CSE 431/531: Algorithm Analysis and Design (Spring 2019)

Dynamic Programming

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Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many independent sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Recall: Computing the n-th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

Fib(n)

1. $F[0] \leftarrow 0$
2. $F[1] \leftarrow 1$
3. for $i \leftarrow 2$ to n do
4. $\quad F[i] \leftarrow F[i-1] + F[i-2]$
5. return $F[n]$

- Store each $F[i]$ for future use.

Outline

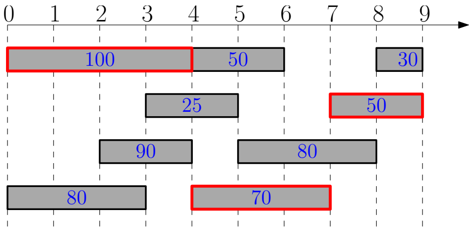
Recall: Interval Schduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs

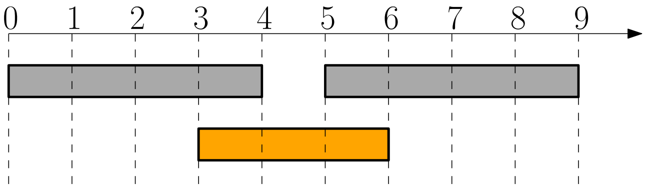


Hard to Design a Greedy Algorithm

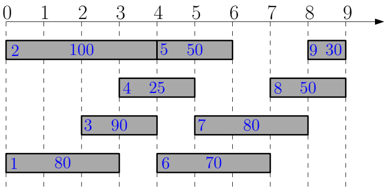
Q: Which job is safe to schedule?

- Job with the earliest finish time? No, we are ignoring weights
- Job with the largest weight? No, we are ignoring times
- Job with the largest $\frac{\text{weight}}{\text{length}}$?

  No, when weights are equal, this is the shortest job



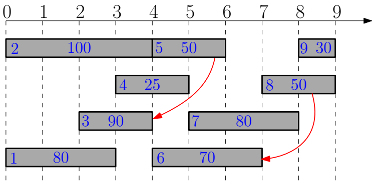
Designing a Dynamic Programming Algorithm



i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

Designing a Dynamic Programming Algorithm

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

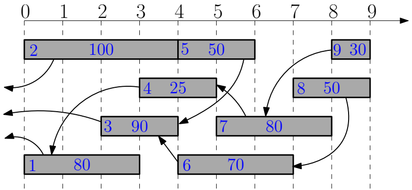
Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

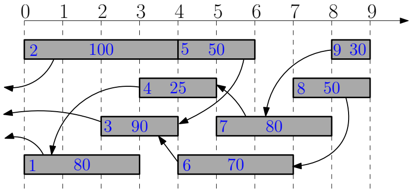


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\} = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



- $opt[0] = 0, \quad opt[1] = 80, \quad opt[2] = 100$
- $opt[3] = 100, \quad opt[4] = 105, \quad opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Recursive Algorithm to Compute $opt[n]$

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. return compute-opt(n)

compute-opt(i)

1. if $i = 0$ then
2.   return 0
3. else
4.   return $\max\{$compute-opt$(i-1), v_i +$ compute-opt$(p_i)\}$

- Running time can be exponential in n
- Reason: we are computed each $opt[i]$ many times
- Solution: store the value of $opt[i]$, so it's computed only once

Memoized Recursive Algorithm

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. $opt[0] \leftarrow 0$ and $opt[i] \leftarrow \perp$ for every $i = 1, 2, 3, \cdots, n$
4. return compute-opt(n)

compute-opt(i)

1. if $opt[i] = \perp$ then
2. $\quad opt[i] \leftarrow \max\{$compute-opt$(i-1), v_i +$ compute-opt$(p_i)\}$
3. return $opt[i]$

- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

Dynamic Programming

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. $opt[0] \leftarrow 0$
4. for $i \leftarrow 1$ to n
5. $\quad opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$

- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

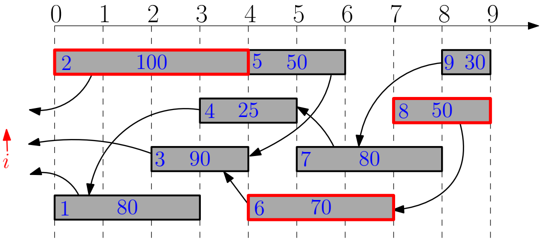
How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n
    if $opt[i-1] \geq v_i + opt[p_i]$
        $opt[i] \leftarrow opt[i-1]$
        $b[i] \leftarrow$ N
    else
        $opt[i] \leftarrow v_i + opt[p_i]$
        $b[i] \leftarrow$ Y

$i \leftarrow n, S \leftarrow \emptyset$
while $i \neq 0$
    if $b[i] =$ N
        $i \leftarrow i - 1$
    else
        $S \leftarrow S \cup \{i\}$
        $i \leftarrow p_i$
return S

Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Outline

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Q: What if we change "non-increasing" to "non-decreasing"?

A: No. $W = 100, n = 3, w = (1, 50, 50)$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Q: The value of the optimum solution that contains i?

A: $opt[i-1, W'-w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$
opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max\left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W' - w_i] + w_i \end{array} \right\} & i > 0, w_i \le W' \end{cases}
$$

Dynamic Programming

for $W' \leftarrow 0$ to W
    $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n
    for $W' \leftarrow 0$ to W
        $opt[i, W'] \leftarrow opt[i-1, W']$
        if $w_i \geq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ then
            $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
return $opt[n, W]$

Recover the Optimum Set

```
for W' ← 0 to W
    opt[0, W'] ← 0
for i ← 1 to n
    for W' ← 0 to W
        opt[i, W'] ← opt[i − 1, W']
        b[i, W'] ← N
        if w_i ≤ W' and opt[i − 1, W' − w_i] + w_i ≥ opt[i, W'] then
            opt[i, W'] ← opt[i − 1, W' − w_i] + w_i
            b[i, W'] ← Y
return opt[n, W]
```

Recover the Optimum Set

$i \leftarrow n, W' \leftarrow W, S \leftarrow \emptyset$
while $i > 0$
    if $b[i, W'] =$ Y then
        $W' \leftarrow W' - w_i$
        $S \leftarrow S \cup \{i\}$
    $i \leftarrow i - 1$
return S

Running Time of Algorithm

for $W' \leftarrow 0$ to W
    $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n
    for $W' \leftarrow 0$ to W
        $opt[i, W'] \leftarrow opt[i-1, W']$
        if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ then
            $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
return $opt[n, W]$

- Running time is $O(nW)$
- Running time is pseudo-polynomial because it depends on value of the input integers.

Avoiding Unncessary Computation and Memory Using Memoized Algorithm and Hash Map

compute-opt(i, W')

if $opt[i, W'] \neq \bot$ return $opt[i, W']$
if $i = 0$ then $r \leftarrow 0$
else
    $r \leftarrow$ compute-opt$(i-1, W')$
    if $w_i \leq W'$ then
        $r' \leftarrow$ compute-opt$(i-1, W'-w_i) + w_i$
        if $r' > r$ then $r \leftarrow r'$
$opt[i, W'] \leftarrow r$
return r

- Use hash map for opt

Outline

Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

- Motivation: you have budget W, and want to buy a subset of items of maximum total value

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.
- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \cdots, W$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max\left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W'-w_i] + v_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

Exercise: Items with 3 Parameters

Input: integer bounds $W > 0, Z > 0$,

a set of n items, each with an integer weight $w_i > 0$

a size $z_i > 0$ for each item i

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} v_i \qquad \text{s.t.}$$

$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} z_i \leq Z$$

Outline

Subsequence

- $A = bacdca$
- $C = adca$
- C is a subsequence of A

Def. Given two sequences $A[1 .. n]$ and $C[1 .. t]$ of letters, C is called a subsequence of A if there exists integers $1 \le i_1 < i_2 < i_3 < \ldots < i_t \le n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \cdots, t$.

- Exercise: how to check if sequence C is a subsequence of A?

Longest Common Subsequence

Input: $A[1 .. n]$ and $B[1 .. m]$

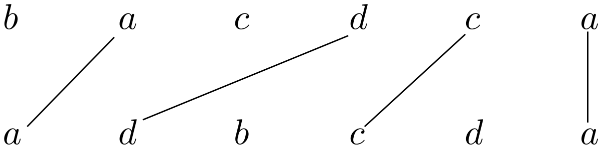
Output: the longest common subsequence of A and B

Example:

- $A = `bacdca'$
- $B = `adbcda'$
- $\text{LCS}(A, B) = `adca'$

- Applications: edit distance (diff), similarity of DNAs

Matching View of LCS



- Goal of LCS: find a maximum-size non-crossing matching between letters in A and letters in B.

Reduce to Subproblems

- $A = `bacdca'$
- $B = `adbcda'$
- either the last letter of A is not matched:
- need to compute LCS($`bacdc', `adbc'$)
- or the last letter of B is not matched:
- need to compute LCS($`bacd', `adbcd'$)

Dynamic Programming for LCS

- $opt[i,j], 0 \le i \le n, 0 \le j \le m$: length of longest common sub-sequence of $A[1..i]$ and $B[1..j]$.
- if $i = 0$ or $j = 0$, then $opt[i,j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i,j] = \begin{cases} opt[i-1,j-1]+1 & \text{if } A[i] = B[j] \\ \max\begin{cases} opt[i-1,j] \\ opt[i,j-1] \end{cases} & \text{if } A[i] \ne B[j] \end{cases}$$

Dynamic Programming for LCS

for $j \leftarrow 0$ to m do

  $opt[0, j] \leftarrow 0$

for $i \leftarrow 1$ to n

  $opt[i, 0] \leftarrow 0$

  for $j \leftarrow 1$ to m

    if $A[i] = B[j]$ then

      $opt[i, j] \leftarrow opt[i-1, j-1] + 1$, $\pi[i, j] \leftarrow$ "$\nwarrow$"

    elseif $opt[i, j-1] \geq opt[i-1, j]$ then

      $opt[i, j] \leftarrow opt[i, j-1]$, $\pi[i, j] \leftarrow$ "$\leftarrow$"

    else

      $opt[i, j] \leftarrow opt[i-1, j]$, $\pi[i, j] \leftarrow$ "$\uparrow$"

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	4 $\nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	4 $\nwarrow$

Find Common Subsequence

```
i ← n, j ← m, S ← ""
while i > 0 and j > 0
    if π[i, j] = "↖" then
        S ← A[i] ⋈ S, i ← i − 1, j ← j − 1
    else if π[i, j] = "↑"
        i ← i − 1
    else
        j ← j − 1
return S
```

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B?

Example:

- $A =$ ocurrance, $B =$ occurrence
- 3 operations: insert 'c', remove 'a' and insert 'e'

Obs. $\#$OPs $=$ length(A) $+$ length(B) - 2 $\cdot$ length(LCS(A, B))

Variants of Problem

Edit Distance with Insertions, Deletions and Replacing

Input: a string A,

each time we can delete a letter from A, insert a letter to A or change a letter

Output: how many operations do we need to change A to B?

Example:

- $A =$ ocurrance, $B =$ occurrence.
- 2 operations: insert 'c', change 'a' to 'e'

- Not related to LCS any more

Edit Distance (with Replacing)

- $opt[i,j], 0 \leq i \leq n, 0 \leq j \leq m$: edit distance between $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ then $opt[i,j] = j$; if $j = 0$ then $opt[i,j] = i$.
- if $i > 0, j > 0$, then

$$opt[i,j] = \begin{cases} opt[i-1,j-1] & \text{if } A[i] = B[j] \\ \min \begin{cases} opt[i-1,j]+1 \\ opt[i,j-1]+1 \\ opt[i-1,j-1]+1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Exercise: Longest Palindrome

Def. A palindrome is a string which reads the same backward or forward.

- example: "racecar", "wasitacaroracatisaw", "putitup"

Longest Palindrome Subsequence

Input: a sequence A

Output: the longest subsequence C of A that is a palindrome.

Example:

- Input: acbcedeacab
- Output: acedeca

Outline

Computing the Length of LCS

for $j \leftarrow 0$ to m do
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n
    $opt[i, 0] \leftarrow 0$
    for $j \leftarrow 1$ to m
        if $A[i] = B[j]$
            $opt[i, j] \leftarrow opt[i-1, j-1] + 1$
        elseif $opt[i, j-1] \geq opt[i-1, j]$
            $opt[i, j] \leftarrow opt[i, j-1]$
        else
            $opt[i, j] \leftarrow opt[i-1, j]$

Obs. The i-th row of table only depends on $(i-1)$-th row.

Reducing Space to $O(n+m)$

Obs. The i-th row of table only depends on $(i-1)$-th row.

Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i-1)$-th row and the i-th row.

Linear Space Algorithm to Compute Length of LCS

for $j \leftarrow 0$ to m do
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n
    $opt[i \text{ mod } 2, 0] \leftarrow 0$
    for $j \leftarrow 1$ to m
        if $A[i] = B[j]$
            $opt[i \text{ mod } 2, j] \leftarrow opt[i - 1 \text{ mod } 2, j - 1] + 1$
        elseif $opt[i \text{ mod } 2, j - 1] \geq opt[i - 1 \text{ mod } 2, j]$
            $opt[i \text{ mod } 2, j] \leftarrow opt[i \text{ mod } 2, j - 1]$
        else
            $opt[i \text{ mod } 2, j] \leftarrow opt[i - 1 \text{ mod } 2, j]$
return $opt[n \text{ mod } 2, m]$

How to Recover LCS Using Linear Space?

- Only keep the last two rows: only know how to match $A[n]$
- Can recover the LCS using n rounds: time $= O(n^2m)$
- Using Divide and Conquer + Dynamic Programming:
  - Space: $O(m+n)$
  - Time: $O(nm)$

Outline

Recall: Single Source Shortest Path Problem

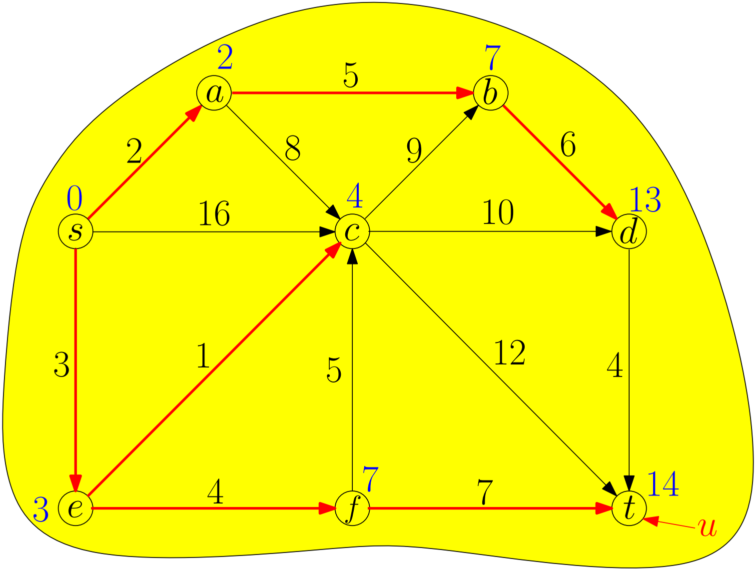
Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

- Algorithm for the problem: Dijkstra's algorithm


2
7
5
a
b
2
8
9
6
0
4
13
16
10
s
c
d
3
1
5
12
4
7
4
7
14
3
e
f
t
u

Dijkstra's Algorithm Using Priorty Queue

Dijkstra(G, w, s)

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $d(u) + w(u, v) < d(v)$ then
            $d(v) \leftarrow d(u) + w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return (π, d)

- Running time $= O(m + n \lg n)$.

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

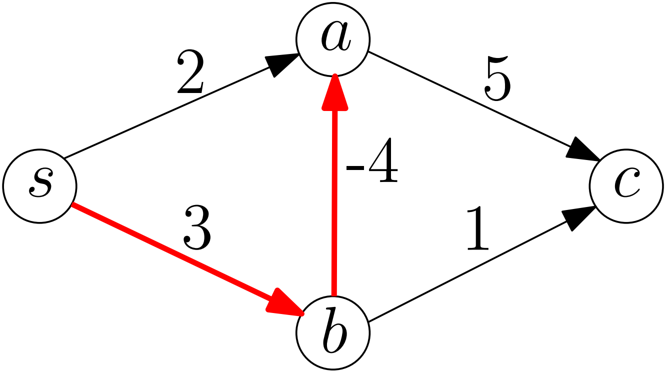
assume all vertices are reachable from s

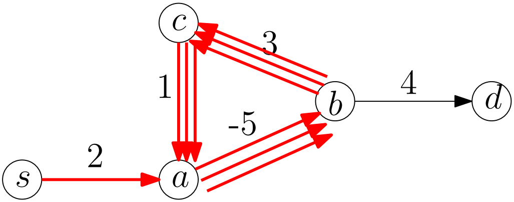
$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

Dijkstra's Algorithm Fails if We Have Negative Weights





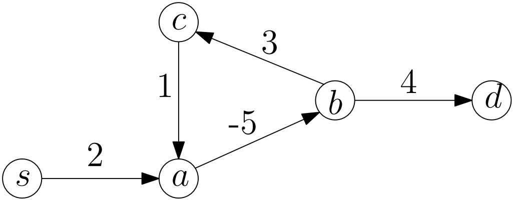
Q: What is the length of the shortest path from s to d?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"



Q: What is the length of the shortest simple path from s to d?

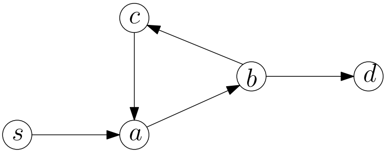
A: 1

- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

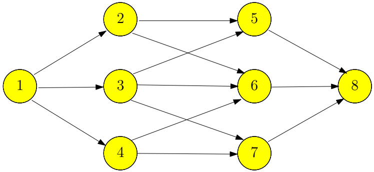
Outline

Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



not a DAG



a DAG

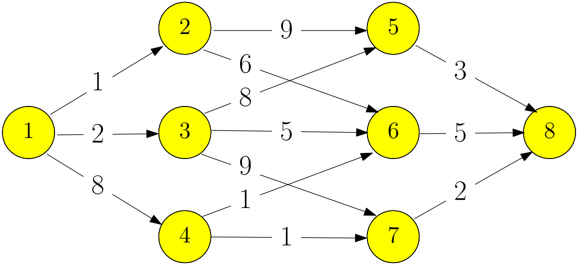
Lemma A directed graph is a DAG if and only its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i, for every $i \in V$



Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i)\in E} \{f(j) + w(j,i)\} & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

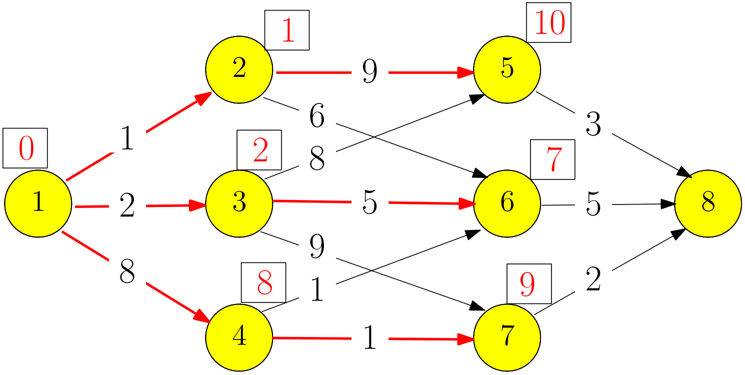
Shortest Paths in DAG

$f[1] \leftarrow 0$
for $i \leftarrow 2$ to n do
    $f[i] \leftarrow \infty$
    for each incoming edge $(j, i) \in E$ of i
        if $f[j] + w(j, i) < f[i]$
            $f[i] \leftarrow f[j] + w(j, i)$
            $\pi(i) \leftarrow j$

print-path(t)

if $t = 1$ then
    print(1)
    return
print-path$(\pi(t))$
print(",", t)

Example



Outline

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

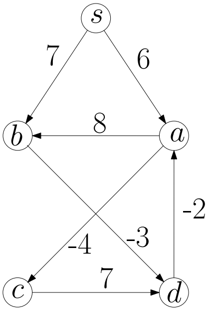
Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

- first try: $f[v]$: length of shortest path from s to v
- issue: do not know in which order we compute $f[v]$'s
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges



- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges

- $f^2[a] = 6$
- $f^3[a] = 2$

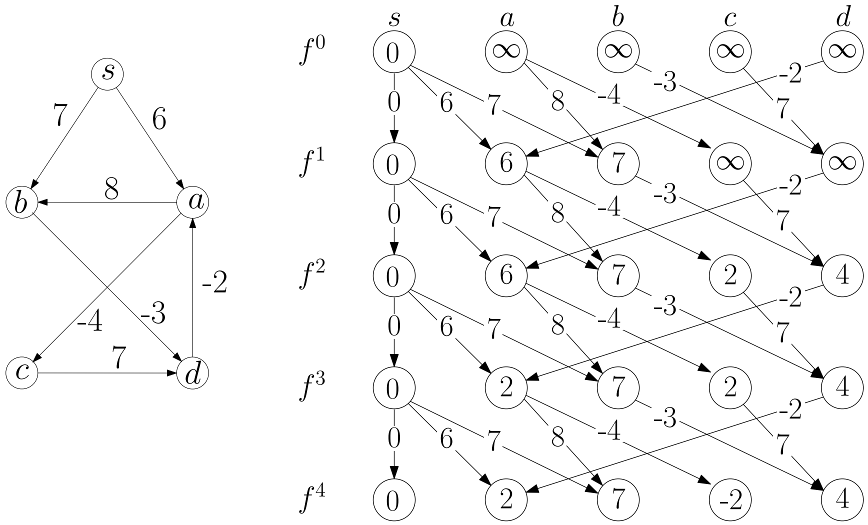
$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \begin{cases} f^{\ell-1}[v] \\ \min_{u:(u,v)\in E} \left(f^{\ell-1}[u] + w(u,v)\right) \end{cases} & \ell > 0 \end{cases}$$

dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ do
    copy $f^{\ell-1} \rightarrow f^{\ell}$
    for each $(u, v) \in E$
        if $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$
            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Dynamic Programming: Example



dynamic-programming(G, w, s)

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n-1$ do
3.   copy $f^{\ell-1} \rightarrow f^\ell$
4.   for each $(u, v) \in E$
5.   if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$
6. $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
7. return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Q: What if there are negative cycles?

Dynamic Programming With Negative Cycle Detection

dynamic-programming(G, w, s)

```
f^0[s] ← 0 and f^0[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^{ℓ−1} → f^ℓ
    for each (u, v) ∈ E
        if f^{ℓ−1}[u] + w(u, v) < f^ℓ[v]
            f^ℓ[v] ← f^{ℓ−1}[u] + w(u, v)
for each (u, v) ∈ E
    if f^{n−1}[u] + w(u, v) < f^{n−1}[v]
        report "negative cycle exists" and exit
return (f^{n−1}[v])_{v∈V}
```

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

1. $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n-1$ do
3.   for each $(u, v) \in E$
4.   if $f[u] + w(u,v) < f[v]$
5. $f[v] \leftarrow f[u] + w(u,v)$
6. return f

- Issue: when we compute $f[u] + w(u,v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!

- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ do
    for each $(u, v) \in E$
        if $f[u] + w(u, v) < f[v]$
            $f[v] \leftarrow f[u] + w(u, v)$
return f

- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v
- Assuming there are no negative cycles, after iteration $n-1$, $f[v] =$ length of shortest path from s to v

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

1. $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to n do
3. $updated \leftarrow$ false
4.   for each $(u, v) \in E$
5.   if $f[u] + w(u, v) < f[v]$
6. $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
7. $updated \leftarrow$ true
8.   if not $updated$, then return f
9. output "negative cycle exists"

- $\pi[v]$: the parent of v in the shortest path tree
- Running time $= O(nm)$

Outline

Matrix Chain Multiplication

Matrix Chain Multiplication

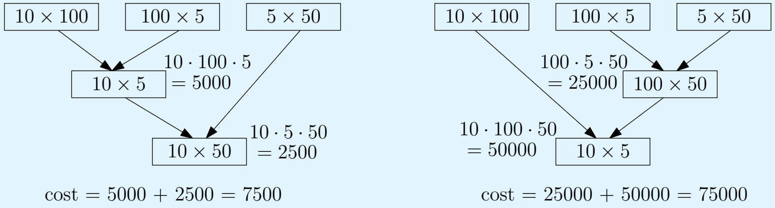
Input: n matrices $A_1, A_2, \cdots, A_n$ of sizes $r_1 \times c_1, r_2 \times c_2, \cdots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \cdots, n-1$.

Output: the order of computing $A_1 A_2 \cdots A_n$ with the minimum number of multiplications

Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$



- $(A_1 A_2) A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1 (A_2 A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Matrix Chain Multiplication: Design DP

- Assume the last step is $(A_1A_2\cdots A_i)(A_{i+1}A_{i+2}\cdots A_n)$
- Cost of last step: $r_1 \times c_i \times c_n$
- Optimality for sub-instances: we need to compute $A_1A_2\cdots A_i$ and $A_{i+1}A_{i+2}\cdots A_n$ optimally
- $opt[i,j]$: the minimum cost of computing $A_iA_{i+1}\cdots A_j$

$$opt[i,j] = \begin{cases} 0 & i = j \\ \min_{k:i\le k<j}\left(opt[i,k] + opt[k+1,j] + r_ic_kc_j\right) & i < j \end{cases}$$

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

```
let opt[i, i] ← 0 for every i = 1, 2, ··· , n
for ℓ ← 2 to n
    for i ← 1 to n − ℓ + 1
        j ← i + ℓ − 1
        opt[i, j] ← ∞
        for k ← i to j − 1
            if opt[i, k] + opt[k + 1, j] + r_i c_k c_j < opt[i, j]
                opt[i, j] ← opt[i, k] + opt[k + 1, j] + r_i c_k c_j
return opt[1, n]
```

Outline

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j]$ = value of instance defined by $A[1..i]$ and $B[1..j]$
- Matrix chain multiplication: $opt[i, j]$ = value of instances defined by matrices i to j
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Bellman-Ford: $f^\ell[v]$ = length of shortest path from s to v that uses at most ℓ edges