# CSE 431/531: Algorithm Analysis and Design (Spring 2018)
# NP-Completeness

Lecturer: Shi Li

*Department of Computer Science and Engineering*
*University at Buffalo*

3.36pt

# NP-Completeness Theory

- The topics we discussed so far are positive results: how to design efficient algorithms for solving a given problem.
- NP-Completeness provides negative results: some problems can not be solved efficiently.

**Q:** Why do we study negative results?

# NP-Completeness Theory

- The topics we discussed so far are positive results: how to design efficient algorithms for solving a given problem.
- NP-Completeness provides negative results: some problems can not be solved efficiently.

**Q:** Why do we study negative results?

- A given problem $X$ cannot be solved in polynomial time.

# NP-Completeness Theory

- The topics we discussed so far are positive results: how to design efficient algorithms for solving a given problem.
- NP-Completeness provides negative results: some problems can not be solved efficiently.

**Q:** Why do we study negative results?

- A given problem $X$ cannot be solved in polynomial time.
- Without knowing it, you will have to keep trying to find polynomial time algorithm for solving $X$. All our efforts are doomed!

# Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$

# Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5} \log n), O(n^{100})$

# Efficient $=$ Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5}\log n), O(n^{100})$
- Not polynomial time: $O(2^n), O(n^{\log n})$

# Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5}\log n), O(n^{100})$
- Not polynomial time: $O(2^n), O(n^{\log n})$
- Almost all algorithms we learnt so far run in polynomial time

# Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5}\log n), O(n^{100})$
- Not polynomial time: $O(2^n), O(n^{\log n})$
- Almost all algorithms we learnt so far run in polynomial time

**Reason for Efficient = Polynomial Time**

# Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5}\log n), O(n^{100})$
- Not polynomial time: $O(2^n), O(n^{\log n})$
- Almost all algorithms we learnt so far run in polynomial time

**Reason for Efficient = Polynomial Time**

- For natural problems, if there is an $O(n^k)$-time algorithm, then $k$ is small, say 4

# Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5} \log n), O(n^{100})$
- Not polynomial time: $O(2^n), O(n^{\log n})$
- Almost all algorithms we learnt so far run in polynomial time

**Reason for Efficient = Polynomial Time**

- For natural problems, if there is an $O(n^k)$-time algorithm, then $k$ is small, say 4
- A good cut separating problems: for most natural problems, either we have a polynomial time algorithm, or the best algorithm runs in time $\Omega(2^{n^c})$ for some $c$

# Efficient = Polynomial Time

- Polynomial time: $O(n^k)$ for any constant $k > 0$
- Example: $O(n), O(n^2), O(n^{2.5}\log n), O(n^{100})$
- Not polynomial time: $O(2^n), O(n^{\log n})$
- Almost all algorithms we learnt so far run in polynomial time

**Reason for Efficient = Polynomial Time**

- For natural problems, if there is an $O(n^k)$-time algorithm, then $k$ is small, say 4
- A good cut separating problems: for most natural problems, either we have a polynomial time algorithm, or the best algorithm runs in time $\Omega(2^{n^c})$ for some $c$
- Do not need to worry about the computational model

# Pseudo-Polynomial Is not Polynomial!

Polynomial:

- Kruskal's algorithm for minimum spanning tree: $O(n \lg n + m)$
- Floyd-Warshall for all-pair shortest paths: $O(n^3)$

Reason: we need to specify $m \geq n - 1$ edges in the input

# Pseudo-Polynomial Is not Polynomial!

Polynomial:

- Kruskal's algorithm for minimum spanning tree: $O(n \lg n + m)$
- Floyd-Warshall for all-pair shortest paths: $O(n^3)$

Reason: we need to specify $m \geq n - 1$ edges in the input

Pseudo-Polynomial:

- Knapsack Problem: $O(nW)$, where $W$ is the maximum weight the Knapsack can hold

Reason: to specify integer in $[0, W]$, we only need $O(\lg W)$ bits.

# Outline

3.36pt

## Recall: Knapsack Problem

**Input:** $n$ items, each item $i$ with a weight $w_i$, and a value $v_i$; a bound $W$ on the total weight the knapsack can hold

**Output:** the maximum value of items the knapsack can hold, i.e, a set $S \subseteq \{1, 2, \cdots, n\}$:

$$\max \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W$$

- DP is $O(nW)$-time algorithm, not a real polynomial time
- Knapsack is NP-hard: it is unlikely that the problem can be solved in polynomial time

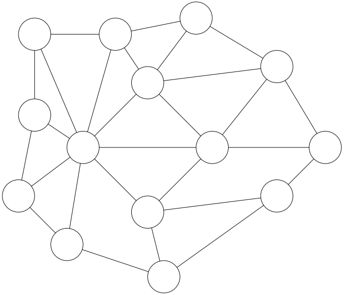
# Example: Hamiltonian Cycle Problem

**Def.** Let $G$ be an undirected graph. A Hamiltonian Cycle (HC) of $G$ is a cycle $C$ in $G$ that passes each vertex of $G$ exactly once.

**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle



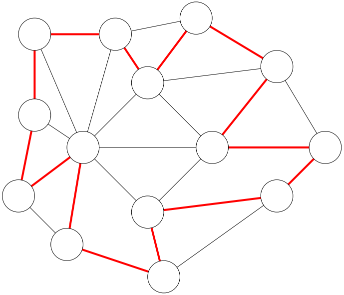
# Example: Hamiltonian Cycle Problem

**Def.** Let $G$ be an undirected graph. A Hamiltonian Cycle (HC) of $G$ is a cycle $C$ in $G$ that passes each vertex of $G$ exactly once.

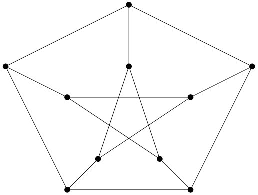
**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle



# Example: Hamiltonian Cycle Problem



- The graph is called the Petersen Graph. It has no HC.

# Example: Hamiltonian Cycle Problem

**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle

# Example: Hamiltonian Cycle Problem

**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle

Algorithm for Hamiltonian Cycle Problem:

- Enumerate all possible permutations, and check if it corresponds to a Hamiltonian Cycle

# Example: Hamiltonian Cycle Problem

**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle

Algorithm for Hamiltonian Cycle Problem:

- Enumerate all possible permutations, and check if it corresponds to a Hamiltonian Cycle
- Running time: $O(n!m) = 2^{O(n \lg n)}$

# Example: Hamiltonian Cycle Problem

**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle

Algorithm for Hamiltonian Cycle Problem:

- Enumerate all possible permutations, and check if it corresponds to a Hamiltonian Cycle
- Running time: $O(n!m) = 2^{O(n \lg n)}$
- Better algorithm: $2^{O(n)}$

# Example: Hamiltonian Cycle Problem

**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle

Algorithm for Hamiltonian Cycle Problem:

- Enumerate all possible permutations, and check if it corresponds to a Hamiltonian Cycle
- Running time: $O(n!m) = 2^{O(n \lg n)}$
- Better algorithm: $2^{O(n)}$
- Far away from polynomial time

# Example: Hamiltonian Cycle Problem

**Hamiltonian Cycle (HC) Problem**

**Input:** graph $G = (V, E)$

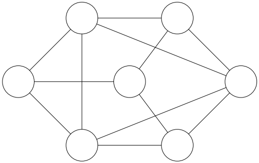
**Output:** whether $G$ contains a Hamiltonian cycle

Algorithm for Hamiltonian Cycle Problem:

- Enumerate all possible permutations, and check if it corresponds to a Hamiltonian Cycle
- Running time: $O(n!m) = 2^{O(n \lg n)}$
- Better algorithm: $2^{O(n)}$
- Far away from polynomial time
- HC is NP-hard: it is unlikely that it can be solved in polynomial time.

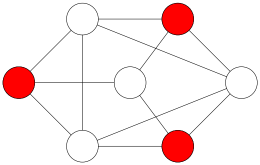
# Maximum Independent Set Problem

**Def.** An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in $I$ are adjacent in $G$.



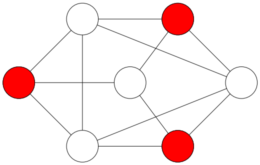
# Maximum Independent Set Problem

**Def.** An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in $I$ are adjacent in $G$.



# Maximum Independent Set Problem

**Def.** An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in $I$ are adjacent in $G$.



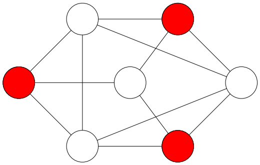
**Maximum Independent Set Problem**

**Input:** graph $G = (V, E)$

**Output:** the size of the maximum independent set of $G$

# Maximum Independent Set Problem

**Def.** An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in $I$ are adjacent in $G$.



**Maximum Independent Set Problem**

**Input:** graph $G = (V, E)$

**Output:** the size of the maximum independent set of $G$

- Maximum Independent Set is NP-hard

# Formula Satisfiability

**Formula Satisfiability**

**Input:** boolean formula with $n$ variables, with $\vee, \wedge, \neg$ operators.

**Output:** whether the boolean formula is satisfiable

- Example: $\neg((\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3))$ is not satisfiable
- Trivial algorithm: enumerate all possible assignments, and check if each assignment satisfies the formula

# Formula Satisfiability

**Formula Satisfiability**

**Input:** boolean formula with $n$ variables, with $\vee, \wedge, \neg$ operators.

**Output:** whether the boolean formula is satisfiable

- Example: $\neg((\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3))$ is not satisfiable
- Trivial algorithm: enumerate all possible assignments, and check if each assignment satisfies the formula
- Formula Satisfiablity is NP-hard

# Outline

3.36pt

# Decision Problem Vs Optimization Problem

**Def.** A problem $X$ is called a decision problem if the output is either 0 or 1 (yes/no).

# Decision Problem Vs Optimization Problem

**Def.** A problem $X$ is called a decision problem if the output is either 0 or 1 (yes/no).

- When we define the P and NP, we only consider decision problems.

# Decision Problem Vs Optimization Problem

**Def.** A problem $X$ is called a decision problem if the output is either 0 or 1 (yes/no).

- When we define the P and NP, we only consider decision problems.

**Fact** For each optimization problem $X$, there is a decision version $X'$ of the problem. If we have a polynomial time algorithm for the decision version $X'$, we can solve the original problem $X$ in polynomial time.

# Optimization to Decision

**Shortest Path**

**Input:** graph $G = (V, E)$, weight $w, s, t$ and a bound $L$

**Output:** whether there is a path from $s$ to $t$ of length at most $L$

# Optimization to Decision

**Shortest Path**

**Input:** graph $G = (V, E)$, weight $w, s, t$ and a bound $L$

**Output:** whether there is a path from $s$ to $t$ of length at most $L$

**Maximum Independent Set**

**Input:** a graph $G$ and a bound $k$

**Output:** whether there is an independent set of size at least $k$

# Encoding

The input of a problem will be encoded as a binary string.

# Encoding

The input of a problem will be encoded as a binary string.

Example: Sorting problem

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)
- String:

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)
- String: 111101

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)
- String: 11110111110001

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)
- String: 111101111110001111100001100000l

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)
- String: 111101111100011111000011000001 1100001101

# Encoding

The input of a problem will be encoded as a binary string.

**Example: Sorting problem**

- Input: (3, 6, 100, 9, 60)
- Binary: (11, 110, 1100100, 1001, 111100)
- String: 111101111100011111000011000001 110000110111111111000001

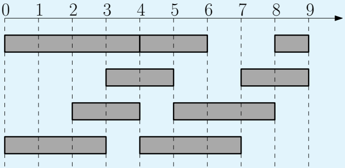
# Encoding

The input of an problem will be encoded as a binary string.

# Encoding

The input of an problem will be encoded as a binary string.

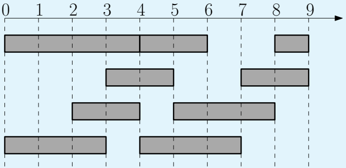
## Example: Interval Scheduling Problem



# Encoding

The input of an problem will be encoded as a binary string.

Example: Interval Scheduling Problem

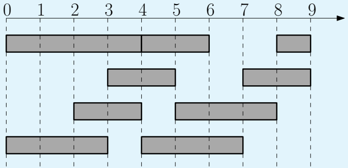



- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$

# Encoding

The input of an problem will be encoded as a binary string.

**Example: Interval Scheduling Problem**




- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$
- Encode the sequence into a binary string as before

# Encoding

**Def.** The size of an input is the length of the encoded string $s$ for the input, denoted as $|s|$.

**Q:** Does it matter how we encode the input instances?

# Encoding

**Def.** The size of an input is the length of the encoded string $s$ for the input, denoted as $|s|$.

**Q:** Does it matter how we encode the input instances?

**A:** No! As long as we are using a "natural" encoding. We only care whether the running time is polynomial or not

# Define Problem as a Set

**Def.** A decision problem $X$ is the set of strings on which the output is yes. i.e, $s \in X$ if and only if the correct output for the input $s$ is 1 (yes).

# Define Problem as a Set

**Def.** A decision problem $X$ is the set of strings on which the output is yes. i.e, $s \in X$ if and only if the correct output for the input $s$ is 1 (yes).

**Def.** An algorithm $A$ solves a problem $X$ if, $A(s) = 1$ if and only if $s \in X$.

# Define Problem as a Set

**Def.** A decision problem $X$ is the set of strings on which the output is yes. i.e, $s \in X$ if and only if the correct output for the input $s$ is 1 (yes).

**Def.** An algorithm $A$ solves a problem $X$ if, $A(s) = 1$ if and only if $s \in X$.

**Def.** $A$ has a polynomial running time if there is a polynomial function $p(\cdot)$ so that for every string $s$, the algorithm $A$ terminates on $s$ in at most $p(|s|)$ steps.

# Complexity Class P

**Def.** The complexity class P is the set of decision problems $X$ that can be solved in polynomial time.

# Complexity Class P

**Def.** The complexity class P is the set of decision problems $X$ that can be solved in polynomial time.

- The decision versions of interval scheduling, shortest path and minimum spanning tree all in P.

# Certifier for Hamiltonian Cycle (HC)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for HC

# Certifier for Hamiltonian Cycle (HC)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for HC
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

# Certifier for Hamiltonian Cycle (HC)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for HC
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

**Q:** Given a graph $G = (V, E)$ with a HC, how can Alice convince Bob that $G$ contains a Hamiltonian cycle?

# Certifier for Hamiltonian Cycle (HC)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for HC
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

**Q:** Given a graph $G = (V, E)$ with a HC, how can Alice convince Bob that $G$ contains a Hamiltonian cycle?

**A:** Alice gives a Hamiltonian cycle to Bob, and Bob checks if it is really a Hamiltonian cycle of $G$

# Certifier for Hamiltonian Cycle (HC)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for HC
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

**Q:** Given a graph $G = (V, E)$ with a HC, how can Alice convince Bob that $G$ contains a Hamiltonian cycle?

**A:** Alice gives a Hamiltonian cycle to Bob, and Bob checks if it is really a Hamiltonian cycle of $G$

**Def.** The message Alice sends to Bob is called a certificate, and the algorithm Bob runs is called a certifier.

# Certifier for Independent Set (Ind-Set)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for Ind-Set
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

# Certifier for Independent Set (Ind-Set)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for Ind-Set
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

**Q:** Given graph $G = (V, E)$ and integer $k$, such that there is an independent set of size $k$ in $G$, how can Alice convince Bob that there is such a set?

# Certifier for Independent Set (Ind-Set)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for Ind-Set
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

**Q:** Given graph $G = (V, E)$ and integer $k$, such that there is an independent set of size $k$ in $G$, how can Alice convince Bob that there is such a set?

**A:** Alice gives a set of size $k$ to Bob and Bob checks if it is really a independent set in $G$.

# Certifier for Independent Set (Ind-Set)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for Ind-Set
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

**Q:** Given graph $G = (V, E)$ and integer $k$, such that there is an independent set of size $k$ in $G$, how can Alice convince Bob that there is such a set?

**A:** Alice gives a set of size $k$ to Bob and Bob checks if it is really a independent set in $G$.

- Certificate: a set of size $k$

# Certifier for Independent Set (Ind-Set)

- Alice has a supercomputer, fast enough to run the $2^{O(n)}$ time algorithm for Ind-Set
- Bob has a slow computer, which can only run an $O(n^3)$-time algorithm

**Q:** Given graph $G = (V, E)$ and integer $k$, such that there is an independent set of size $k$ in $G$, how can Alice convince Bob that there is such a set?

**A:** Alice gives a set of size $k$ to Bob and Bob checks if it is really a independent set in $G$.

- Certificate: a set of size $k$
- Certifier: check if the given set is really an independent set

# Graph Isomorphism

**Graph Isomorphism**

**Input:** two graphs $G_1$ and $G_2$,

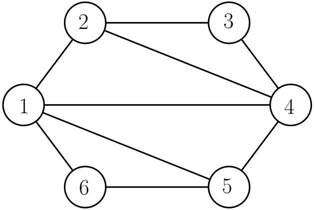
**Output:** whether two graphs are isomorphic to each other

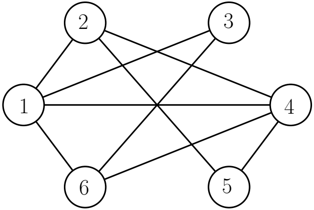
# Graph Isomorphism

## Graph Isomorphism

**Input:** two graphs $G_1$ and $G_2$,

**Output:** whether two graphs are isomorphic to each other



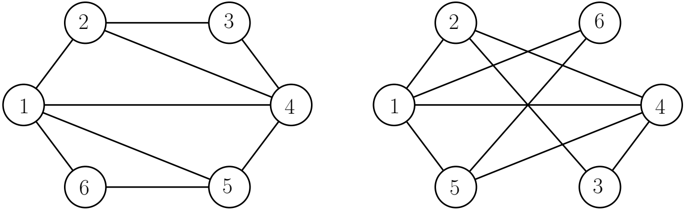



# Graph Isomorphism

**Graph Isomorphism**

**Input:** two graphs $G_1$ and $G_2$,

**Output:** whether two graphs are isomorphic to each other

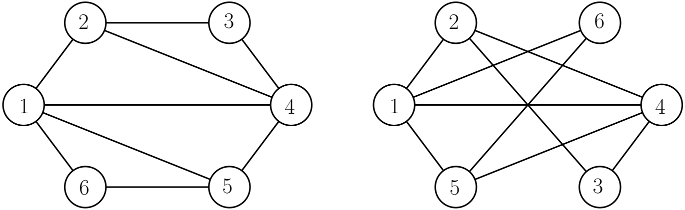


# Graph Isomorphism

**Graph Isomorphism**

**Input:** two graphs $G_1$ and $G_2$,

**Output:** whether two graphs are isomorphic to each other



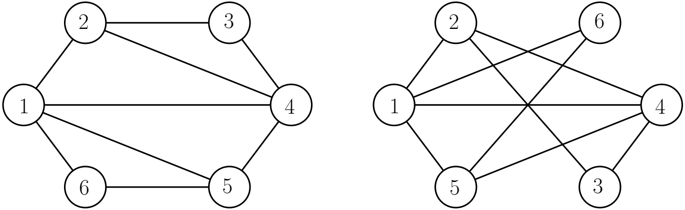

- What is the certificate?

# Graph Isomorphism

**Graph Isomorphism**

**Input:** two graphs $G_1$ and $G_2$,

**Output:** whether two graphs are isomorphic to each other




- What is the certificate?
- What is the certifier?

## The Complexity Class NP

**Def.** $B$ is an efficient certifier for a problem $X$ if

- $B$ is a polynomial-time algorithm that takes two input strings $s$ and $t$
- there is a polynomial function $p$ such that, $s \in X$ if and only if there is string $t$ such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

The string $t$ such that $B(s,t) = 1$ is called a certificate.

# The Complexity Class NP

**Def.** $B$ is an efficient certifier for a problem $X$ if

- $B$ is a polynomial-time algorithm that takes two input strings $s$ and $t$
- there is a polynomial function $p$ such that, $s \in X$ if and only if there is string $t$ such that $|t| \leq p(|s|)$ and $B(s, t) = 1$.

The string $t$ such that $B(s, t) = 1$ is called a certificate.

**Def.** The complexity class NP is the set of all problems for which there exists an efficient certifier.

# Hamiltonian Cycle $\in$ NP

- Input: Graph $G$

# Hamiltonian Cycle $\in$ NP

- Input: Graph $G$
- Certificate: a sequence $S$ of edges in $G$

# Hamiltonian Cycle $\in$ NP

- Input: Graph $G$
- Certificate: a sequence $S$ of edges in $G$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G)|)$ for some polynomial function $p$

# Hamiltonian Cycle $\in$ NP

- Input: Graph $G$
- Certificate: a sequence $S$ of edges in $G$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G)|)$ for some polynomial function $p$
- Certifier $B$: $B(G, S) = 1$ if and only if $S$ is an HC in $G$

# Hamiltonian Cycle $\in$ NP

- Input: Graph $G$
- Certificate: a sequence $S$ of edges in $G$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G)|)$ for some polynomial function $p$
- Certifier $B$: $B(G, S) = 1$ if and only if $S$ is an HC in $G$
- Clearly, $B$ runs in polynomial time

# Hamiltonian Cycle $\in$ NP

- Input: Graph $G$
- Certificate: a sequence $S$ of edges in $G$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G)|)$ for some polynomial function $p$
- Certifier $B$: $B(G, S) = 1$ if and only if $S$ is an HC in $G$
- Clearly, $B$ runs in polynomial time
- $G \in \text{HC} \qquad \Longleftrightarrow \qquad \exists S,\ B(G, S) = 1$

# Graph Isomorphism $\in$ NP

- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on $V$

# Graph Isomorphism $\in$ NP

- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on $V$
- Certificate: a 1-1 function $f : V \to V$

# Graph Isomorphism $\in$ NP

- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on $V$
- Certificate: a 1-1 function $f : V \to V$
- $|\text{encoding}(f)| \leq p(|\text{encoding}(G_1, G_2)|)$ for some polynomial function $p$

# Graph Isomorphism $\in$ NP

- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on $V$
- Certificate: a 1-1 function $f : V \to V$
- $|\text{encoding}(f)| \leq p(|\text{encoding}(G_1, G_2)|)$ for some polynomial function $p$
- Certifier $B$: $B((G_1, G_2), f) = 1$ if and only if for every $u, v \in V$, we have $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.

# Graph Isomorphism $\in$ NP

- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on $V$
- Certificate: a 1-1 function $f : V \to V$
- $|\text{encoding}(f)| \leq p(|\text{encoding}(G_1, G_2)|)$ for some polynomial function $p$
- Certifier $B$: $B((G_1, G_2), f) = 1$ if and only if for every $u, v \in V$, we have $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.
- Clearly, $B$ runs in polynomial time

# Graph Isomorphism $\in$ NP

- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on $V$
- Certificate: a 1-1 function $f : V \to V$
- $|\text{encoding}(f)| \leq p(|\text{encoding}(G_1, G_2)|)$ for some polynomial function $p$
- Certifier $B$: $B((G_1, G_2), f) = 1$ if and only if for every $u, v \in V$, we have $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.
- Clearly, $B$ runs in polynomial time

- $(G_1, G_2) \in \text{GI} \qquad \Longleftrightarrow \qquad \exists f,\ B((G_1, G_2), f) = 1$

# Maximum Independent Set $\in$ NP

- Input: graph $G = (V, E)$ and integer $k$

# Maximum Independent Set $\in$ NP

- Input: graph $G = (V, E)$ and integer $k$
- Certificate: a set $S \subseteq V$ of size $k$

# Maximum Independent Set $\in$ NP

- Input: graph $G = (V, E)$ and integer $k$
- Certificate: a set $S \subseteq V$ of size $k$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G, k)|)$ for some polynomial function $p$

# Maximum Independent Set $\in$ NP

- Input: graph $G = (V, E)$ and integer $k$
- Certificate: a set $S \subseteq V$ of size $k$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G, k)|)$ for some polynomial function $p$
- Certifier $B$: $B((G, k), S) = 1$ if and only if $S$ is an independent set in $G$

# Maximum Independent Set $\in$ NP

- Input: graph $G = (V, E)$ and integer $k$
- Certificate: a set $S \subseteq V$ of size $k$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G, k)|)$ for some polynomial function $p$
- Certifier $B$: $B((G, k), S) = 1$ if and only if $S$ is an independent set in $G$
- Clearly, $B$ runs in polynomial time

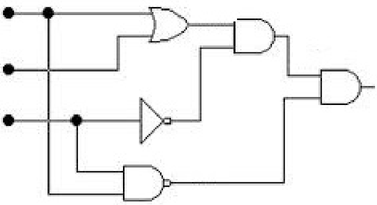
# Maximum Independent Set $\in$ NP

- Input: graph $G = (V, E)$ and integer $k$
- Certificate: a set $S \subseteq V$ of size $k$
- $|\text{encoding}(S)| \leq p(|\text{encoding}(G, k)|)$ for some polynomial function $p$
- Certifier $B$: $B((G, k), S) = 1$ if and only if $S$ is an independent set in $G$
- Clearly, $B$ runs in polynomial time
- $(G, k) \in \text{MIS} \quad \Longleftrightarrow \quad \exists S,\ B((G, k), S) = 1$

## Circuit Satisfiablity (Circuit-Sat) Problem

**Input:** a circuit with and/or/not gates

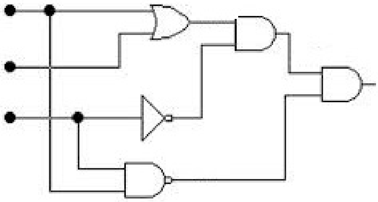
**Output:** whether there is an assignment such that the output is 1?



## Circuit Satisfiablity (Circuit-Sat) Problem

**Input:** a circuit with and/or/not gates

**Output:** whether there is an assignment such that the output is $1$?



- Is Circuit-Sat $\in$ NP?

$\overline{\text{HC}}$

**Input:** graph $G = (V, E)$

**Output:** whether $G$ does not contain a Hamiltonian cycle

### $\overline{\text{HC}}$

**Input:** graph $G = (V, E)$

**Output:** whether $G$ does not contain a Hamiltonian cycle

- Is $\overline{\text{HC}} \in \text{NP}$?

## $\overline{\text{HC}}$

**Input:** graph $G = (V, E)$

**Output:** whether $G$ does not contain a Hamiltonian cycle

- Is $\overline{\text{HC}} \in \text{NP}$?
- Can Alice convince Bob that $G$ is a yes-instance (i.e, $G$ does not contain a HC), if this is true.

## $\overline{\text{HC}}$

**Input:** graph $G = (V, E)$

**Output:** whether $G$ does not contain a Hamiltonian cycle

- Is $\overline{\text{HC}} \in \text{NP}$?
- Can Alice convince Bob that $G$ is a yes-instance (i.e, $G$ does not contain a HC), if this is true.
- Unlikely

$\overline{\text{HC}}$

**Input:** graph $G = (V, E)$

**Output:** whether $G$ does not contain a Hamiltonian cycle

- Is $\overline{\text{HC}} \in$ NP?
- Can Alice convince Bob that $G$ is a yes-instance (i.e, $G$ does not contain a HC), if this is true.
- Unlikely
- Alice can only convince Bob that $G$ is a no-instance

## $\overline{\text{HC}}$

**Input:** graph $G = (V, E)$

**Output:** whether $G$ does not contain a Hamiltonian cycle

- Is $\overline{\text{HC}} \in \text{NP}$?
- Can Alice convince Bob that $G$ is a yes-instance (i.e, $G$ does not contain a HC), if this is true.
- Unlikely
- Alice can only convince Bob that $G$ is a no-instance
- $\overline{\text{HC}} \in \text{Co-NP}$

# The Complexity Class Co-NP

**Def.** For a problem $X$, the problem $\overline{X}$ is the problem such that $s \in \overline{X}$ if and only if $s \notin X$.

**Def.** Co-NP is the set of decision problems $X$ such that $\overline{X} \in$ NP.

**Def.** A tautology is a boolean formula that always evaluates to 1.

### Tautology Problem

**Input:** a boolean formula

**Output:** whether the formula is a tautology

- e.g. $(\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3)$ is a tautology

**Def.** A tautology is a boolean formula that always evaluates to 1.

### Tautology Problem

**Input:** a boolean formula

**Output:** whether the formula is a tautology

- e.g. $(\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3)$ is a tautology
- Bob can certify that a formula is not a tautology

**Def.** A tautology is a boolean formula that always evaluates to 1.

**Tautology Problem**

**Input:** a boolean formula

**Output:** whether the formula is a tautology

- e.g. $(\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3)$ is a tautology
- Bob can certify that a formula is not a tautology
- Thus Tautology $\in$ Co-NP

**Def.** A tautology is a boolean formula that always evaluates to 1.

### Tautology Problem

**Input:** a boolean formula

**Output:** whether the formula is a tautology

- e.g. $(\neg x_1 \land x_2) \lor (\neg x_1 \land \neg x_3) \lor x_1 \lor (\neg x_2 \land x_3)$ is a tautology
- Bob can certify that a formula is not a tautology
- Thus Tautology $\in$ Co-NP
- Indeed, Tautology $= \overline{\text{Formula-Unsat}}$

# Prime

**Prime**

**Input:** an integer $q \geq 2$

**Output:** whether $q$ is a prime

# Prime

**Prime**

**Input:** an integer $q \geq 2$

**Output:** whether $q$ is a prime

- It is easy to certify that $q$ is not a prime

# Prime

**Prime**

**Input:** an integer $q \geq 2$

**Output:** whether $q$ is a prime

- It is easy to certify that $q$ is not a prime
- Prime $\in$ Co-NP

# Prime

**Prime**

**Input:** an integer $q \geq 2$

**Output:** whether $q$ is a prime

- It is easy to certify that $q$ is not a prime
- Prime $\in$ Co-NP
- [Pratt 1970] Prime $\in$ NP

# Prime

**Prime**

**Input:** an integer $q \geq 2$

**Output:** whether $q$ is a prime

- It is easy to certify that $q$ is not a prime
- Prime $\in$ Co-NP
- [Pratt 1970] Prime $\in$ NP
- P $\subseteq$ NP $\cap$ Co-NP (see soon)

# Prime

**Prime**

**Input:** an integer $q \geq 2$

**Output:** whether $q$ is a prime

- It is easy to certify that $q$ is not a prime
- Prime $\in$ Co-NP
- [Pratt 1970] Prime $\in$ NP
- P $\subseteq$ NP $\cap$ Co-NP (see soon)
- If a natural problem $X$ is in NP $\cap$ Co-NP, then it is likely that $X \in P$

# Prime

**Prime**

**Input:** an integer $q \geq 2$

**Output:** whether $q$ is a prime

- It is easy to certify that $q$ is not a prime
- Prime $\in$ Co-NP
- [Pratt 1970] Prime $\in$ NP
- P $\subseteq$ NP $\cap$ Co-NP (see soon)
- If a natural problem $X$ is in NP $\cap$ Co-NP, then it is likely that $X \in P$
- [AKS 2002] Prime $\in$ P

# $P \subseteq NP$

# P $\subseteq$ NP

- Let $X \in$ P and $s \in X$

**Q:** How can Alice convince Bob that $s$ is a yes instance?

# P $\subseteq$ NP

- Let $X \in$ P and $s \in X$

**Q:** How can Alice convince Bob that $s$ is a yes instance?

**A:** Since $X \in$ P, Bob can check whether $s \in X$ by himself, without Alice's help.

# P $\subseteq$ NP

- Let $X \in$ P and $s \in X$

**Q:** How can Alice convince Bob that $s$ is a yes instance?

**A:** Since $X \in$ P, Bob can check whether $s \in X$ by himself, without Alice's help.

- The certificate is an empty string

# P $\subseteq$ NP

- Let $X \in$ P and $s \in X$

**Q:** How can Alice convince Bob that $s$ is a yes instance?

**A:** Since $X \in$ P, Bob can check whether $s \in X$ by himself, without Alice's help.

- The certificate is an empty string
- Thus, $X \in$ NP and P $\subseteq$ NP

# P $\subseteq$ NP

- Let $X \in$ P and $s \in X$

**Q:** How can Alice convince Bob that $s$ is a yes instance?

**A:** Since $X \in$ P, Bob can check whether $s \in X$ by himself, without Alice's help.

- The certificate is an empty string
- Thus, $X \in$ NP and P $\subseteq$ NP
- Similarly, P $\subseteq$ Co-NP, thus P $\subseteq$ NP $\cap$ Co-NP

# Is P = NP?

# Is P = NP?

- A famous, big, and fundamental open problem in computer science

# Is P = NP?

- A famous, big, and fundamental open problem in computer science
- Little progress has been made

# Is P = NP?

- A famous, big, and fundamental open problem in computer science
- Little progress has been made
- General belief is $P \neq NP$

# Is P = NP?

- A famous, big, and fundamental open problem in computer science
- Little progress has been made
- General belief is P $\neq$ NP
- It would be too amazing if P = NP: if one can check a solution efficiently, then one can find a solution efficiently

# Is P = NP?

- A famous, big, and fundamental open problem in computer science
- Little progress has been made
- General belief is P $\neq$ NP
- It would be too amazing if P = NP: if one can check a solution efficiently, then one can find a solution efficiently
- Complexity assumption: P $\neq$ NP

# Is P = NP?

- A famous, big, and fundamental open problem in computer science
- Little progress has been made
- General belief is P $\neq$ NP
- It would be too amazing if P $=$ NP: if one can check a solution efficiently, then one can find a solution efficiently
- Complexity assumption: P $\neq$ NP
- We said it is unlikely that Hamiltonian Cycle can be solved in polynomial time:

# Is P = NP?

- A famous, big, and fundamental open problem in computer science
- Little progress has been made
- General belief is P $\neq$ NP
- It would be too amazing if P $=$ NP: if one can check a solution efficiently, then one can find a solution efficiently

- Complexity assumption: P $\neq$ NP
- We said it is unlikely that Hamiltonian Cycle can be solved in polynomial time:
  - if P $\neq$ NP, then HC $\notin$ P

# Is P $=$ NP?

- A famous, big, and fundamental open problem in computer science
- Little progress has been made
- General belief is P $\neq$ NP
- It would be too amazing if P $=$ NP: if one can check a solution efficiently, then one can find a solution efficiently
- Complexity assumption: P $\neq$ NP
- We said it is unlikely that Hamiltonian Cycle can be solved in polynomial time:
  - if P $\neq$ NP, then HC $\notin$ P
  - HC $\notin$ P, unless P $=$ NP

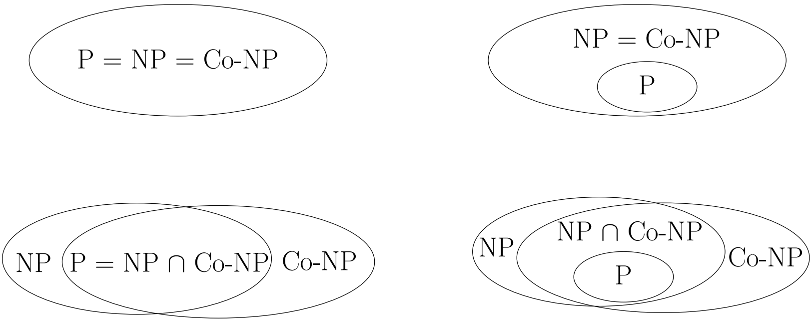
# Is NP = Co-NP?

- Again, a big open problem

# Is NP $=$ Co-NP?

- Again, a big open problem
- General belief: NP $\neq$ Co-NP.

Notice that $X \in \text{NP} \iff \overline{X} \in \text{Co-NP}$ and $\text{P} \subseteq \text{NP} \cap \text{Co-NP}$




- General belief: we are in the 4th scenario

# Outline

3.36pt

# Polynomial-Time Reducations

**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

# Polynomial-Time Reducations

**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

To prove positive results:

Suppose $Y \leq_P X$. If $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.

# Polynomial-Time Reducations

**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

To prove positive results:

Suppose $Y \leq_P X$. If $X$ can be solved in polynomial time, then $Y$ can be solved in polynomial time.

To prove negative results:

Suppose $Y \leq_P X$. If $Y$ cannot be solved in polynomial time, then $X$ cannot be solved in polynomial time.

# Polynomial-Time Reduction: Example

**Hamiltonian-Path (HP) problem**

**Input:** $G = (V, E)$ and $s, t \in V$

**Output:** whether there is a Hamiltonian path from $s$ to $t$ in $G$

# Polynomial-Time Reduction: Example

**Hamiltonian-Path (HP) problem**

**Input:** $G = (V, E)$ and $s, t \in V$

**Output:** whether there is a Hamiltonian path from $s$ to $t$ in $G$

**Lemma** HP $\leq_{\mathsf{P}}$ HC.

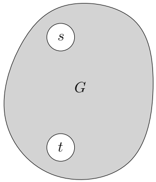
# Polynomial-Time Reduction: Example

**Hamiltonian-Path (HP) problem**

**Input:** $G = (V, E)$ and $s, t \in V$

**Output:** whether there is a Hamiltonian path from $s$ to $t$ in $G$

**Lemma** HP $\leq_{\mathsf{P}}$ HC.



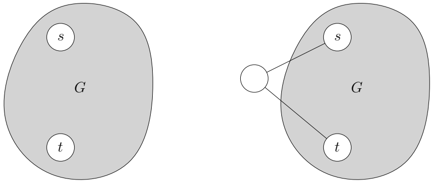
# Polynomial-Time Reduction: Example

**Hamiltonian-Path (HP) problem**

**Input:** $G = (V, E)$ and $s, t \in V$

**Output:** whether there is a Hamiltonian path from $s$ to $t$ in $G$

**Lemma** HP $\leq_P$ HC.



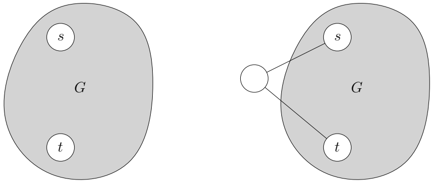
# Polynomial-Time Reduction: Example

**Hamiltonian-Path (HP) problem**

**Input:** $G = (V, E)$ and $s, t \in V$

**Output:** whether there is a Hamiltonian path from $s$ to $t$ in $G$

**Lemma** HP $\leq_{\text{P}}$ HC.




**Obs.** $G$ has a HP from $s$ to $t$ if and only if graph on right side has a HC.

# NP-Completeness

**Def.** A problem $X$ is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_\text{P} X$ for every $Y \in \text{NP}$.

# NP-Completeness

**Def.** A problem $X$ is called NP-hard if

2. $Y \leq_{\mathsf{P}} X$ for every $Y \in \mathsf{NP}$.

# NP-Completeness

**Def.** A problem $X$ is called NP-complete if

1. $X \in \mathsf{NP}$, and
2. $Y \le_{\mathsf{P}} X$ for every $Y \in \mathsf{NP}$.

# NP-Completeness

**Def.** A problem $X$ is called NP-complete if

1. $X \in \mathrm{NP}$, and
2. $Y \leq_{\mathrm{P}} X$ for every $Y \in \mathrm{NP}$.

**Theorem** If $X$ is NP-complete and $X \in \mathrm{P}$, then $\mathrm{P} = \mathrm{NP}$.

# NP-Completeness

**Def.** A problem $X$ is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_\text{P} X$ for every $Y \in \text{NP}$.

**Theorem** If $X$ is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-complete problems are the hardest problems in NP

# NP-Completeness

**Def.** A problem $X$ is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_\text{P} X$ for every $Y \in \text{NP}$.

**Theorem** If $X$ is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-complete problems are the hardest problems in NP
- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)

# NP-Completeness

**Def.** A problem $X$ is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

**Theorem** If $X$ is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-complete problems are the hardest problems in NP
- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)
- To prove $\text{P} = \text{NP}$ (if you believe it), you only need to give an efficient algorithm for any NP-complete problem

# NP-Completeness

**Def.** A problem $X$ is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

**Theorem** If $X$ is NP-complete and $X \in \text{P}$, then $\text{P} = \text{NP}$.

- NP-complete problems are the hardest problems in NP
- NP-hard problems are at least as hard as NP-complete problems (a NP-hard problem is not required to be in NP)
- To prove $\text{P} = \text{NP}$ (if you believe it), you only need to give an efficient algorithm for any NP-complete problem
- If you believe $\text{P} \neq \text{NP}$, and proved that a problem $X$ is NP-complete (or NP-hard), stop trying to design efficient algorithms for $X$

# Outline

3.36pt

**Def.** A problem $X$ is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

**Def.** A problem $X$ is called NP-complete if

1. $X \in$ NP, and
2. $Y \leq_{\mathsf{P}} X$ for every $Y \in$ NP.

- How can we find a problem $X \in$ NP such that every problem $Y \in$ NP is polynomial time reducible to $X$? Are we asking for too much?

**Def.** A problem $X$ is called NP-complete if

1. $X \in \text{NP}$, and
2. $Y \leq_{\text{P}} X$ for every $Y \in \text{NP}$.

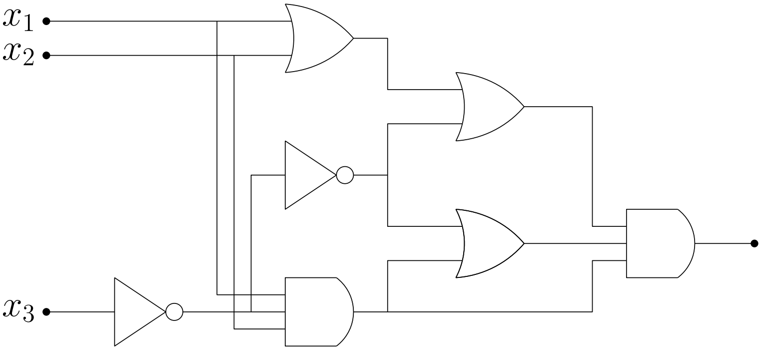
- How can we find a problem $X \in \text{NP}$ such that every problem $Y \in \text{NP}$ is polynomial time reducible to $X$? Are we asking for too much?
- No! There is indeed a large family of natural NP-complete problems

# The First NP-Complete Problem: Circuit-Sat

**Circuit Satisfiability (Circuit-Sat)**

**Input:** a circuit

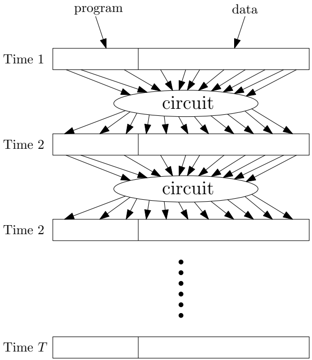
**Output:** whether the circuit is satisfiable



# Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

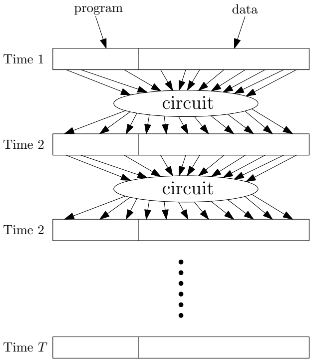
**Fact** Any algorithm that takes $n$ bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.



# Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

**Fact** Any algorithm that takes $n$ bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.

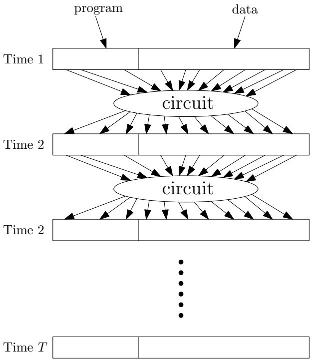



- Then, we can show that any problem $Y \in$ NP can be reduced to Circuit-Sat.

# Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

**Fact** Any algorithm that takes $n$ bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.




- Then, we can show that any problem $Y \in$ NP can be reduced to Circuit-Sat.
- We prove HC $\leq_P$ Circuit-Sat as an example.

# HC $\leq_P$ Circuit-Sat

check-HC$(G, S)$

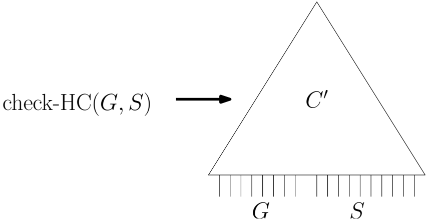
- Let check-HC$(G, S)$ be the certifier for the Hamiltonian cycle problem: check-HC$(G, S)$ returns 1 if $S$ is a Hamiltonian cycle is $G$ and $0$ otherwise.

# HC $\leq_P$ Circuit-Sat

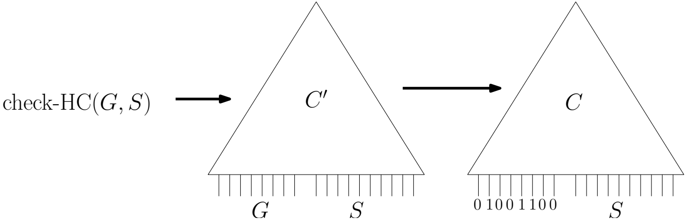
check-HC$(G, S)$

- Let check-HC$(G, S)$ be the certifier for the Hamiltonian cycle problem: check-HC$(G, S)$ returns 1 if $S$ is a Hamiltonian cycle is $G$ and $0$ otherwise.
- $G$ is a yes-instance if and only if there is an $S$ such that check-HC$(G, S)$ returns 1

# HC $\leq_P$ Circuit-Sat

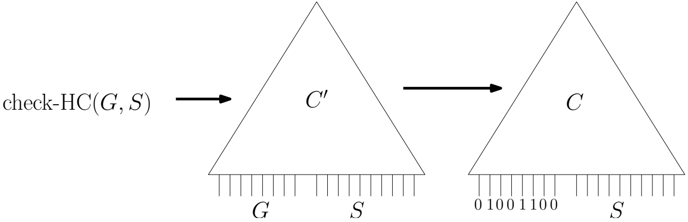



- Let check-HC$(G, S)$ be the certifier for the Hamiltonian cycle problem: check-HC$(G, S)$ returns 1 if $S$ is a Hamiltonian cycle is $G$ and $0$ otherwise.
- $G$ is a yes-instance if and only if there is an $S$ such that check-HC$(G, S)$ returns 1
- Construct a circuit $C'$ for the algorithm check-HC




- Let check-HC$(G, S)$ be the certifier for the Hamiltonian cycle problem: check-HC$(G, S)$ returns 1 if $S$ is a Hamiltonian cycle is $G$ and $0$ otherwise.
- $G$ is a yes-instance if and only if there is an $S$ such that check-HC$(G, S)$ returns 1
- Construct a circuit $C'$ for the algorithm check-HC
- hard-wire the instance $G$ to the circuit $C'$ to obtain the circuit $C$

# HC $\leq_P$ Circuit-Sat




- Let check-HC$(G, S)$ be the certifier for the Hamiltonian cycle problem: check-HC$(G, S)$ returns 1 if $S$ is a Hamiltonian cycle is $G$ and $0$ otherwise.
- $G$ is a yes-instance if and only if there is an $S$ such that check-HC$(G, S)$ returns 1
- Construct a circuit $C'$ for the algorithm check-HC
- hard-wire the instance $G$ to the circuit $C'$ to obtain the circuit $C$
- $G$ is a yes-instance if and only if $C$ is satisfiable

# $Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

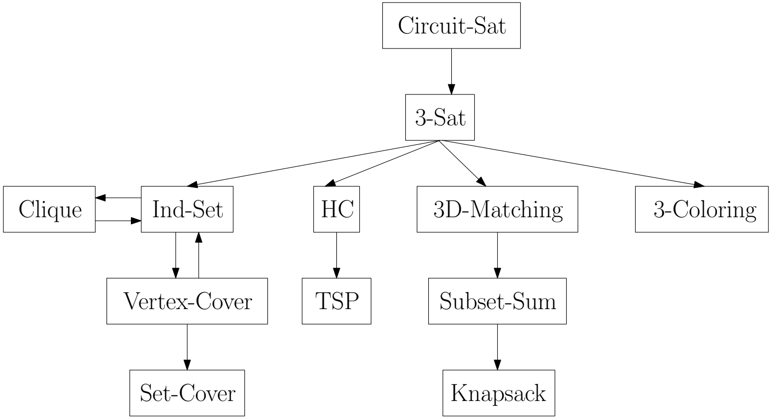
- Let check-Y$(s, t)$ be the certifier for problem $Y$:
  check-Y$(s, t)$ returns 1 if $t$ is a valid certificate for $s$.
- $s$ is a yes-instance if and only if there is a $t$ such that check-Y$(s, t)$ returns 1
- Construct a circuit $C'$ for the algorithm check-Y
- hard-wire the instance $s$ to the circuit $C'$ to obtain the circuit $C$
- $s$ is a yes-instance if and only if $C$ is satisfiable $\square$

# $Y \le_P$ Circuit-Sat, For Every $Y \in$NP

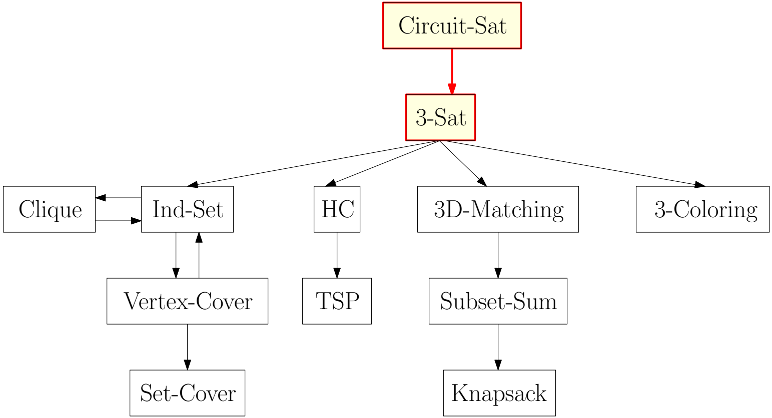
- Let check-Y$(s,t)$ be the certifier for problem $Y$:
  check-Y$(s,t)$ returns 1 if $t$ is a valid certificate for $s$.
- $s$ is a yes-instance if and only if there is a $t$ such that check-Y$(s,t)$ returns 1
- Construct a circuit $C'$ for the algorithm check-Y
- hard-wire the instance $s$ to the circuit $C'$ to obtain the circuit $C$
- $s$ is a yes-instance if and only if $C$ is satisfiable $\square$

**Theorem** Circuit-Sat is NP-complete.

# Reductions of NP-Complete Problems



# Reductions of NP-Complete Problems



# 3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

# 3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$

# 3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: $x_i$ or $\neg x_i$

# 3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: $x_i$ or $\neg x_i$
- Clause: disjunction ("or") of at most $3$ literals: $x_3 \vee \neg x_4$, $x_1 \vee x_8 \vee \neg x_9$, $\neg x_2 \vee \neg x_5 \vee x_7$

3-CNF (conjunctive normal form) is a special case of formula:

- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: $x_i$ or $\neg x_i$
- Clause: disjunction ("or") of at most $3$ literals: $x_3 \vee \neg x_4$, $x_1 \vee x_8 \vee \neg x_9, \quad \neg x_2 \vee \neg x_5 \vee x_7$
- 3-CNF formula: conjunction ("and") of clauses: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

# 3-Sat

**3-Sat**

**Input:** a 3-CNF formula

**Output:** whether the 3-CNF is satisfiable

# 3-Sat

**3-Sat**

**Input:** a 3-CNF formula

**Output:** whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses

# 3-Sat

**3-Sat**

**Input:** a 3-CNF formula

**Output:** whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal

# 3-Sat

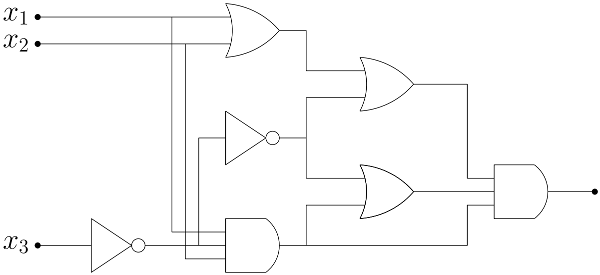
**3-Sat**

**Input:** a 3-CNF formula

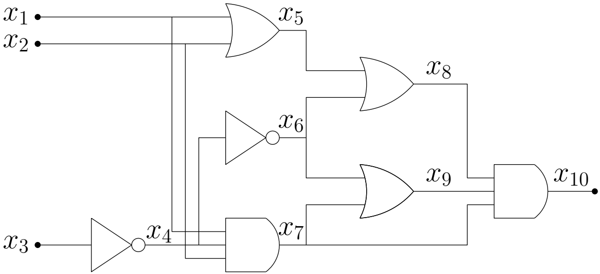
**Output:** whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal
- Assignment $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$ satisfies $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

# Circuit-Sat $\leq_P$ 3-Sat

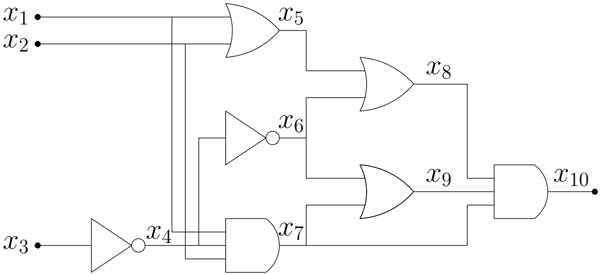


# Circuit-Sat $\leq_P$ 3-Sat




- Associate every wire with a new variable

# Circuit-Sat $\leq_P$ 3-Sat




- Associate every wire with a new variable
- The circuit is equivalent to the following formula:

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

# Circuit-Sat $\leq_P$ 3-Sat

$$
\begin{aligned}
&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4) \\
&\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6) \\
&\wedge (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}
\end{aligned}
$$

Convert each clause to a 3-CNF

# Circuit-Sat $\leq_P$ 3-Sat

$$
\begin{aligned}
&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4) \\
&\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6) \\
&\wedge (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}
\end{aligned}
$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\ (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\ (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\ (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\ (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\ (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\ (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$
\begin{aligned}
&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4) \\
&\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6) \\
&\wedge (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}
\end{aligned}
$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$
\begin{aligned}
&(x_1 \vee x_2 \vee \neg x_5) \quad \wedge \\
&(x_1 \vee \neg x_2 \vee x_5) \quad \wedge \\
&(\neg x_1 \vee x_2 \vee x_5) \quad \wedge
\end{aligned}
$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\ (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\ (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee x_2 \vee x_5) \quad \wedge$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\ (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\ (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee \neg x_2 \vee x_5)$$

| $x_1$ | $x_2$ | $x_5$ | $x_5 \leftrightarrow x_1 \vee x_2$ |
|---|---|---|---|
| 0 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 |
| 0 | 1 | 0 | 0 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 1 |

# Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF

# Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable

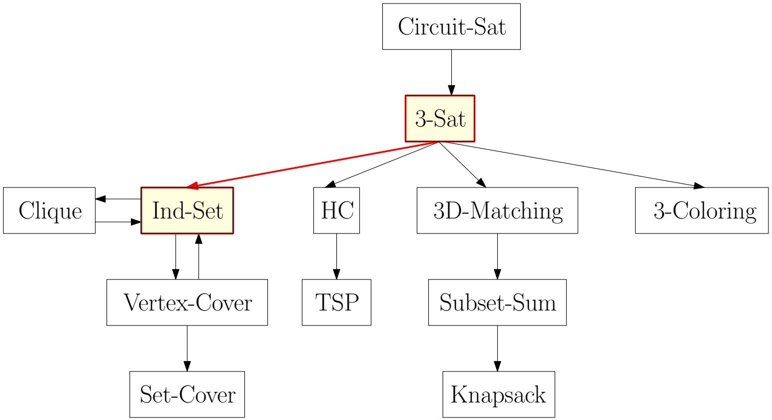
# Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit

# Circuit-Sat $\leq_P$ 3-Sat

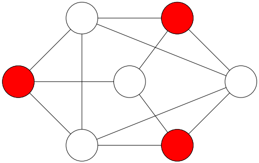
- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit
- Thus, Circuit-Sat $\leq_P$ 3-Sat

# Reductions of NP-Complete Problems



# Recall: Independent Set Problem

**Def.** An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in $I$ are adjacent in $G$.



**Independent Set (Ind-Set) Problem**

**Input:** $G = (V, E), k$

**Output:** whether there is an independent set of size $k$ in $G$

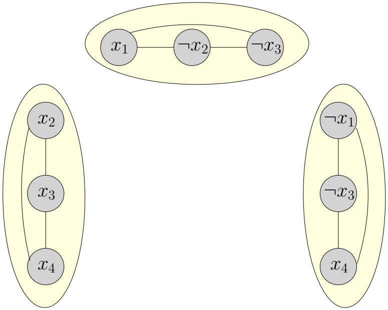
# 3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

# 3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

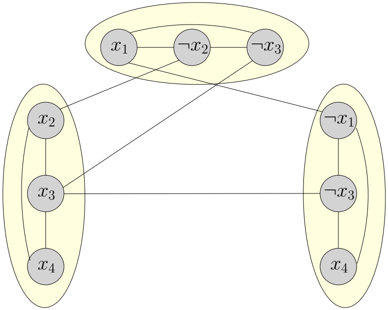
- A clause $\Rightarrow$ a group of $3$ vertices, one for each literal
- An edge between every pair of vertices in same group



# 3-Sat $\leq_P$ Ind-Set

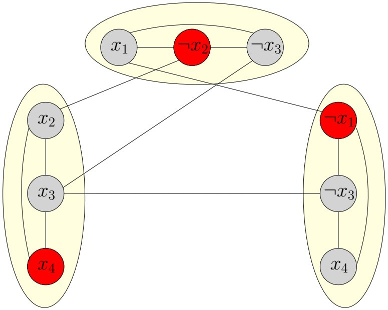
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- A clause $\Rightarrow$ a group of $3$ vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals



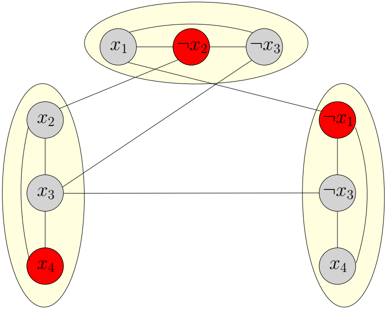
# 3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$
- A clause $\Rightarrow$ a group of $3$ vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals
- Problem: whether there is an IS of size $k = \#$clauses



# 3-Sat $\leq_P$ Ind-Set

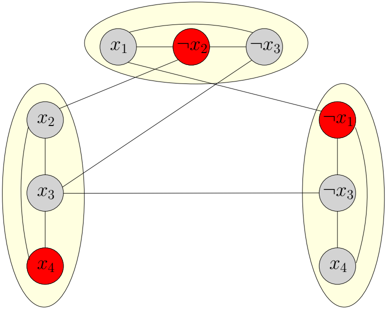
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$
- A clause $\Rightarrow$ a group of $3$ vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals
- Problem: whether there is an IS of size $k = \#$clauses




3-Sat instance is yes-instance $\Leftrightarrow$ clique instance is yes-instance:

# 3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$
- A clause $\Rightarrow$ a group of $3$ vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals
- Problem: whether there is an IS of size $k = \#$clauses

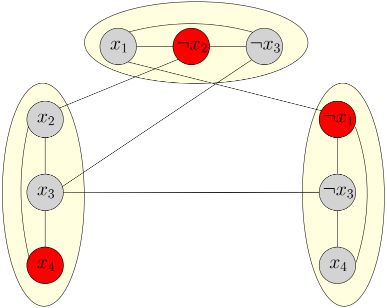



3-Sat instance is yes-instance $\Leftrightarrow$ clique instance is yes-instance:

- satisfying assignment $\Rightarrow$ independent set of size $k$
- independent set of size $k$ $\Rightarrow$ satisfying assignment

# Satisfying Assignment $\Rightarrow$ IS of Size $k$

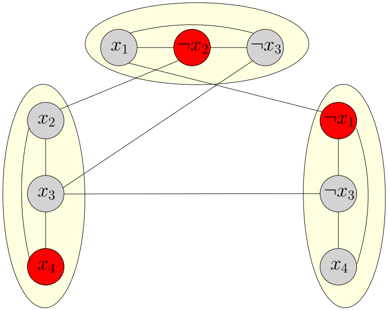
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$



# Satisfying Assignment $\Rightarrow$ IS of Size $k$

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

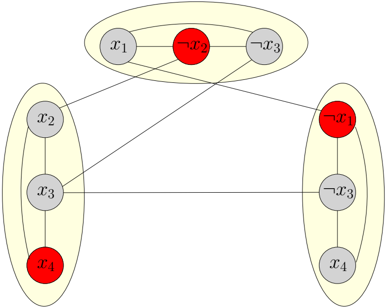
- For every clause, at least 1 literal is satisfied



# Satisfying Assignment $\Rightarrow$ IS of Size $k$

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

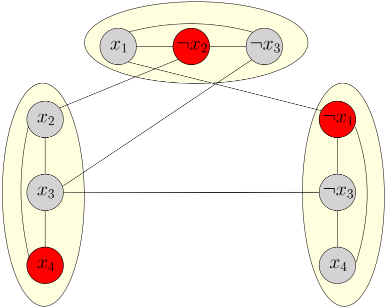
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal



# Satisfying Assignment $\Rightarrow$ IS of Size $k$

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

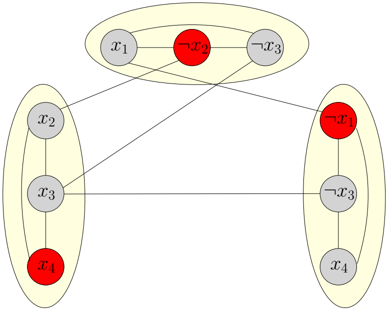
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group



# Satisfying Assignment $\Rightarrow$ IS of Size $k$

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

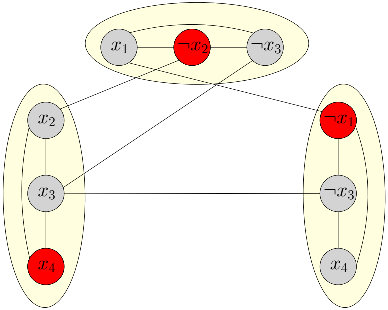
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group
- No contradictions among the selected literals



# Satisfying Assignment $\Rightarrow$ IS of Size $k$

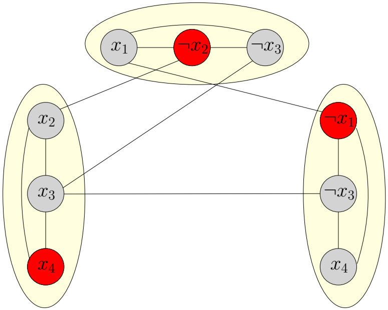
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group
- No contradictions among the selected literals
- An IS of size $k$



# IS of Size $k \Rightarrow$ Satisfying Assignment

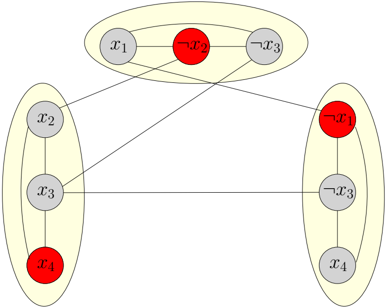
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$



# IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

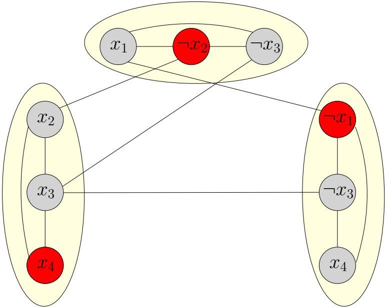
- For every group, exactly one literal is selected in IS



# IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

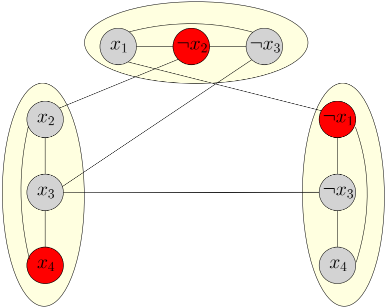
- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals



# IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

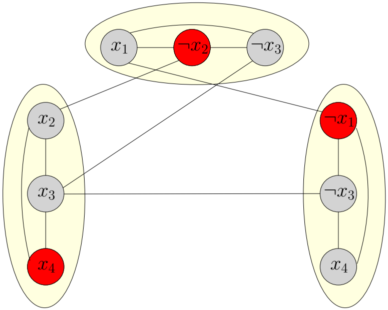
- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
- If $x_i$ is selected in IS, set $x_i = 1$



# IS of Size $k$ $\Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

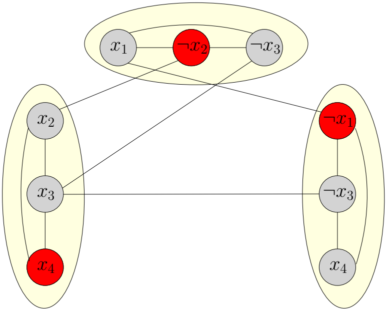
- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
- If $x_i$ is selected in IS, set $x_i = 1$
- If $\neg x_i$ is selected in IS, set $x_i = 0$



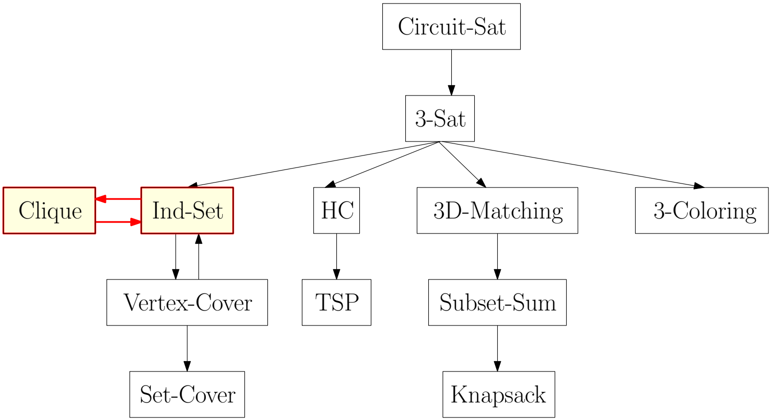
# IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

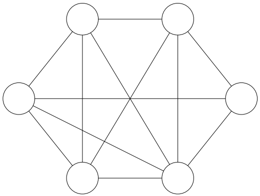
- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
- If $x_i$ is selected in IS, set $x_i = 1$
- If $\neg x_i$ is selected in IS, set $x_i = 0$
- Otherwise, set $x_i$ arbitrarily



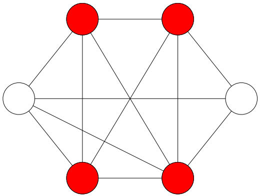
# Reductions of NP-Complete Problems



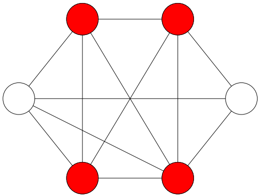
**Def.** A clique in an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall u, v \in S$ we have $(u, v) \in E$



**Def.** A clique in an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall u, v \in S$ we have $(u, v) \in E$



**Def.** A clique in an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall u, v \in S$ we have $(u, v) \in E$

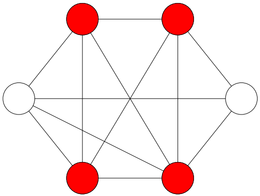


### Clique Problem

**Input:** $G = (V, E)$ and integer $k > 0$,

**Output:** whether there exists a clique of size $k$ in $G$

**Def.** A clique in an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall u, v \in S$ we have $(u, v) \in E$



**Clique Problem**

**Input:** $G = (V, E)$ and integer $k > 0$,

**Output:** whether there exists a clique of size $k$ in $G$

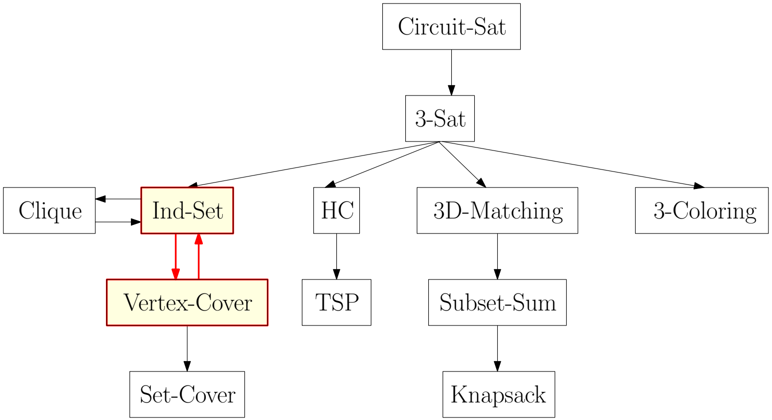
- What is the relationship between Clique and Ind-Set?

# Clique $=_P$ Ind-Set

**Def.** Given a graph $G = (V, E)$, define $\overline{G} = (V, \overline{E})$ be the graph such that $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$.

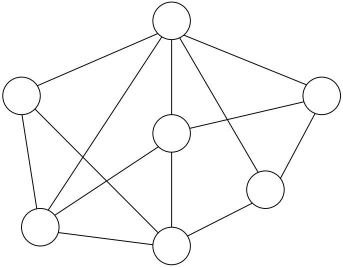
**Obs.** $S$ is an independent set in $G$ if and only if $S$ is a clique in $\overline{G}$.

# Reductions of NP-Complete Problems



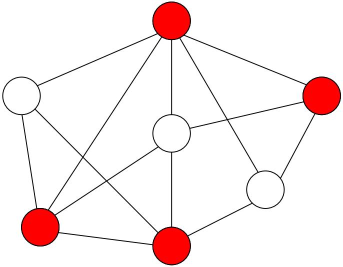
# Vertex-Cover

**Def.** Given a graph $G = (V, E)$, a vertex cover of $G$ is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$ .



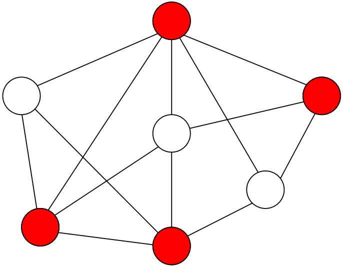
# Vertex-Cover

**Def.** Given a graph $G = (V, E)$, a vertex cover of $G$ is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$ .



# Vertex-Cover

**Def.** Given a graph $G = (V, E)$, a vertex cover of $G$ is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$ .



Vertex-Cover Problem

**Input:** $G = (V, E)$ and integer $k$

**Output:** whether there is a vertex cover of $G$ of size at most $k$

# Vertex-Cover $=_P$ Ind-Set

**Q:** What is the relationship between Vertex-Cover and Ind-Set?

# Vertex-Cover $=_P$ Ind-Set

**Q:** What is the relationship between Vertex-Cover and Ind-Set?

**A:** $S$ is a vertex-cover of $G = (V, E)$ if and only if $V \setminus S$ is an independent set of $G$.

# A Strategy of Polynomial Reduction

Recall the definition of polynomial time reductions:

**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

# A Strategy of Polynomial Reduction

Recall the definition of polynomial time reductions:

**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

- In general, algorithm for $Y$ can call the algorithm for $X$ many times.

# A Strategy of Polynomial Reduction

Recall the definition of polynomial time reductions:

**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

- In general, algorithm for $Y$ can call the algorithm for $X$ many times.
- However, for most reductions, we call algorithm for $X$ only once

# A Strategy of Polynomial Reduction

Recall the definition of polynomial time reductions:

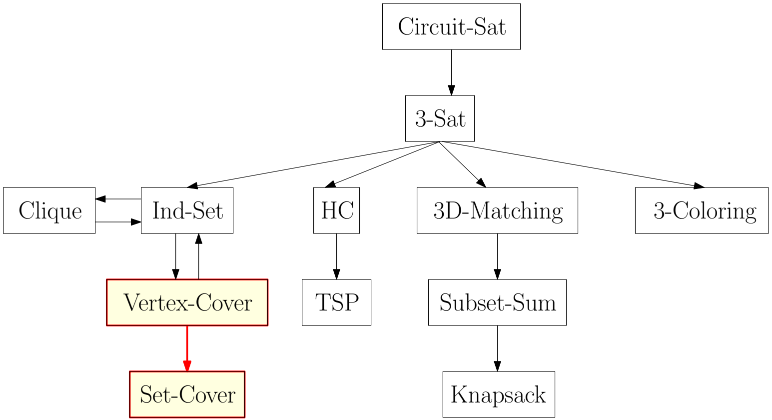
**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

- In general, algorithm for $Y$ can call the algorithm for $X$ many times.
- However, for most reductions, we call algorithm for $X$ only once
- That is, for a given instance $s_Y$ for $Y$, we only construct one instance $s_X$ for $X$

# A Strategy of Polynomial Reduction

- Given an instance $s_Y$ of problem $Y$, show how to construct in polynomial time an instance $s_X$ of problem such that:
  - $s_Y$ is a yes-instance of $Y \Rightarrow s_X$ is a yes-instance of $X$
  - $s_X$ is a yes-instance of $X \Rightarrow s_Y$ is a yes-instance of $Y$

# Reductions of NP-Complete Problems



**Set-Cover Problem**

**Input:** ground set $U$ and $m$ subsets $S_1, S_2, \cdots, S_m$ of $U$ and an integer $k$

**Output:** whether there is a set $I \subseteq \{1, 2, 3, \cdots, m\}$ of size $\leq k$ such that $\bigcup_{i \in I} S_i = U$

## Set-Cover Problem

**Input:** ground set $U$ and $m$ subsets $S_1, S_2, \cdots, S_m$ of $U$ and an integer $k$

**Output:** whether there is a set $I \subseteq \{1, 2, 3, \cdots, m\}$ of size $\leq k$ such that $\bigcup_{i \in I} S_i = U$

### Example:

- $U = \{1, 2, 3, 4, 5, 6\}$, $S_1 = \{1, 3, 4\}, S_2 = \{2, 3\}, S_3 = \{3, 6\}, S_4 = \{2, 5\}, S_5 = \{1, 2, 6\}$
- Then $S_1 \cup S_4 \cup S_5 = U$; we need 3 subsets to cover $U$

## Set-Cover Problem

**Input:** ground set $U$ and $m$ subsets $S_1, S_2, \cdots, S_m$ of $U$ and an integer $k$

**Output:** whether there is a set $I \subseteq \{1, 2, 3, \cdots, m\}$ of size $\leq k$ such that $\bigcup_{i \in I} S_i = U$

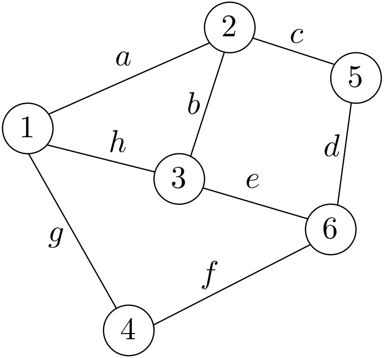
## Example:

- $U = \{1, 2, 3, 4, 5, 6\}$, $S_1 = \{1, 3, 4\}, S_2 = \{2, 3\}, S_3 = \{3, 6\}, S_4 = \{2, 5\}, S_5 = \{1, 2, 6\}$
- Then $S_1 \cup S_4 \cup S_5 = U$; we need 3 subsets to cover $U$

## Sample Application

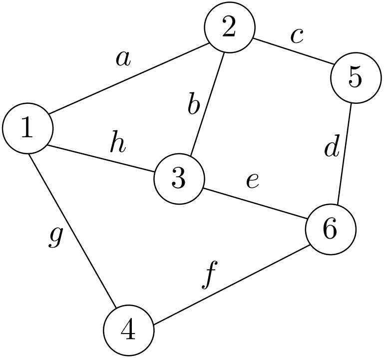
- $m$ available packages for a software
- $U$ is the set of features
- The package $i$ covers the set $S_i$ of features
- want to cover all features using fewest number of packages

# Vertex-Cover $\leq_P$ Set-Cover




$$U = \{a, b, c, d, e, f, g\}$$
$$S_1 = \{a, g, h\}$$
$$S_2 = \{a, b, c\}$$
$$S_3 = \{b, e, h\}$$
$$S_4 = \{g, h\}$$
$$S_5 = \{c, d\}$$
$$S_6 = \{d, e, f\}$$

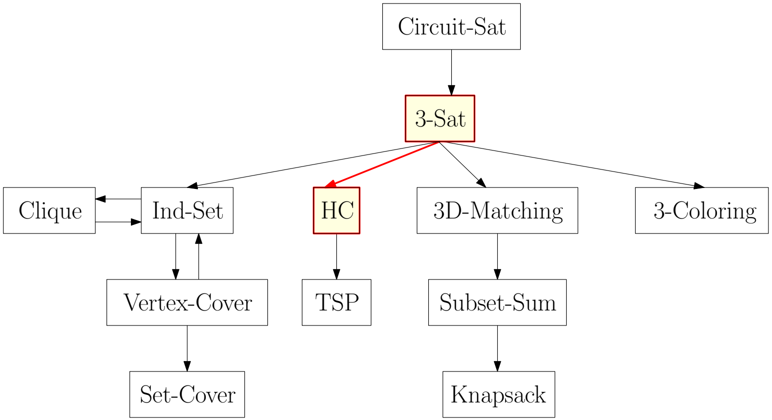
# Vertex-Cover $\leq_P$ Set-Cover




$$U = \{a, b, c, d, e, f, g\}$$
$$S_1 = \{a, g, h\}$$
$$S_2 = \{a, b, c\}$$
$$S_3 = \{b, e, h\}$$
$$S_4 = \{g, h\}$$
$$S_5 = \{c, d\}$$
$$S_6 = \{d, e, f\}$$

- edges $\Longrightarrow$ elements in $U$
- vertices $\Longrightarrow$ sets
- edge incident on vertex $\Longrightarrow$ element contained in set
- use vertices to cover edges $\Longrightarrow$ use sets to cover elements

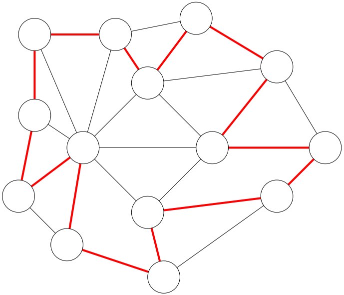
# Reductions of NP-Complete Problems



## Recall: Hamiltonian Cycle (HC) Problem

**Input:** graph $G = (V, E)$

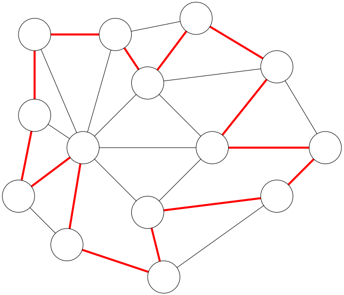
**Output:** whether $G$ contains a Hamiltonian cycle



## Recall: Hamiltonian Cycle (HC) Problem

**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle

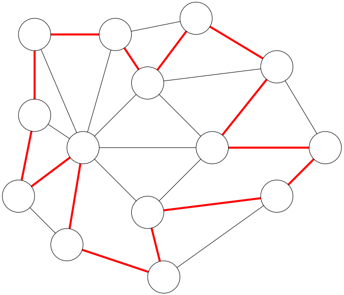


- We consider Hamiltonian Cycle Problem in directed graphs

## Recall: Hamiltonian Cycle (HC) Problem

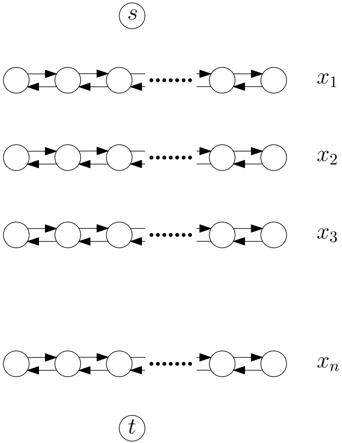
**Input:** graph $G = (V, E)$

**Output:** whether $G$ contains a Hamiltonian cycle



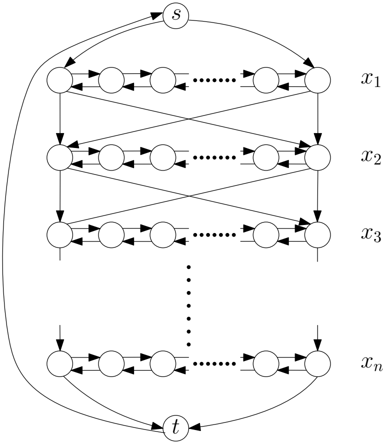
- We consider Hamiltonian Cycle Problem in directed graphs
- Exercise: HC-directed $\leq_P$ HC

# 3-Sat $\leq_P$ Directed-HC



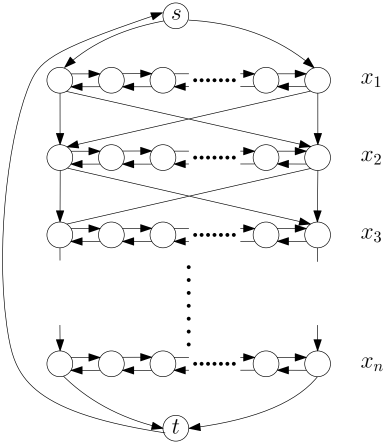

- Vertices $s, t$
- A long enough double-path $P_i$ for each variable $x_i$

# 3-Sat $\leq_P$ Directed-HC



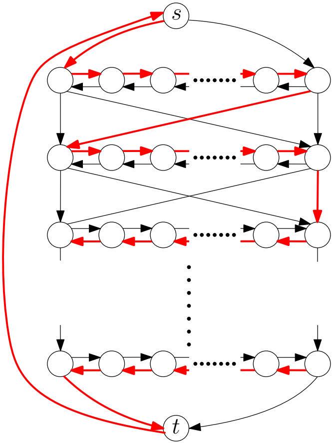

- Vertices $s, t$
- A long enough double-path $P_i$ for each variable $x_i$
- Edges from $s$ to $P_1$
- Edges from $P_n$ to $t$
- Edges from $P_i$ to $P_{i+1}$

# 3-Sat $\leq_P$ Directed-HC



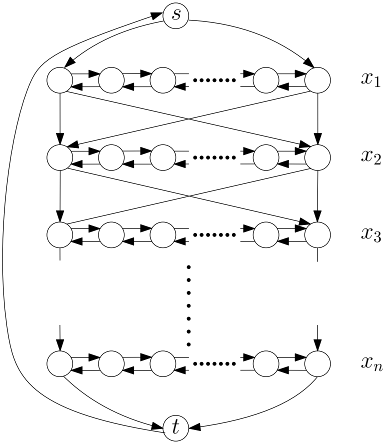

- Vertices $s, t$
- A long enough double-path $P_i$ for each variable $x_i$
- Edges from $s$ to $P_1$
- Edges from $P_n$ to $t$
- Edges from $P_i$ to $P_{i+1}$
- $x_i = 1 \iff$ traverse $P_i$ from left to right

# 3-Sat $\leq_P$ Directed-HC



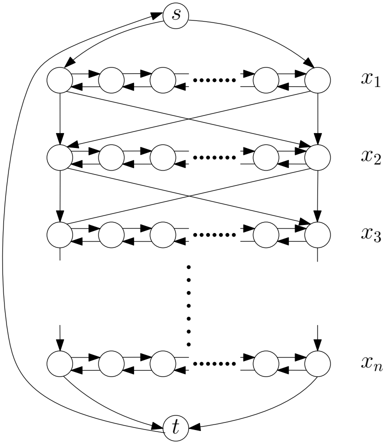

- Vertices $s, t$
- A long enough double-path $P_i$ for each variable $x_i$
- Edges from $s$ to $P_1$
- Edges from $P_n$ to $t$
- Edges from $P_i$ to $P_{i+1}$
- $x_i = 1 \iff$ traverse $P_i$ from left to right
- e.g, $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$

# 3-Sat $\leq_P$ Directed-HC



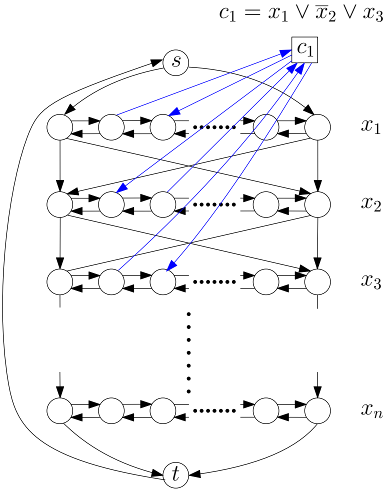

- Vertices $s, t$
- A long enough double-path $P_i$ for each variable $x_i$
- Edges from $s$ to $P_1$
- Edges from $P_n$ to $t$
- Edges from $P_i$ to $P_{i+1}$
- $x_i = 1 \iff$ traverse $P_i$ from left to right
- e.g, $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$

# 3-Sat $\leq_P$ Directed-HC



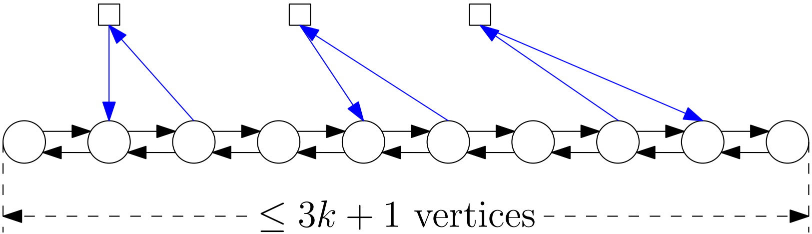

- There are exactly $2^n$ different Hamiltonian cycles, each correspondent to one assignment of variables

# 3-Sat $\leq_P$ Directed-HC




- There are exactly $2^n$ different Hamiltonian cycles, each correspondent to one assignment of variables
- Add a vertex for each clause, so that the vertex can be visited only if one of the literals is satisfied.

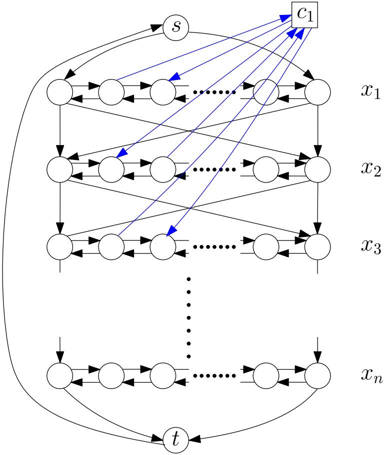
# A Path Should Be Long Enough




- $k$: number of clauses

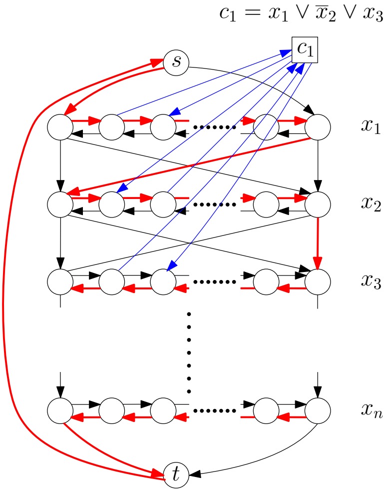
# Yes-Instance for 3-Sat ⇒ Yes-Instance for Di-HC

$c_1 = x_1 \vee \overline{x}_2 \vee x_3$



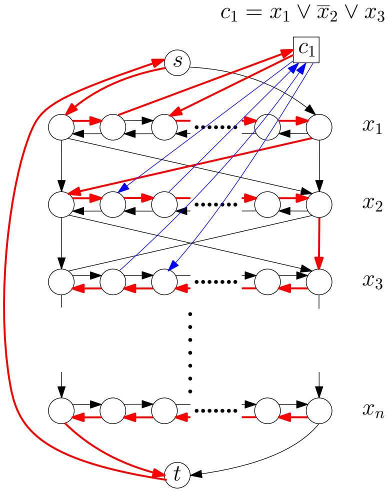

- In base graph, construct an HC according to the satisfying assignment

# Yes-Instance for 3-Sat ⇒ Yes-Instance for Di-HC



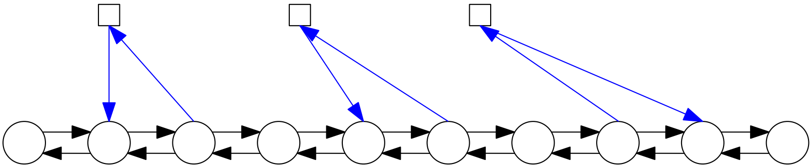

- In base graph, construct an HC according to the satisfying assignment
- For every clause, one literal is satisfied

# Yes-Instance for 3-Sat ⇒ Yes-Instance for Di-HC



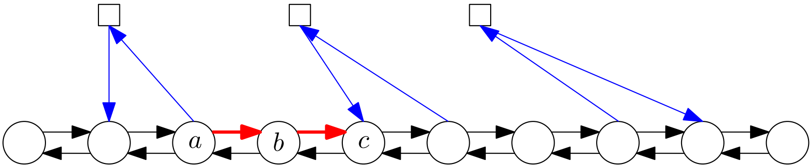

- In base graph, construct an HC according to the satisfying assignment
- For every clause, one literal is satisfied
- Visit the vertex for the clause by taking a “detour” from the path for the literal

# Yes-Instance for Di-HC ⇒ Yes-Instance for 3-Sat



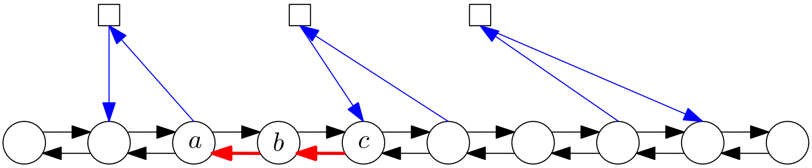
- Idea: for each path $P_i$, must follow the left-to-right or right-to-right pattern.

# Yes-Instance for Di-HC $\Rightarrow$ Yes-Instance for 3-Sat



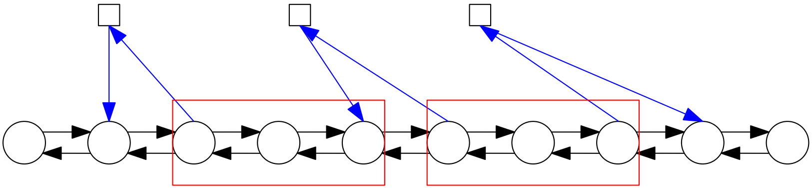

- Idea: for each path $P_i$, must follow the left-to-right or right-to-right pattern.
- To visit vertex $b$, can either go $a$-$b$-$c$ or $b$-$c$-$a$

# Yes-Instance for Di-HC ⇒ Yes-Instance for 3-Sat



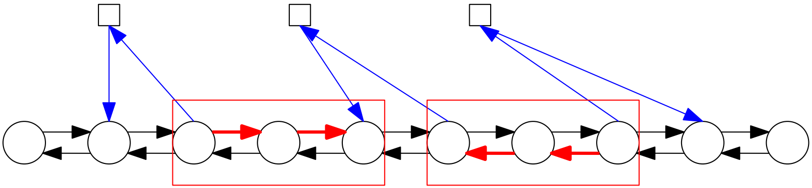

- Idea: for each path $P_i$, must follow the left-to-right or right-to-right pattern.
- To visit vertex $b$, can either go $a$-$b$-$c$ or $b$-$c$-$a$

# Yes-Instance for Di-HC $\Rightarrow$ Yes-Instance for 3-Sat



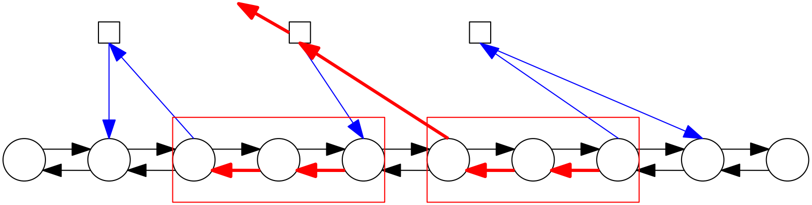
- Idea: for each path $P_i$, must follow the left-to-right or right-to-right pattern.
- To visit vertex $b$, can either go $a$-$b$-$c$ or $b$-$c$-$a$
- Created “chunks” of $3$ vertices.

# Yes-Instance for Di-HC $\Rightarrow$ Yes-Instance for 3-Sat



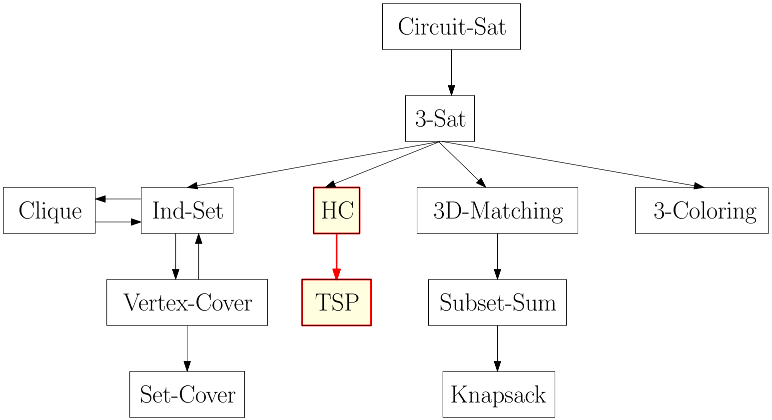
- Idea: for each path $P_i$, must follow the left-to-right or right-to-right pattern.
- To visit vertex $b$, can either go $a$-$b$-$c$ or $b$-$c$-$a$
- Created "chunks" of $3$ vertices.
- Directions of the chunks must be the same

# Yes-Instance for Di-HC ⇒ Yes-Instance for 3-Sat



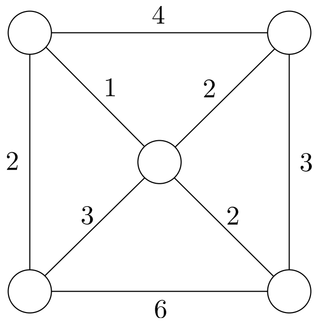
- Idea: for each path $P_i$, must follow the left-to-right or right-to-right pattern.
- To visit vertex $b$, can either go $a$-$b$-$c$ or $b$-$c$-$a$
- Created "chunks" of $3$ vertices.
- Directions of the chunks must be the same
- Can not take a detour to some other path

# Reductions of NP-Complete Problems



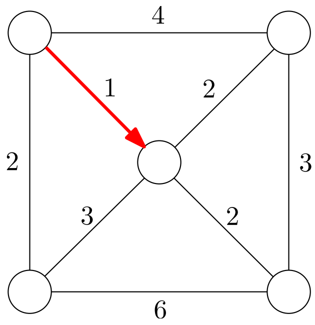
# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost



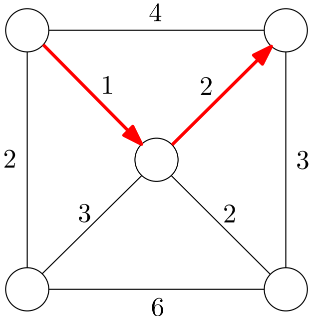
# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost



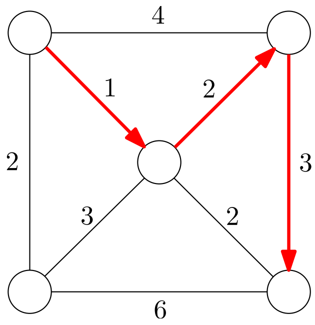
# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost



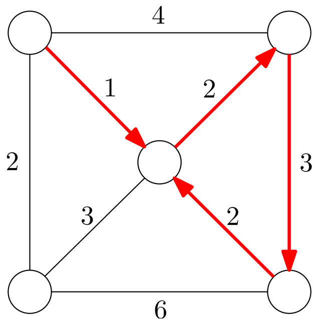
# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost



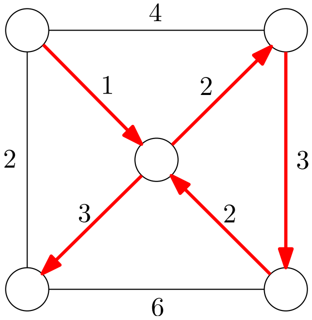
# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost



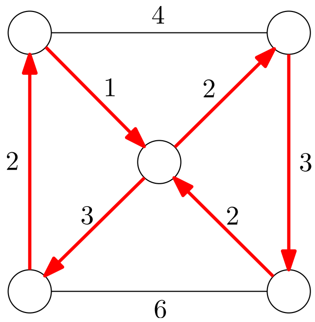
# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost



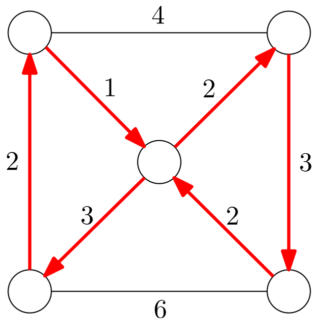
# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost



# Traveling Salesman Problem

- A salesman needs to visit $n$ cities $1, 2, 3, \cdots, n$
- He needs to start from and return to city $1$
- Goal: find a tour with the minimum cost

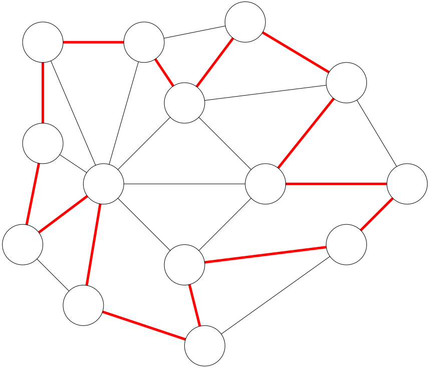



**Travelling Salesman Problem (TSP)**

**Input:** a graph $G = (V, E)$, weights $w : E \to \mathbb{R}_{\geq 0}$, and $L > 0$

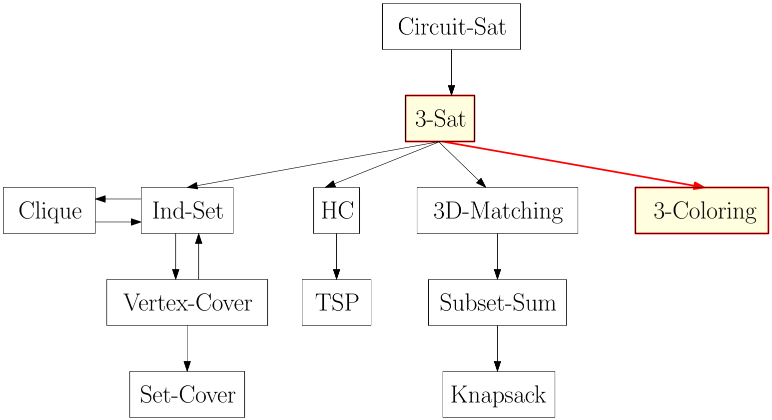
**Output:** whether there is a tour of length at most $L$

# HC $\leq_P$ TSP



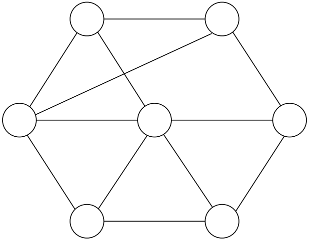
**Obs.** There is a Hamilton cycle in $G$ if and only if there is a tour for the salesman of length $n = |V|$.

# Reductions of NP-Complete Problems



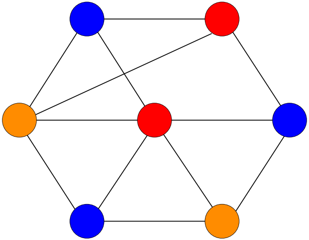
# $k$-coloring problem

**Def.** A $k$-coloring of $G = (V, E)$ is a function $f : V \to \{1, 2, 3, \cdots, k\}$ so that for every edge $(u, v) \in E$, we have $f(u) \neq f(v)$. $G$ is $k$-colorable if there is a $k$-coloring of $G$.



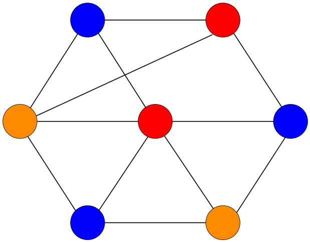
# $k$-coloring problem

**Def.** A $k$-coloring of $G = (V, E)$ is a function $f : V \to \{1, 2, 3, \cdots, k\}$ so that for every edge $(u, v) \in E$, we have $f(u) \neq f(v)$. $G$ is $k$-colorable if there is a $k$-coloring of $G$.



# $k$-coloring problem

**Def.** A $k$-coloring of $G = (V, E)$ is a function $f : V \to \{1, 2, 3, \cdots, k\}$ so that for every edge $(u, v) \in E$, we have $f(u) \neq f(v)$. $G$ is $k$-colorable if there is a $k$-coloring of $G$.



**$k$-coloring problem**

**Input:** a graph $G = (V, E)$

**Output:** whether $G$ is $k$-colorable or not

# 2-Coloring Problem

**Obs.** A graph $G$ is 2-colorable if and only if it is bipartite.

- There is an $O(m+n)$-time algorithm to decide if a graph $G$ is $2$-colorable

# 2-Coloring Problem

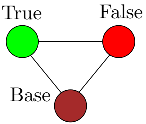
**Obs.** A graph $G$ is 2-colorable if and only if it is bipartite.

- There is an $O(m+n)$-time algorithm to decide if a graph $G$ is $2$-colorable
- Idea: suppose $G$ is connected. If we fix the color of one vertex in $G$, then the colors of all other vertices are fixed.

# 3-SAT $\leq_P$ 3-Coloring

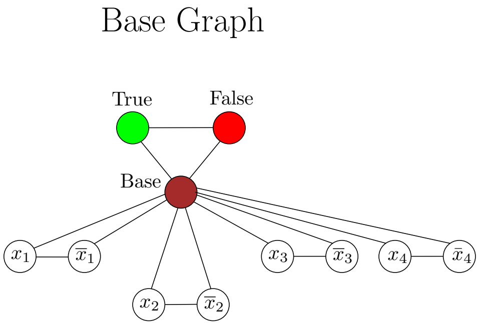
- Construct the base graph

Base Graph



# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph

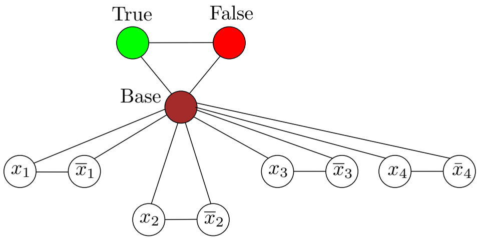


# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

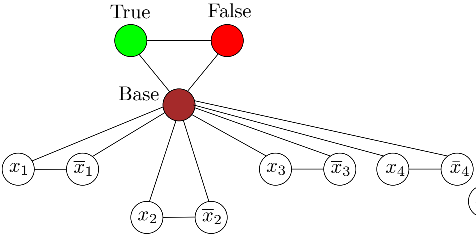
$$x_1 \vee \neg x_2 \vee x_3$$



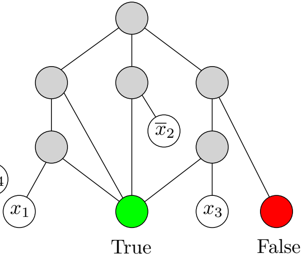
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

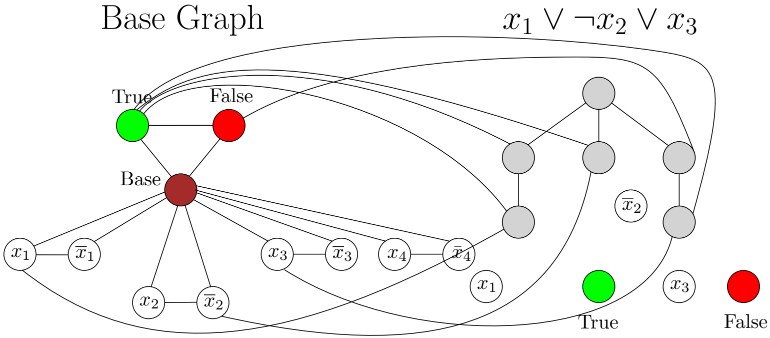



$x_1 \vee \neg x_2 \vee x_3$



# 3-SAT $\leq_P$ 3-Coloring

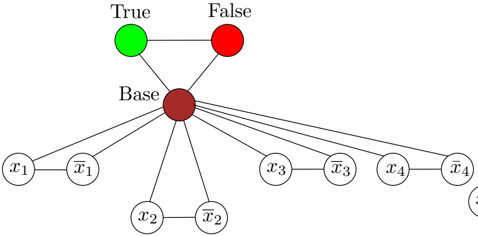
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.



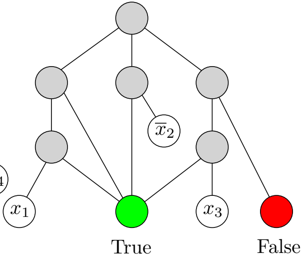
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph



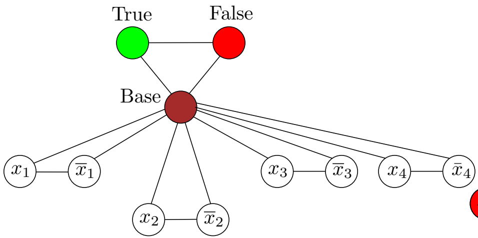

$x_1 \vee \neg x_2 \vee x_3$



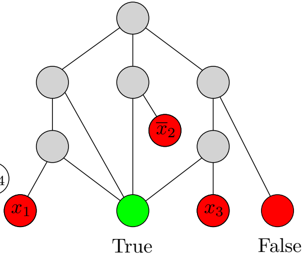
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

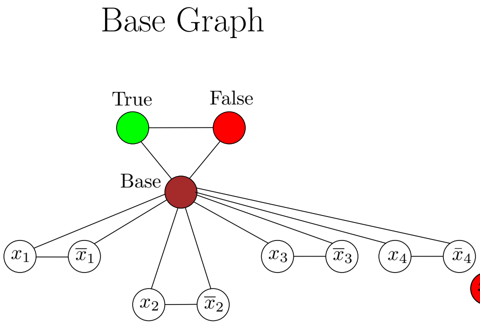


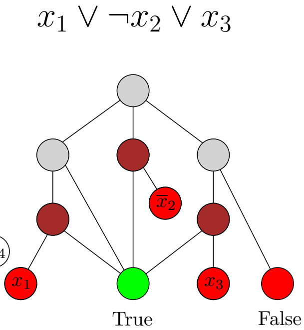

$x_1 \vee \neg x_2 \vee x_3$



# 3-SAT $\leq_P$ 3-Coloring

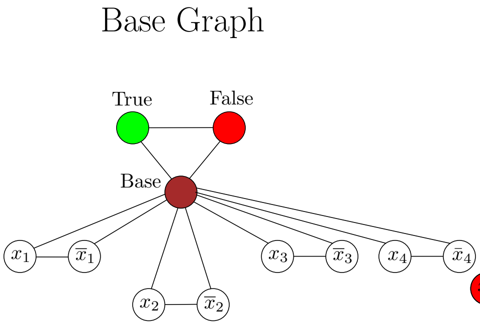
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

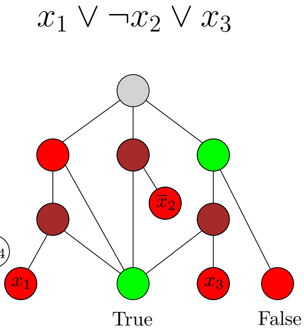


# 3-SAT $\leq_P$ 3-Coloring

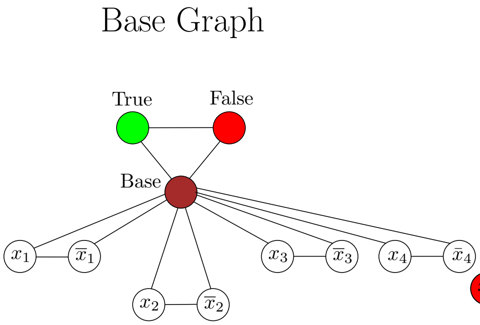
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

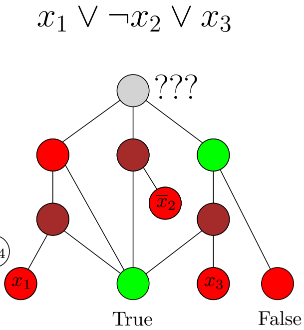


# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

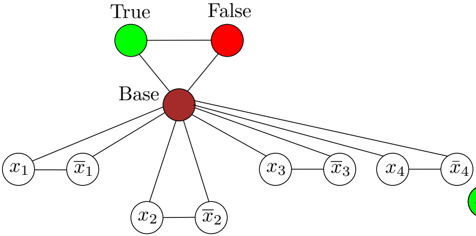


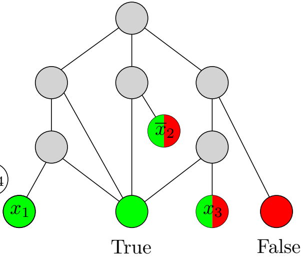
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph



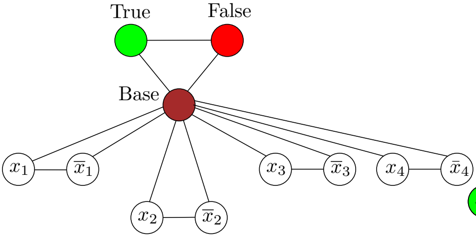

$x_1 \vee \neg x_2 \vee x_3$



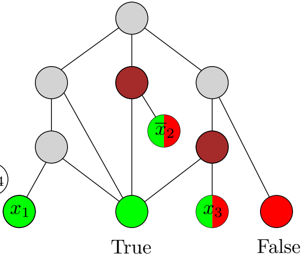
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph



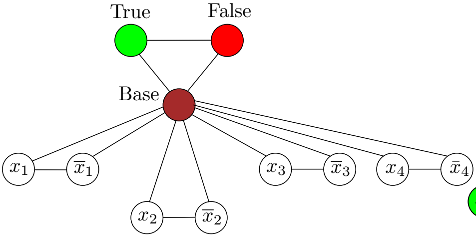

$x_1 \vee \neg x_2 \vee x_3$



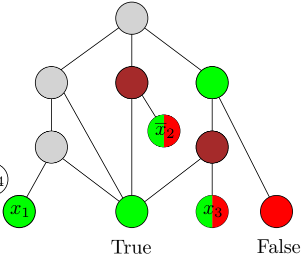
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

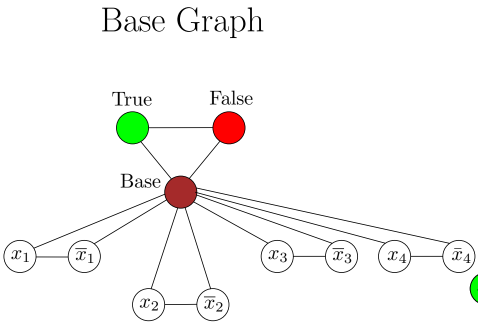


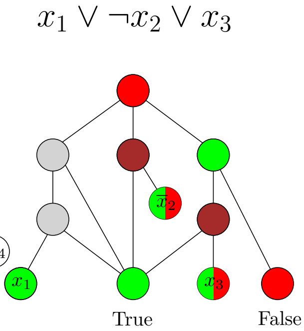

$x_1 \vee \neg x_2 \vee x_3$



# 3-SAT $\leq_P$ 3-Coloring

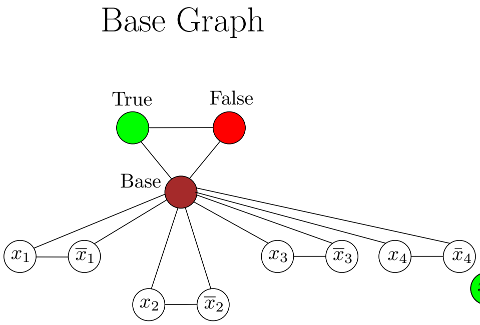
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

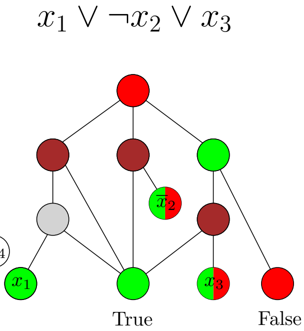


# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

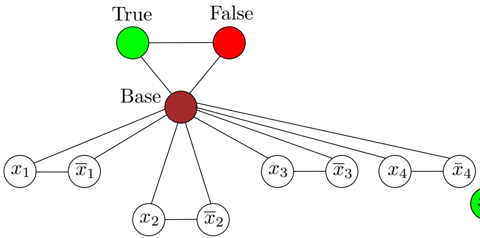


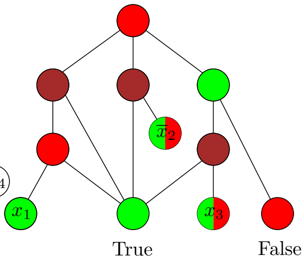
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

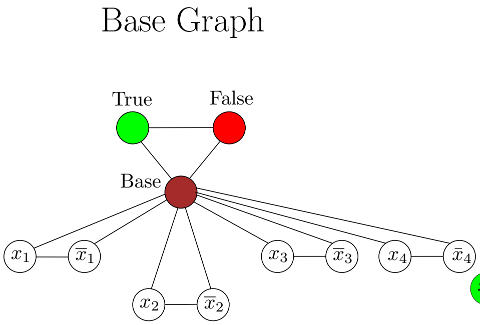


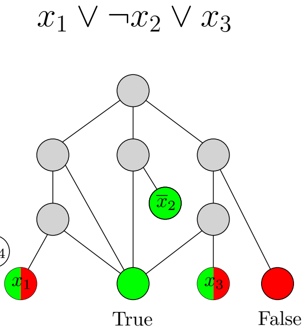

$x_1 \vee \neg x_2 \vee x_3$



# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

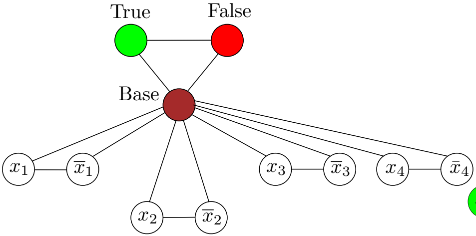


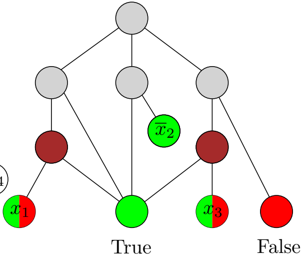
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

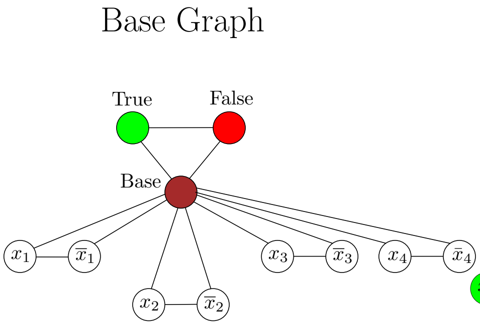


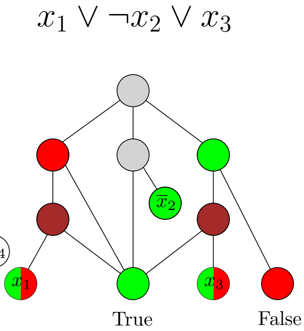

$x_1 \vee \neg x_2 \vee x_3$



# 3-SAT $\leq_P$ 3-Coloring

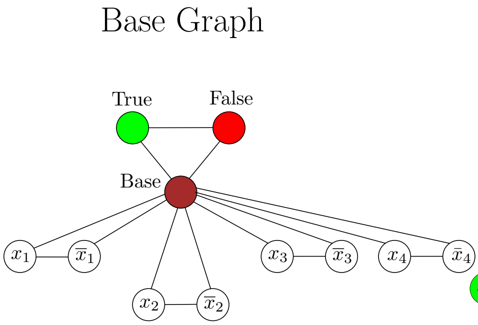
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

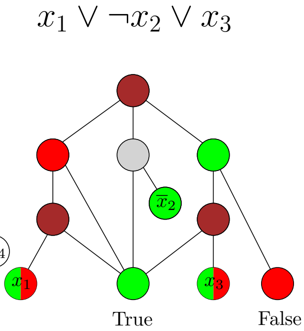


# 3-SAT $\leq_P$ 3-Coloring

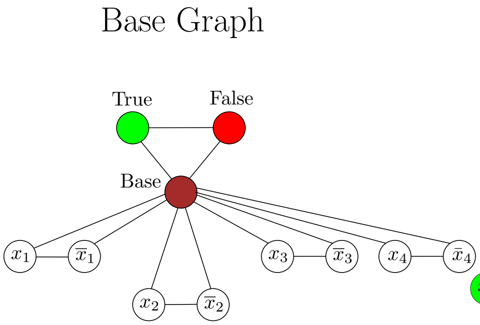
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

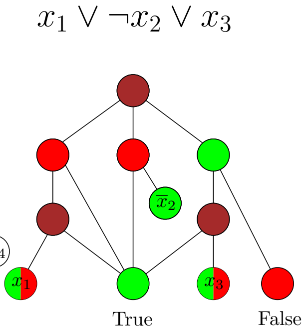


# 3-SAT $\leq_P$ 3-Coloring

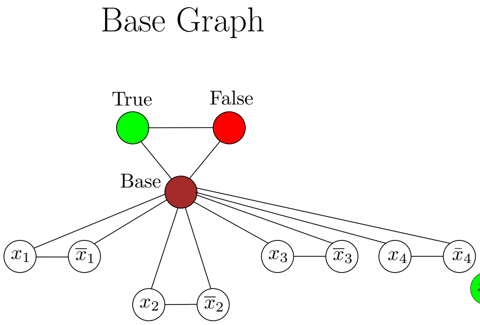
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

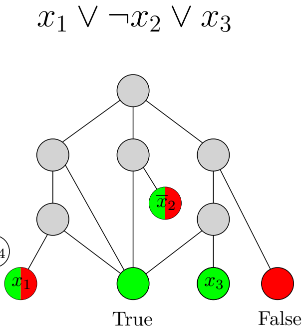


# 3-SAT $\leq_P$ 3-Coloring

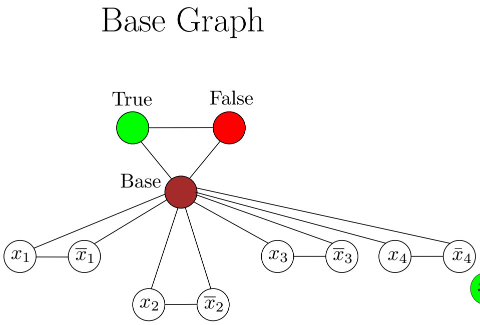
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

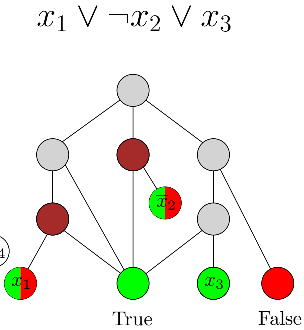


# 3-SAT $\leq_P$ 3-Coloring

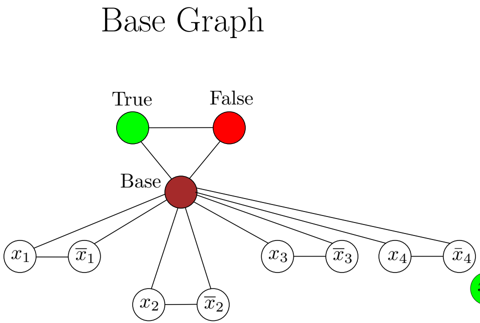
- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

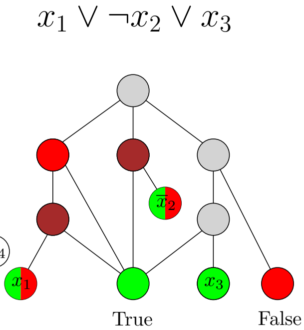


# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

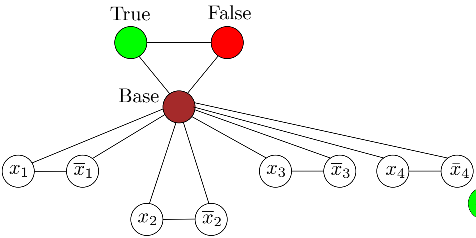


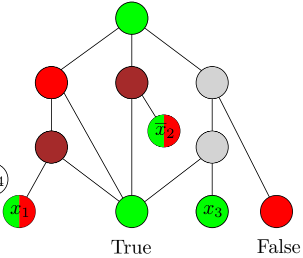
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph

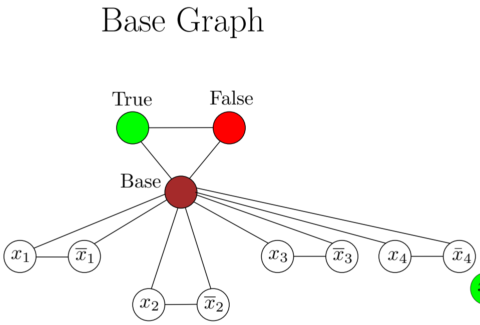


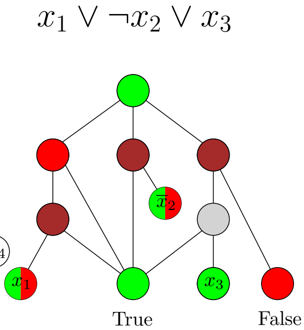

$x_1 \vee \neg x_2 \vee x_3$



# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

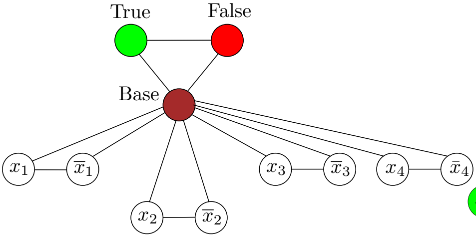


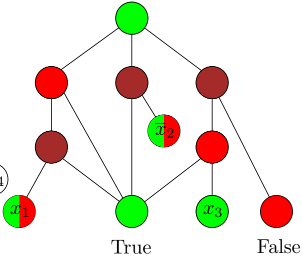
# 3-SAT $\leq_P$ 3-Coloring

- Construct the base graph
- Construct a gadget from each clause: gadget is 3-colorable if and only if the clause is satisfied.

Base Graph




$x_1 \vee \neg x_2 \vee x_3$



# Outline

3.36pt

**Q:** How far away are we from proving or disproving $\mathsf{P} = \mathsf{NP}$?

**Q:** How far away are we from proving or disproving P $=$ NP?

- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.

**Q:** How far away are we from proving or disproving $P = NP$?

- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.
- For 3-Sat problem:

**Q:** How far away are we from proving or disproving P = NP?

- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.
- For 3-Sat problem:
  - Assume the number of clauses is $\Theta(n)$, $n =$ number variables

**Q:** How far away are we from proving or disproving P $=$ NP?

- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.
- For 3-Sat problem:
  - Assume the number of clauses is $\Theta(n)$, $n =$ number variables
  - Best algorithm runs in time $O(c^n)$ for some constant $c > 1$

**Q:** How far away are we from proving or disproving P $=$ NP?

- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.
- For 3-Sat problem:
  - Assume the number of clauses is $\Theta(n)$, $n =$ number variables
  - Best algorithm runs in time $O(c^n)$ for some constant $c > 1$
  - Best lower bound is $\Omega(n)$

**Q:** How far away are we from proving or disproving P $=$ NP?

- Try to prove an "unconditional" lower bound on running time of algorithm solving a NP-complete problem.
- For 3-Sat problem:
  - Assume the number of clauses is $\Theta(n)$, $n =$ number variables
  - Best algorithm runs in time $O(c^n)$ for some constant $c > 1$
  - Best lower bound is $\Omega(n)$
- Essentially we have no techniques for proving lower bound for running time

# Dealing with NP-Hard Problems

- Faster exponential time algorithms
- Solving the problem for special cases
- Fixed parameter tractability
- Approximation algorithms

# Faster Exponential Time Algorithms

3-SAT:

# Faster Exponential Time Algorithms

3-SAT:

- Brute-force: $O(2^n \cdot \text{poly}(n))$

# Faster Exponential Time Algorithms

3-SAT:

- Brute-force: $O(2^n \cdot \text{poly}(n))$
- $2^n \to 1.844^n \to 1.3334^n$

# Faster Exponential Time Algorithms

3-SAT:

- Brute-force: $O(2^n \cdot \text{poly}(n))$
- $2^n \to 1.844^n \to 1.3334^n$
- Practical SAT Solver: solves real-world sat instances with more than 10,000 variables

# Faster Exponential Time Algorithms

3-SAT:

- Brute-force: $O(2^n \cdot \text{poly}(n))$
- $2^n \to 1.844^n \to 1.3334^n$
- Practical SAT Solver: solves real-world sat instances with more than 10,000 variables

Travelling Salesman Problem:

# Faster Exponential Time Algorithms

3-SAT:

- Brute-force: $O(2^n \cdot \text{poly}(n))$
- $2^n \to 1.844^n \to 1.3334^n$
- Practical SAT Solver: solves real-world sat instances with more than 10,000 variables

Travelling Salesman Problem:

- Brute-force: $O(n! \cdot \text{poly}(n))$

# Faster Exponential Time Algorithms

3-SAT:

- Brute-force: $O(2^n \cdot \text{poly}(n))$
- $2^n \to 1.844^n \to 1.3334^n$
- Practical SAT Solver: solves real-world sat instances with more than 10,000 variables

Travelling Salesman Problem:

- Brute-force: $O(n! \cdot \text{poly}(n))$
- Better algorithm: $O(2^n \cdot \text{poly}(n))$

# Faster Exponential Time Algorithms

3-SAT:

- Brute-force: $O(2^n \cdot \text{poly}(n))$
- $2^n \to 1.844^n \to 1.3334^n$
- Practical SAT Solver: solves real-world sat instances with more than 10,000 variables

Travelling Salesman Problem:

- Brute-force: $O(n! \cdot \text{poly}(n))$
- Better algorithm: $O(2^n \cdot \text{poly}(n))$
- In practice: TSP Solver can solve Euclidean TSP instances with more than 100,000 vertices

# Solving the problem for special cases

Maximum independent set problem is NP-hard on general graphs, but easy on

# Solving the problem for special cases

Maximum independent set problem is NP-hard on general graphs, but easy on

- trees

# Solving the problem for special cases

Maximum independent set problem is NP-hard on general graphs, but easy on

- trees
- bounded tree-width graphs

# Solving the problem for special cases

Maximum independent set problem is NP-hard on general graphs, but easy on

- trees
- bounded tree-width graphs
- interval graphs

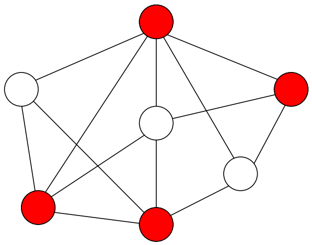
# Solving the problem for special cases

Maximum independent set problem is NP-hard on general graphs, but easy on

- trees
- bounded tree-width graphs
- interval graphs
- $\ldots$

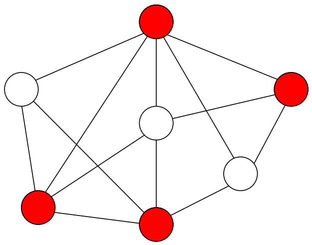
# Fixed Parameter Tractability

- Problem: whether there is a vertex cover of size $k$, for a small $k$ (number of nodes is $n$, number of edges is $\Theta(n)$.)



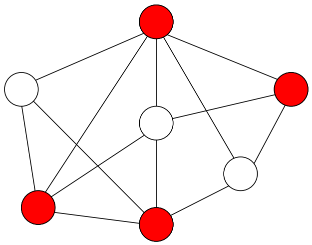
# Fixed Parameter Tractability

- Problem: whether there is a vertex cover of size $k$, for a small $k$ (number of nodes is $n$, number of edges is $\Theta(n)$.)
- Brute-force algorithm: $O(kn^{k+1})$



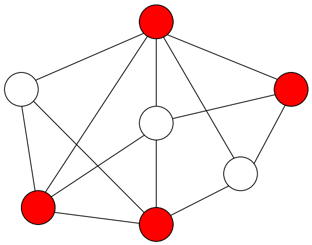
# Fixed Parameter Tractability

- Problem: whether there is a vertex cover of size $k$, for a small $k$ (number of nodes is $n$, number of edges is $\Theta(n)$.)
- Brute-force algorithm: $O(kn^{k+1})$
- Better running time : $O(2^k \cdot kn)$



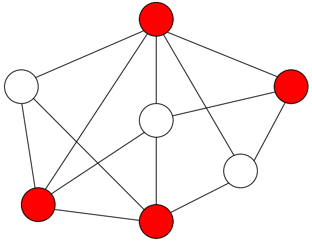
# Fixed Parameter Tractability

- Problem: whether there is a vertex cover of size $k$, for a small $k$ (number of nodes is $n$, number of edges is $\Theta(n)$.)
- Brute-force algorithm: $O(kn^{k+1})$
- Better running time : $O(2^k \cdot kn)$
- Running time is $f(k)n^c$ for some $c$ independent of $k$



# Fixed Parameter Tractability

- Problem: whether there is a vertex cover of size $k$, for a small $k$ (number of nodes is $n$, number of edges is $\Theta(n)$.)
- Brute-force algorithm: $O(kn^{k+1})$
- Better running time : $O(2^k \cdot kn)$
- Running time is $f(k)n^c$ for some $c$ independent of $k$
- Vertex-Cover is fixed-parameter tractable.



# Approximation Algorithms

- For optimization problems, approximation algorithms will find sub-optimal solutions in polynomial time

# Approximation Algorithms

- For optimization problems, approximation algorithms will find sub-optimal solutions in polynomial time
- Approximation ratio is the ratio between the quality of the solution output by the algorithm and the quality of the optimal solution

# Approximation Algorithms

- For optimization problems, approximation algorithms will find sub-optimal solutions in polynomial time
- Approximation ratio is the ratio between the quality of the solution output by the algorithm and the quality of the optimal solution
- We want to make the approximation ratio as small as possible, while maintaining the property that the algorithm runs in polynomial time

# Approximation Algorithms

- For optimization problems, approximation algorithms will find sub-optimal solutions in polynomial time
- Approximation ratio is the ratio between the quality of the solution output by the algorithm and the quality of the optimal solution
- We want to make the approximation ratio as small as possible, while maintaining the property that the algorithm runs in polynomial time

- There is an 1.5-approximation for travelling salesman problem: we can efficiently find a tour whose length is at most 1.5 times the length of the optimal tour

# Approximation Algorithms

- For optimization problems, approximation algorithms will find sub-optimal solutions in polynomial time
- Approximation ratio is the ratio between the quality of the solution output by the algorithm and the quality of the optimal solution
- We want to make the approximation ratio as small as possible, while maintaining the property that the algorithm runs in polynomial time

- There is an 1.5-approximation for travelling salesman problem: we can efficiently find a tour whose length is at most 1.5 times the length of the optimal tour
- 2-approximation for vertex-cover

# Approximation Algorithms

- For optimization problems, approximation algorithms will find sub-optimal solutions in polynomial time
- Approximation ratio is the ratio between the quality of the solution output by the algorithm and the quality of the optimal solution
- We want to make the approximation ratio as small as possible, while maintaining the property that the algorithm runs in polynomial time

- There is an 1.5-approximation for travelling salesman problem: we can efficiently find a tour whose length is at most 1.5 times the length of the optimal tour
- 2-approximation for vertex-cover
- $O(\lg n)$-approximation for set-cover

# Outline

3.36pt

# Summary

- We consider decision problems
- Inputs are encoded as $\{0, 1\}$-strings

**Def.** The complexity class P is the set of decision problems $X$ that can be solved in polynomial time.

- Alice has a supercomputer, fast enough to run an exponential time algorithm
- Bob has a slow computer, which can only run a polynomial-time algorithm

**Def.** (Informal) The complexity class NP is the set of problems for which Alice can convince Bob a yes instance is a yes instance

# Summary

**Def.** $B$ is an efficient certifier for a problem $X$ if

- $B$ is a polynomial-time algorithm that takes two input strings $s$ and $t$
- there is a polynomial function $p$ such that, $s \in X$ if and only if there is string $t$ such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

The string $t$ such that $B(s,t) = 1$ is called a certificate.

**Def.** The complexity class NP is the set of all problems for which there exists an efficient certifier.

# Summary

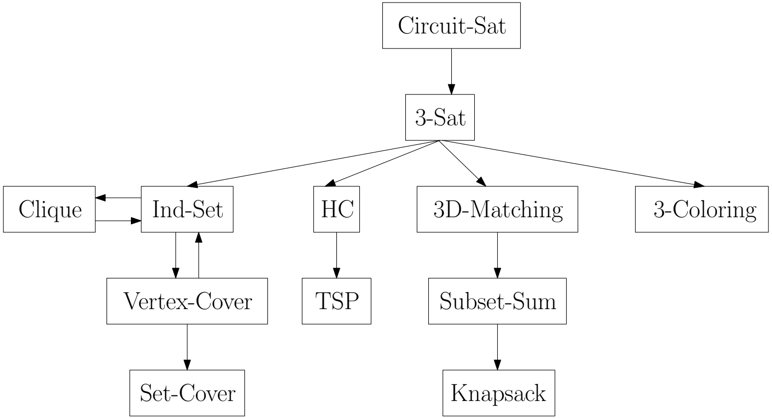
**Def.** Given a black box algorithm $A$ that solves a problem $X$, if any instance of a problem $Y$ can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to $A$, then we say $Y$ is polynomial-time reducible to $X$, denoted as $Y \leq_P X$.

**Def.** A problem $X$ is called NP-complete if

1. $X \in$ NP, and
2. $Y \leq_\text{P} X$ for every $Y \in$ NP.

- If any NP-complete problem can be solved in polynomial time, then $P = NP$
- Unless $P = NP$, a NP-complete problem can not be solved in polynomial time

# Summary


Circuit-Sat
3-Sat
Clique
Ind-Set
HC
3D-Matching
3-Coloring
Vertex-Cover
TSP
Subset-Sum
Set-Cover
Knapsack

# Summary

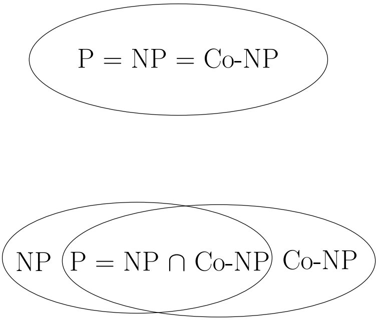
## Proof of NP-Completeness for Circuit-Sat

- Fact 1: a polynomial-time algorithm can be converted to a polynomial-size circuit
- Fact 2: for a problem in NP, there is a efficient certifier.
- Given a problem $X \in$ NP, let $B(s,t)$ be the certifier
- Convert $B(s,t)$ to a circuit and hard-wire $s$ to the input gates
- $s$ is a yes-instance if and only if the resulting circuit is satisfiable

- Proof of NP-Completeness for other problems by reductions

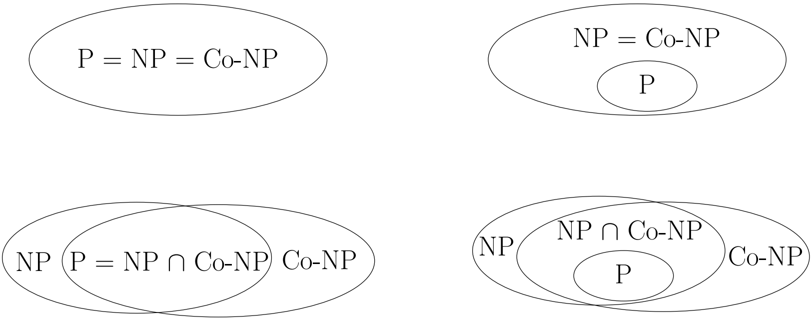
# Exercises

Recall the 4 scenarios:



# Exercises

Recall the 4 scenarios:




- Prove: P = NP if and only if P = CO-NP

# Exercises

For each of the following problem $X$, answer: whether (1) $X \in$ NP, (2) $X \in$ CO-NP. Each answer is either "yes" or "we do not know".

1. Given a graph $G = (V, E)$, whether $G$ is 4-colorable.

# Exercises

For each of the following problem $X$, answer: whether (1) $X \in$ NP, (2) $X \in$ CO-NP. Each answer is either "yes" or "we do not know".

1. Given a graph $G = (V, E)$, whether $G$ is 4-colorable.
2. Given a graph $G = (V, E)$ and an integer $t > 0$, whether the minimum vertex cover of $G$ has size at least $t$.

# Exercises

For each of the following problem $X$, answer: whether (1) $X \in$ NP, (2) $X \in$ CO-NP. Each answer is either "yes" or "we do not know".

1. Given a graph $G = (V, E)$, whether $G$ is 4-colorable.
2. Given a graph $G = (V, E)$ and an integer $t > 0$, whether the minimum vertex cover of $G$ has size at least $t$.
3. Given a directed graph $G = (V, E)$, with weights $w : E \to \mathbb{R}_{>0}$, $s, t \in V$, and a number $L > 0$, whether the length of the shortest path from $s$ to $t$ in $G$ is at most $L$.

# Exercises

For each of the following problem $X$, answer: whether (1) $X \in$ NP, (2) $X \in$ CO-NP. Each answer is either "yes" or "we do not know".

1. Given a graph $G = (V, E)$, whether $G$ is 4-colorable.
2. Given a graph $G = (V, E)$ and an integer $t > 0$, whether the minimum vertex cover of $G$ has size at least $t$.
3. Given a directed graph $G = (V, E)$, with weights $w : E \to \mathbb{R}_{>0}$, $s, t \in V$, and a number $L > 0$, whether the length of the shortest path from $s$ to $t$ in $G$ is at most $L$.
4. Given two boolean formulas, whether the they are equivalent. For example, $(x_1 \vee x_2) \wedge (\neg x_1 \vee x_3)$ and $(\neg x_1 \wedge x_2) \vee (x_1 \wedge x_3)$ are equivalent since they give the same value for every assignment of $(x_1, x_2, x_3)$.

# Exercises

Prove the following reductions:

1. 3-Coloring $\leq_P$ 4-Coloring

# Exercises

Prove the following reductions:

1. 3-Coloring $\leq_P$ 4-Coloring
2. Hamiltonian-Cycle $\leq_P$ Hamiltonian-Path

# Exercises

Prove the following reductions:

1. 3-Coloring $\leq_P$ 4-Coloring
2. Hamiltonian-Cycle $\leq_P$ Hamiltonian-Path
3. Given a graph $G = (V, E)$, the degree-3 spanning tree (D3ST) problem asks whether $G$ contains a spanning tree $T$ of degree at most $3$. Prove Hamiltonian-Path $\leq_P$ D3ST.