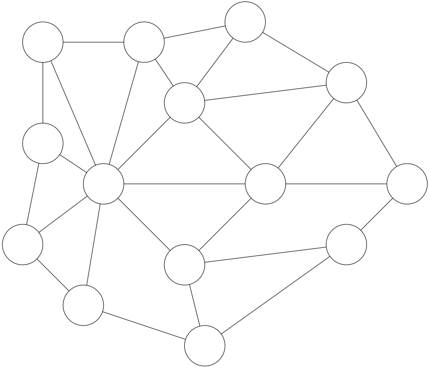
Running time $= O(2^n n^2)$.

Beyond Polynomial Time: $O(n!)$

Hamiltonian Cycle Problem

Input: a graph with n vertices

Output: a cycle that visits each node exactly once,
or say no such cycle exists



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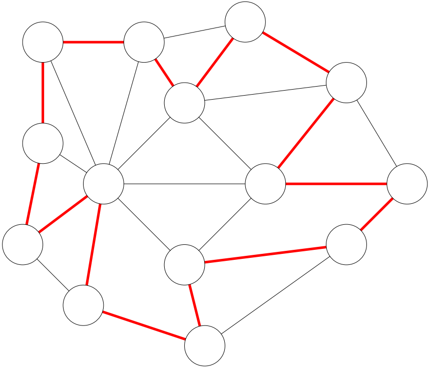

Beyond Polynomial Time: $n!$

Hamiltonian($G = (V, E)$)

1. for every permutation $(p_1, p_2, \cdots, p_n)$ of V
2. $\quad b \leftarrow$ true
3. $\quad$ for $i \leftarrow 1$ to $n - 1$
4. $\quad\quad$ if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
5. $\quad$ if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
6. $\quad$ if b then return $(p_1, p_2, \cdots, p_n)$
7. return "No Hamiltonian Cycle"

Running time $= O(n! \times n)$

$O(\lg n)$ (Logarithmic) Running Time

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- Binary search
  - Input: sorted array A of size n, an integer t;
  - Output: whether t appears in A.

$O(\lg n)$ (Logarithmic) Running Time

- Binary search
  - Input: sorted array A of size n, an integer t;
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- E.g, search 35 in the following array:

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

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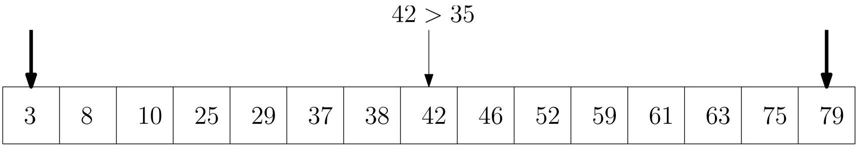

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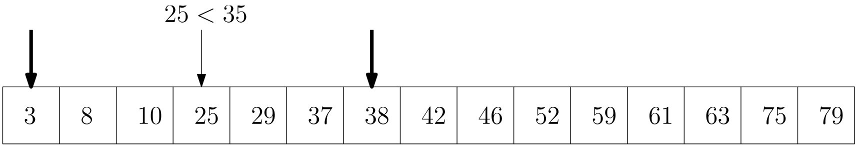

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  - Input: sorted array A of size n, an integer t;
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- E.g, search 35 in the following array:

$37 > 35$

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

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$O(\lg n)$ (Logarithmic) Running Time

Binary search

- Input: sorted array A of size n, an integer t;
- Output: whether t appears in A.

binary-search(A, n, t)

$i \leftarrow 1, j \leftarrow n$
while $i \leq j$ do
    $k \leftarrow \lfloor (i+j)/2 \rfloor$
    if $A[k] = t$ return true
    if $A[k] < t$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
return false

$O(\lg n)$ (Logarithmic) Running Time

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4. $\quad$ if $A[k] = t$ return true
5. $\quad$ if $A[k] < t$ then $j \leftarrow k-1$ else $i \leftarrow k+1$
6. return false

Running time $= O(\lg n)$

Compare the Orders

- Sort the functions from smallest to largest asymptotically
  $n^{\sqrt{n}}, \; \lg n, \; n, \; n^2, \; n \lg n, \; n!, \; 2^n, \; e^n, \; \lg(n!), \; n^n$

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- $f < g$ stands for $f = o(g)$, $f = g$ stands for $f = \Theta(g)$!
- $\lg n < n^{\sqrt{n}}$

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- Sort the functions from smallest to largest asymptotically
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- $f < g$ stands for $f = o(g)$, $f = g$ stands for $f = \Theta(g)$!
- $\lg n < n^{\sqrt{n}}$
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- $\lg n < n < n \lg n = \lg(n!) < n^2 < n^{\sqrt{n}} < 2^n < e^n < n!$
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Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
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When we talk about lower bounds:

- Super-linear time: $\omega(n)$
- Super-quadratic time: $\omega(n^2)$
- Super-polynomial time: $\bigcap_{k>0} \omega(n^k)$

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- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

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- For "natural" algorithms, constants are not so big!
- So, for reasonable n, algorithm with lower order running time beats algorithm with higher order running time.