CSE 431/531: Algorithm Analysis and Design (Spring 2019)

Dynamic Programming

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University at Buffalo

Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many independent sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Recall: Computing the n-th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

Fib(n)

1. $F[0] \leftarrow 0$
2. $F[1] \leftarrow 1$
3. for $i \leftarrow 2$ to n do
4. $\quad F[i] \leftarrow F[i-1] + F[i-2]$
5. return $F[n]$

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- Store each $F[i]$ for future use.

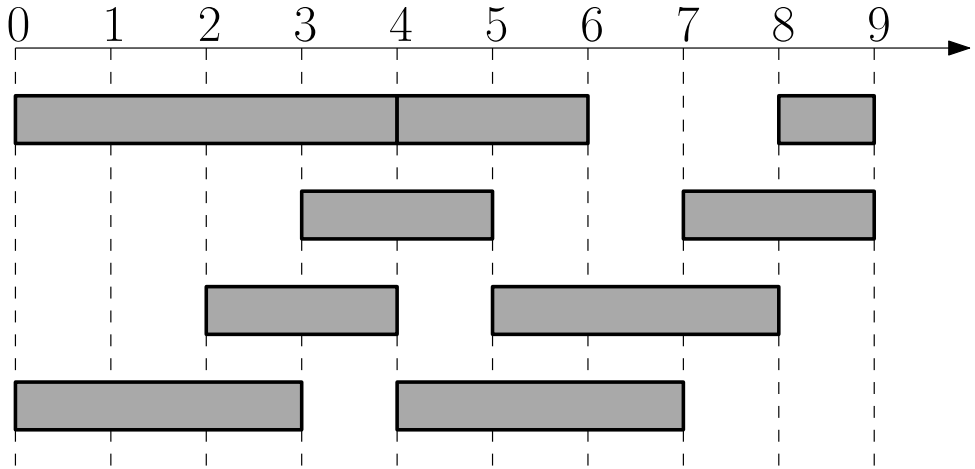
Outline

Recall: Interval Schduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs



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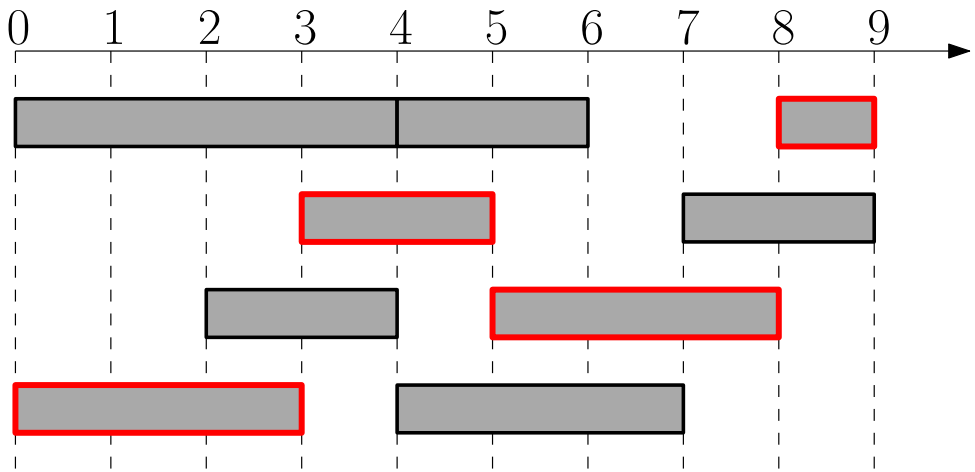

Weighted Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

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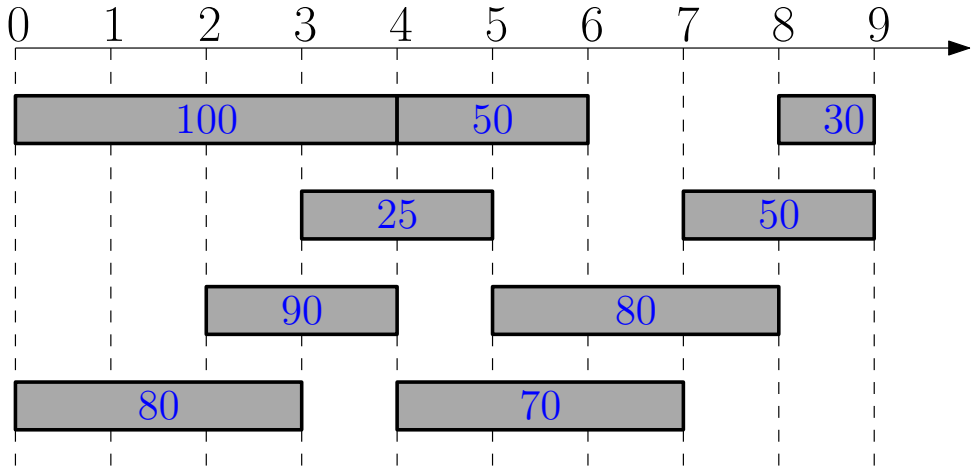

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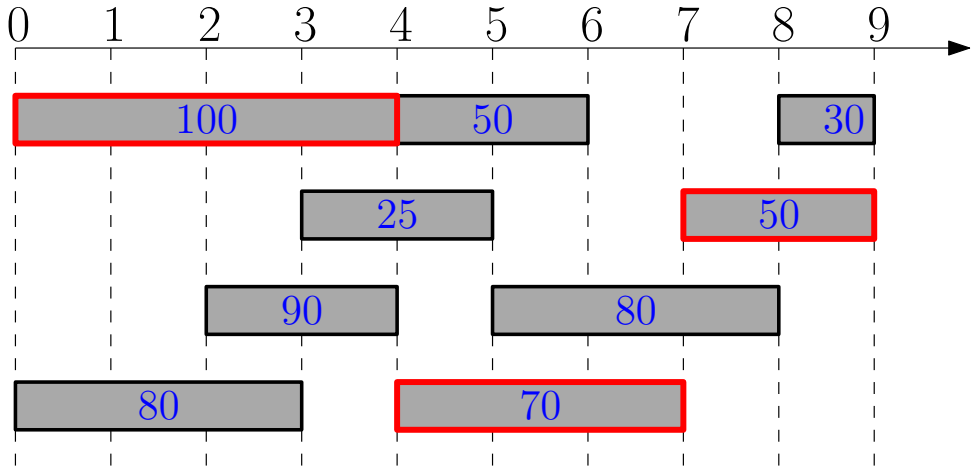

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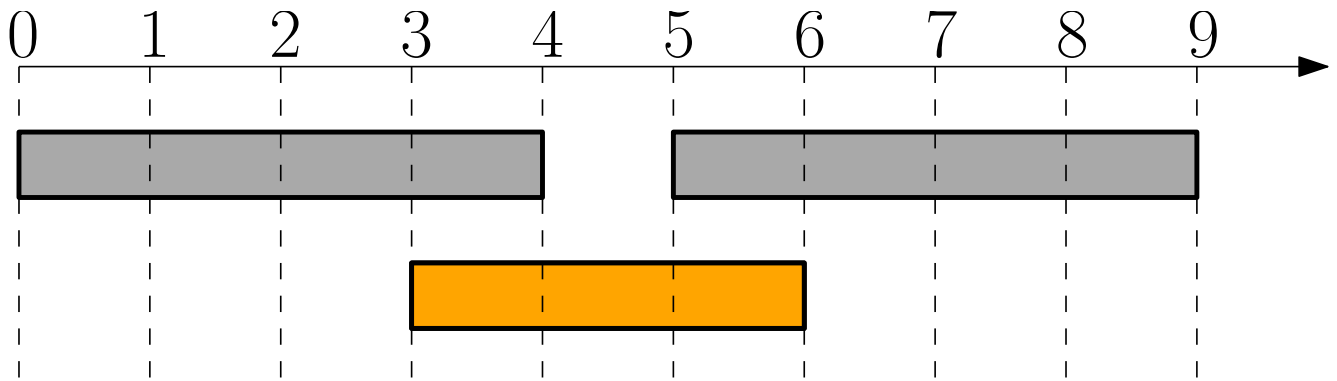

Designing a Dynamic Programming Algorithm

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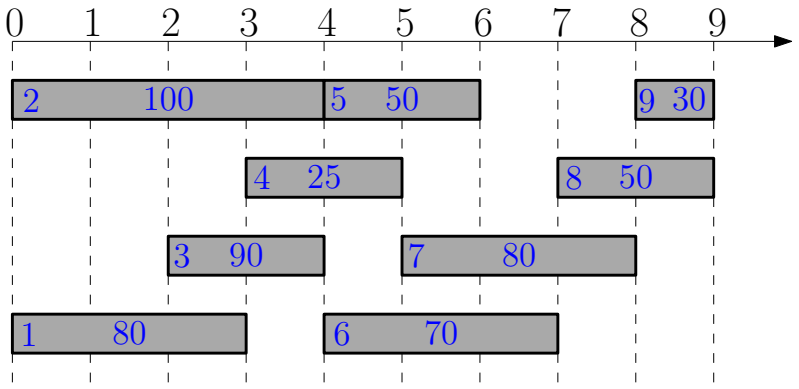

- Sort jobs according to non-decreasing order of finish times

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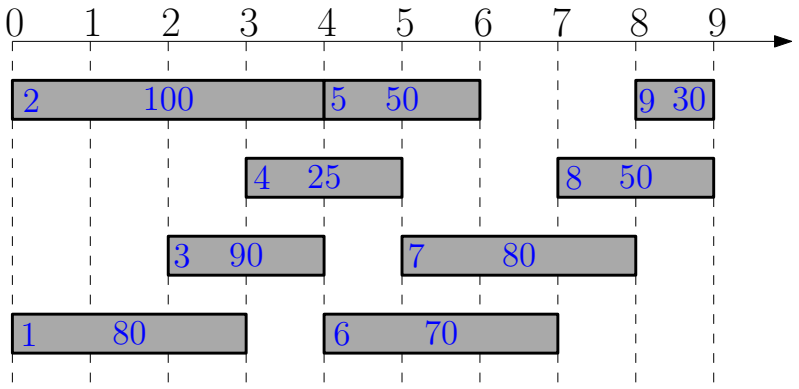

- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

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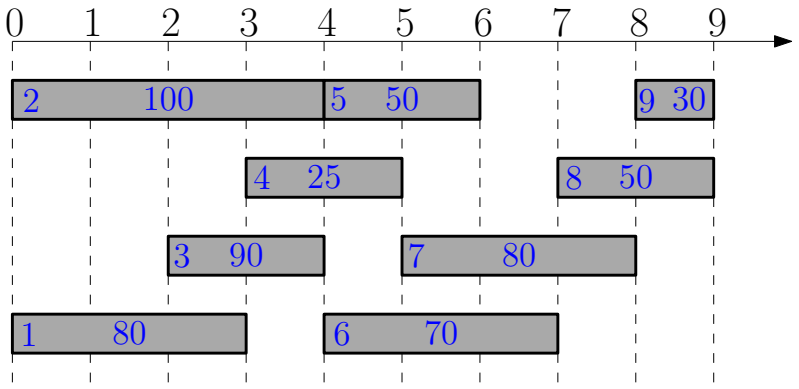

i	$opt[i]$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

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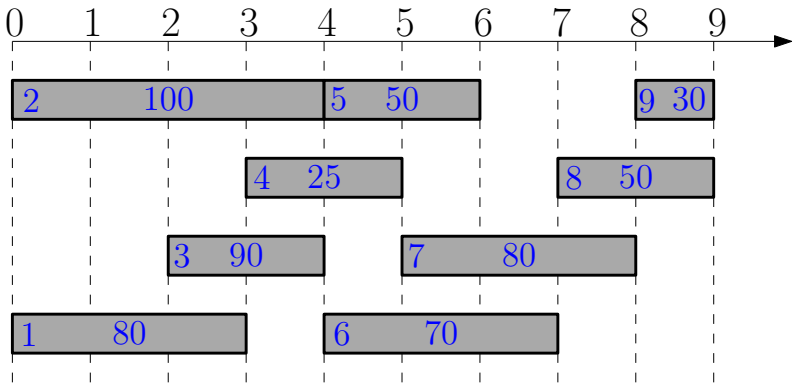

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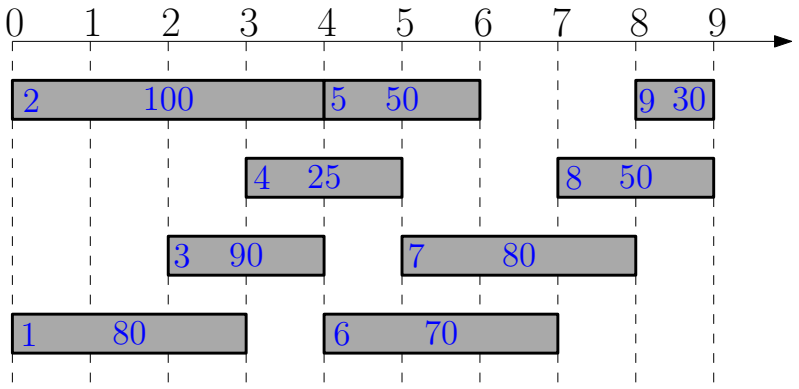

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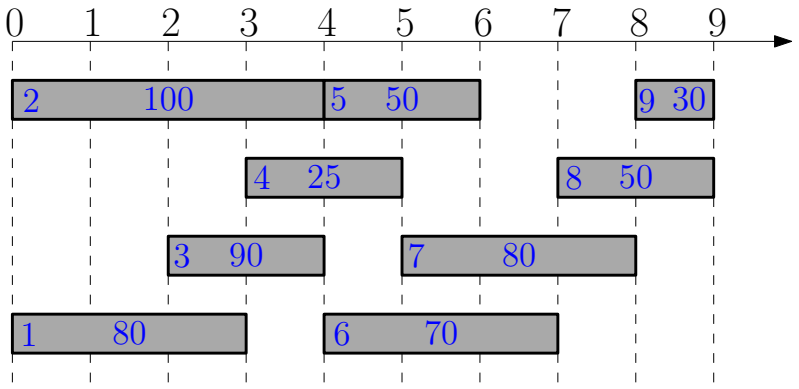
i	$opt[i]$
0	0
1	80
2	
3	
4	
5	
6	
7	
8	
9	

Designing a Dynamic Programming Algorithm



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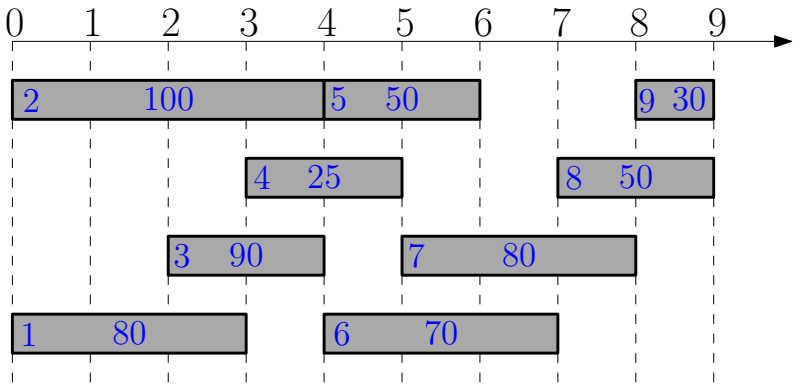

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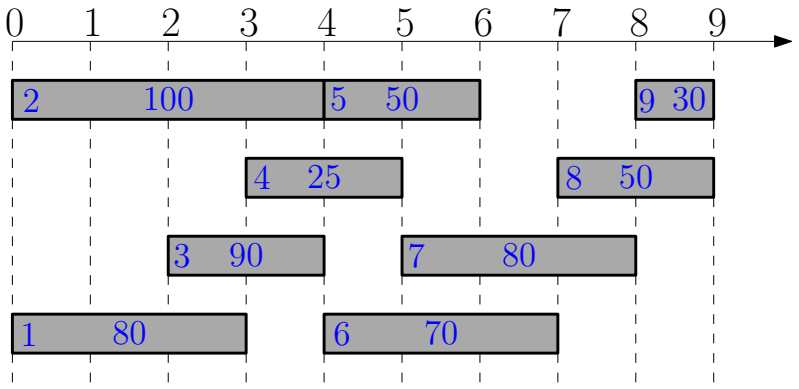

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i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

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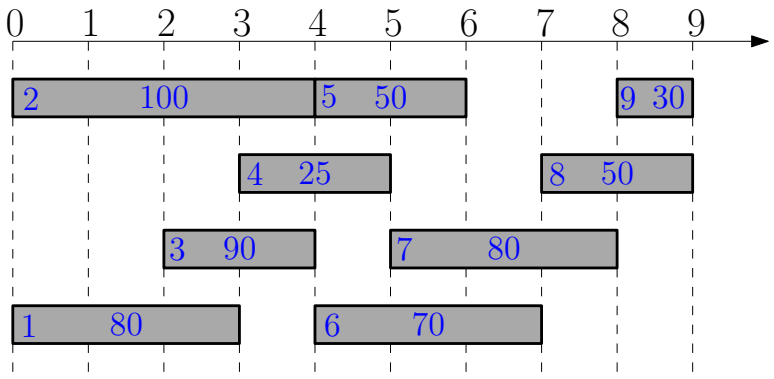

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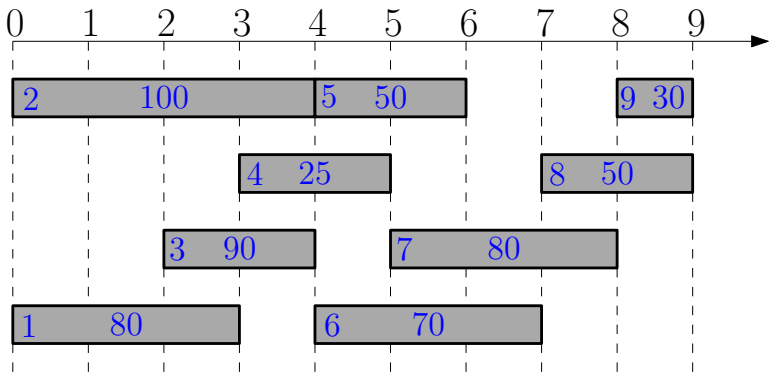

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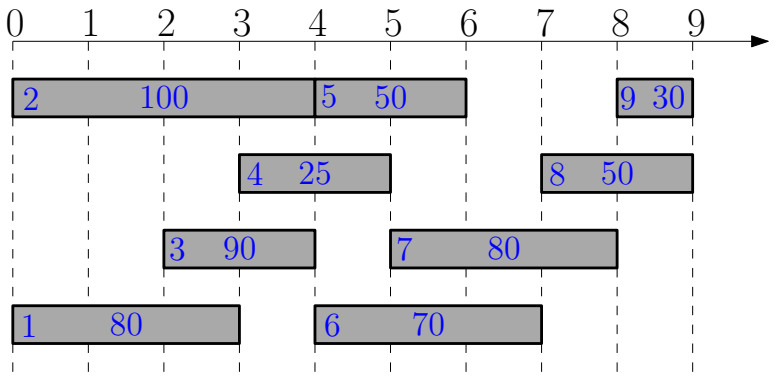

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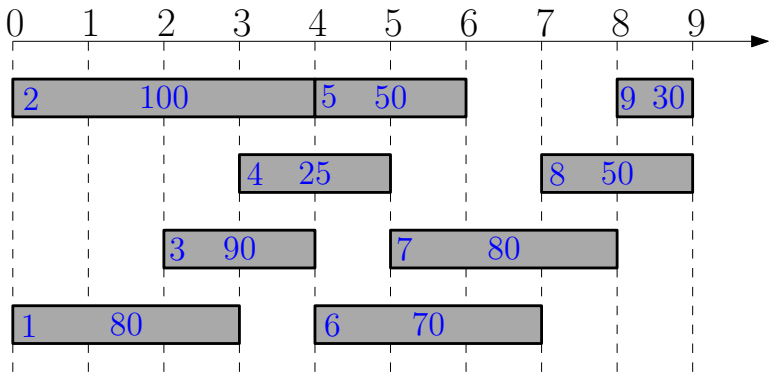

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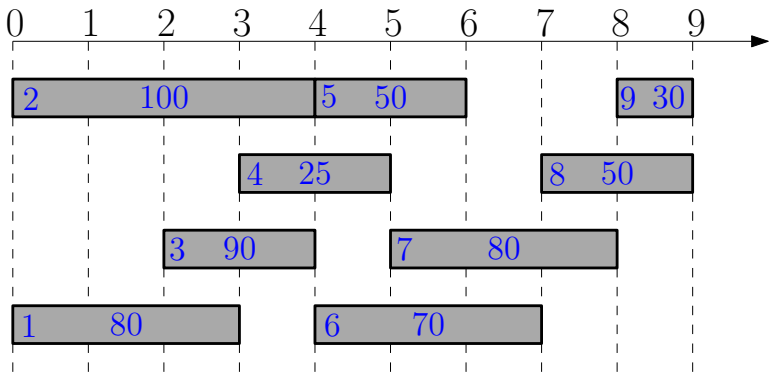

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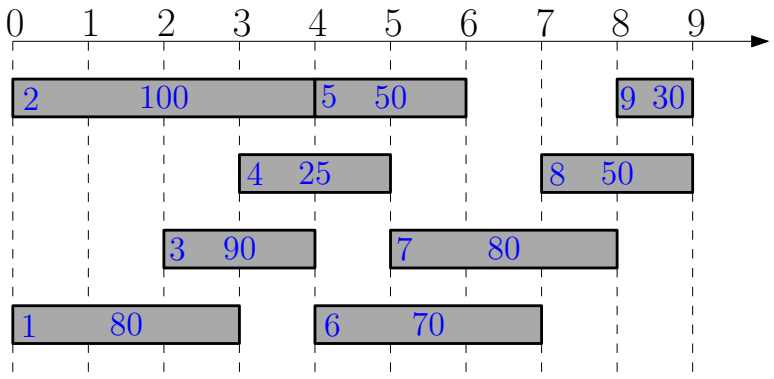
Designing a Dynamic Programming Algorithm



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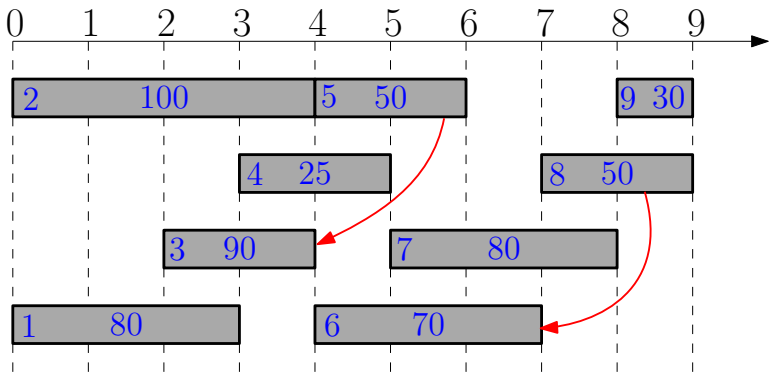
Designing a Dynamic Programming Algorithm



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Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

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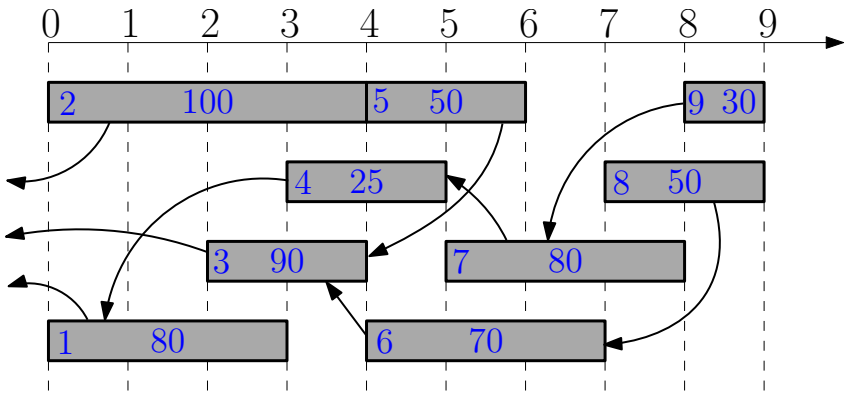

- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] =$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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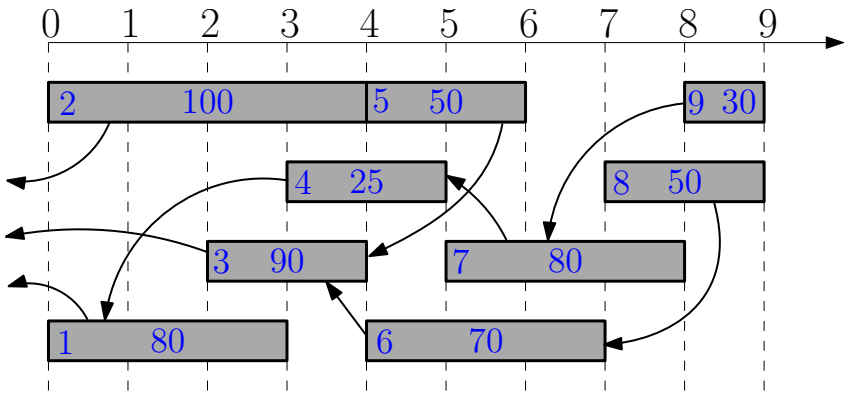

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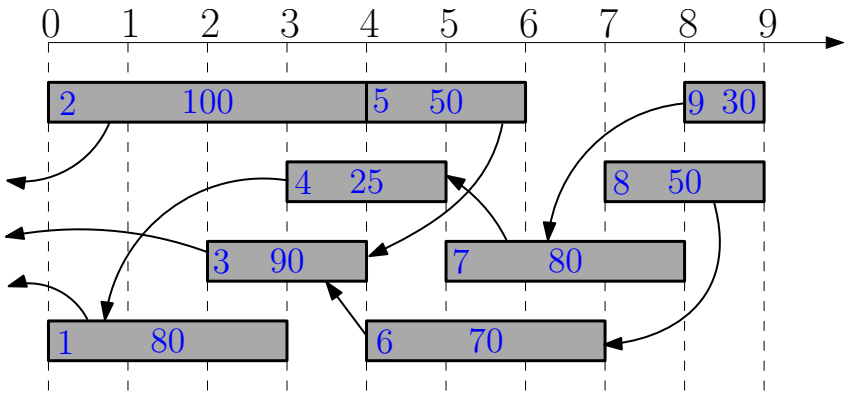

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- $opt[4] =$
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Designing a Dynamic Programming Algorithm

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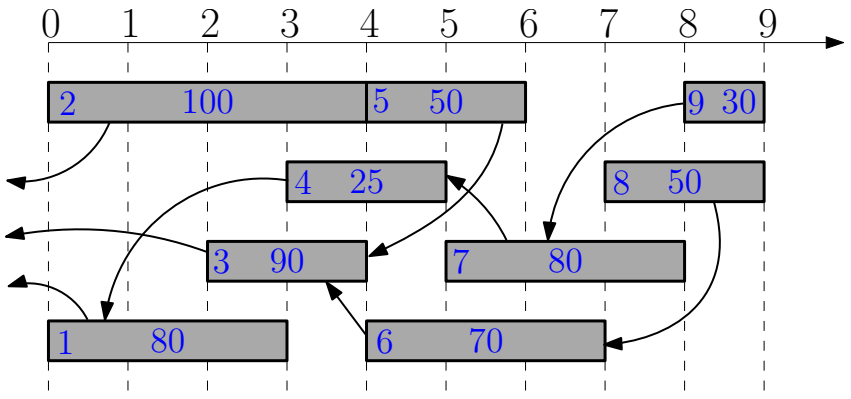

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- $opt[4] =$
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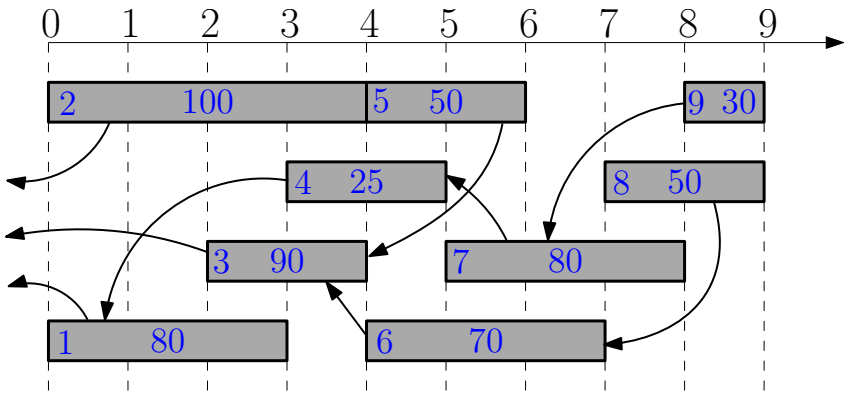

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- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\}$
- $opt[4] =$
- $opt[5] =$

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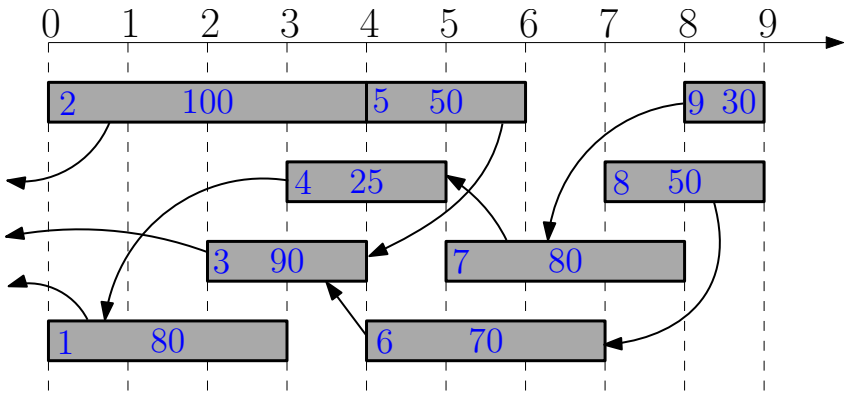

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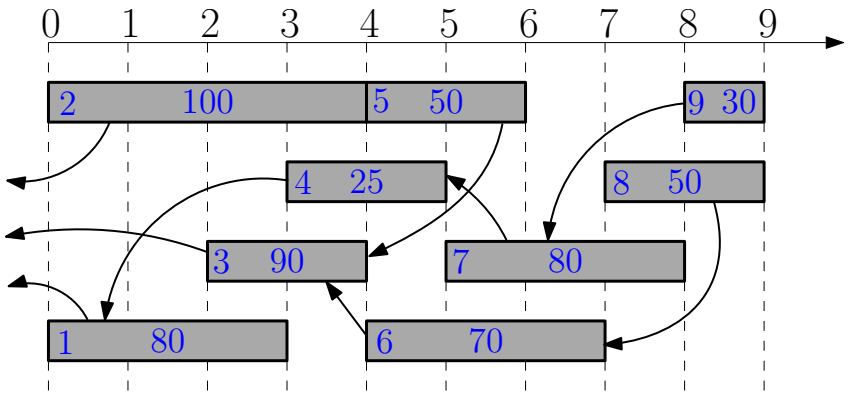

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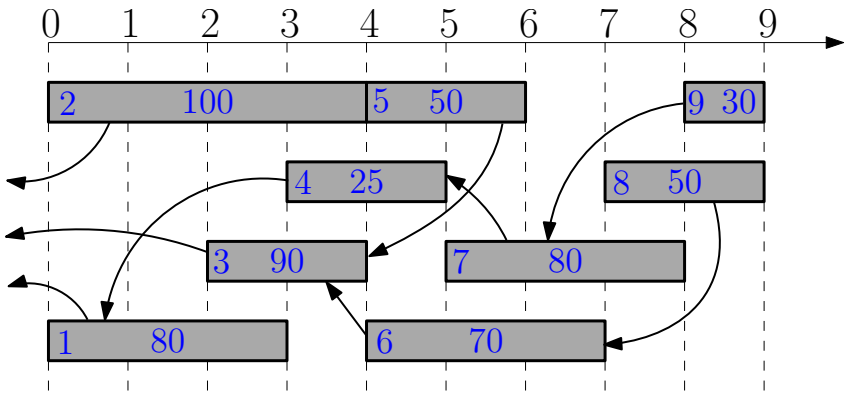
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\}$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



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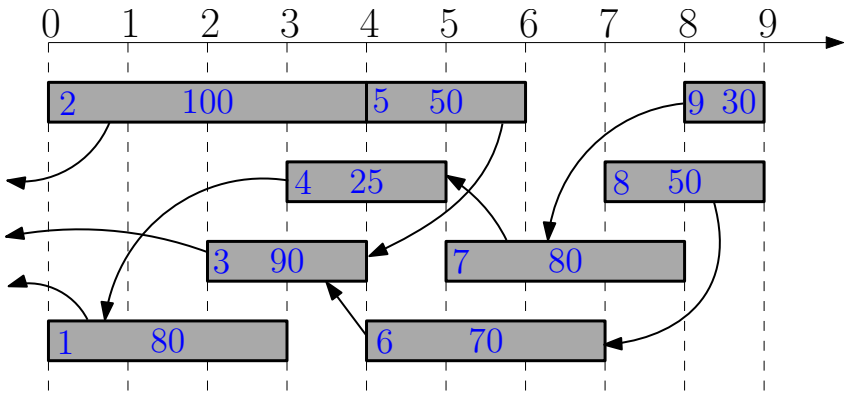

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- $opt[5] = \max\{opt[4], 50 + opt[3]\}$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

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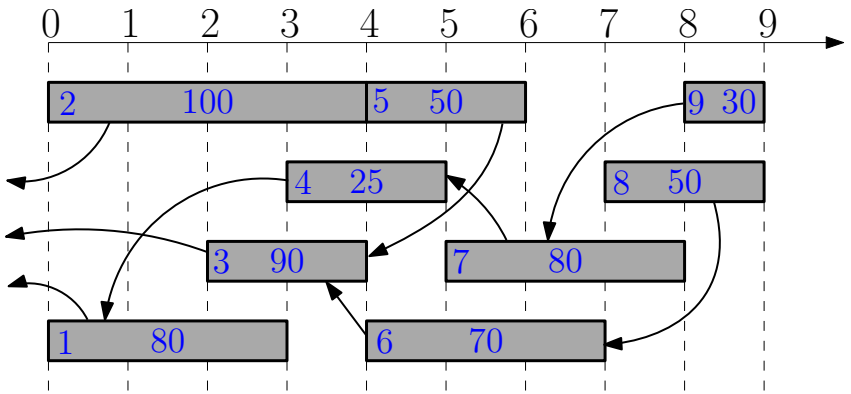

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Designing a Dynamic Programming Algorithm

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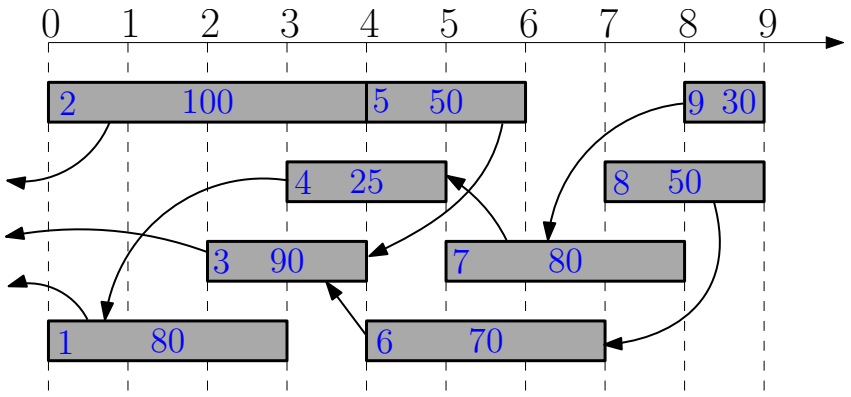

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Designing a Dynamic Programming Algorithm

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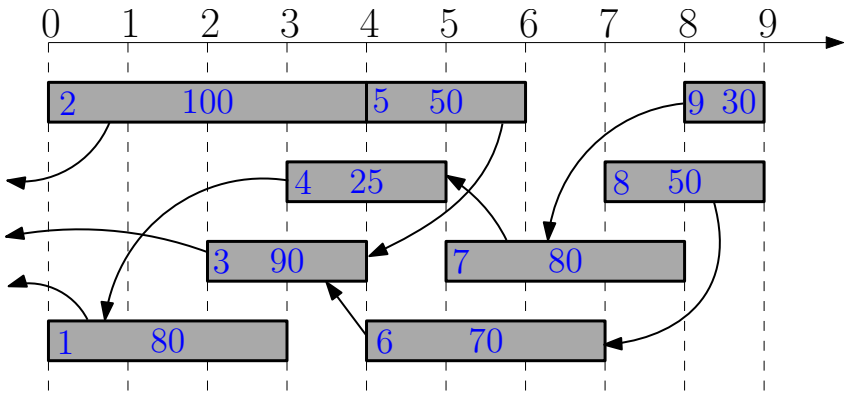

- $opt[0] = 0, \quad opt[1] = 80, \quad opt[2] = 100$
- $opt[3] = 100, \quad opt[4] = 105, \quad opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Recursive Algorithm to Compute $opt[n]$

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. return compute-opt(n)

compute-opt(i)

1. if $i = 0$ then
2. return 0
3. else
4. return $\max\{$compute-opt$(i-1), v_i +$ compute-opt$(p_i)\}$

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- Running time can be exponential in n
- Reason: we are computed each $opt[i]$ many times
- Solution: store the value of $opt[i]$, so it's computed only once

Memoized Recursive Algorithm

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. $opt[0] \leftarrow 0$ and $opt[i] \leftarrow \perp$ for every $i = 1, 2, 3, \cdots, n$
4. return compute-opt(n)

compute-opt(i)

1. if $opt[i] = \perp$ then
2. $\quad opt[i] \leftarrow \max\{$compute-opt$(i-1), v_i +$ compute-opt$(p_i)\}$
3. return $opt[i]$

Memoized Recursive Algorithm

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. $opt[0] \leftarrow 0$ and $opt[i] \leftarrow \perp$ for every $i = 1, 2, 3, \cdots, n$
4. return compute-opt(n)

compute-opt(i)

1. if $opt[i] = \perp$ then
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3. return $opt[i]$

- Running time sorting: $O(n \lg n)$

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- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search

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- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

Dynamic Programming

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. $opt[0] \leftarrow 0$
4. for $i \leftarrow 1$ to n
5. $\quad opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$

Dynamic Programming

1. sort jobs by non-decreasing order of finishing times
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- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

How Can We Recover the Optimum Schedule?

1. sort jobs by non-decreasing order of finishing times
2. compute $p_1, p_2, \cdots, p_n$
3. $opt[0] \leftarrow 0$
4. for $i \leftarrow 1$ to n
5. if $opt[i-1] \geq v_i + opt[p_i]$
6. $opt[i] \leftarrow opt[i-1]$
7.
8. else
9. $opt[i] \leftarrow v_i + opt[p_i]$
10.

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6. $opt[i] \leftarrow opt[i-1]$
7. $b[i] \leftarrow$ N
8. else
9. $opt[i] \leftarrow v_i + opt[p_i]$
10. $b[i] \leftarrow$ Y

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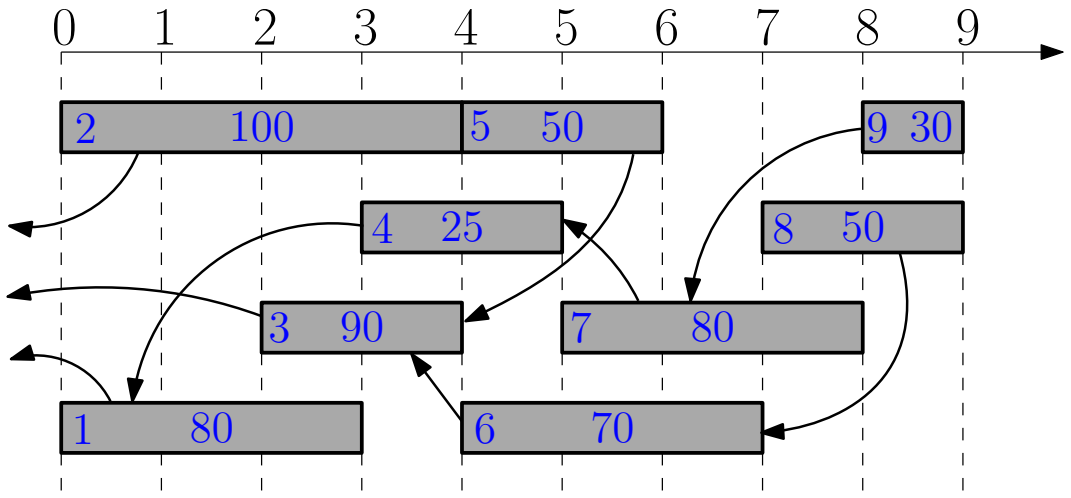
7. $b[i] \leftarrow$ N
8. else
9. $opt[i] \leftarrow v_i + opt[p_i]$
10. $b[i] \leftarrow$ Y

1. $i \leftarrow n, S \leftarrow \emptyset$
2. while $i \neq 0$
3. if $b[i] =$ N
4. $i \leftarrow i - 1$
5. else
6. $S \leftarrow S \cup \{i\}$
7. $i \leftarrow p_i$
8. return S

Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	
2	100	
3	100	
4	105	
5	150	
6	170	
7	185	
8	220	
9	220	



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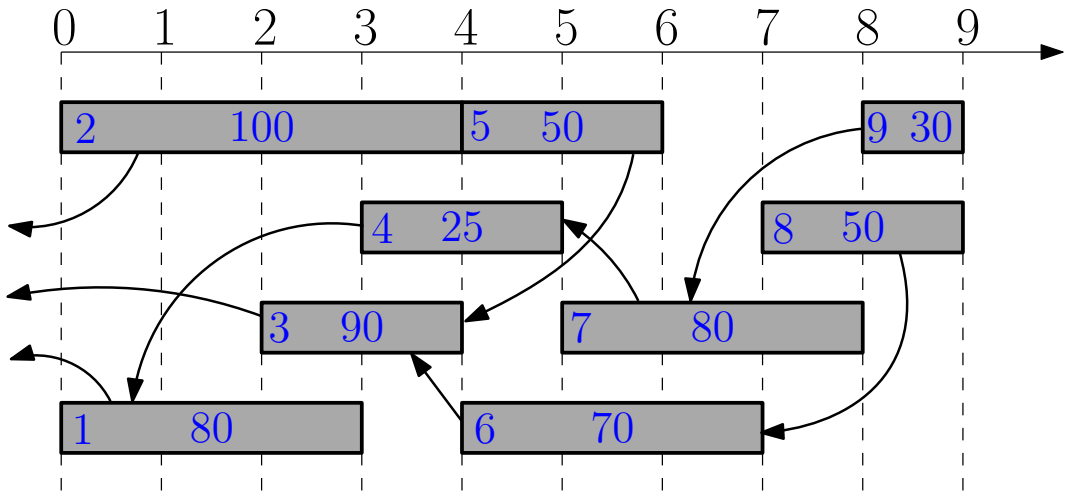

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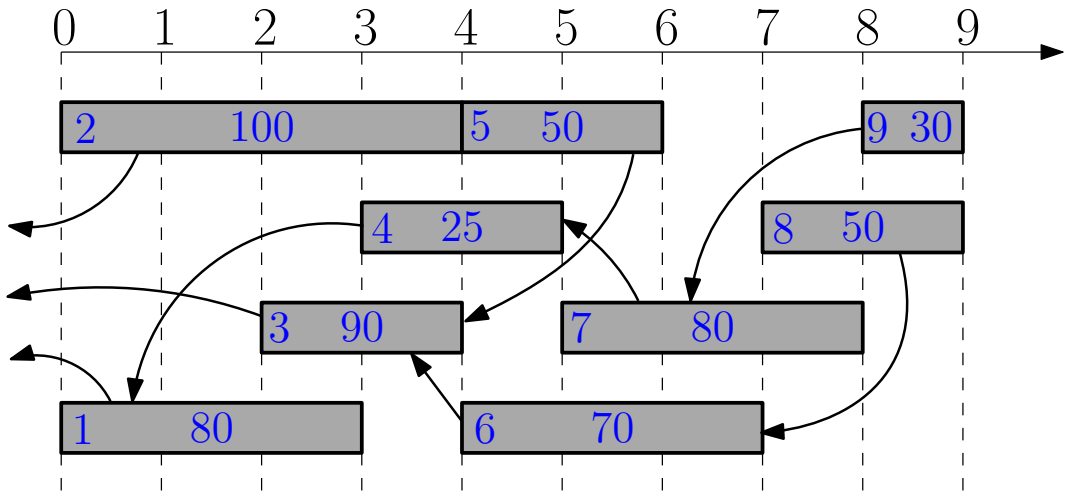

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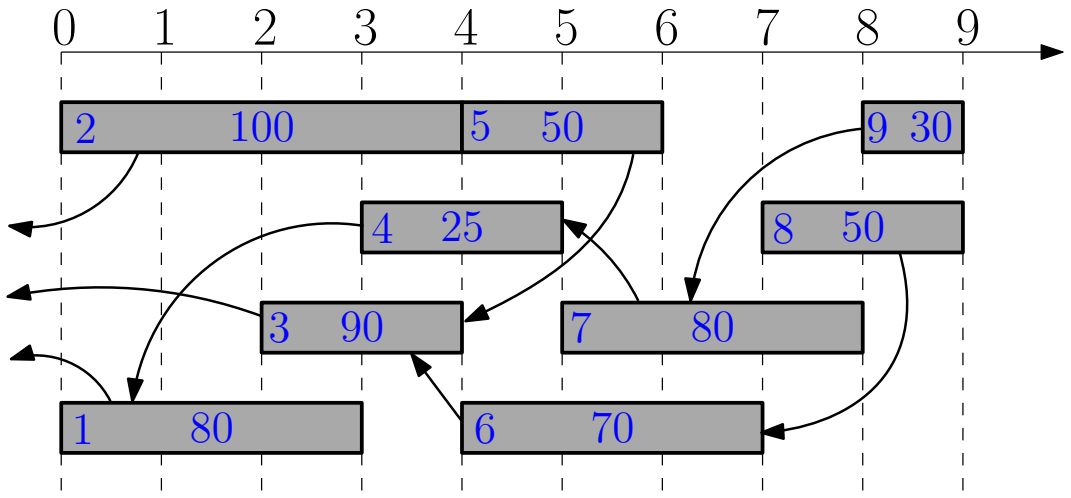

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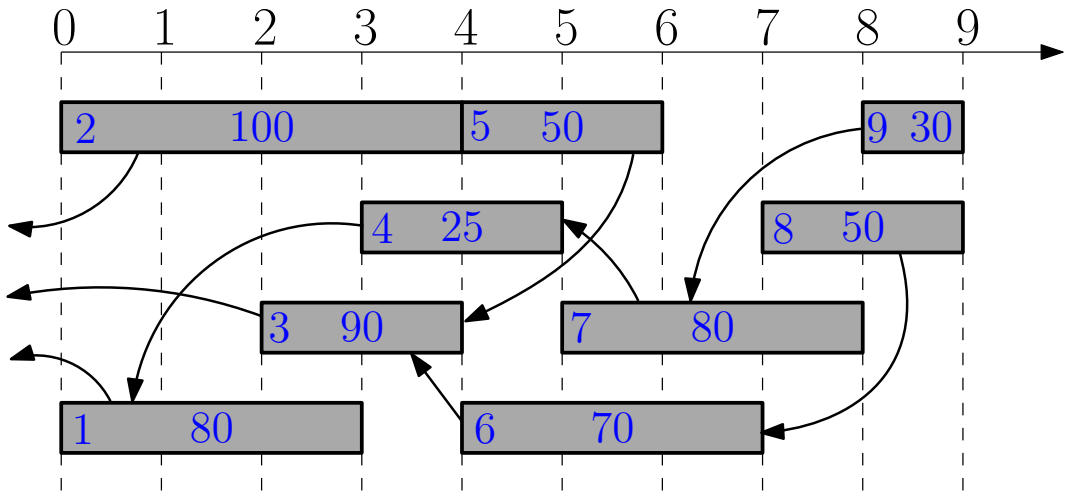

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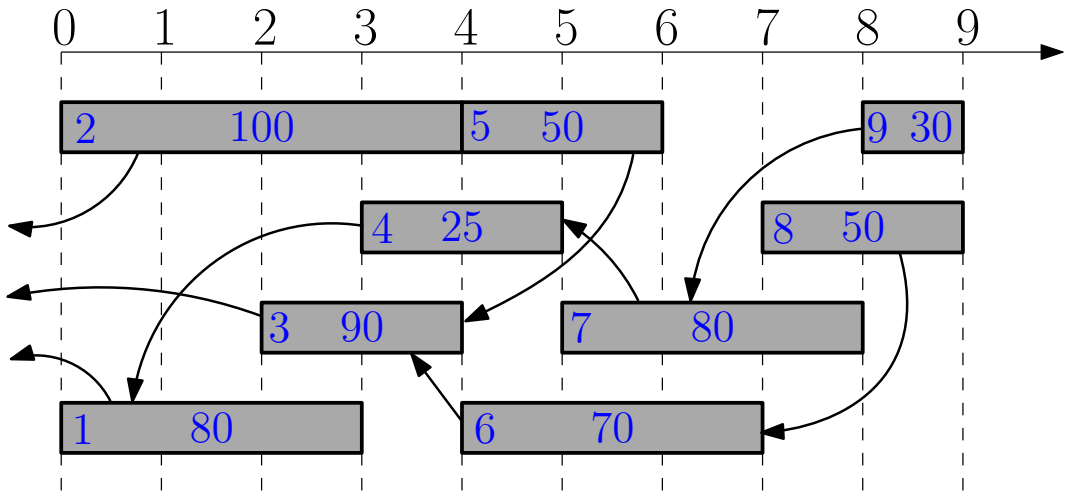

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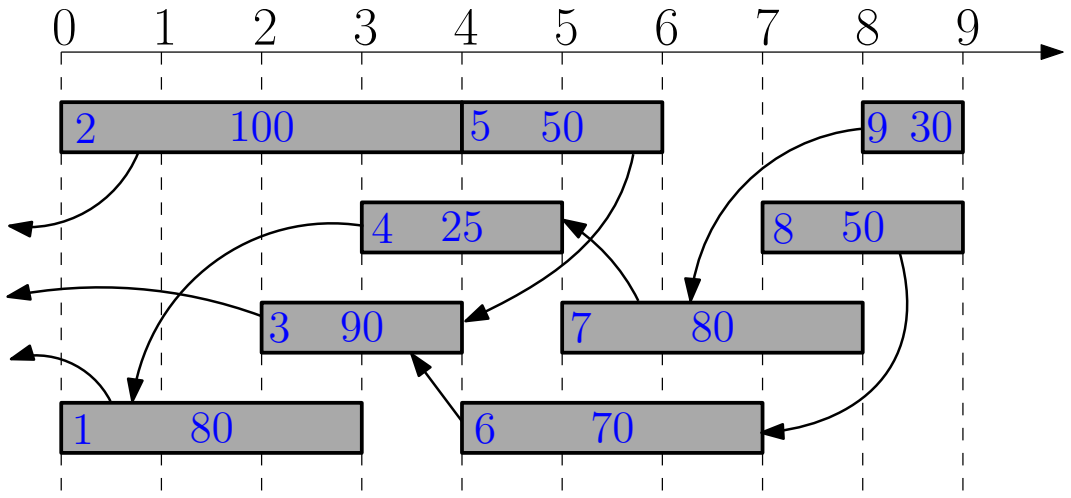

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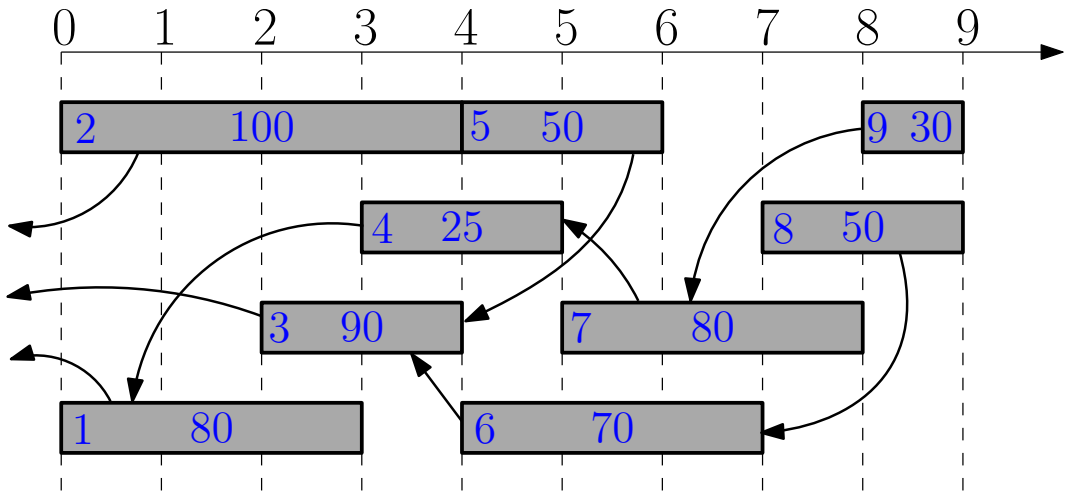

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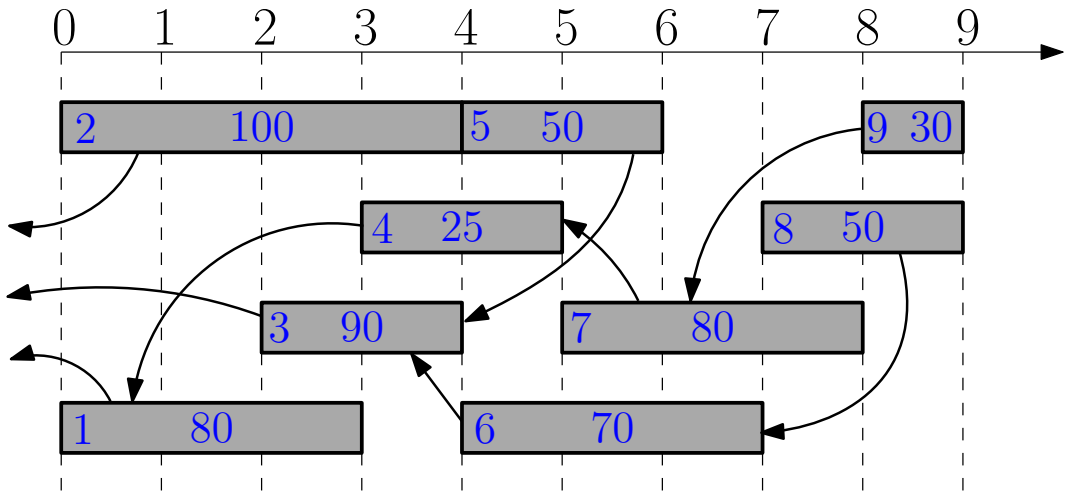

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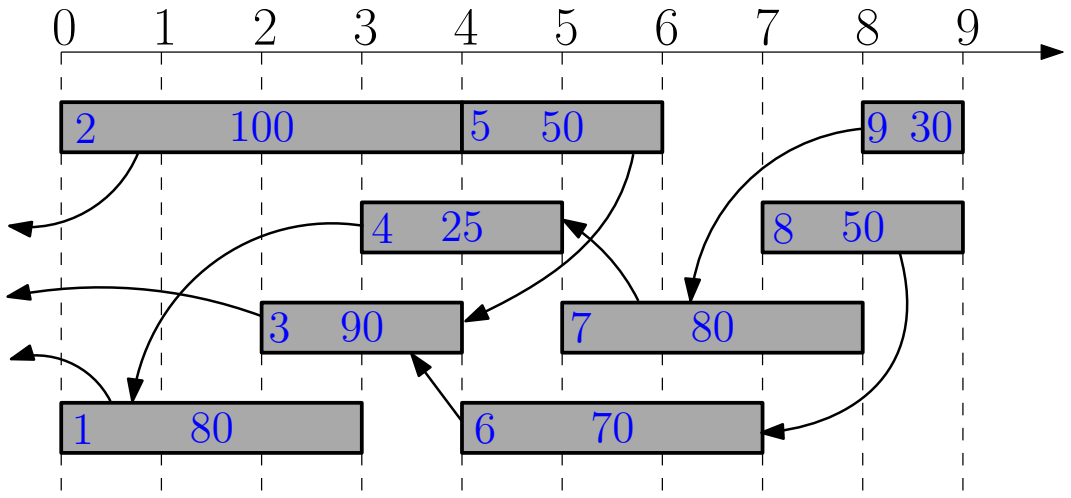

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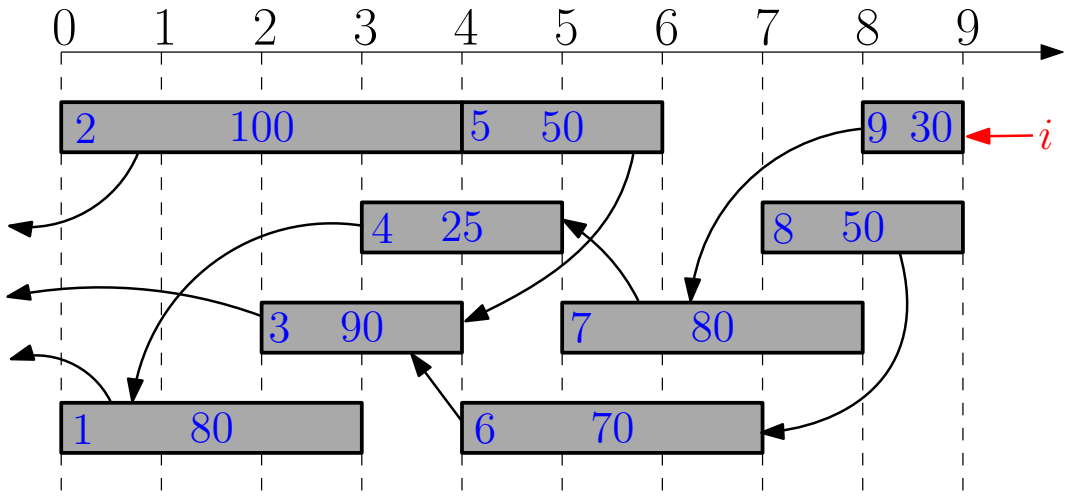

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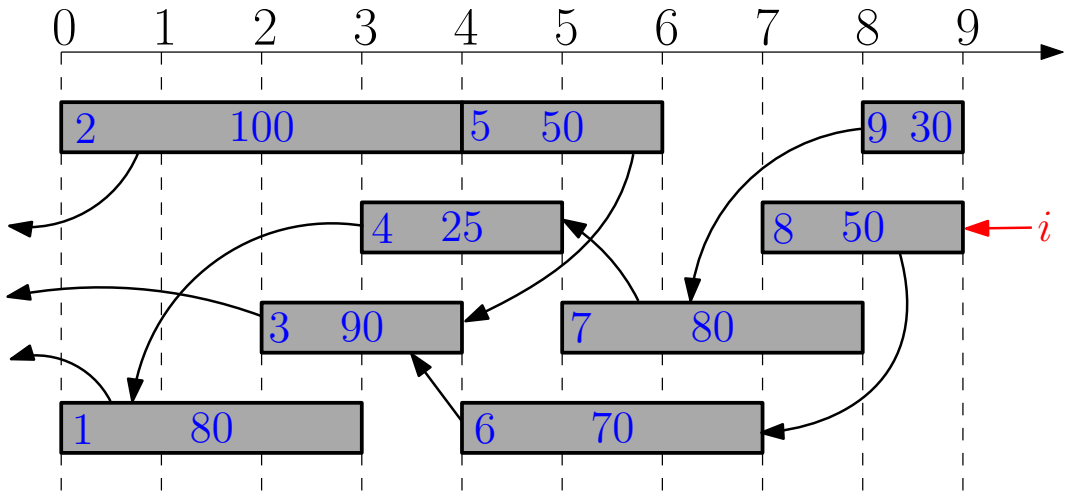

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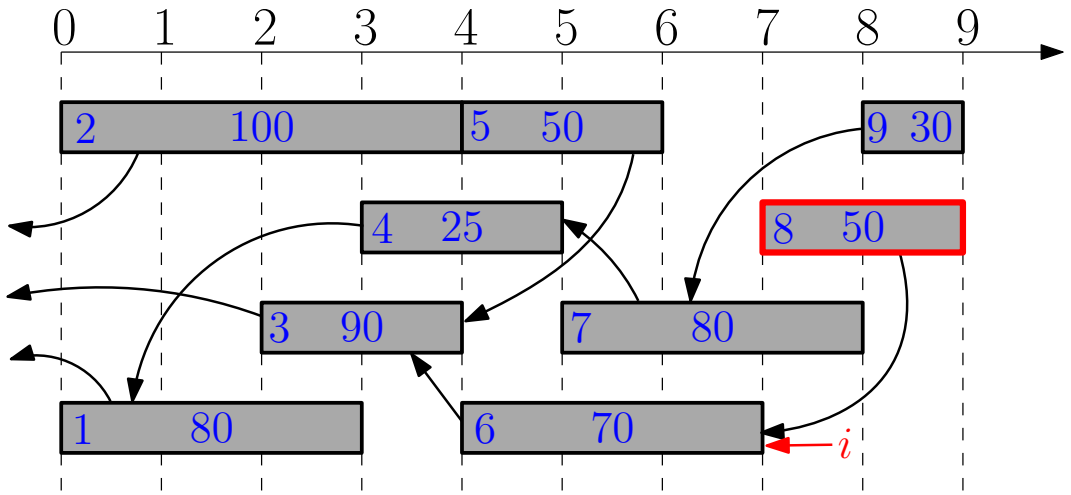

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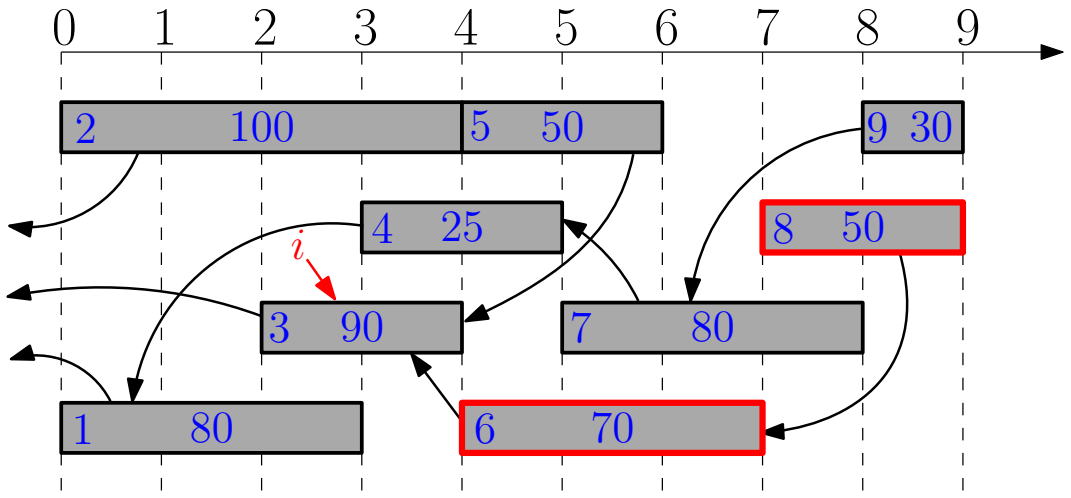

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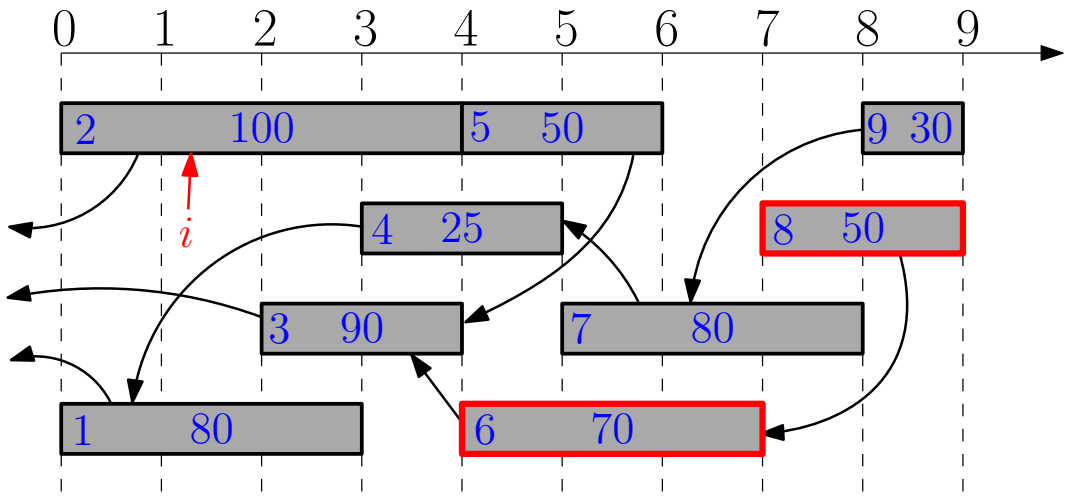

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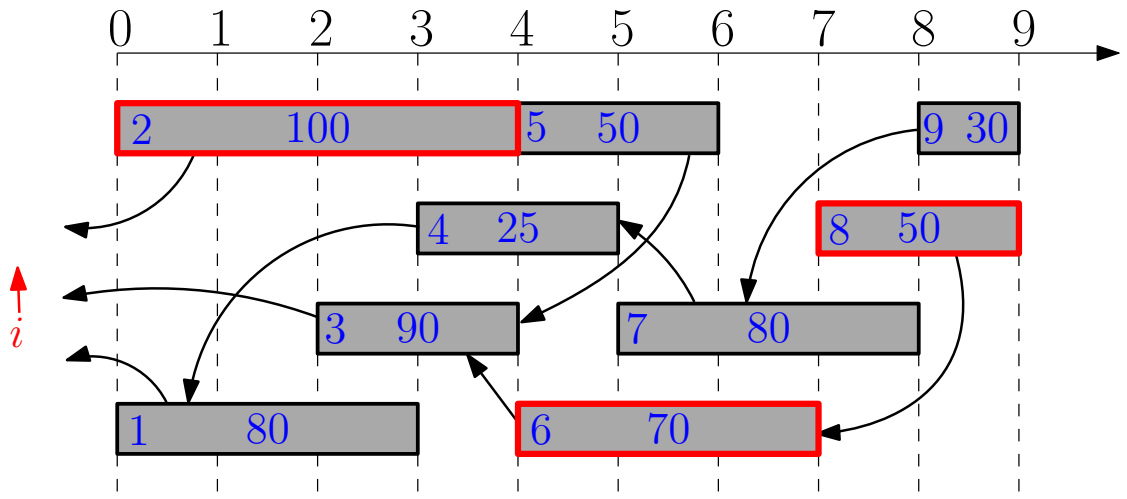

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Outline

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

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- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Subset Sum Problem

Input: an integer bound $W > 0$

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- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

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- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

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- Sort according to non-increasing order of weights
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A: No. $W = 100, n = 3, w = (1, 50, 50)$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

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Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Q: The value of the optimum solution that contains i?

A: $opt[i-1, W'-w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} & i = 0 \\ & i > 0, w_i > W' \\ & i > 0, w_i \leq W' \end{cases}$$

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Dynamic Programming

1. for $W' \leftarrow 0$ to W
2. $\quad opt[0, W'] \leftarrow 0$
3. for $i \leftarrow 1$ to n
4. $\quad$ for $W' \leftarrow 0$ to W
5. $\quad\quad opt[i, W'] \leftarrow opt[i-1, W']$
6. $\quad\quad$ if $w_i \geq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ then
7. $\quad\quad\quad opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
8. return $opt[n, W]$

Recover the Optimum Set

1. for $W' \leftarrow 0$ to W
2. $\quad opt[0, W'] \leftarrow 0$
3. for $i \leftarrow 1$ to n
4. $\quad$ for $W' \leftarrow 0$ to W
5. $\quad\quad opt[i, W'] \leftarrow opt[i-1, W']$
6. $\quad\quad b[i, W'] \leftarrow$ N
7. $\quad\quad$ if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ then
8. $\quad\quad\quad opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
9. $\quad\quad\quad b[i, W'] \leftarrow$ Y
10. return $opt[n, W]$

Recover the Optimum Set

1. $i \leftarrow n, W' \leftarrow W, S \leftarrow \emptyset$
2. while $i > 0$
3. if $b[i, W'] =$ Y then
4. $W' \leftarrow W' - w_i$
5. $S \leftarrow S \cup \{i\}$
6. $i \leftarrow i - 1$
7. return S

Running Time of Algorithm

1. for $W' \leftarrow 0$ to W
2. $\quad opt[0, W'] \leftarrow 0$
3. for $i \leftarrow 1$ to n
4. $\quad$ for $W' \leftarrow 0$ to W
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- Running time is $O(nW)$

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- Running time is $O(nW)$
- Running time is pseudo-polynomial because it depends on value of the input integers.

Avoiding Unncessary Computation and Memory Using Memoized Algorithm and Hash Map

compute-opt(i, W')

1. if $opt[i, W'] \neq \perp$ return $opt[i, W']$
2. if $i = 0$ then $r \leftarrow 0$
3. else
4. $\quad r \leftarrow$ compute-opt$(i - 1, W')$
5. $\quad$ if $w_i \leq W'$ then
6. $\quad\quad r' \leftarrow$ compute-opt$(i - 1, W' - w_i) + w_i$
7. $\quad\quad$ if $r' > r$ then $r \leftarrow r'$
8. $opt[i, W'] \leftarrow r$
9. return r

- Use hash map for opt

Outline

Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

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- Motivation: you have budget W, and want to buy a subset of items of maximum total value

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.
- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \cdots, W$.

$$opt[i, W'] = \begin{cases} & i = 0 \\ & i > 0, w_i > W' \\ & i > 0, w_i \leq W' \end{cases}$$

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Exercise: Items with 3 Parameters

Input: integer bounds $W > 0$, $Z > 0$,

a set of n items, each with an integer weight $w_i > 0$

a size $z_i > 0$ for each item i

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.}$$

$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} z_i \leq Z$$

Outline

Subsequence

- $A = bacdca$
- $C = adca$

Subsequence

- $A = bacdca$
- $C = adca$
- C is a subsequence of A

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Def. Given two sequences $A[1 \,..\, n]$ and $C[1 \,..\, t]$ of letters, C is called a subsequence of A if there exists integers $1 \leq i_1 < i_2 < i_3 < \ldots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \cdots, t$.

Subsequence

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Def. Given two sequences $A[1 \,..\, n]$ and $C[1 \,..\, t]$ of letters, C is called a subsequence of A if there exists integers $1 \leq i_1 < i_2 < i_3 < \ldots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \cdots, t$.

- Exercise: how to check if sequence C is a subsequence of A?

Longest Common Subsequence

Input: $A[1 \,..\, n]$ and $B[1 \,..\, m]$

Output: the longest common subsequence of A and B

Example:

- $A = `bacdca'$
- $B = `adbcda'$

Longest Common Subsequence

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Example:

- $A = `bacdca'$
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- $\mathsf{LCS}(A, B) = `adca'$

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Example:

- $A = `bacdca'$
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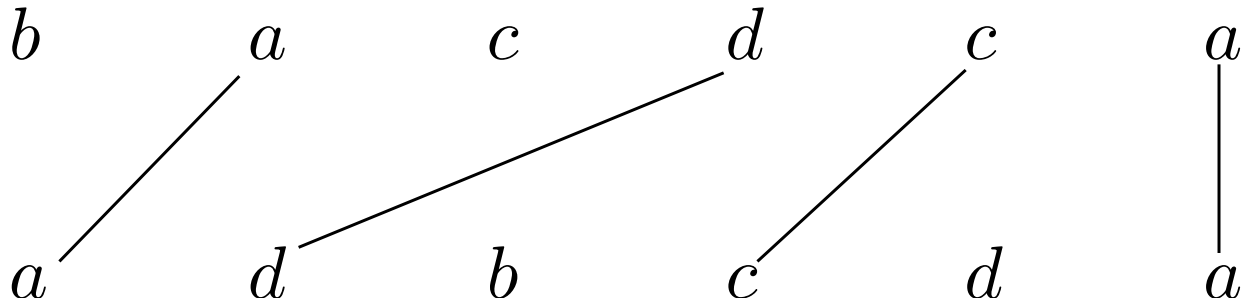
- $\mathsf{LCS}(A, B) = `adca'$

- Applications: edit distance (diff), similarity of DNAs

Matching View of LCS



- Goal of LCS: find a maximum-size non-crossing matching between letters in A and letters in B.

Reduce to Subproblems

- $A = `bacdca'$
- $B = `adbcda'$

Reduce to Subproblems

- $A = `bacdca'$
- $B = `adbcda'$

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$
- either the last letter of A is not matched:
- or the last letter of B is not matched:

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$
- either the last letter of A is not matched:
- need to compute LCS($`bacdc', `adbc'$)
- or the last letter of B is not matched:

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$
- either the last letter of A is not matched:
- need to compute LCS($`bacdc', `adbc'$)
- or the last letter of B is not matched:
- need to compute LCS($`bacd', `adbcd'$)

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 \,..\, i]$ and $B[1 \,..\, j]$.

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Dynamic Programming for LCS

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- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} & \text{if } A[i] = B[j] \\ & \text{if } A[i] \neq B[j] \end{cases}$$

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$$opt[i, j] = \begin{cases} opt[i-1, j-1] + 1 & \text{if } A[i] = B[j] \\ & \\ & \text{if } A[i] \neq B[j] \end{cases}$$

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$$opt[i, j] = \begin{cases} opt[i-1, j-1] + 1 & \text{if } A[i] = B[j] \\ \max \begin{cases} opt[i-1, j] \\ opt[i, j-1] \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

1. for $j \leftarrow 0$ to m do
2. $\quad opt[0, j] \leftarrow 0$
3. for $i \leftarrow 1$ to n
4. $\quad opt[i, 0] \leftarrow 0$
5. $\quad$ for $j \leftarrow 1$ to m
6. $\quad\quad$ if $A[i] = B[j]$ then
7. $\quad\quad\quad opt[i, j] \leftarrow opt[i-1, j-1] + 1$
8. $\quad\quad$ elseif $opt[i, j-1] \geq opt[i-1, j]$ then
9. $\quad\quad\quad opt[i, j] \leftarrow opt[i, j-1]$
10. $\quad\quad$ else
11. $\quad\quad\quad opt[i, j] \leftarrow opt[i-1, j]$

Dynamic Programming for LCS

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2. $\quad opt[0, j] \leftarrow 0$
3. for $i \leftarrow 1$ to n
4. $\quad opt[i, 0] \leftarrow 0$
5. $\quad$ for $j \leftarrow 1$ to m
6. $\quad\quad$ if $A[i] = B[j]$ then
7. $\quad\quad\quad opt[i, j] \leftarrow opt[i-1, j-1] + 1$, $\pi[i, j] \leftarrow$ "$\nwarrow$"
8. $\quad\quad$ elseif $opt[i, j-1] \geq opt[i-1, j]$ then
9. $\quad\quad\quad opt[i, j] \leftarrow opt[i, j-1]$, $\pi[i, j] \leftarrow$"$\leftarrow$"
10. $\quad\quad$ else
11. $\quad\quad\quad opt[i, j] \leftarrow opt[i-1, j]$, $\pi[i, j] \leftarrow$ "$\uparrow$"

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$						
2	0 $\perp$						
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2	0 $\perp$	1 $\nwarrow$					
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Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$		
3	0 $\perp$						
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	
3	0 $\perp$						
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$						
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$						
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$					
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$				
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

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A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$			
4	0 $\perp$						
5	0 $\perp$						
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Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$		
4	0 $\perp$						
5	0 $\perp$						
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Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	
4	0 $\perp$						
5	0 $\perp$						
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Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$						
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$					
5	0 $\perp$						
6	0 $\perp$						

Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$				
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$			
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$		
5	0 $\perp$						
6	0 $\perp$						

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	
5	0 $\perp$						
6	0 $\perp$						

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$						
6	0 $\perp$						

Example

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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$						
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Example

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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$					
6	0 $\perp$						

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$				
6	0 $\perp$						

Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$			
6	0 $\perp$						

Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$		
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	
6	0 $\perp$						

Example

	1	2	3	4	5	6
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$						

Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$						

Example

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$					

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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
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4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$				

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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
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4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$			

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$		

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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	4 $\nwarrow$

Example: Find Common Subsequence

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A	b	a	c	d	c	a
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	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
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Find Common Subsequence

1. $i \leftarrow n, j \leftarrow m, S \leftarrow$ ""
2. while $i > 0$ and $j > 0$
3. if $\pi[i, j] =$ "$\nwarrow$" then
4. $S \leftarrow A[i] \bowtie S, i \leftarrow i - 1, j \leftarrow j - 1$
5. else if $\pi[i, j] =$ "$\uparrow$"
6. $i \leftarrow i - 1$
7. else
8. $j \leftarrow j - 1$
9. return S

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B?

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Obs. $\#\text{OPs} = \text{length}(A) + \text{length}(B) - 2 \cdot \text{length}(\text{LCS}(A, B))$

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Example:

- $A =$ ocurrance, $B =$ occurrence.
- 2 operations: insert 'c', change 'a' to 'e'

- Not related to LCS any more

Edit Distance (with Replacing)

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Def. A palindrome is a string which reads the same backward or forward.

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Example:

- Input: acbcedeacab
- Output: acedeca

Outline

Computing the Length of LCS

1. for $j \leftarrow 0$ to m do
2. $\quad opt[0, j] \leftarrow 0$
3. for $i \leftarrow 1$ to n
4. $\quad opt[i, 0] \leftarrow 0$
5. $\quad$ for $j \leftarrow 1$ to m
6. $\quad\quad$ if $A[i] = B[j]$
7. $\quad\quad\quad opt[i, j] \leftarrow opt[i-1, j-1] + 1$
8. $\quad\quad$ elseif $opt[i, j-1] \geq opt[i-1, j]$
9. $\quad\quad\quad opt[i, j] \leftarrow opt[i, j-1]$
10. $\quad\quad$ else
11. $\quad\quad\quad opt[i, j] \leftarrow opt[i-1, j]$

Obs. The i-th row of table only depends on $(i-1)$-th row.

Reducing Space to $O(n+m)$

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Q: How to use this observation to reduce space?

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Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i-1)$-th row and the i-th row.

Linear Space Algorithm to Compute Length of LCS

1. for $j \leftarrow 0$ to m do
2. $\quad opt[0, j] \leftarrow 0$
3. for $i \leftarrow 1$ to n
4. $\quad opt[i \bmod 2, 0] \leftarrow 0$
5. $\quad$ for $j \leftarrow 1$ to m
6. $\quad\quad$ if $A[i] = B[j]$
7. $\quad\quad\quad opt[i \bmod 2, j] \leftarrow opt[i-1 \bmod 2, j-1] + 1$
8. $\quad\quad$ elseif $opt[i \bmod 2, j-1] \geq opt[i-1 \bmod 2, j]$
9. $\quad\quad\quad opt[i \bmod 2, j] \leftarrow opt[i \bmod 2, j-1]$
10. $\quad\quad$ else
11. $\quad\quad\quad opt[i \bmod 2, j] \leftarrow opt[i-1 \bmod 2, j]$
12. return $opt[n \bmod 2, m]$

How to Recover LCS Using Linear Space?

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- Using Divide and Conquer + Dynamic Programming:
  - Space: $O(m + n)$
  - Time: $O(nm)$

Outline

Recall: Single Source Shortest Path Problem

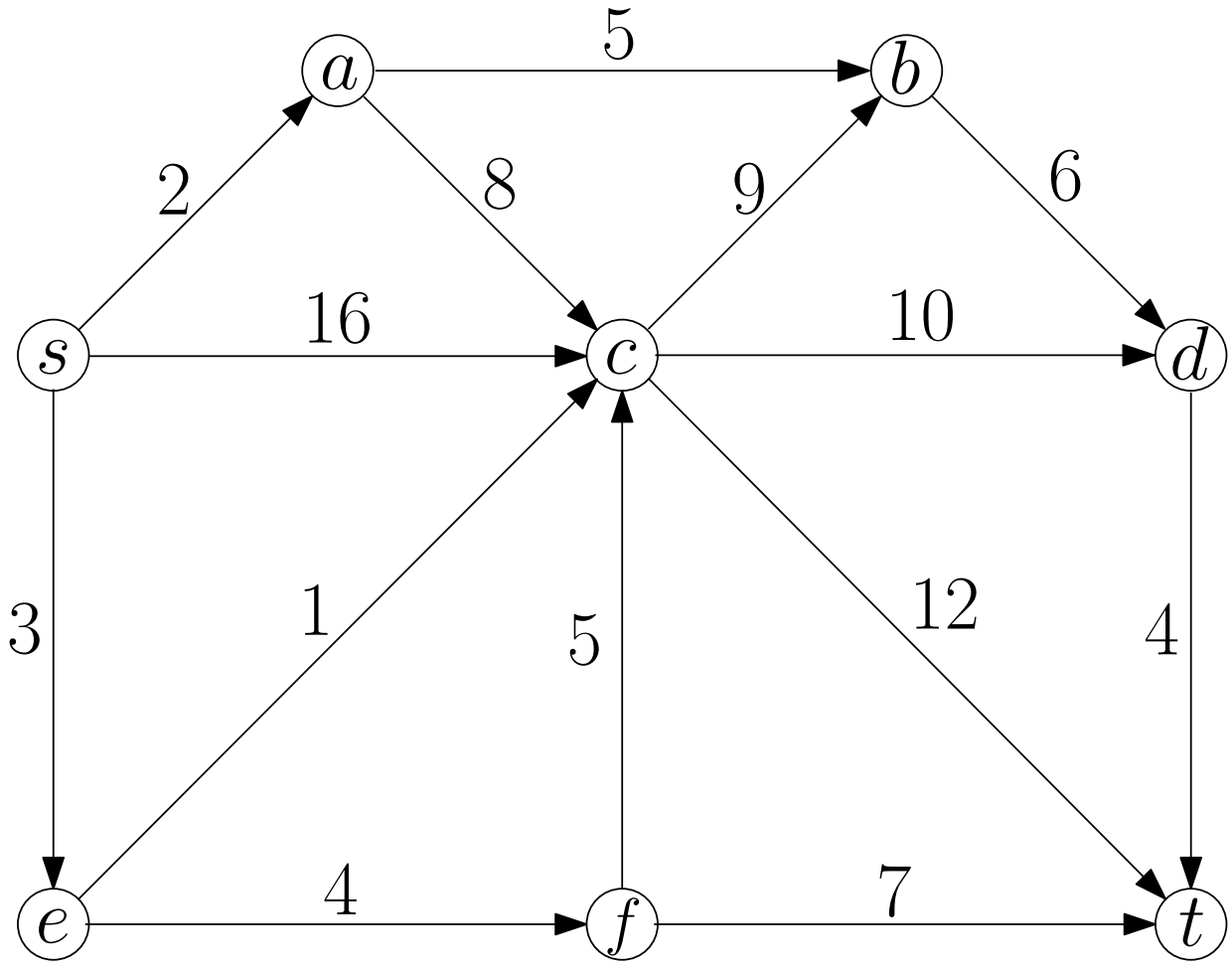
Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

- Algorithm for the problem: Dijkstra's algorithm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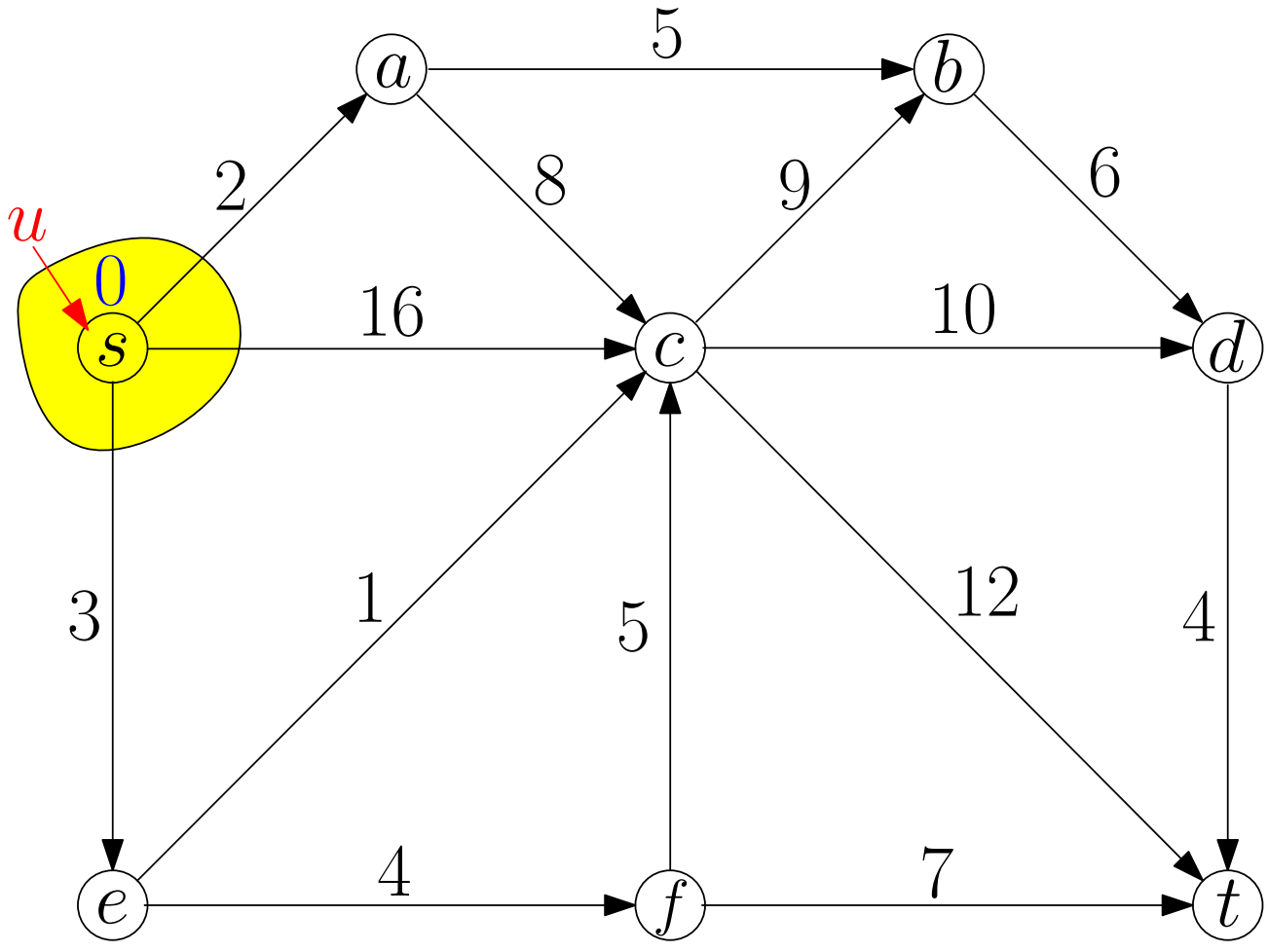
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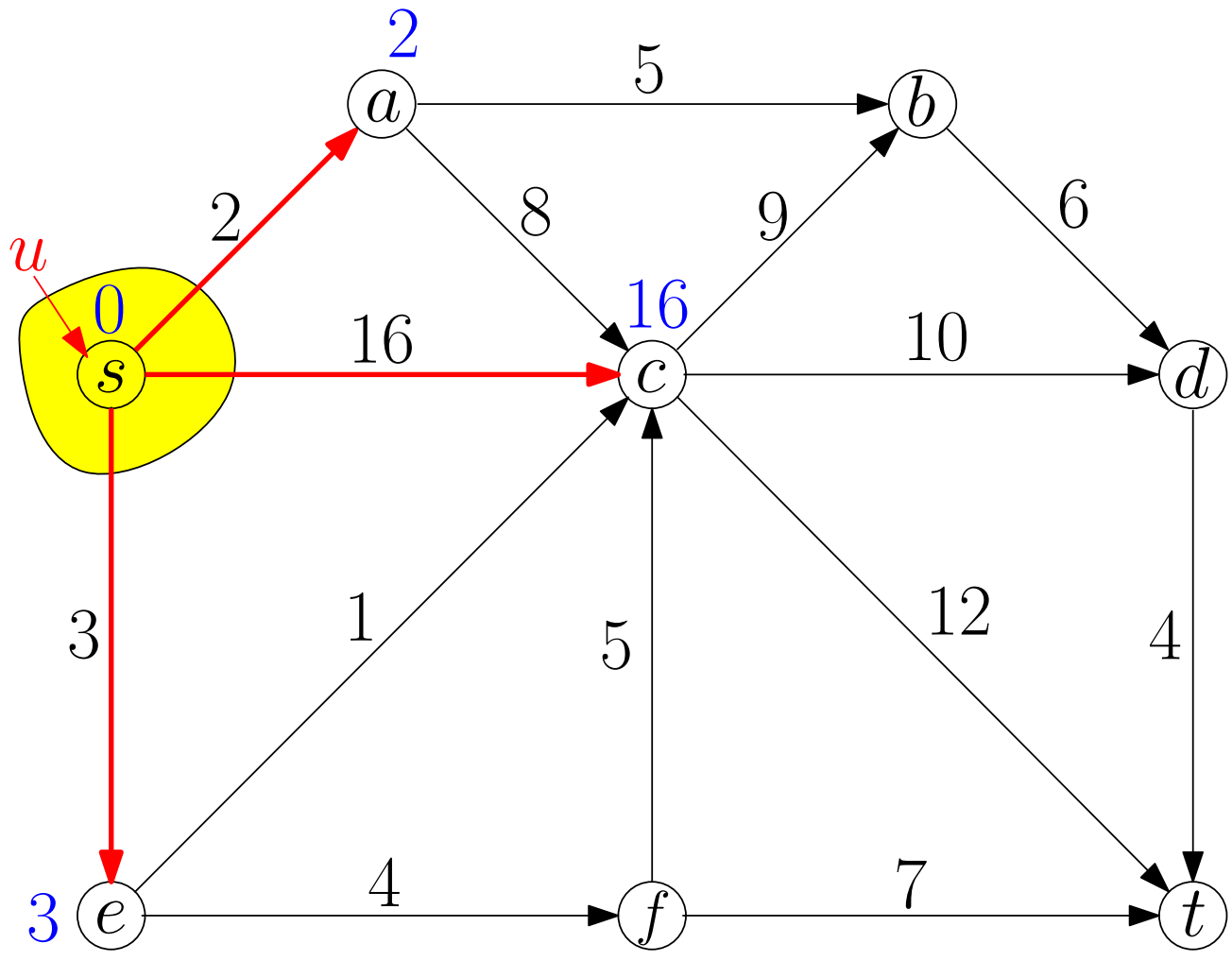
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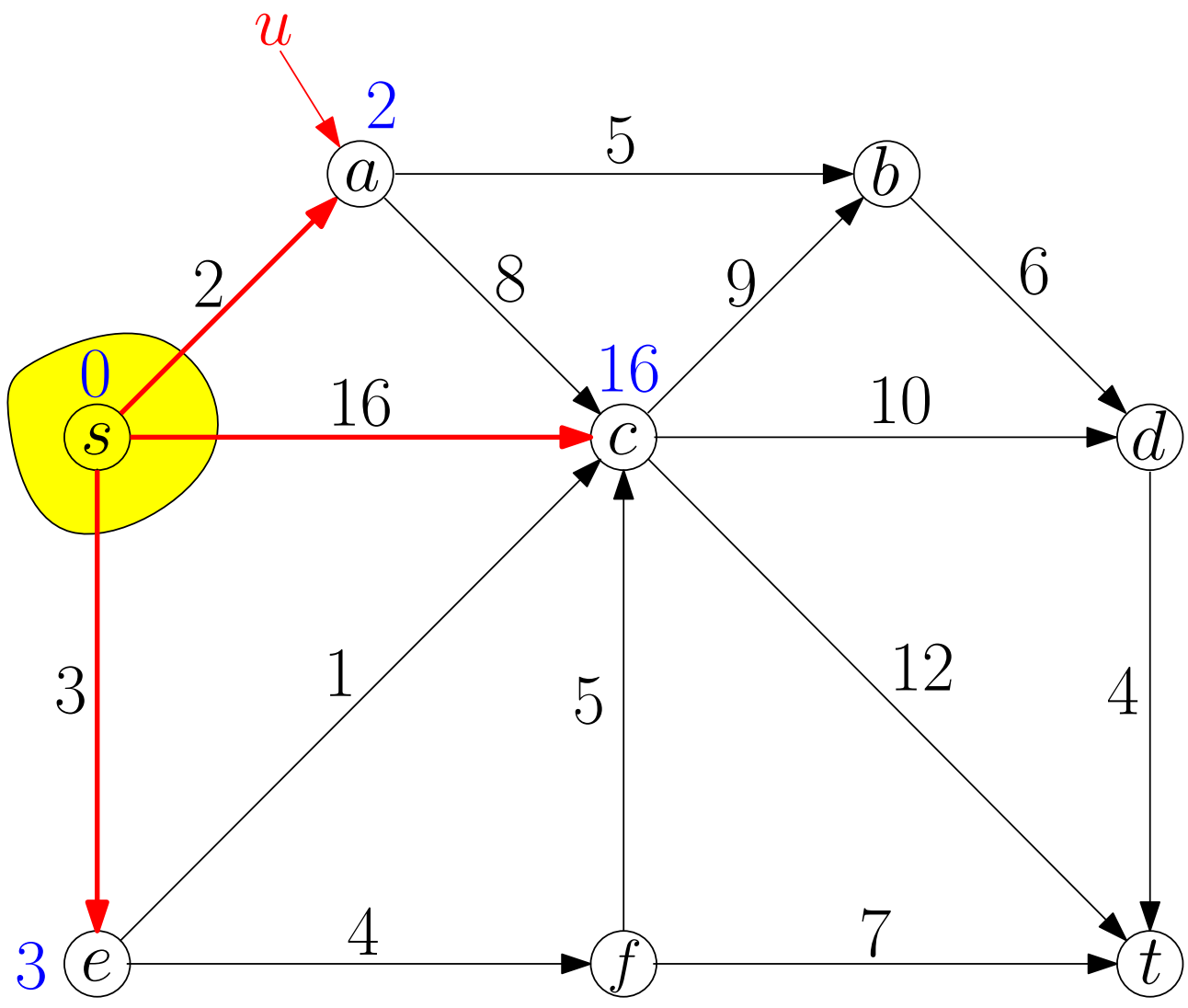
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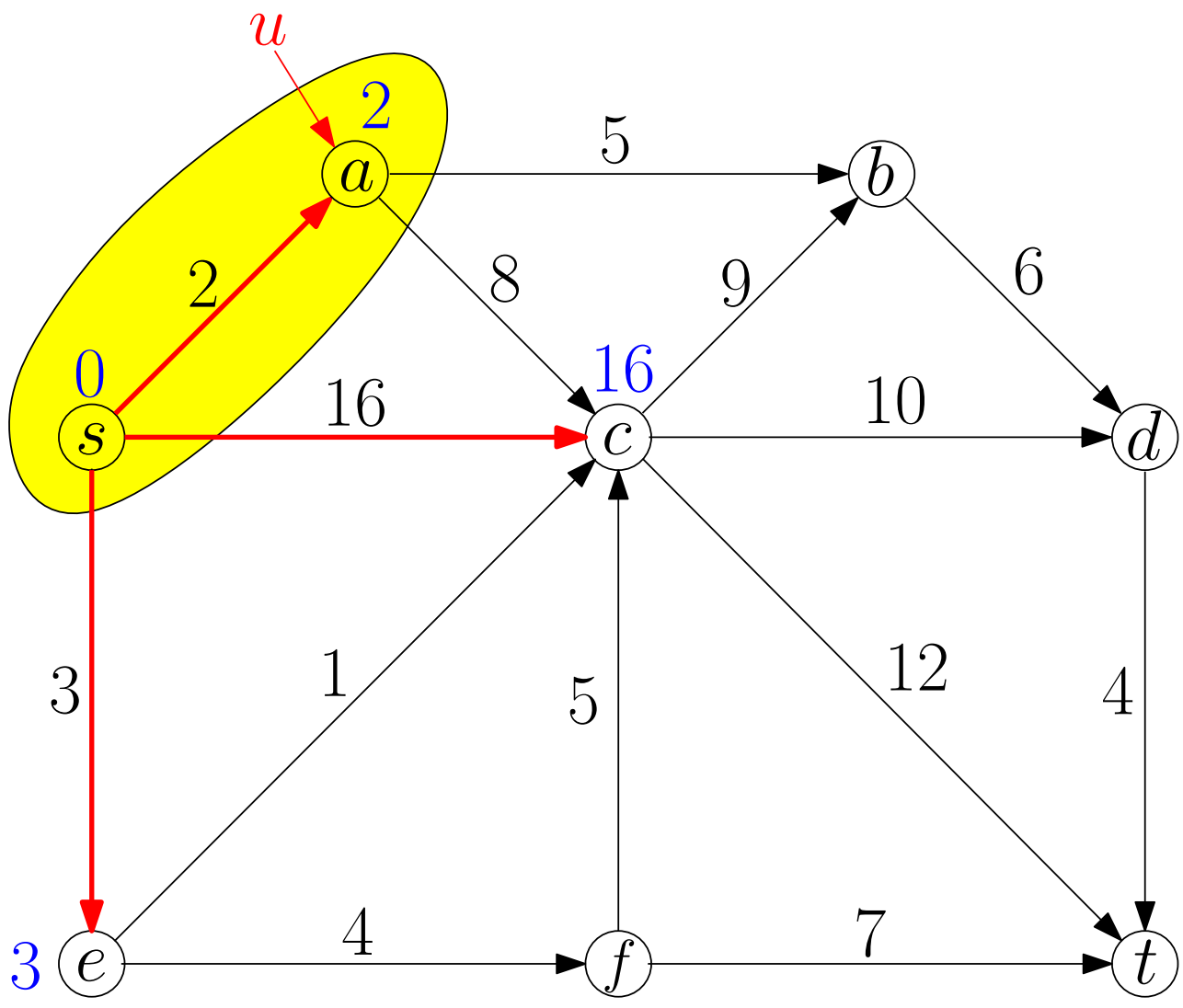
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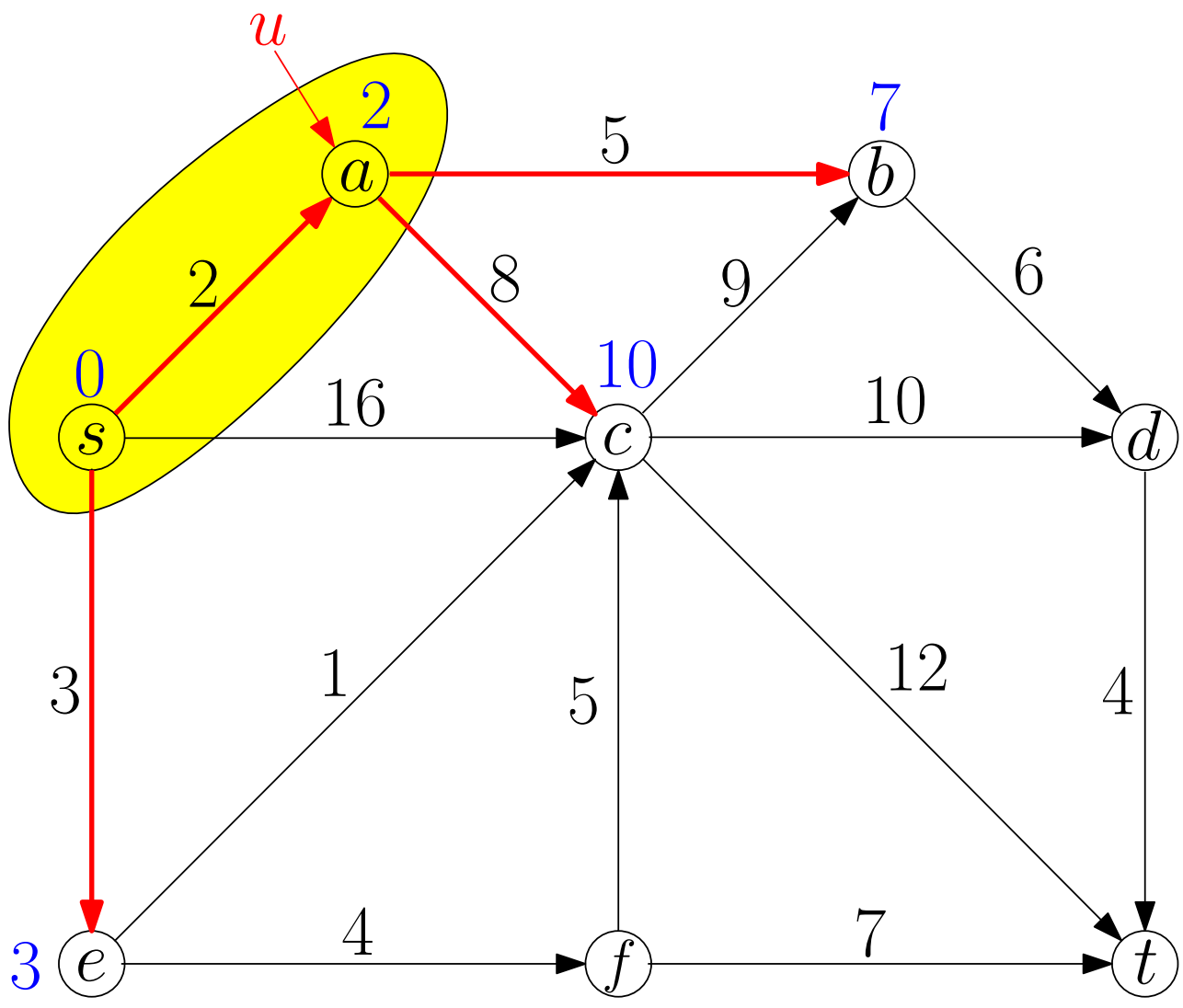
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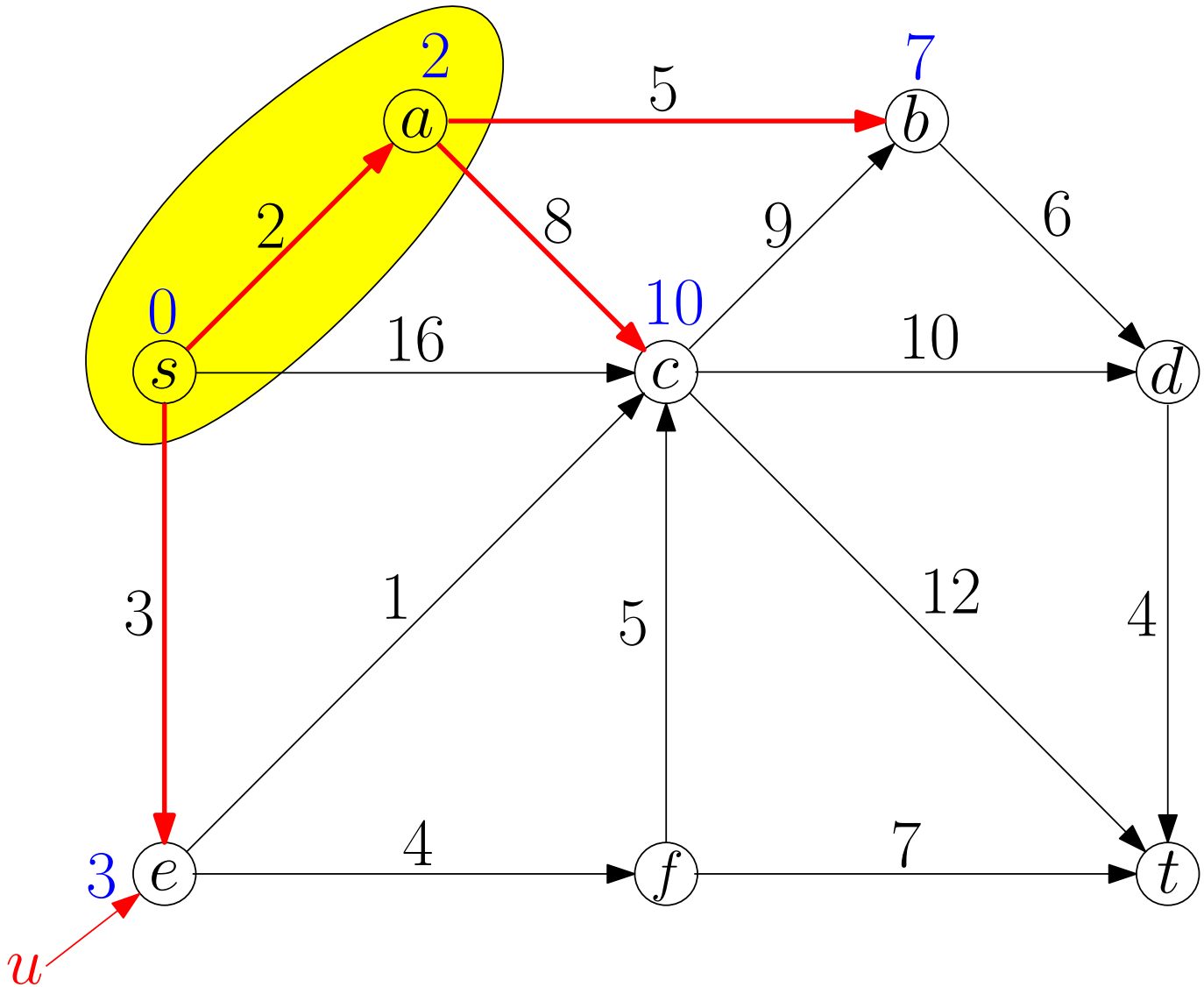
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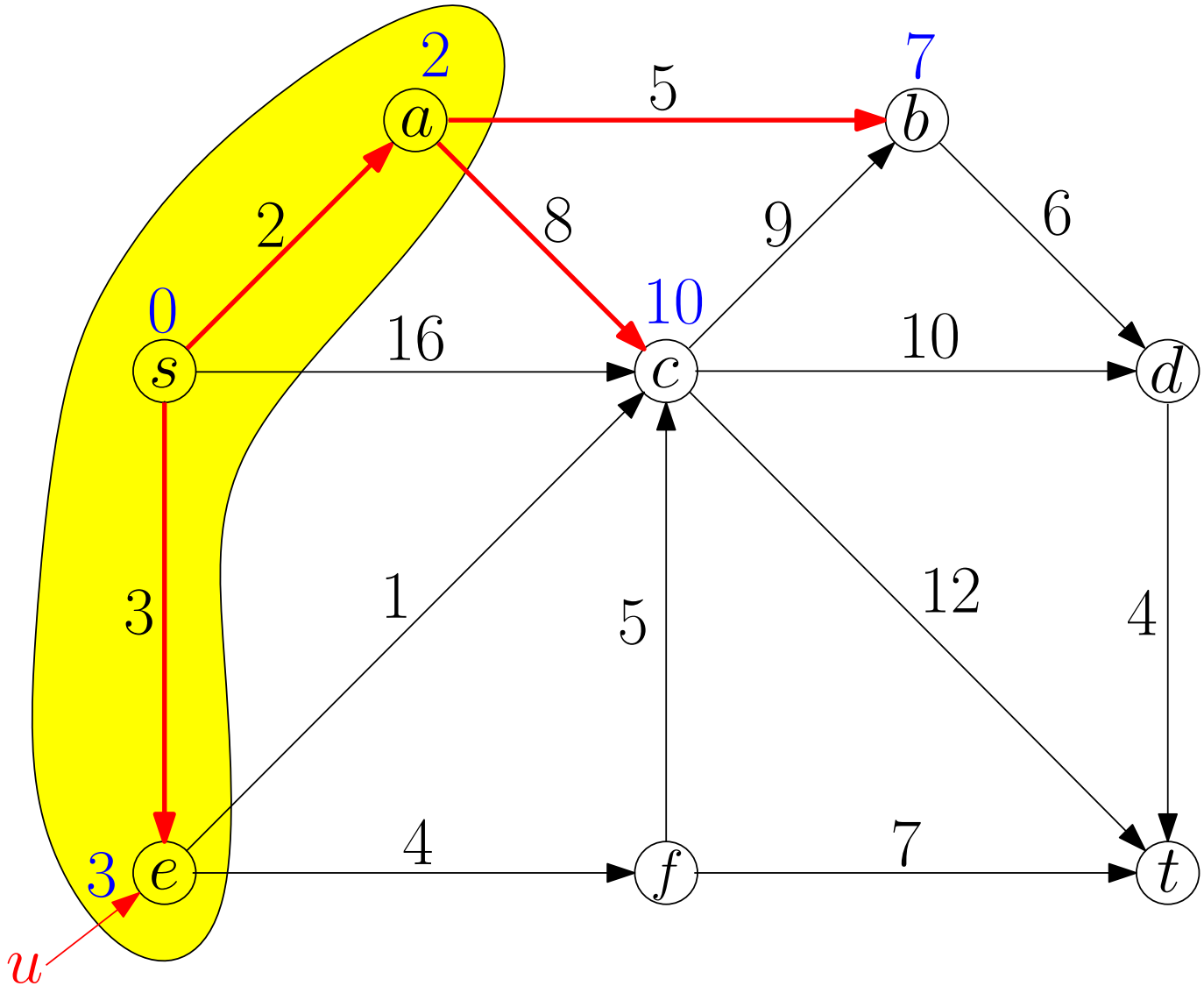
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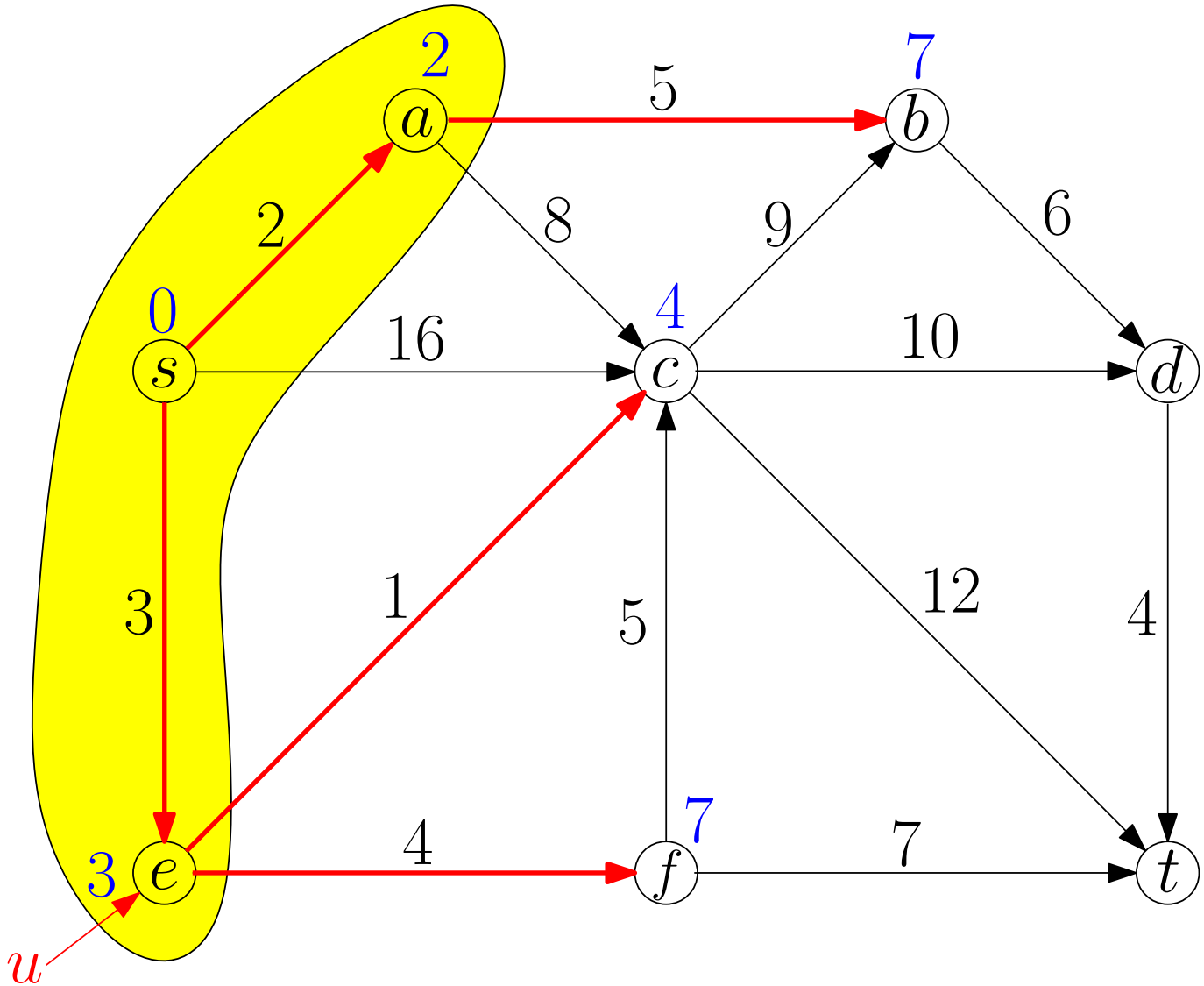
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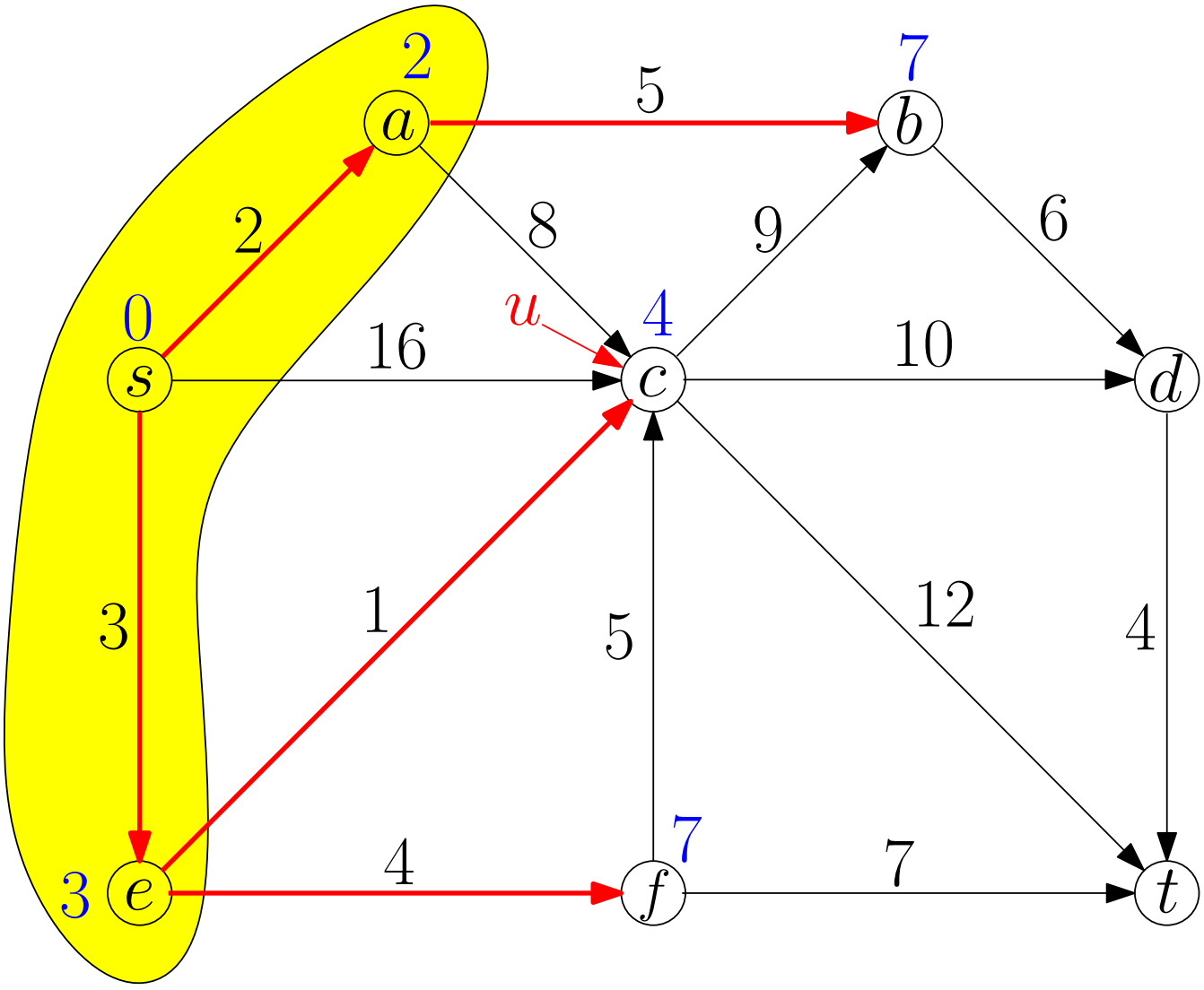
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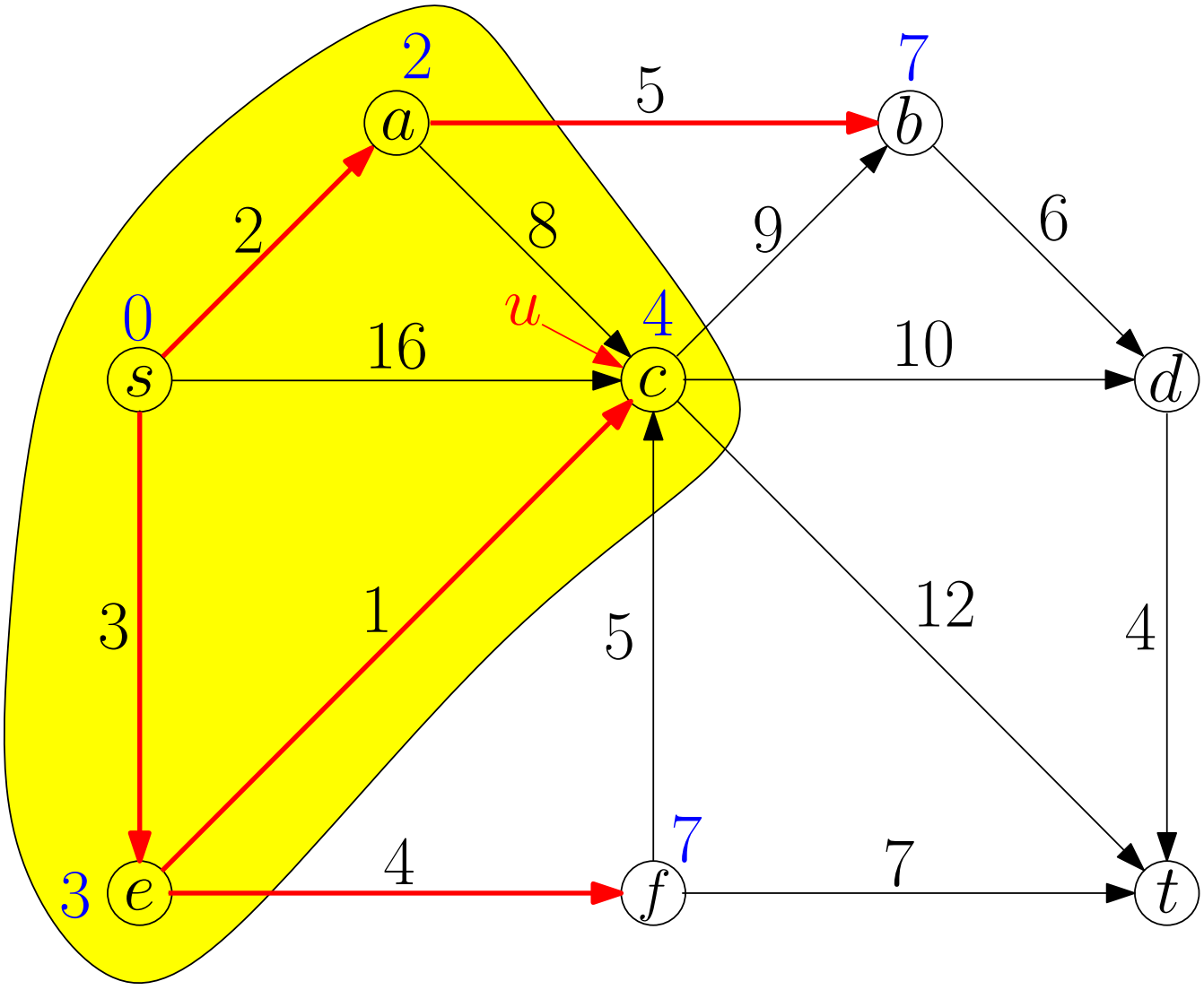
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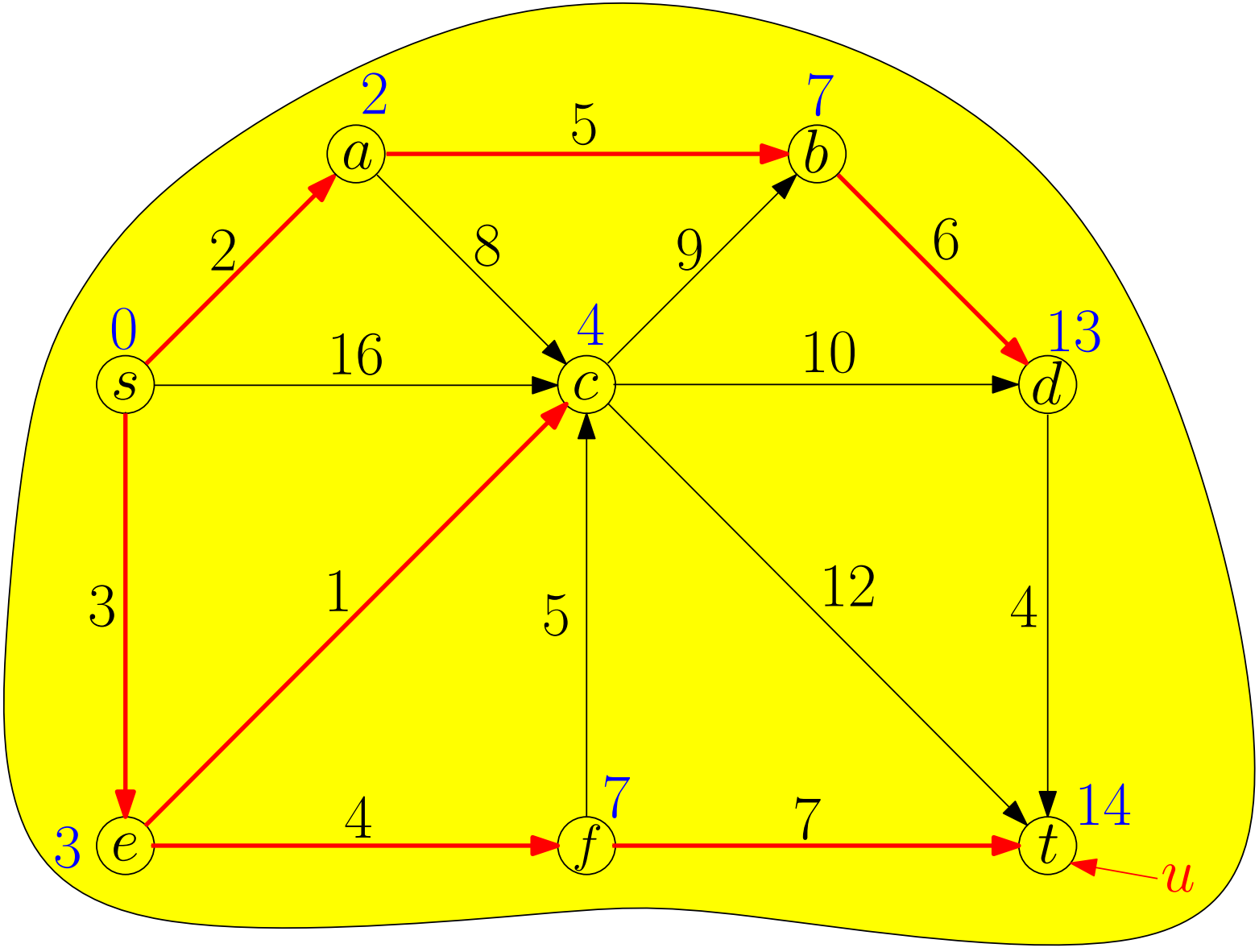
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Dijkstra's Algorithm Using Priorty Queue

Dijkstra(G, w, s)

1. $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
2. $Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
3. while $S \neq V$, do
4. $\quad u \leftarrow Q$.extract_min$()$
5. $\quad S \leftarrow S \cup \{u\}$
6. $\quad$ for each $v \in V \setminus S$ such that $(u, v) \in E$
7. $\quad\quad$ if $d(u) + w(u, v) < d(v)$ then
8. $\quad\quad\quad d(v) \leftarrow d(u) + w(u, v)$, Q.decrease_key$(v, d(v))$
9. $\quad\quad\quad \pi(v) \leftarrow u$
10. return (π, d)

- Running time $= O(m + n \lg n)$.

Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

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- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)

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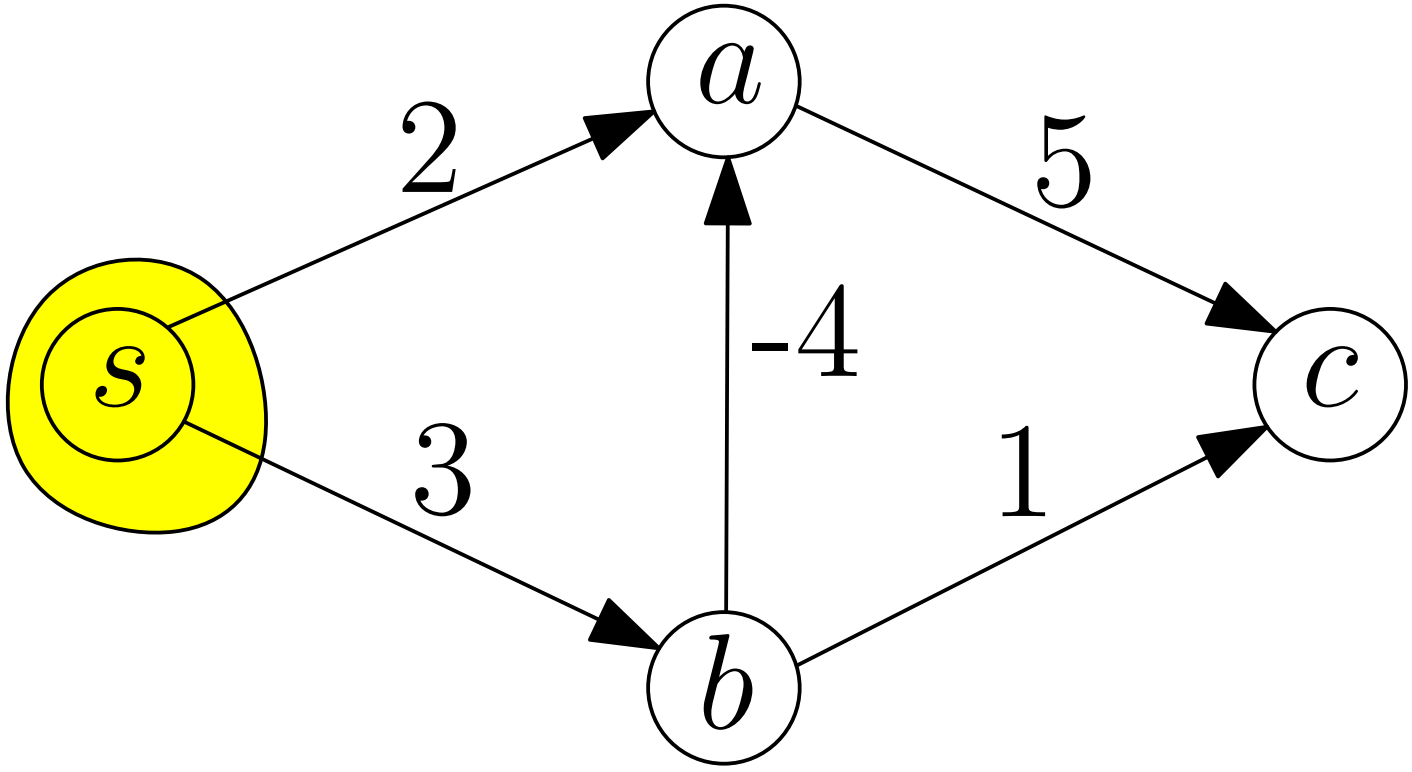
Output: shortest paths from s to all other vertices $v \in V$

- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

Dijkstra's Algorithm Fails if We Have Negative Weights



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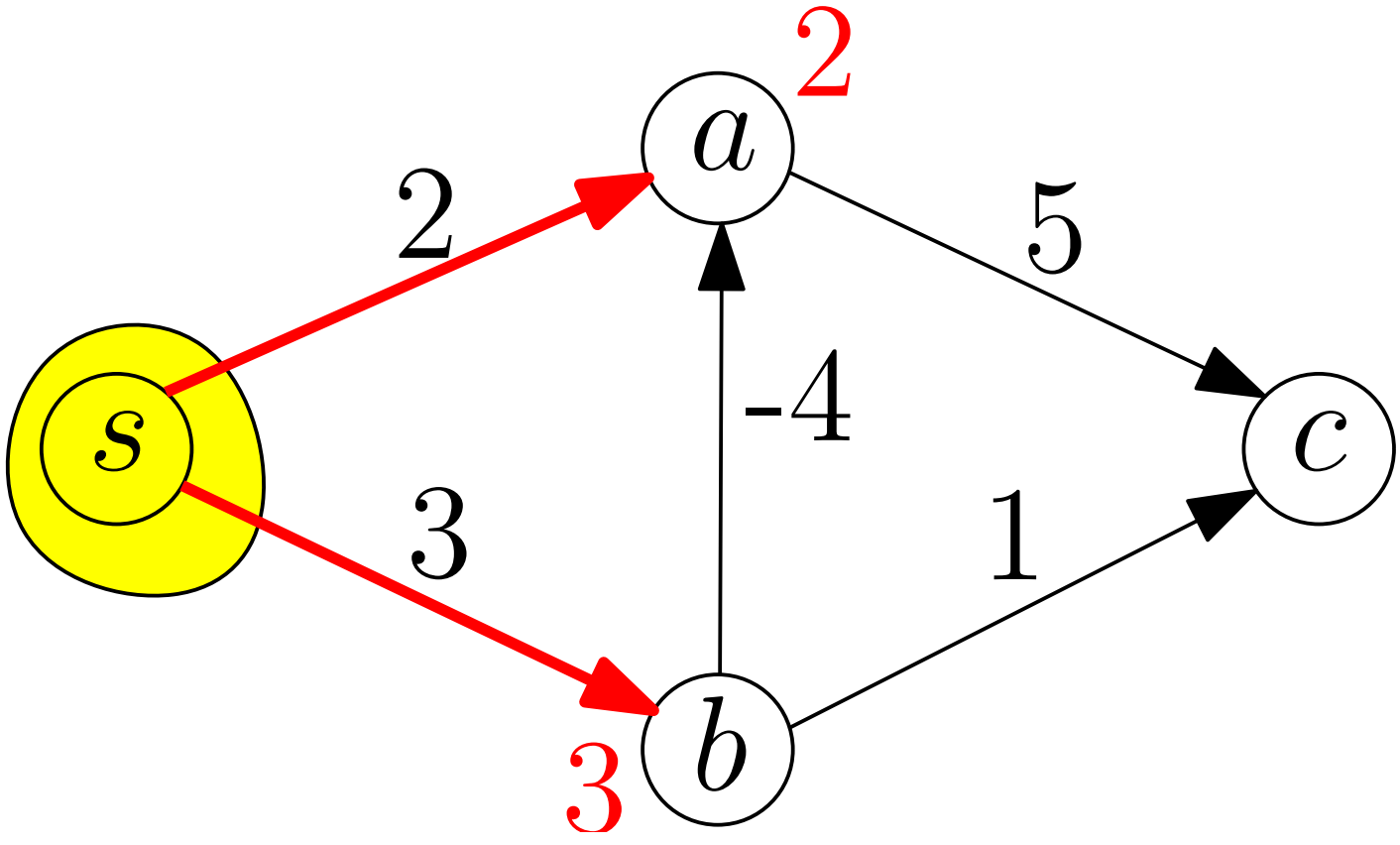

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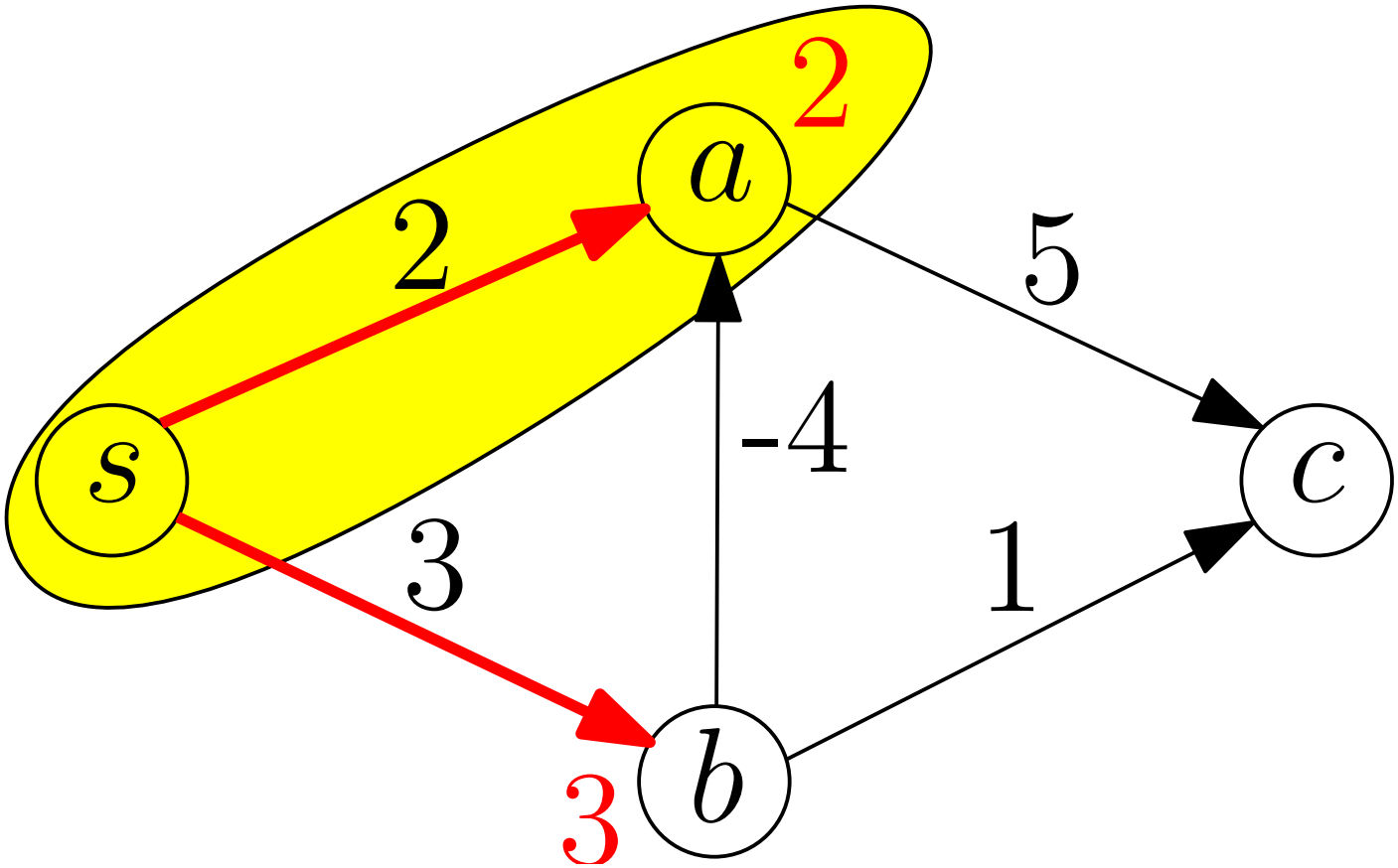

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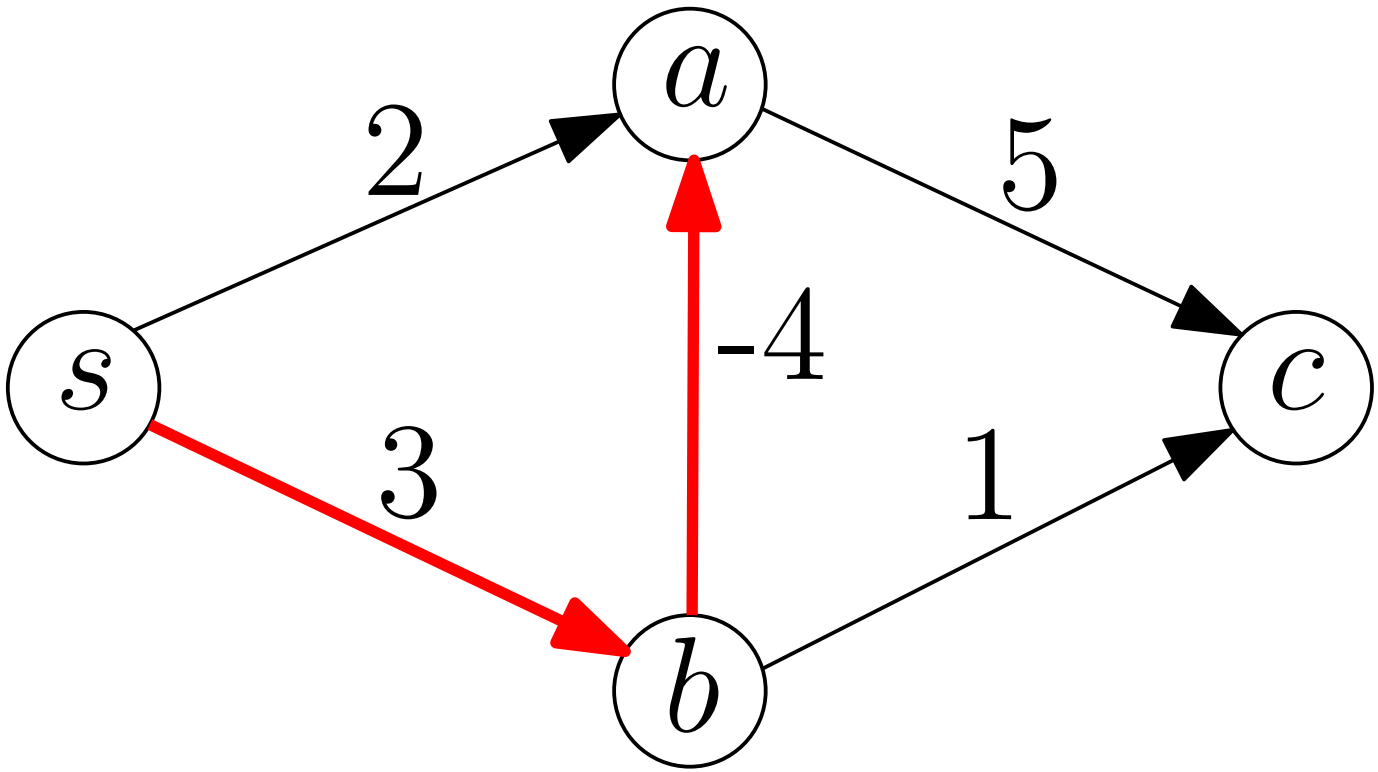

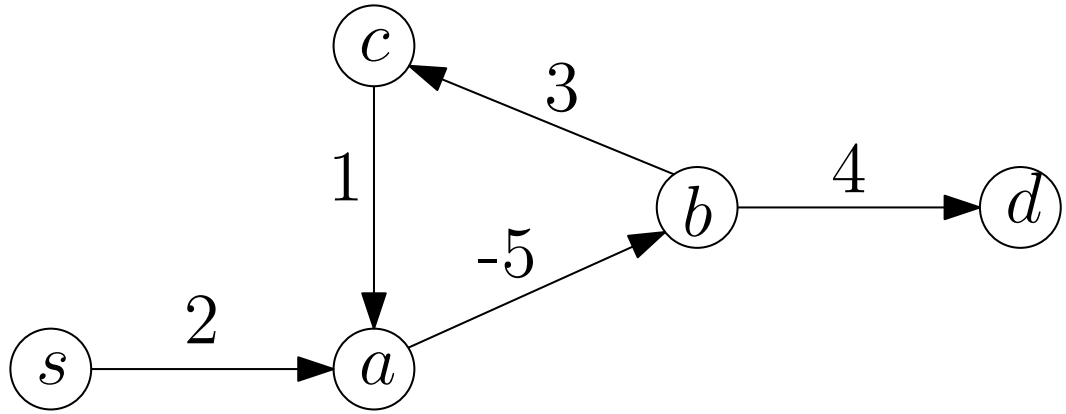
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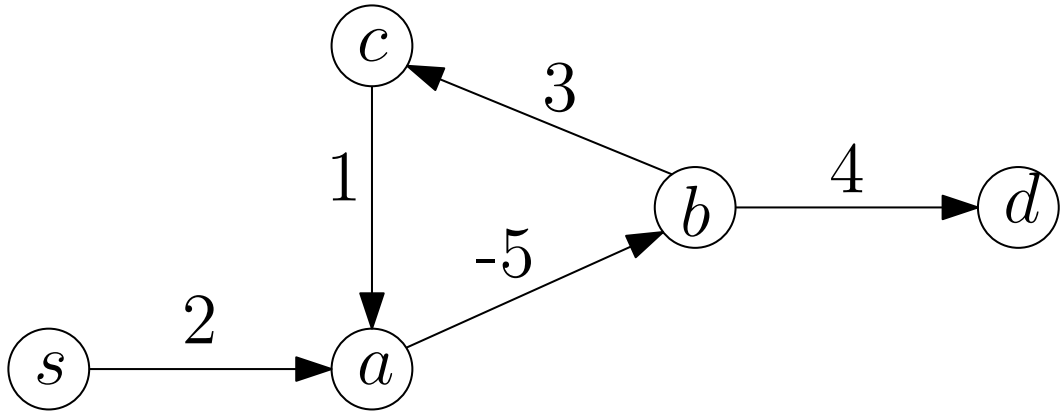
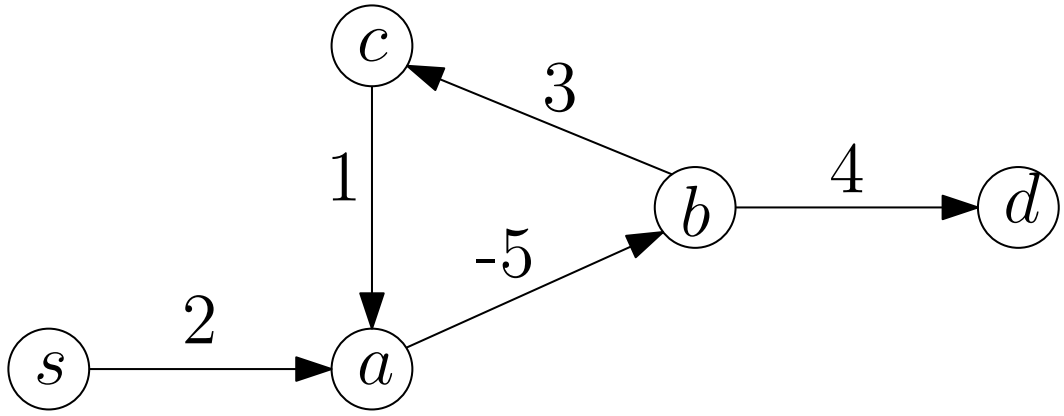


Q: What is the length of the shortest path from s to d?



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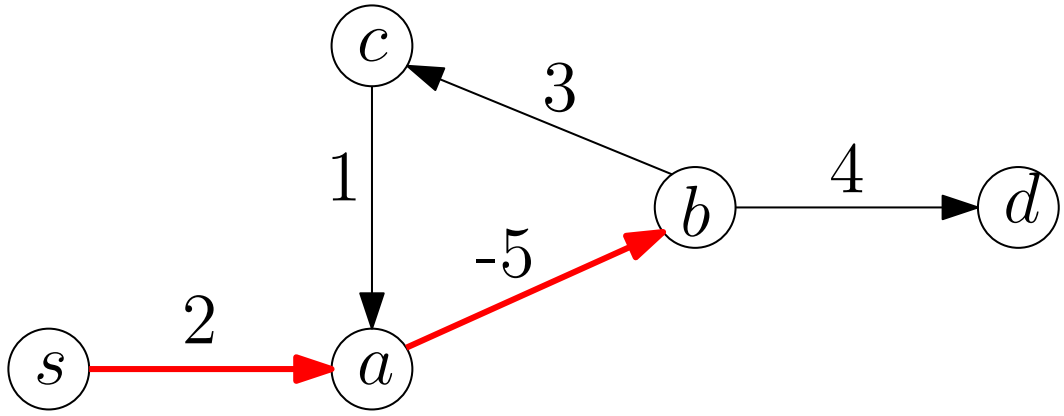

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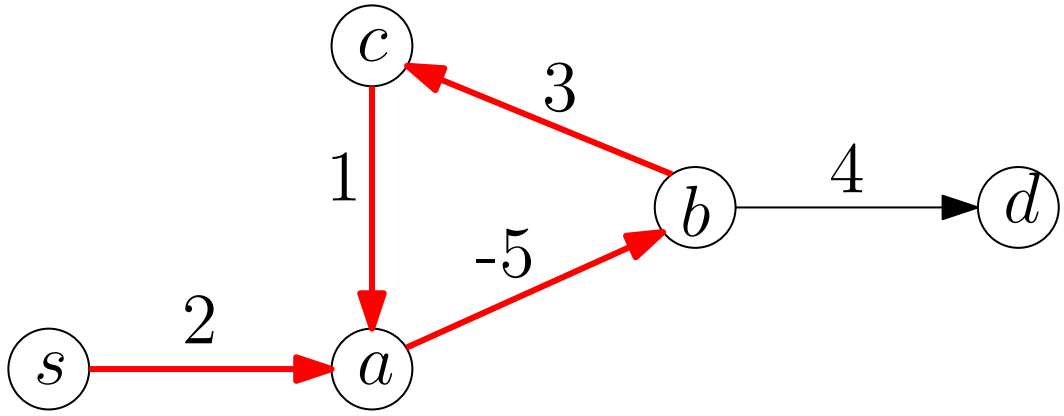

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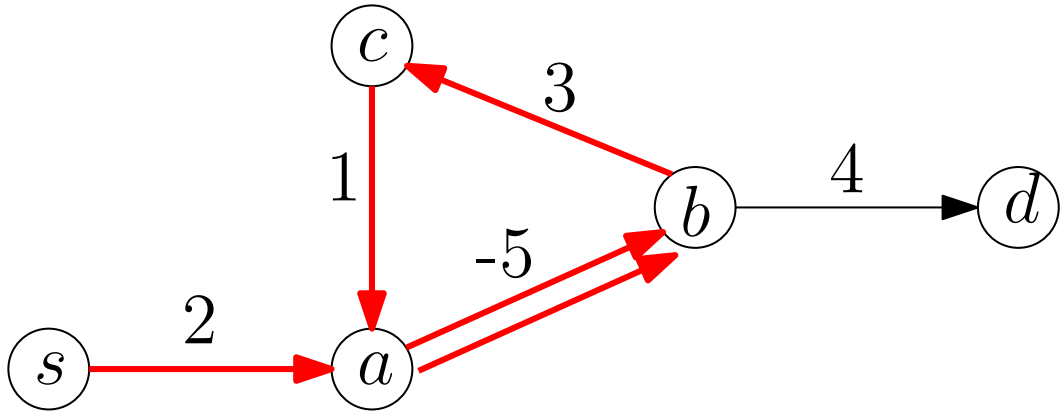

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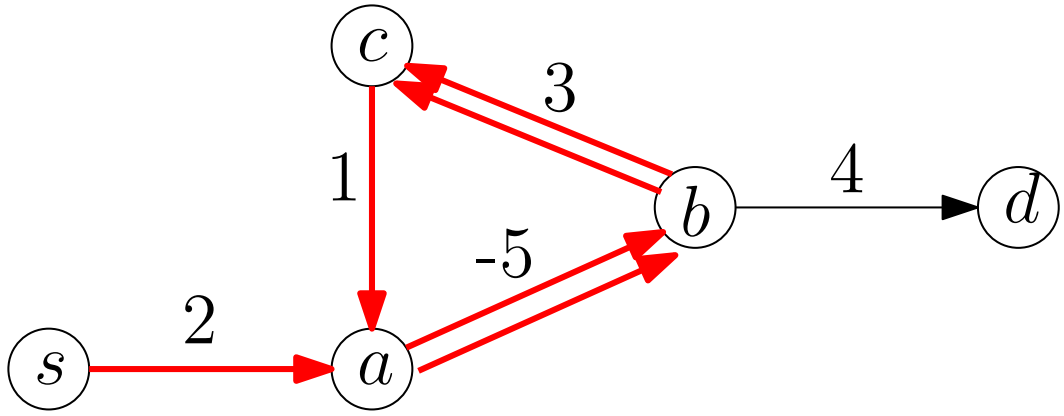

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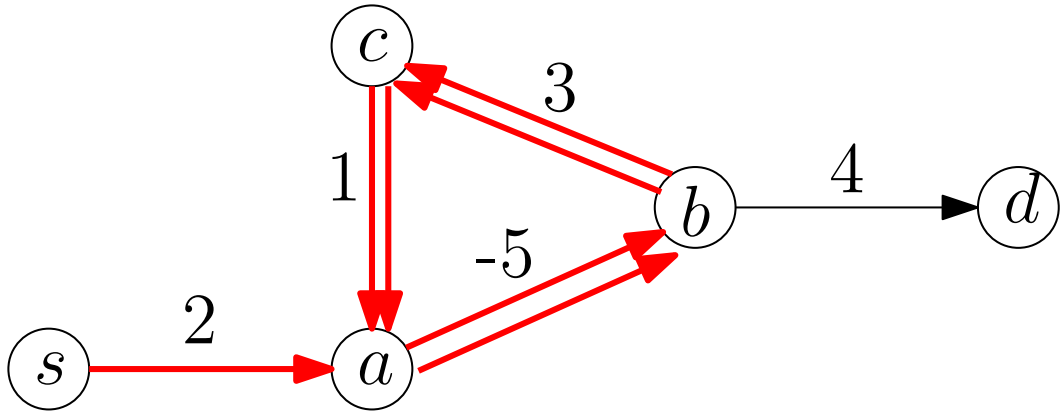

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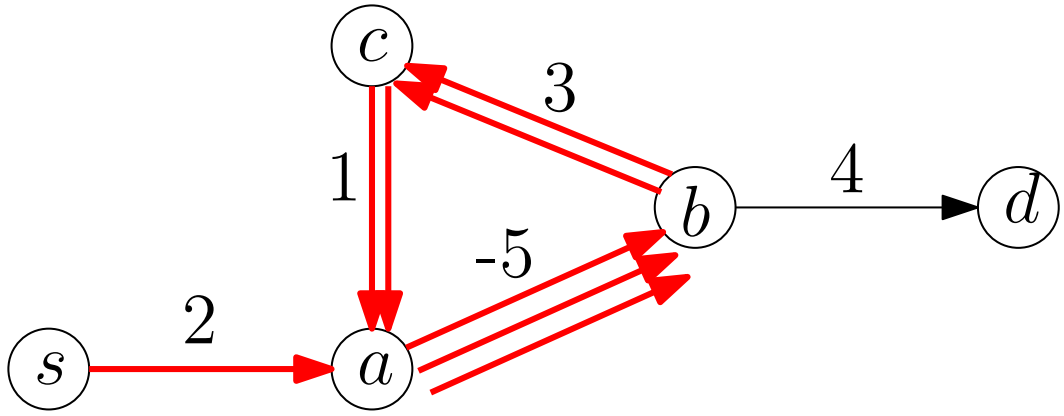

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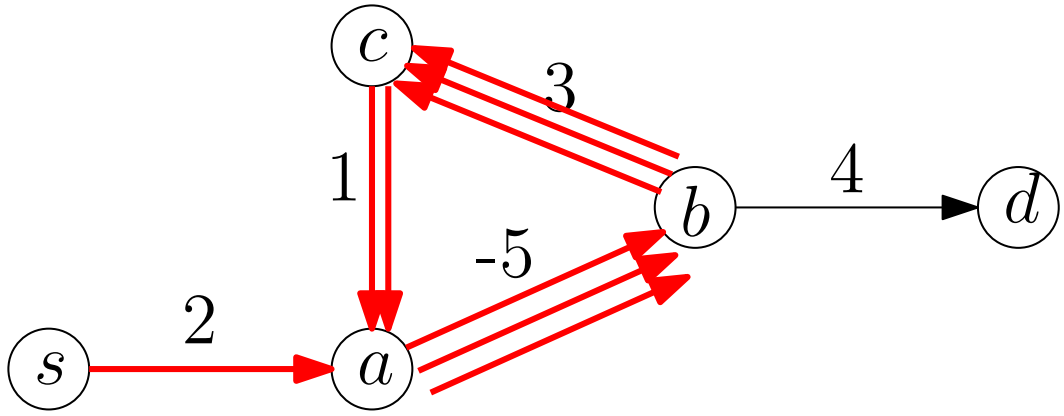

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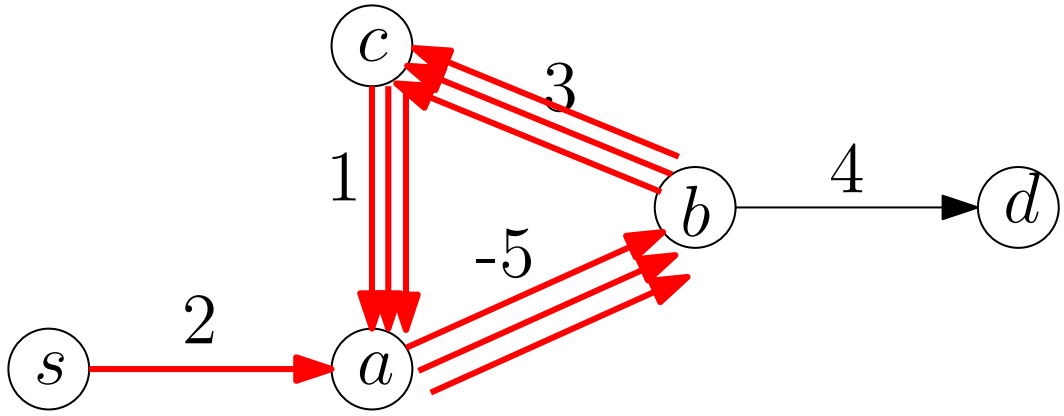

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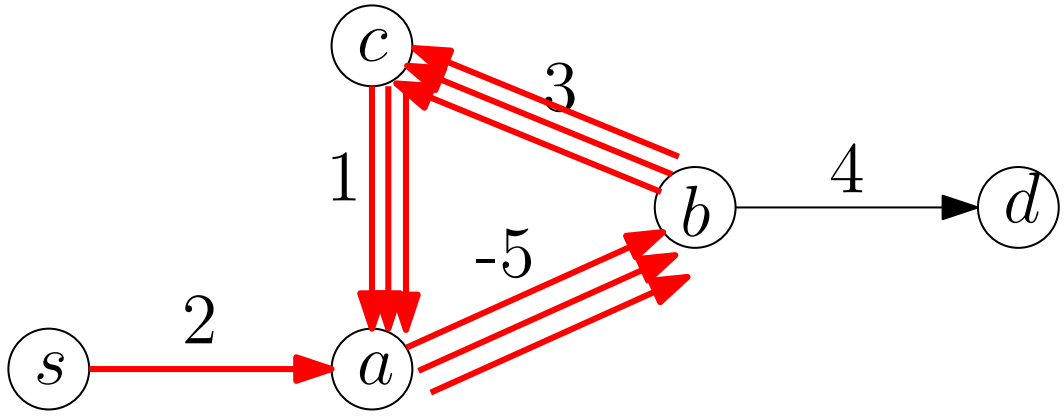

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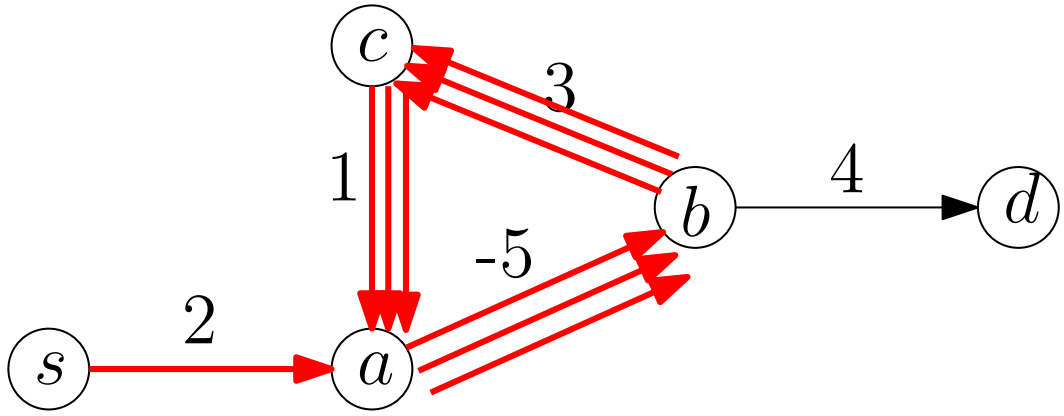
Def. A negative cycle is a cycle in which the total weight of edges is negative.



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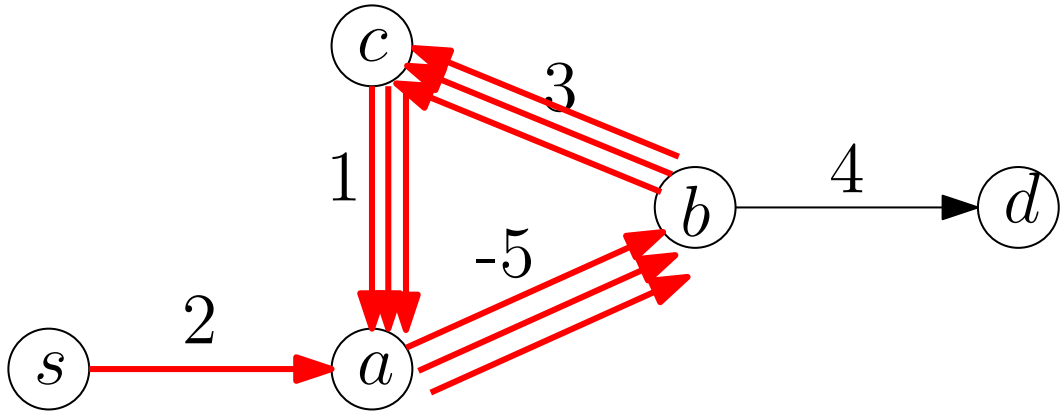
Dealing with Negative Cycles



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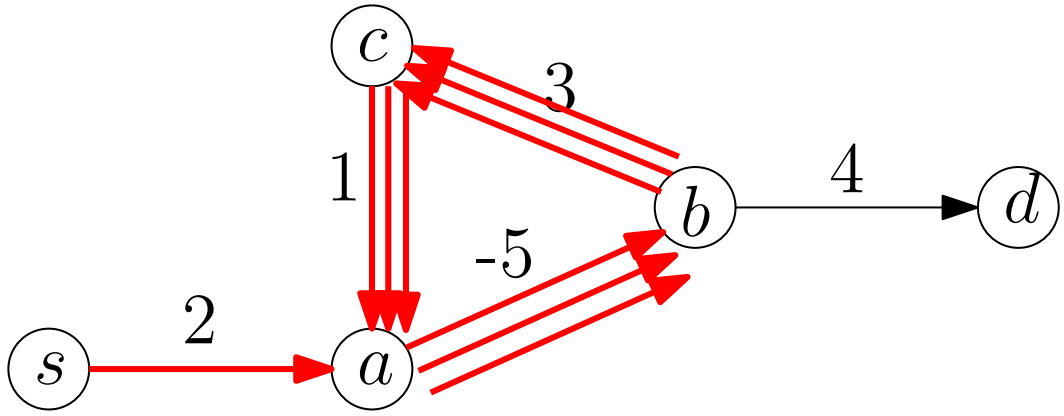
Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or



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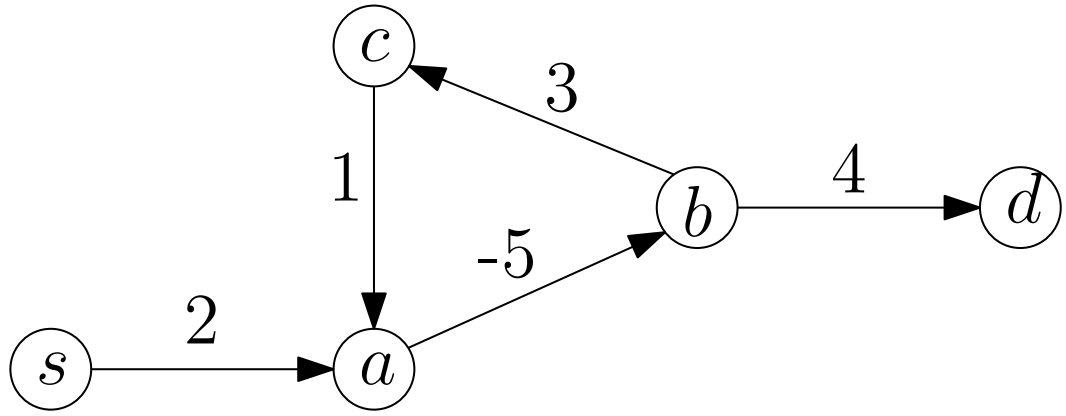
Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"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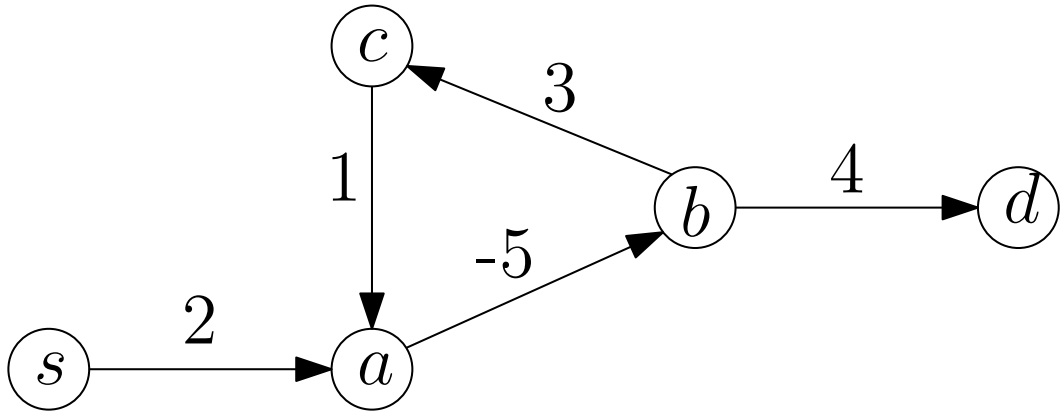
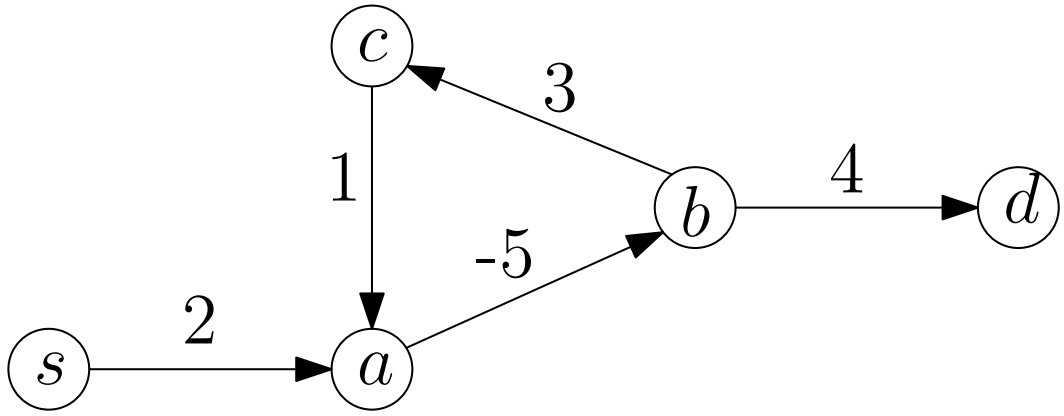


Q: What is the length of the shortest simple path from s to d?



Q: What is the length of the shortest simple path from s to d?

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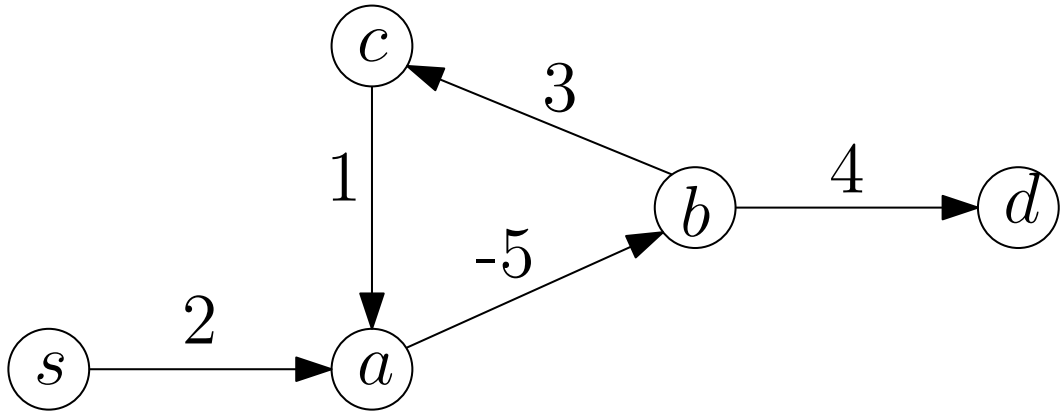


Q: What is the length of the shortest simple path from s to d?

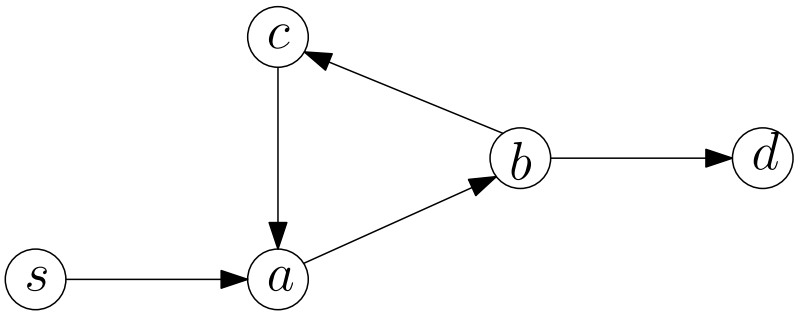
A: 1

- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

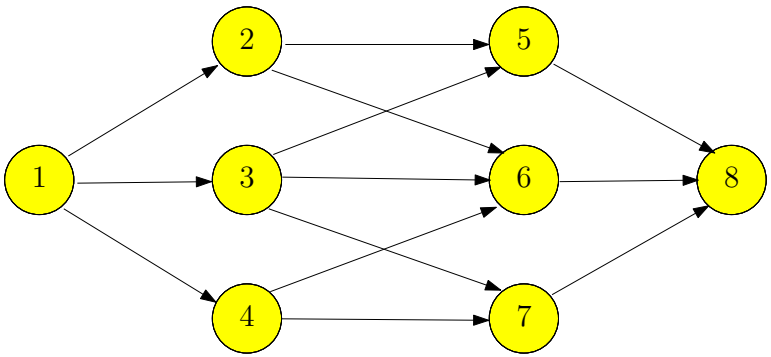
Outline

Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



not a DAG



a DAG

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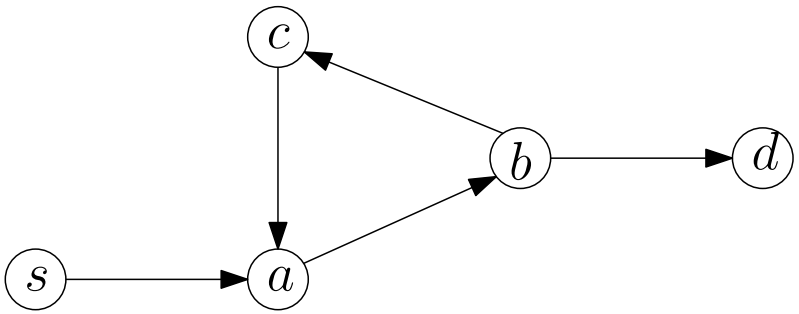

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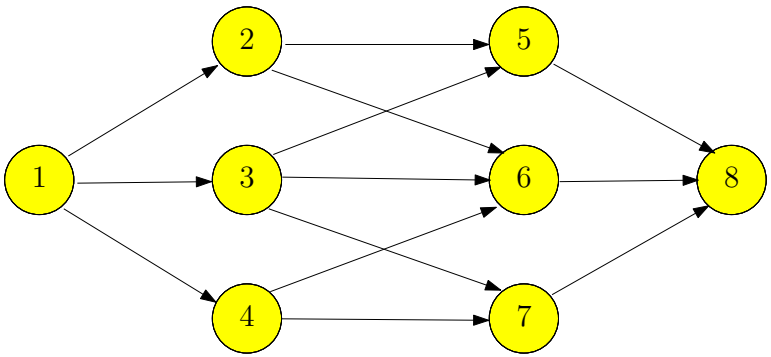


a DAG

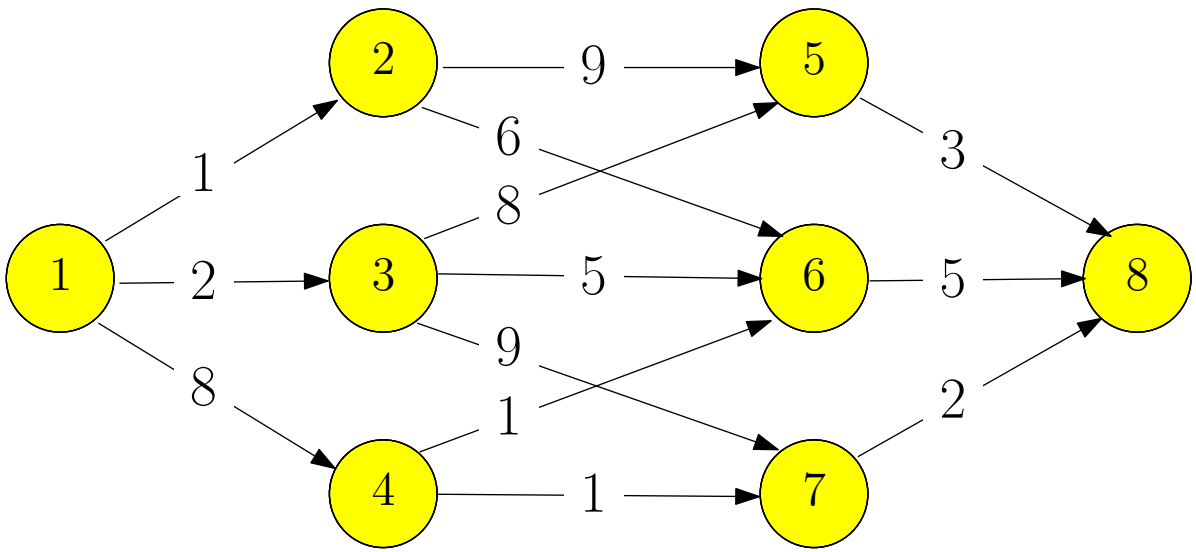
Lemma A directed graph is a DAG if and only its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i, for every $i \in V$



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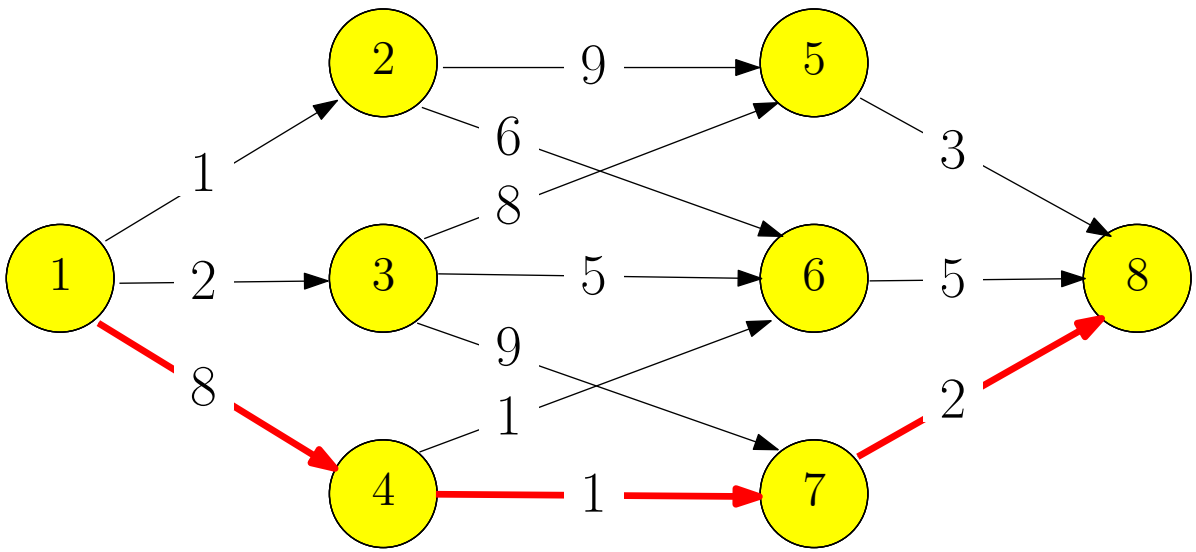

Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} & i = 1 \\ & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i)\in E} \{f(j) + w(j,i)\} & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

Shortest Paths in DAG

1. $f[1] \leftarrow 0$
2. for $i \leftarrow 2$ to n do
3. $\quad f[i] \leftarrow \infty$
4. $\quad$ for each incoming edge $(j, i) \in E$ of i
5. $\quad\quad$ if $f[j] + w(j, i) < f[i]$
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Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

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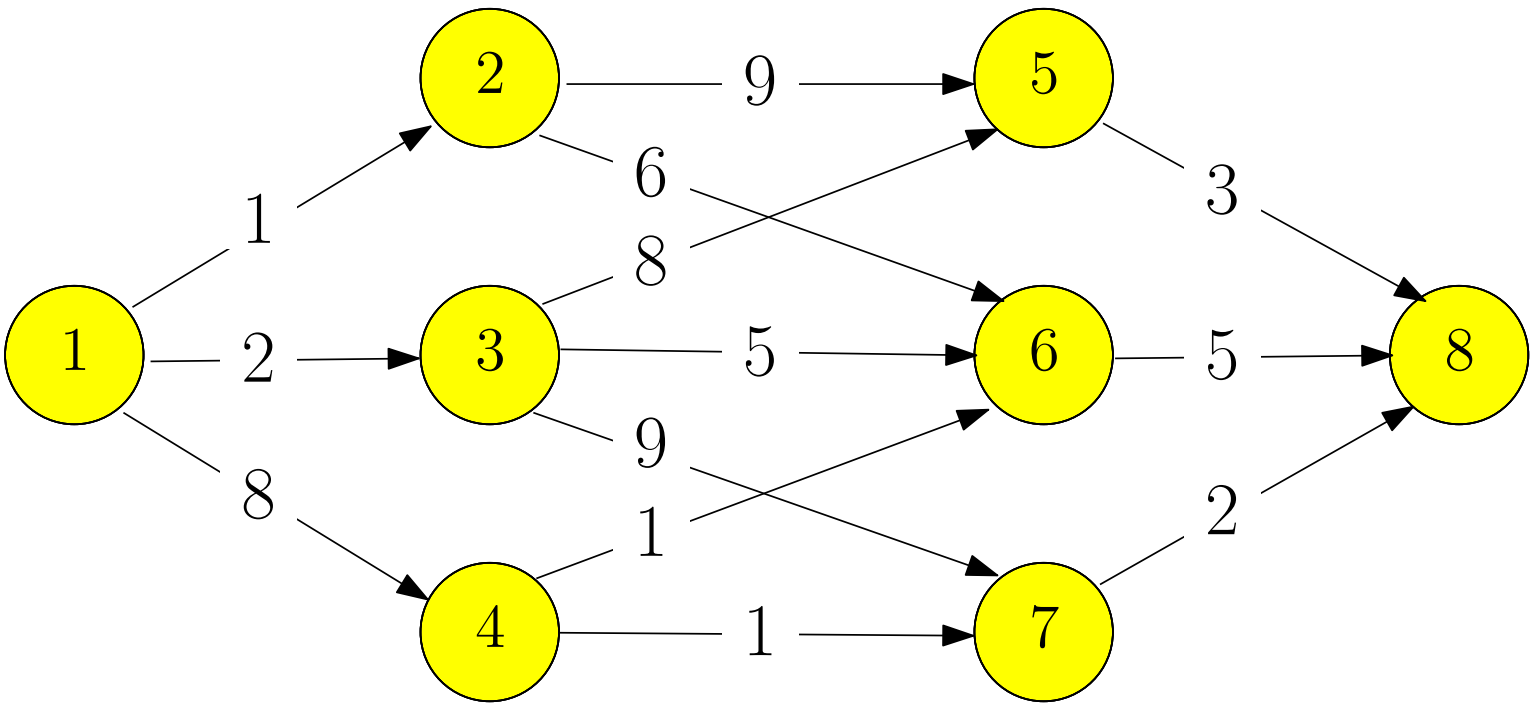
print-path(t)

1. if $t = 1$ then
2. $\quad$ print(1)
3. $\quad$ return
4. print-path$(\pi(t))$
5. print(“,”, t)

Example



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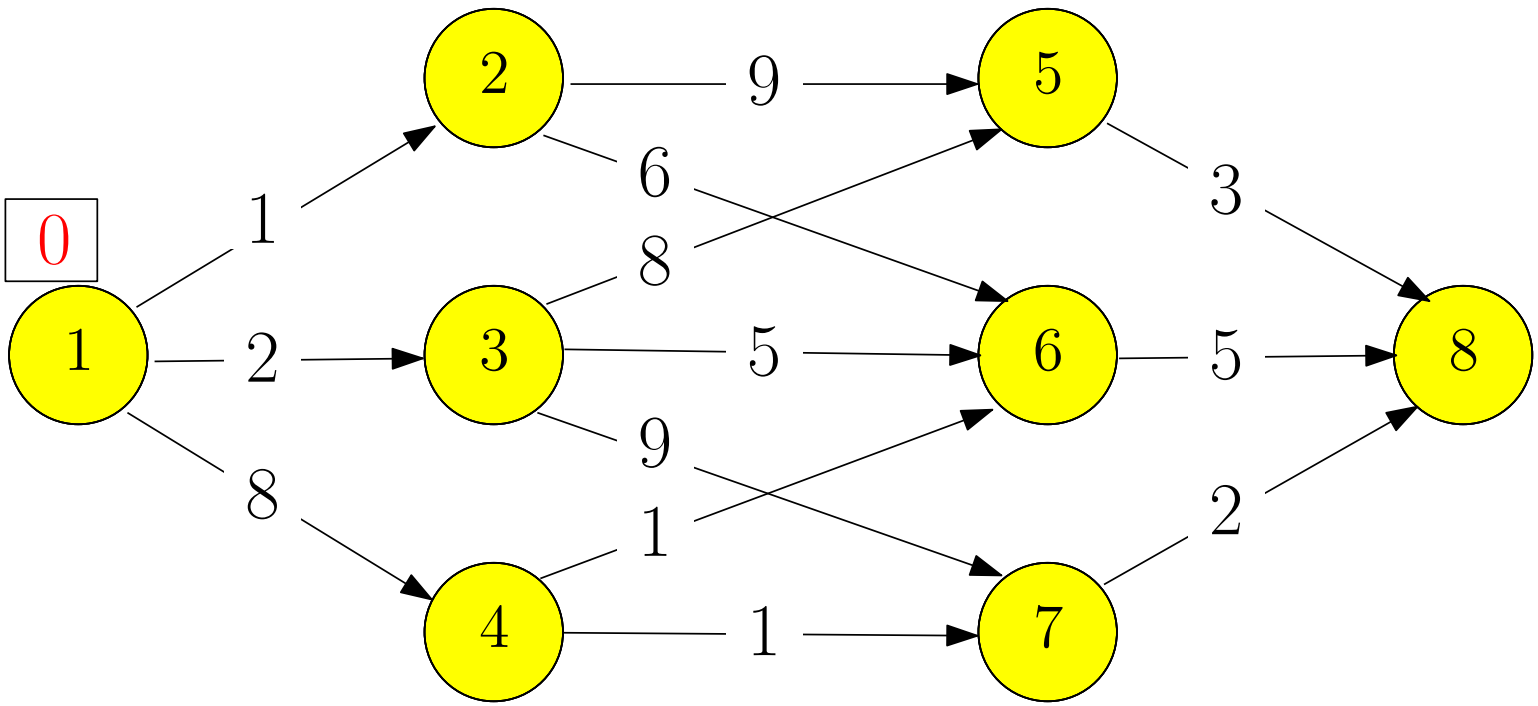

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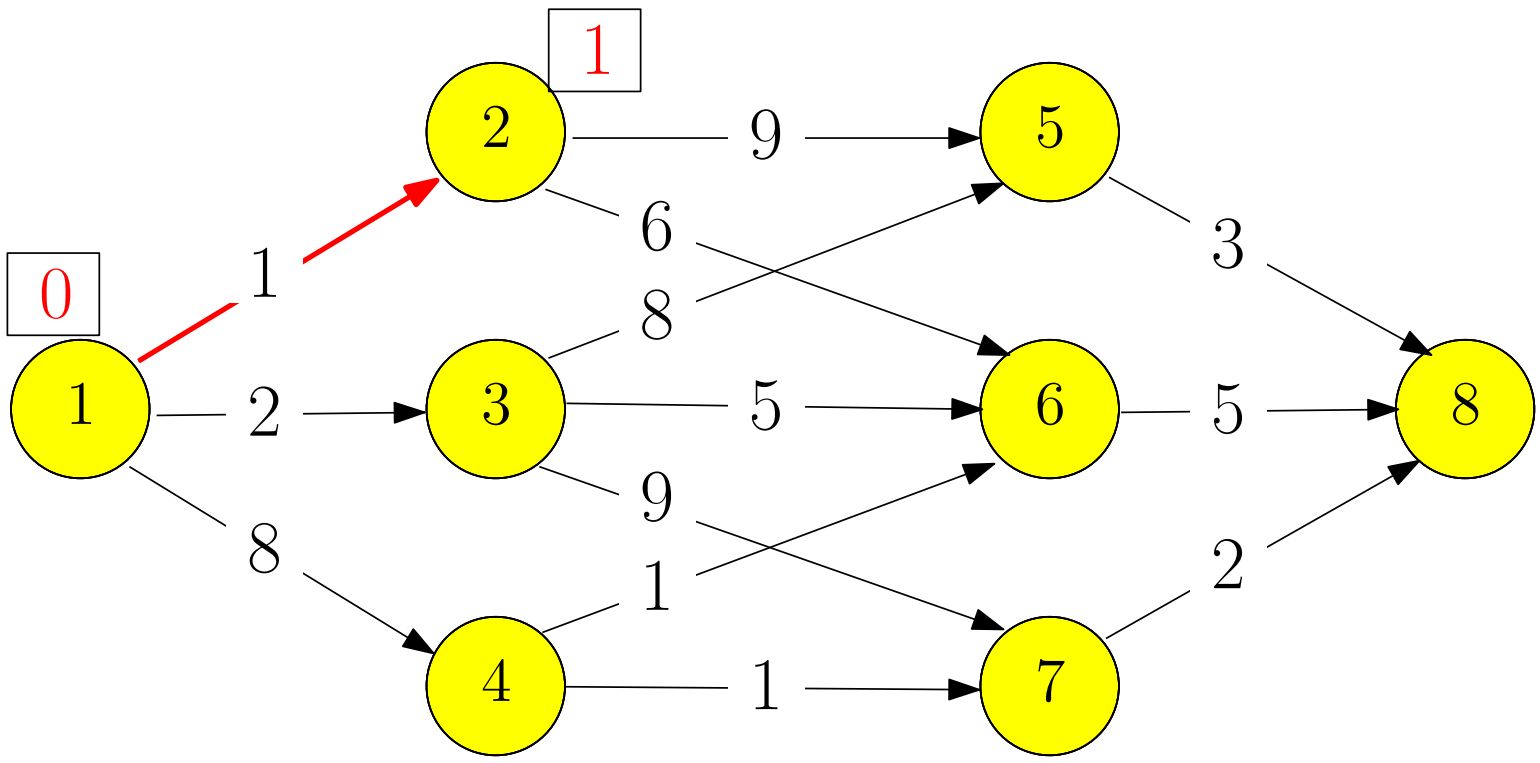

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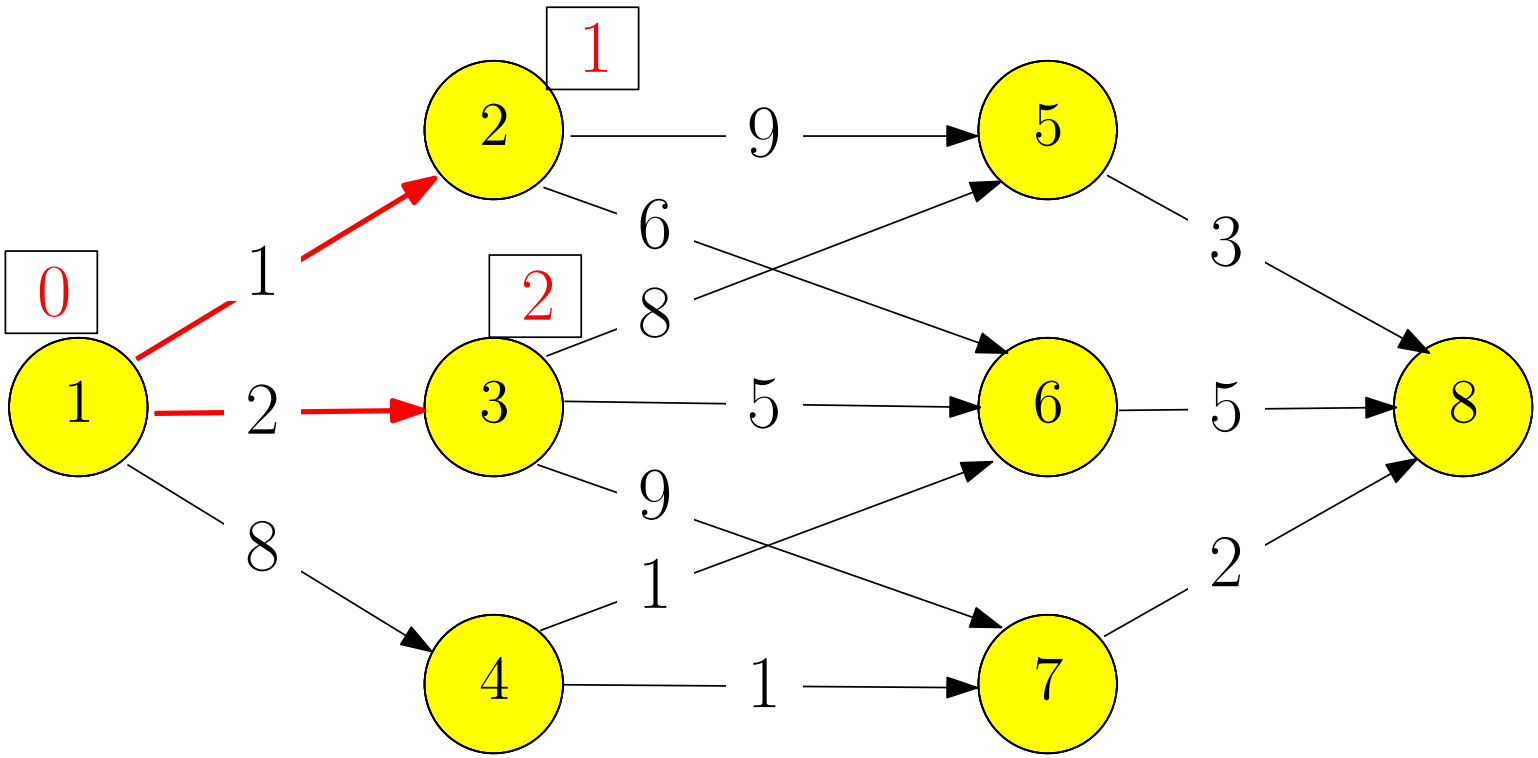

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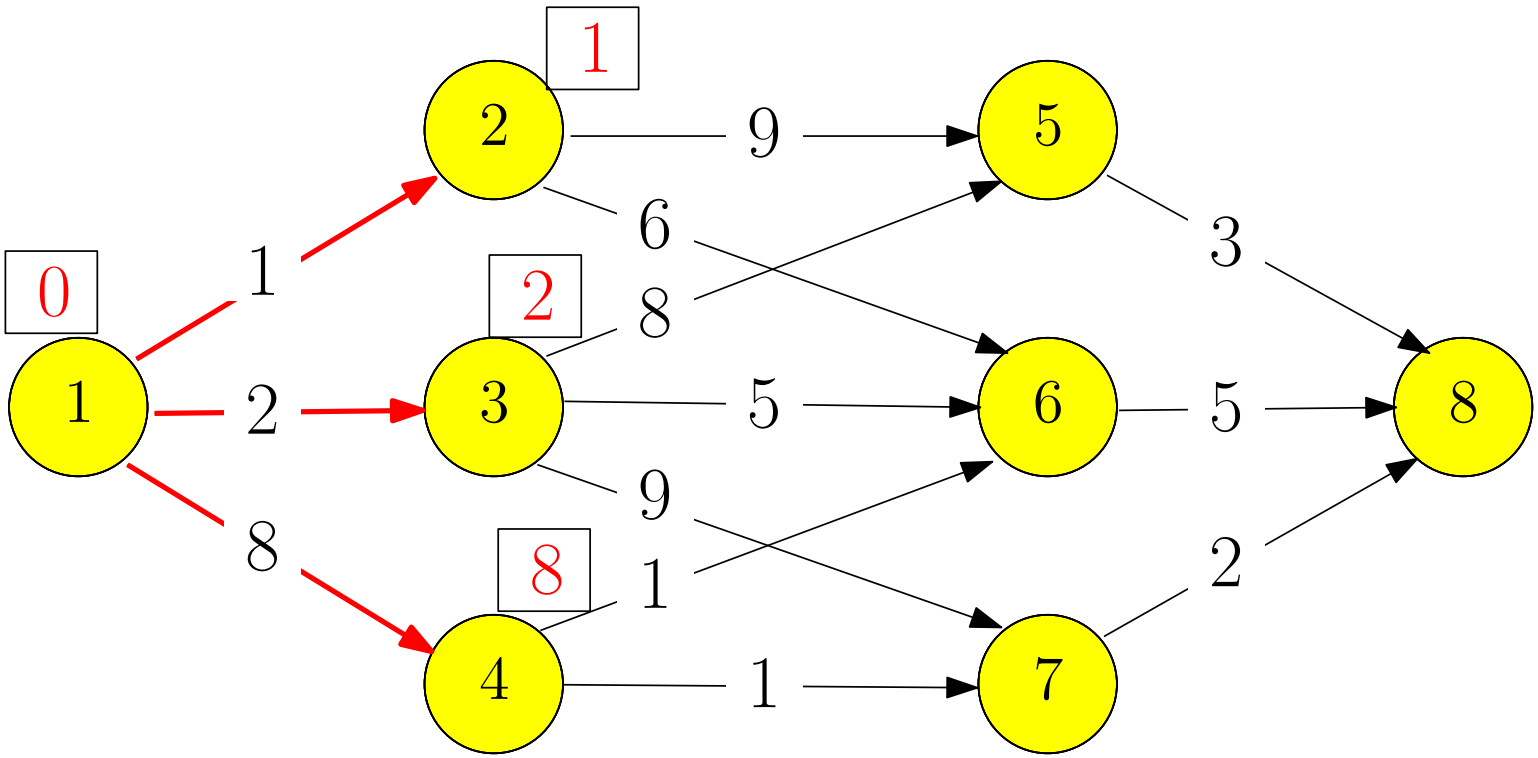

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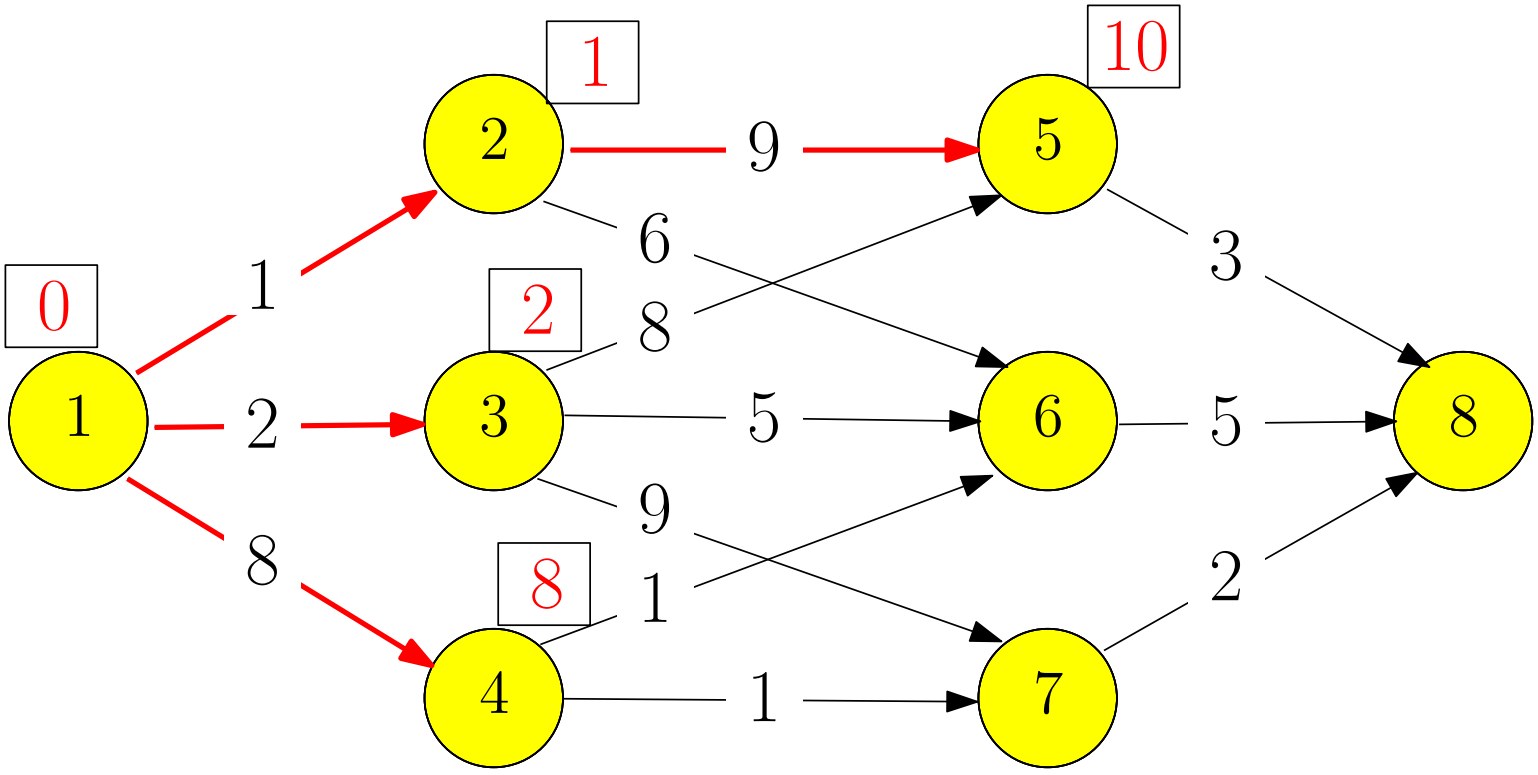

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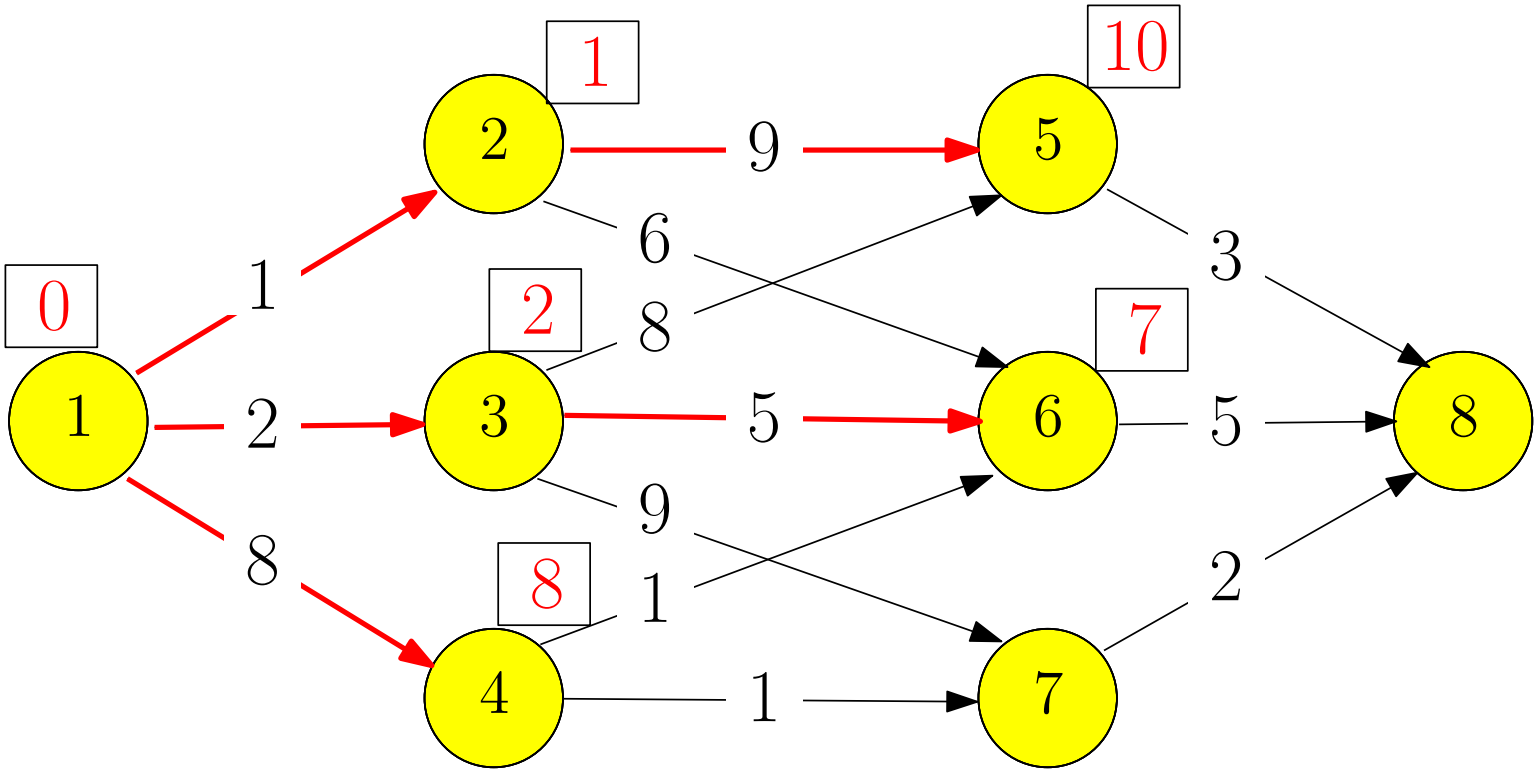

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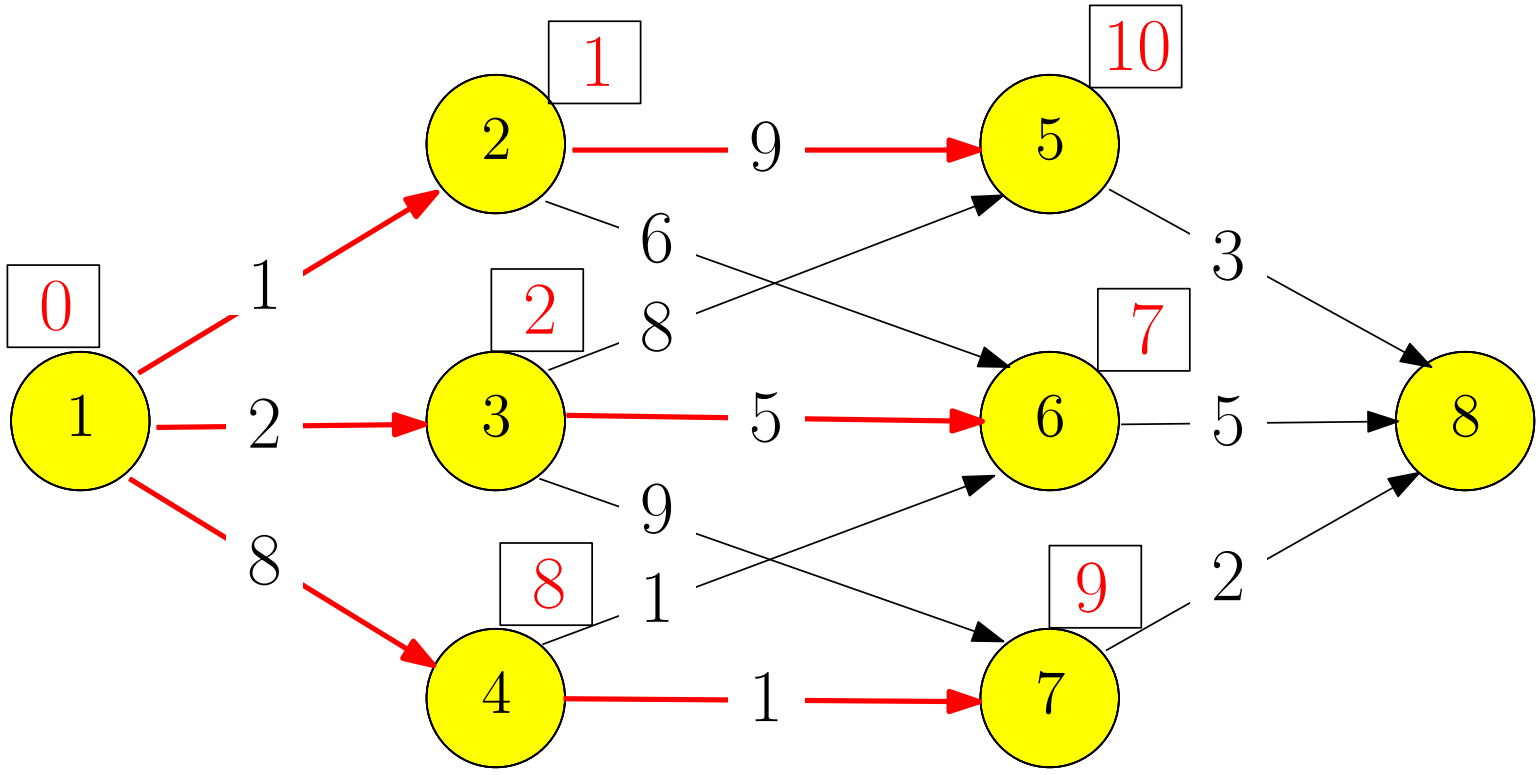

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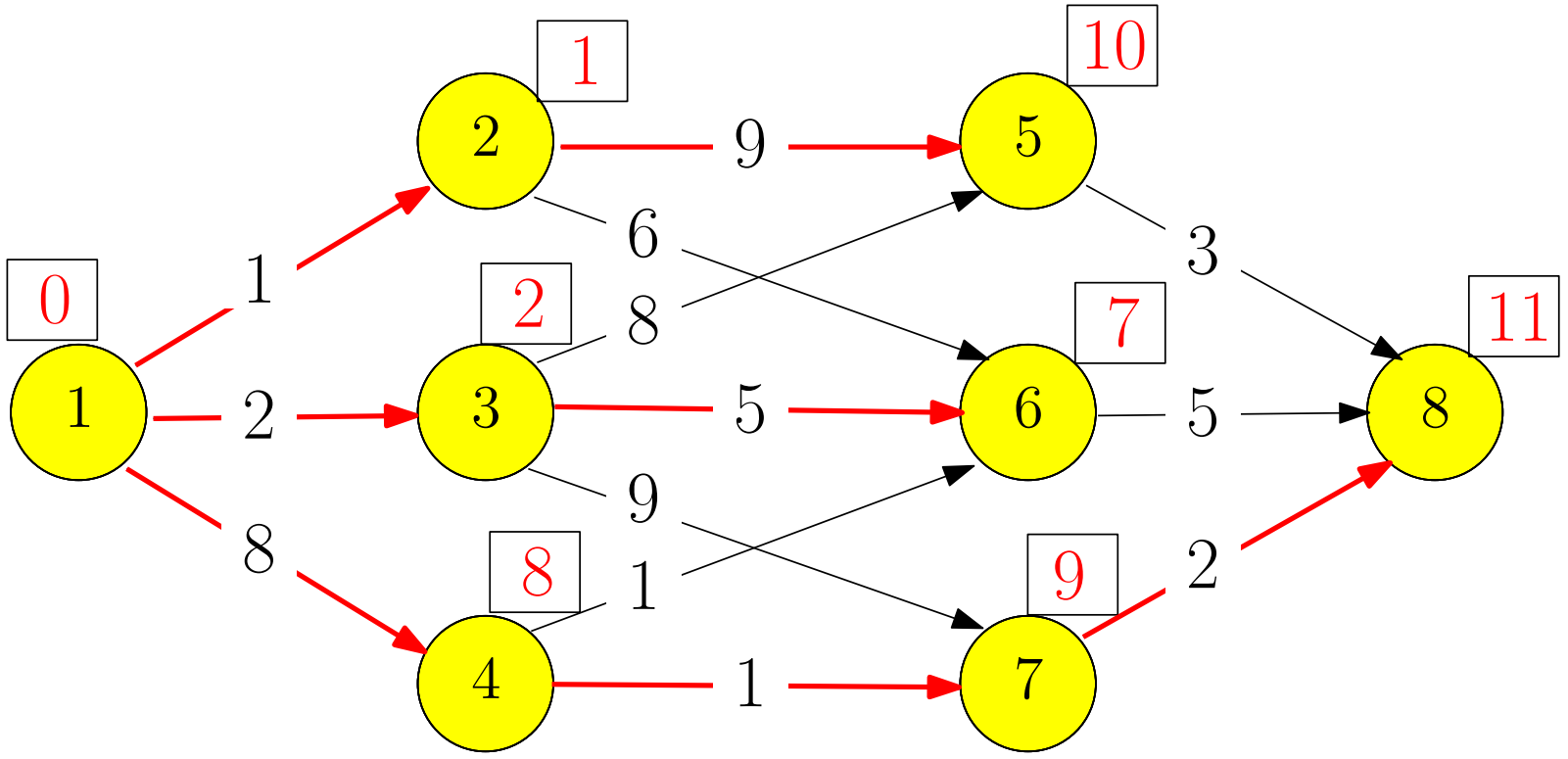

Outline

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

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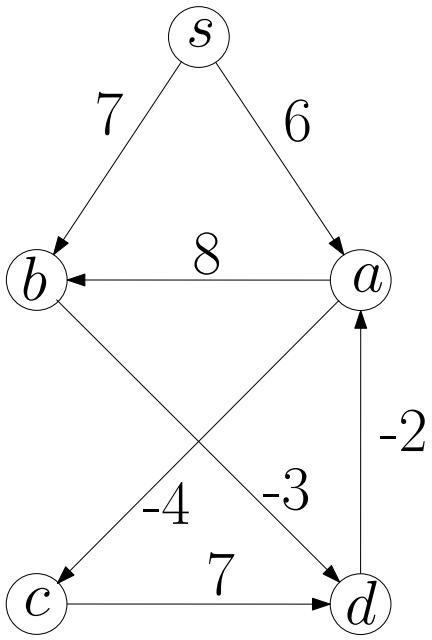

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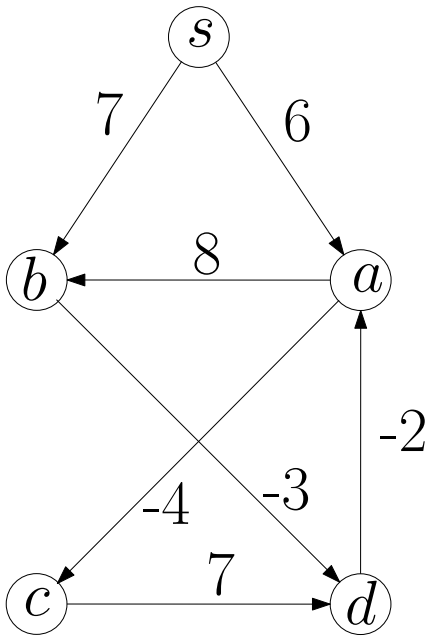

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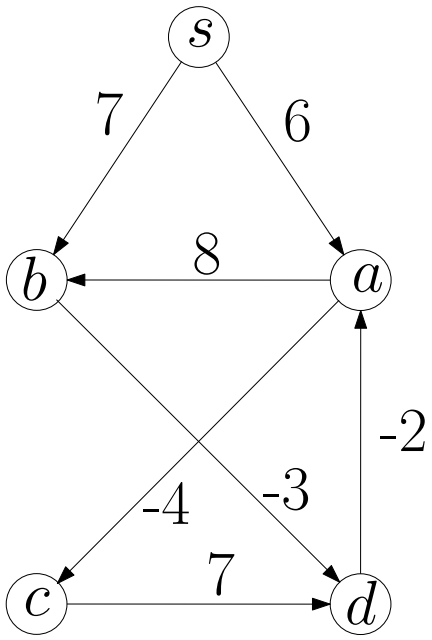

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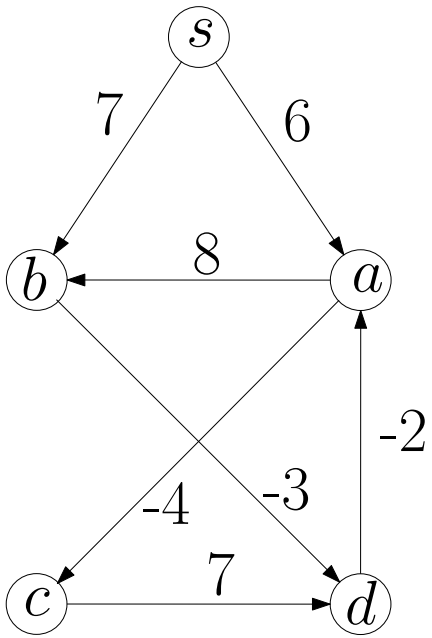

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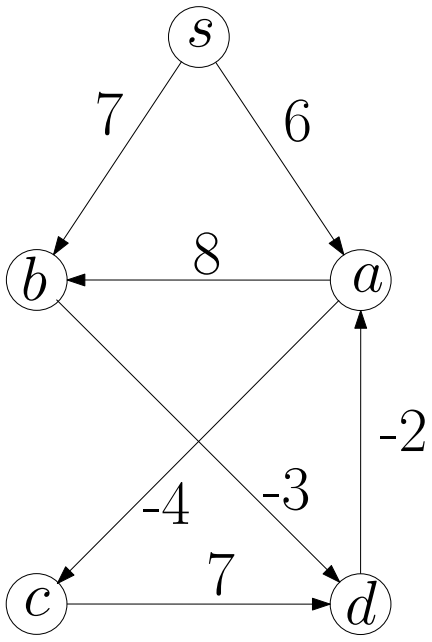

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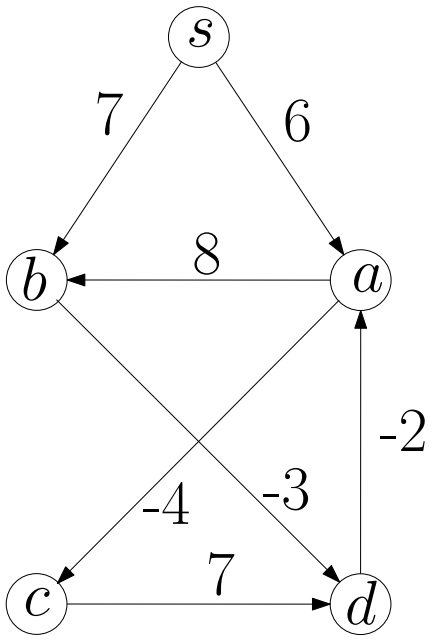

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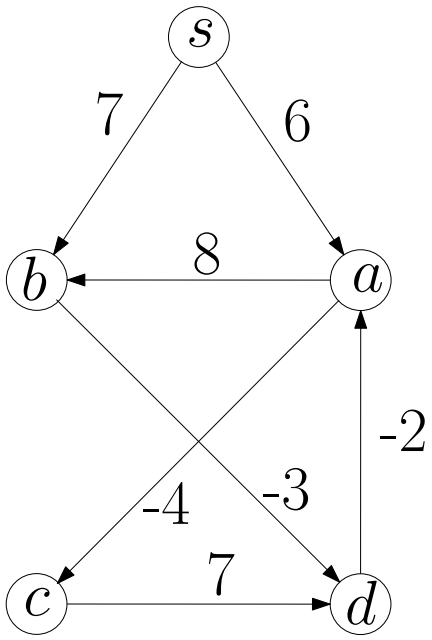

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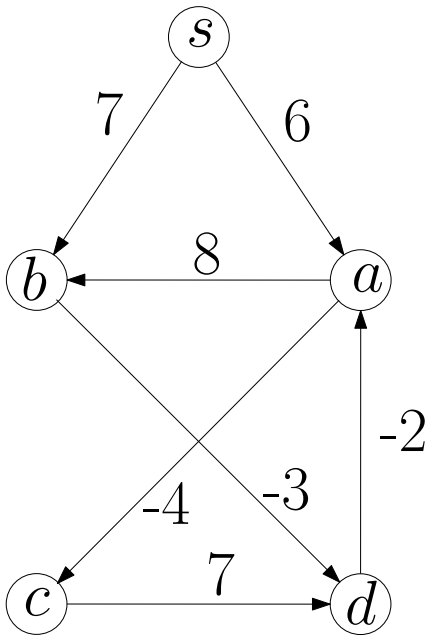

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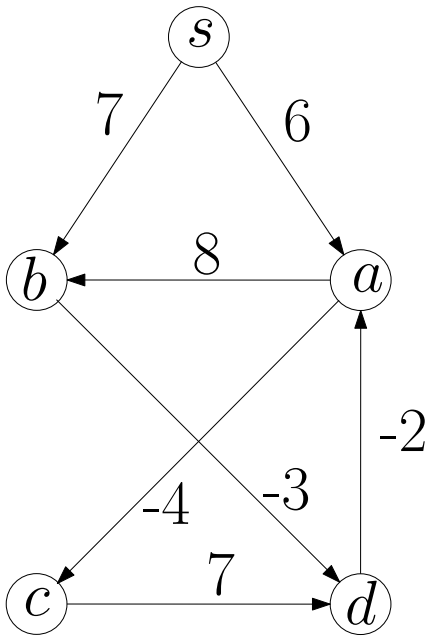

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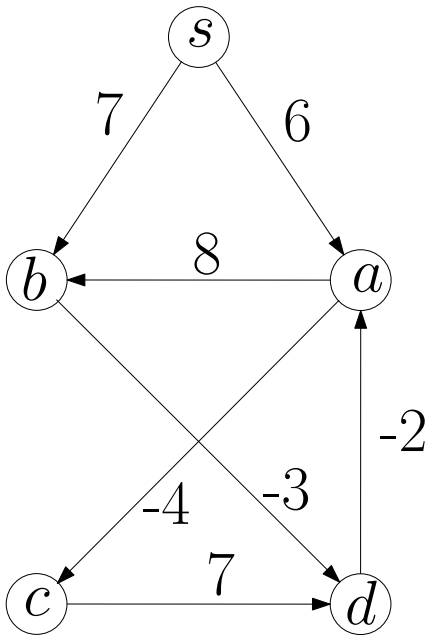

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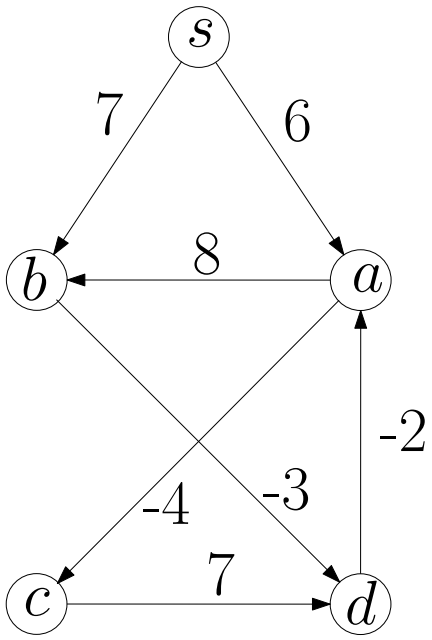
$f^{\ell-1}[v]$



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dynamic-programming(G, w, s)

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n-1$ do
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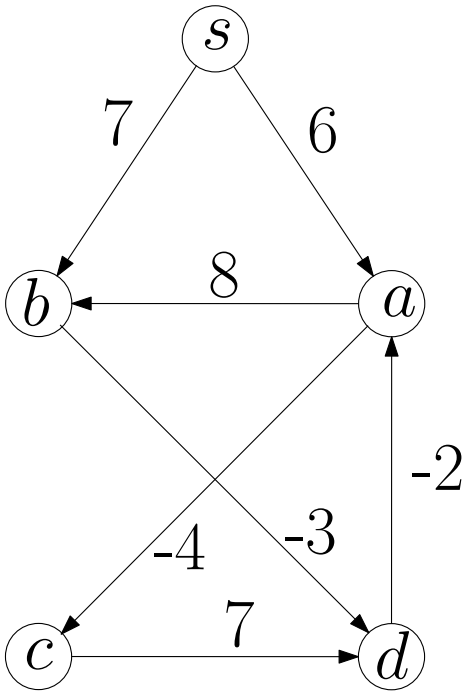
Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Dynamic Programming: Example



	s	a	b	c	d
f^0	0	∞	∞	∞	∞

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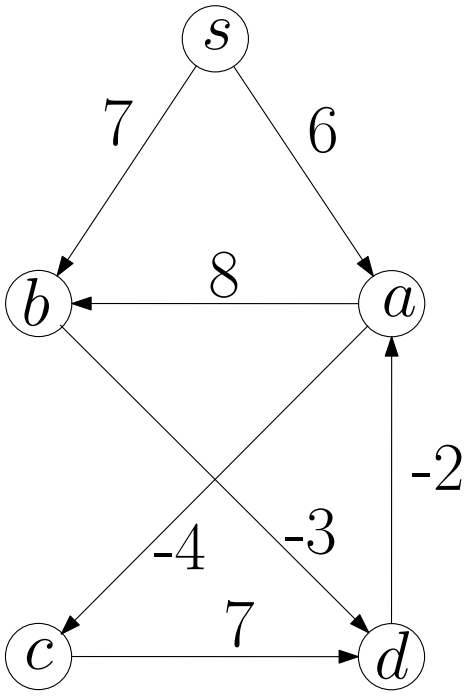

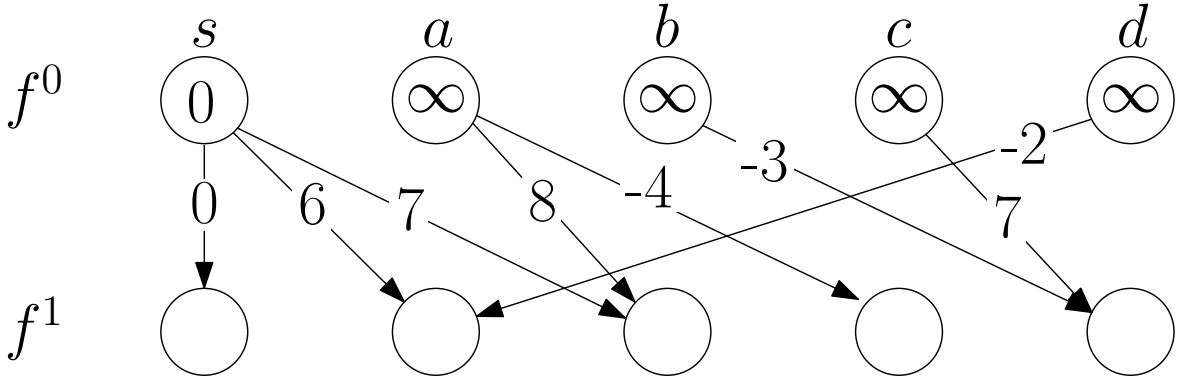

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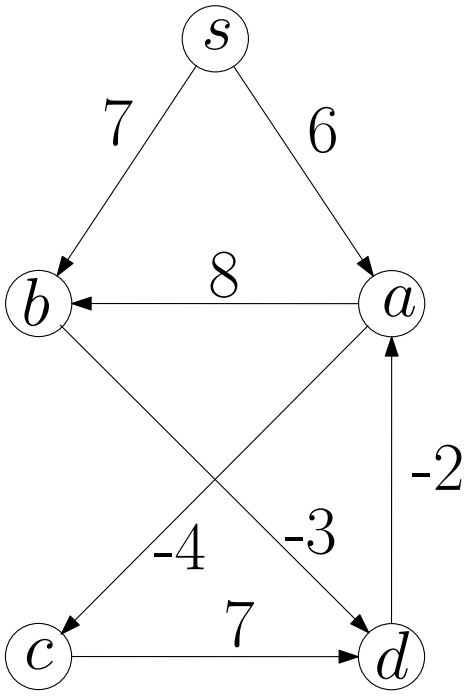

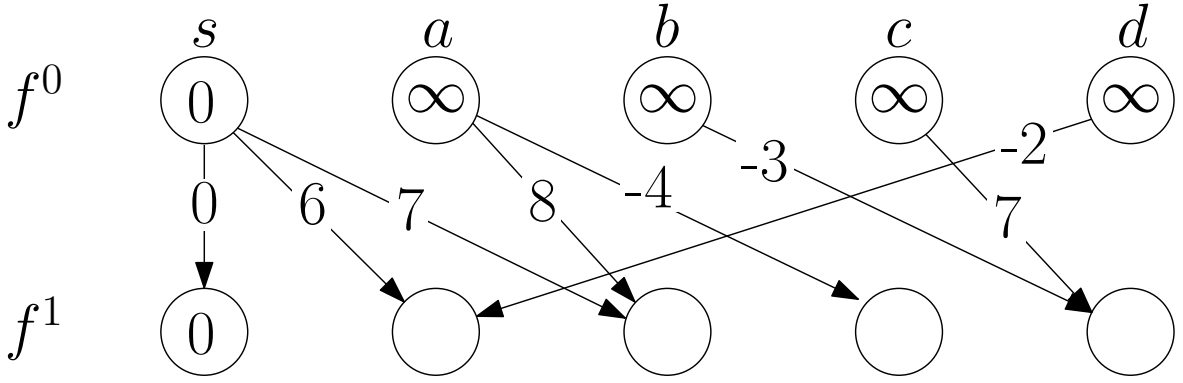

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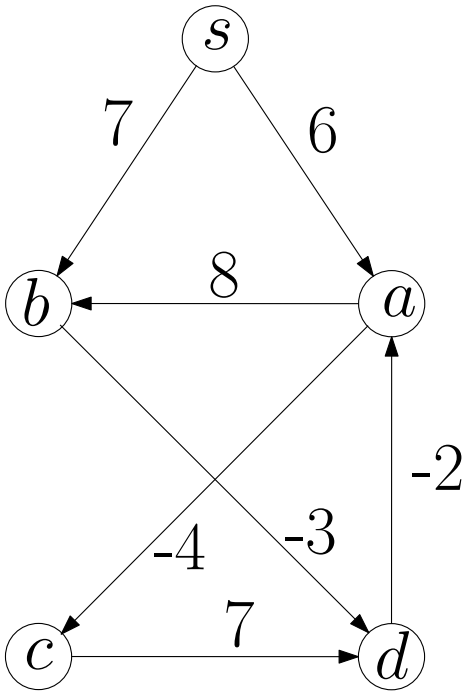

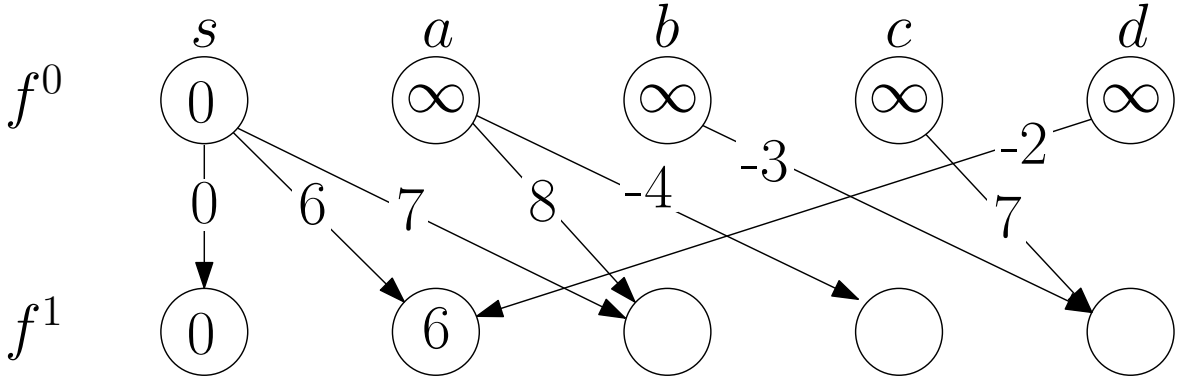

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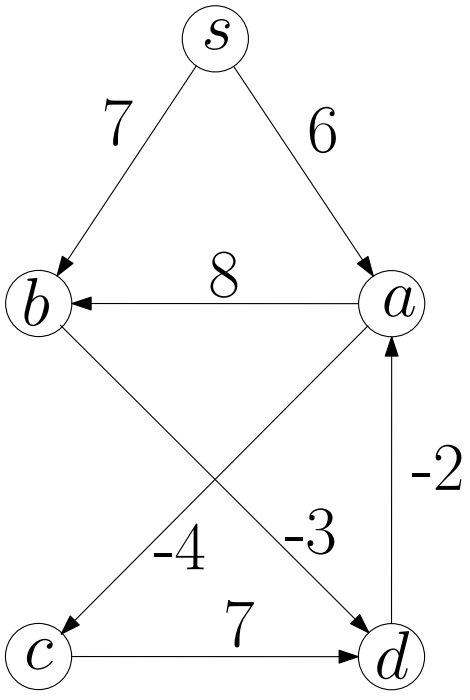

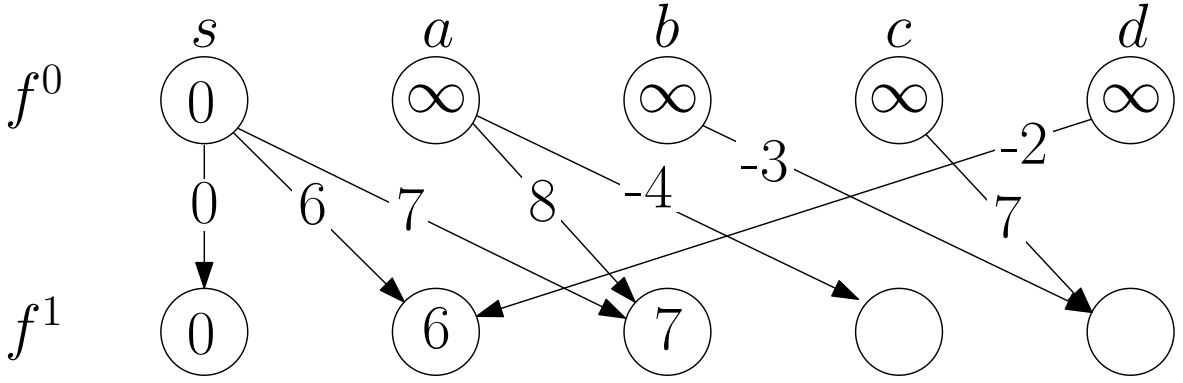

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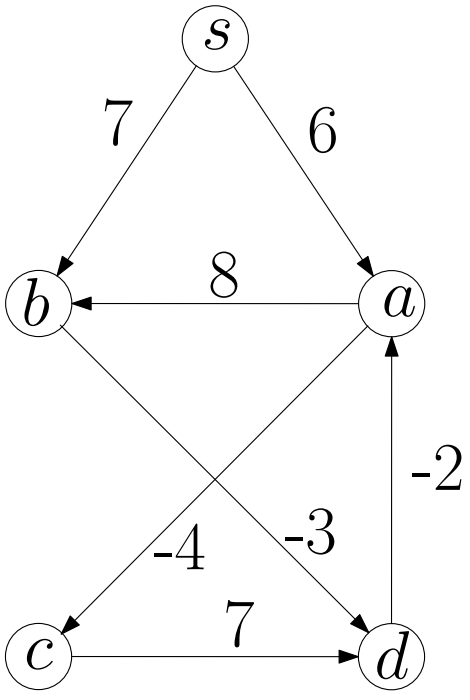

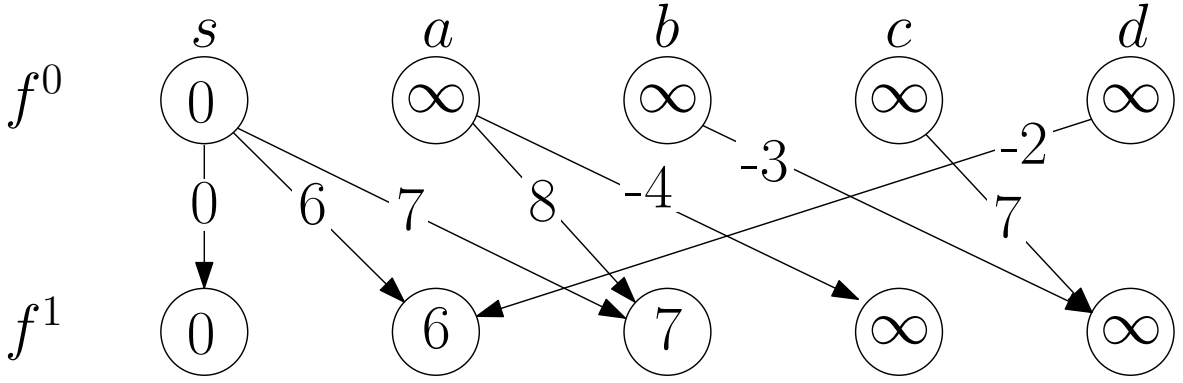

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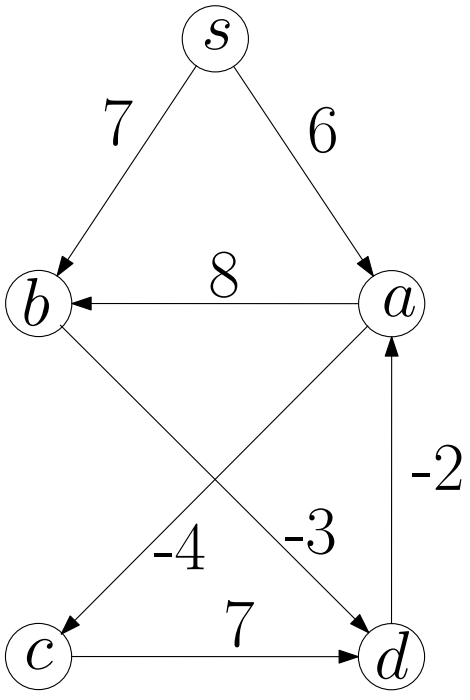
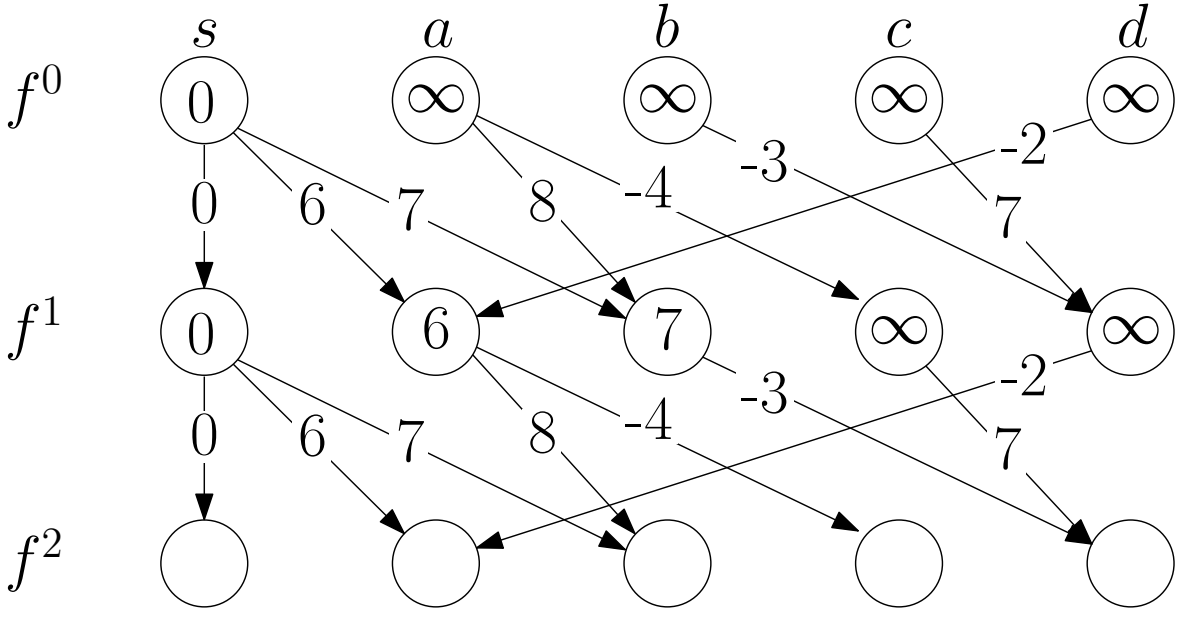
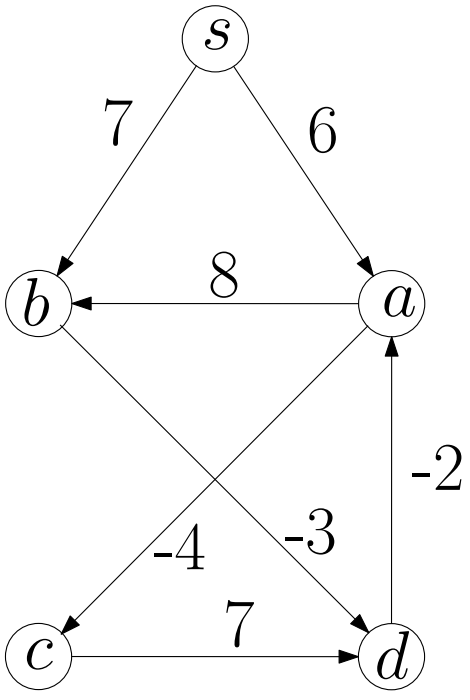
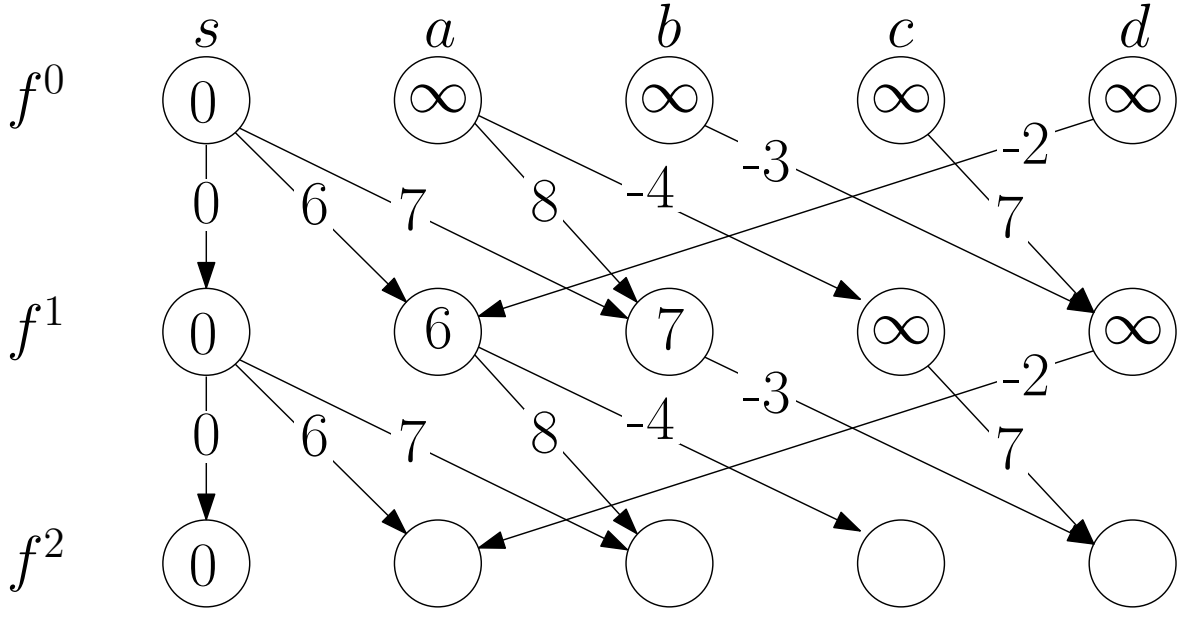




Dynamic Programming: Example





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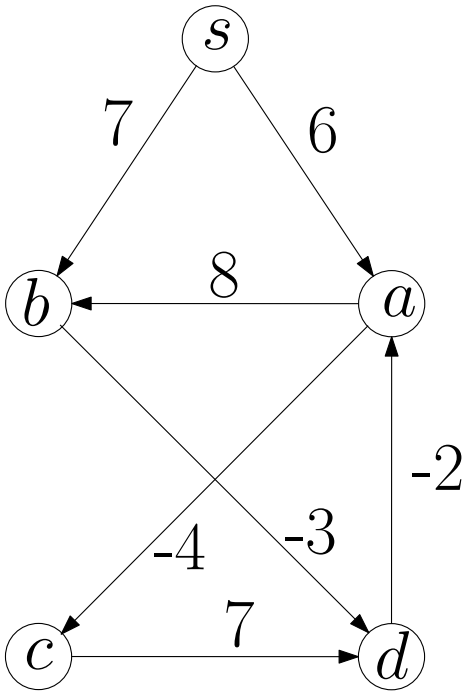

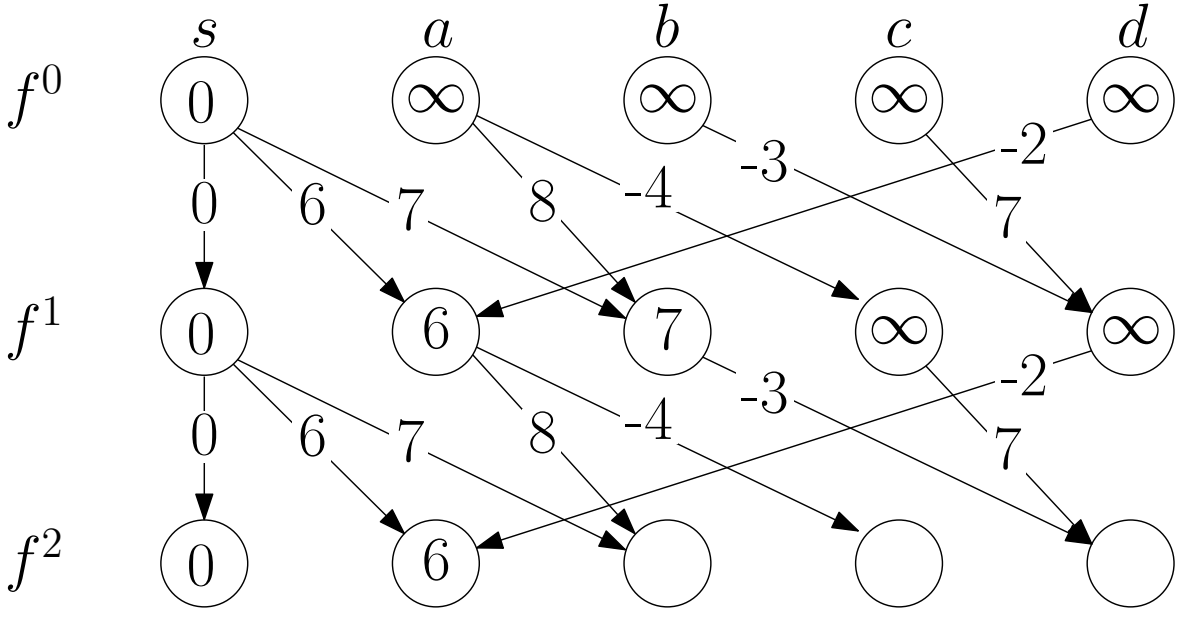

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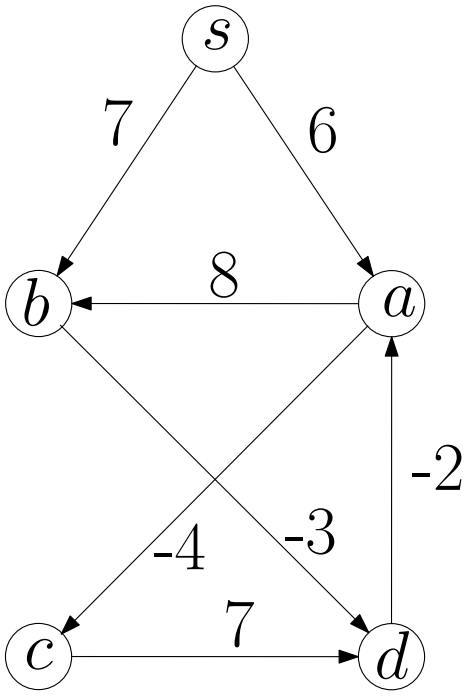

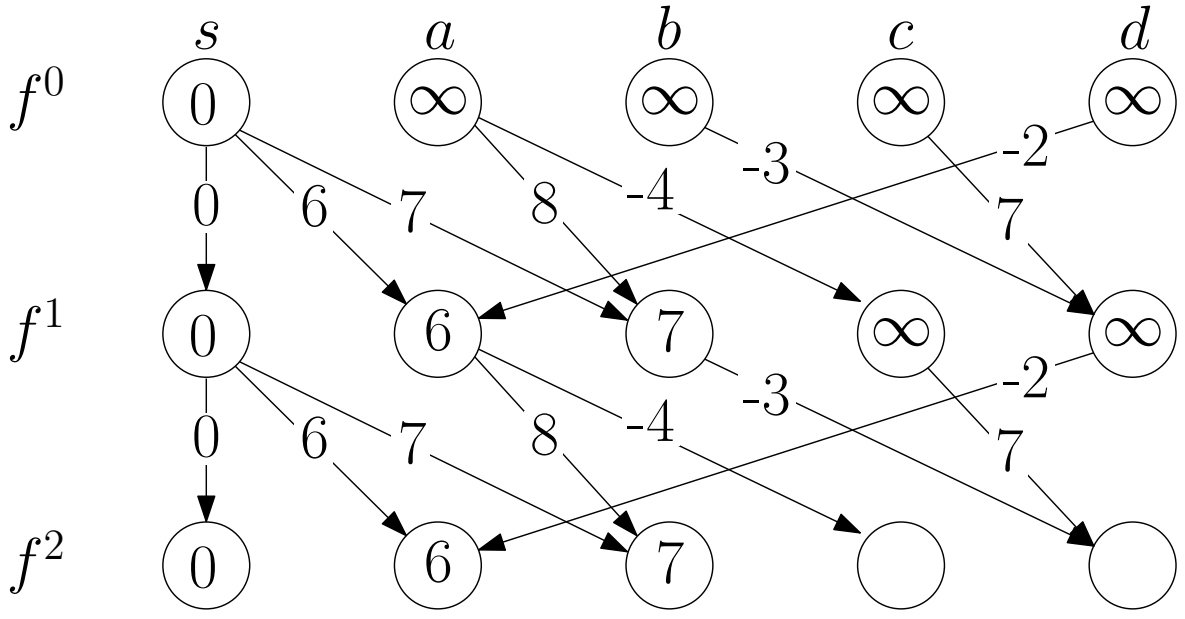

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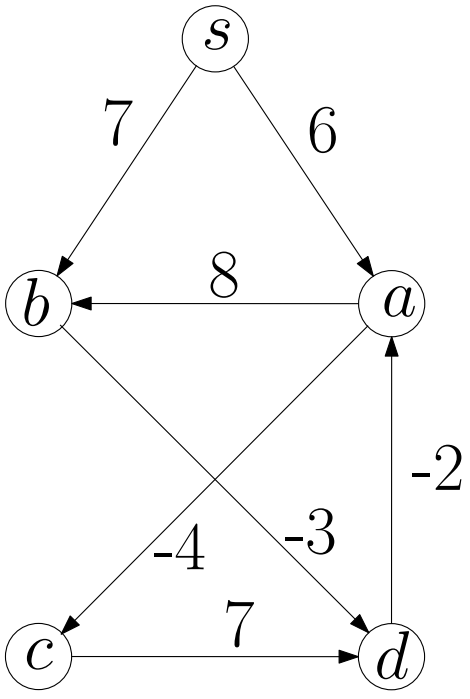

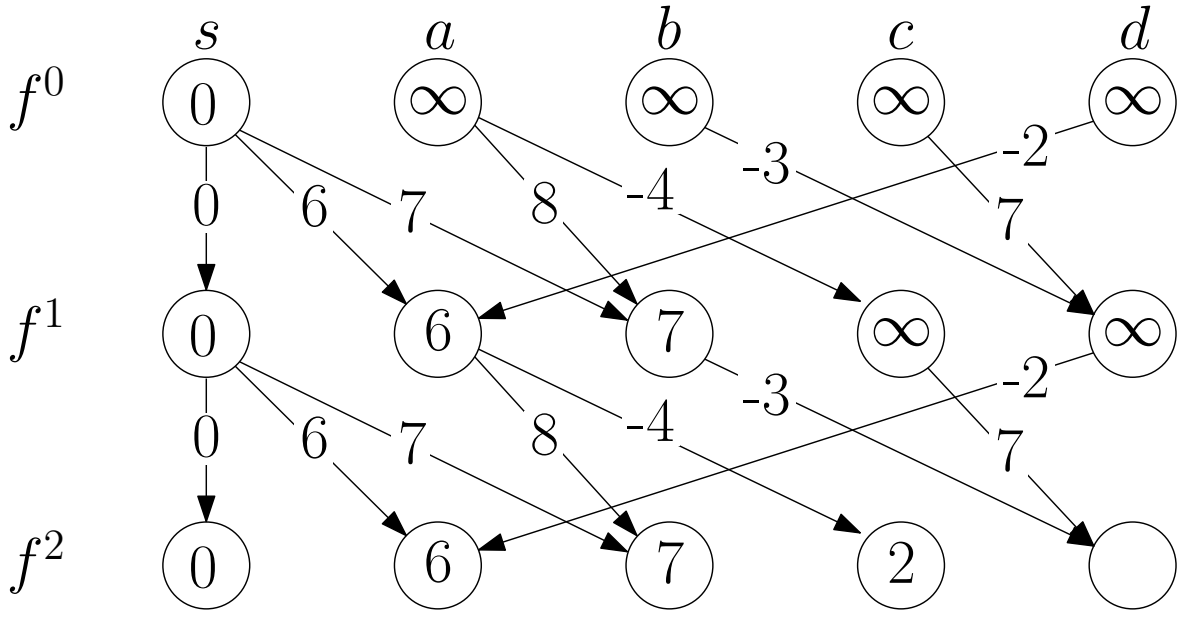

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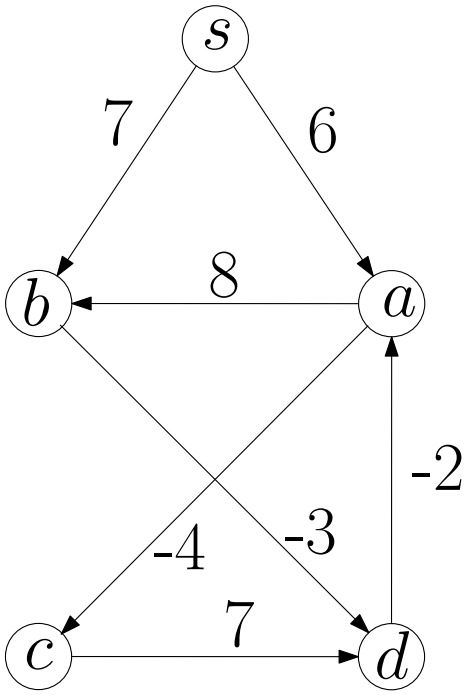

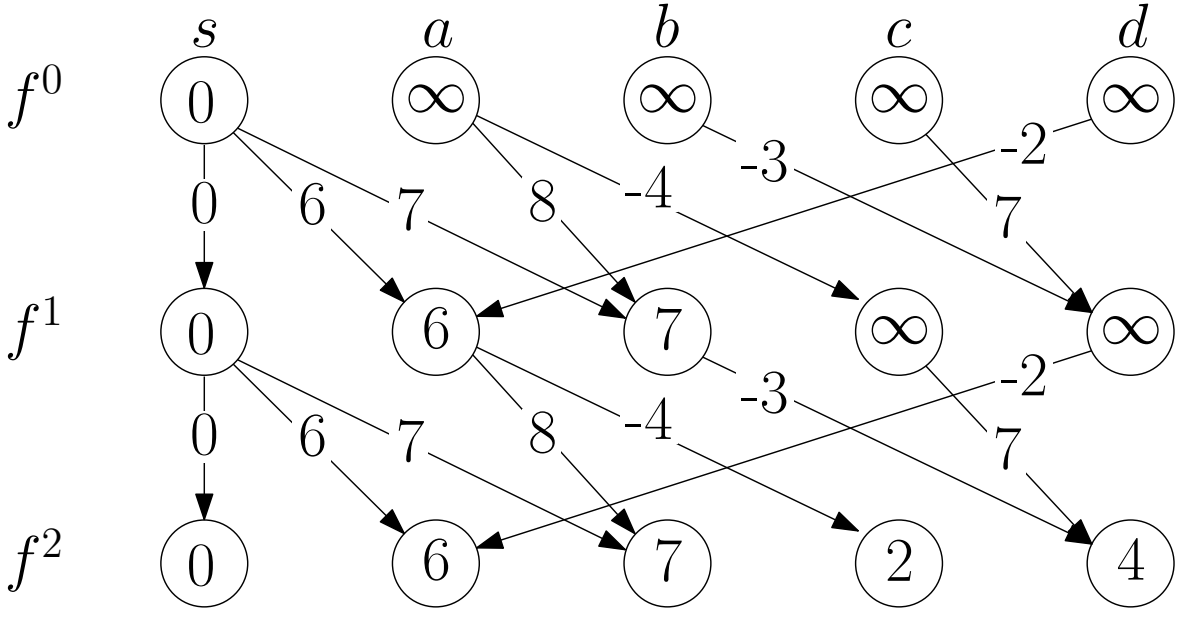

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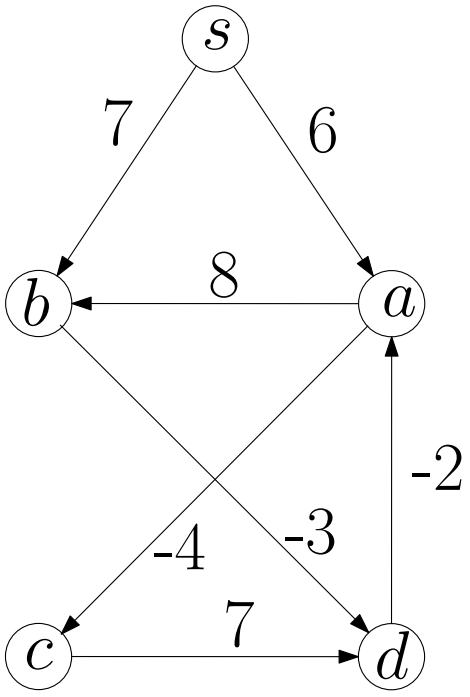

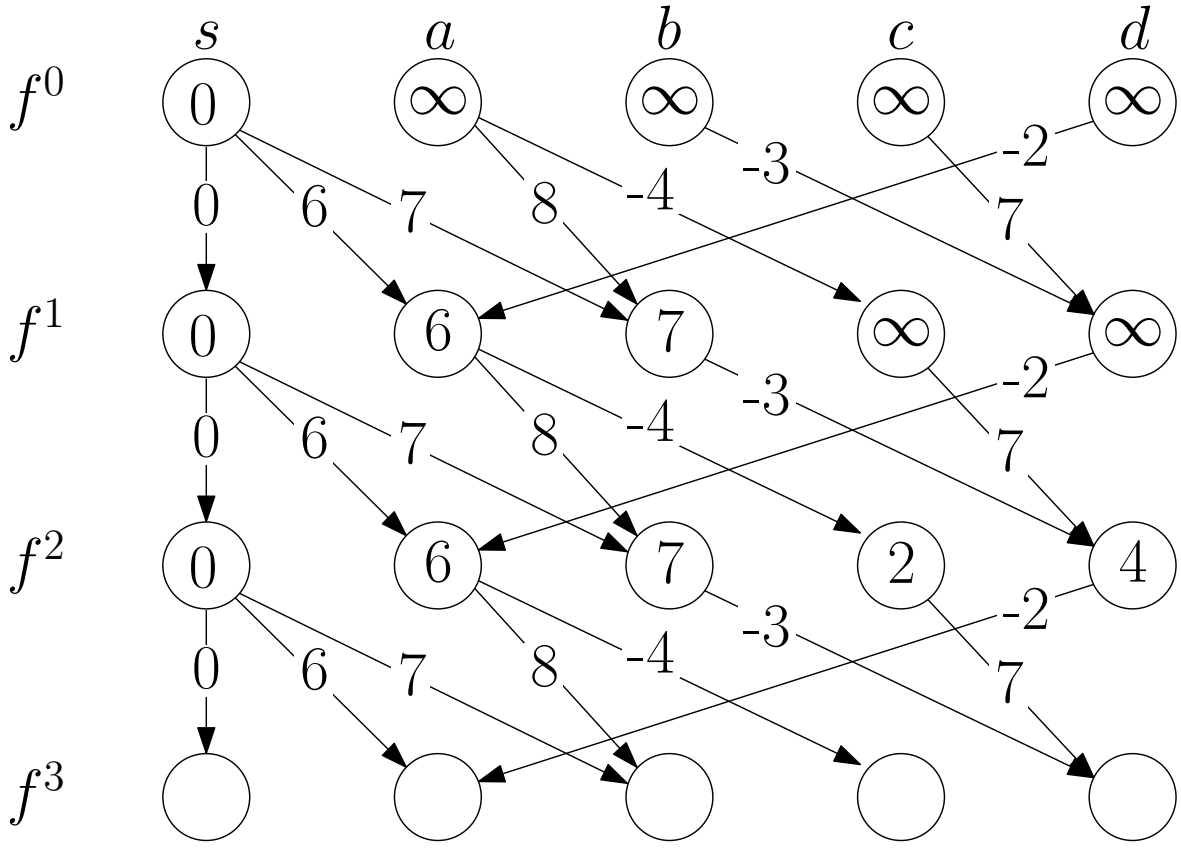

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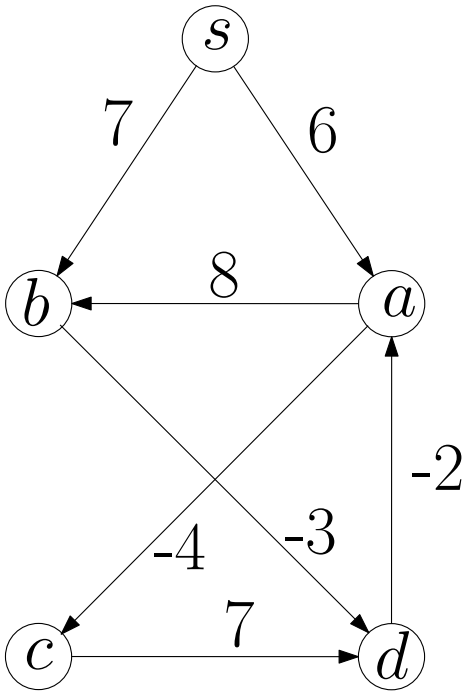

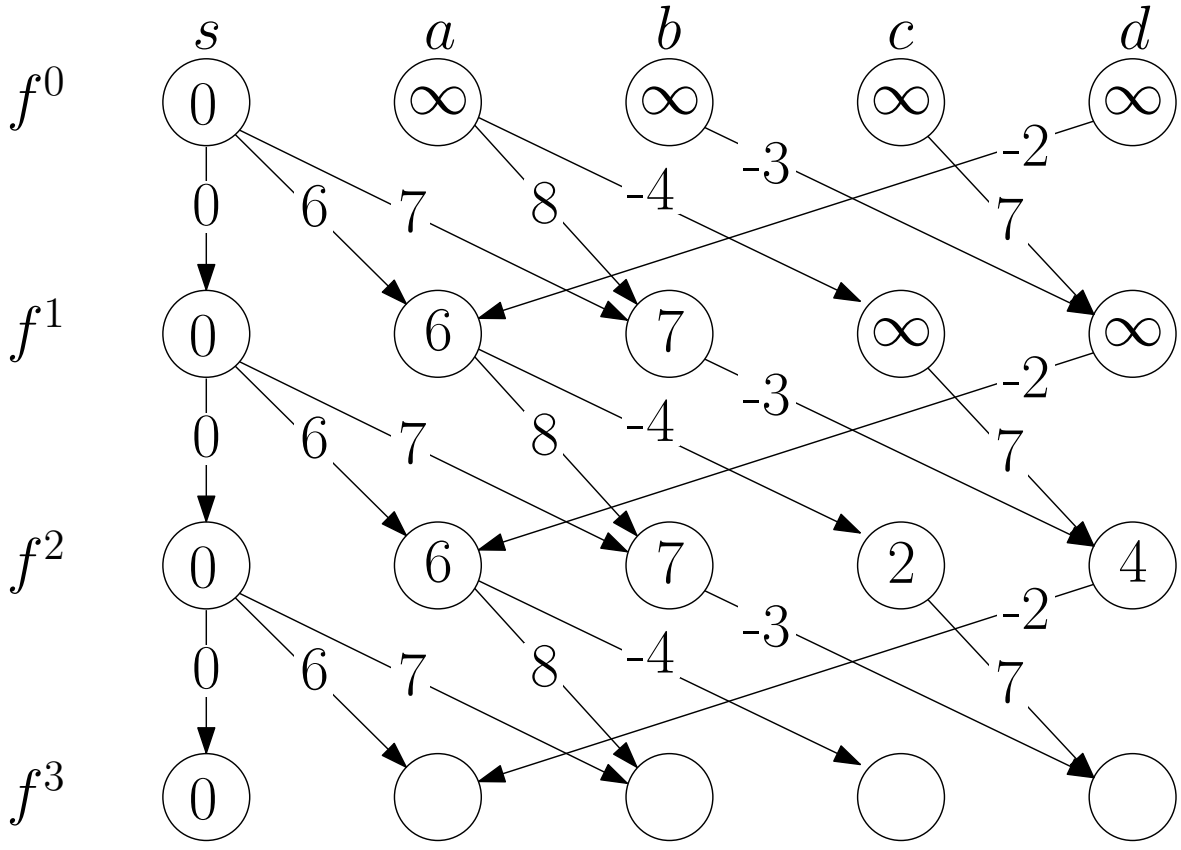

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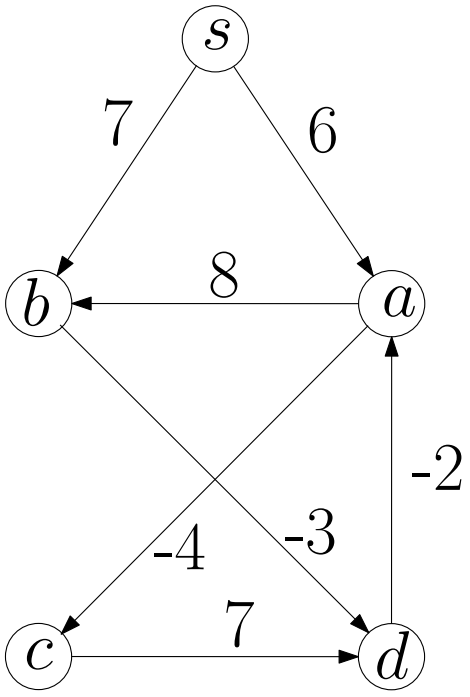

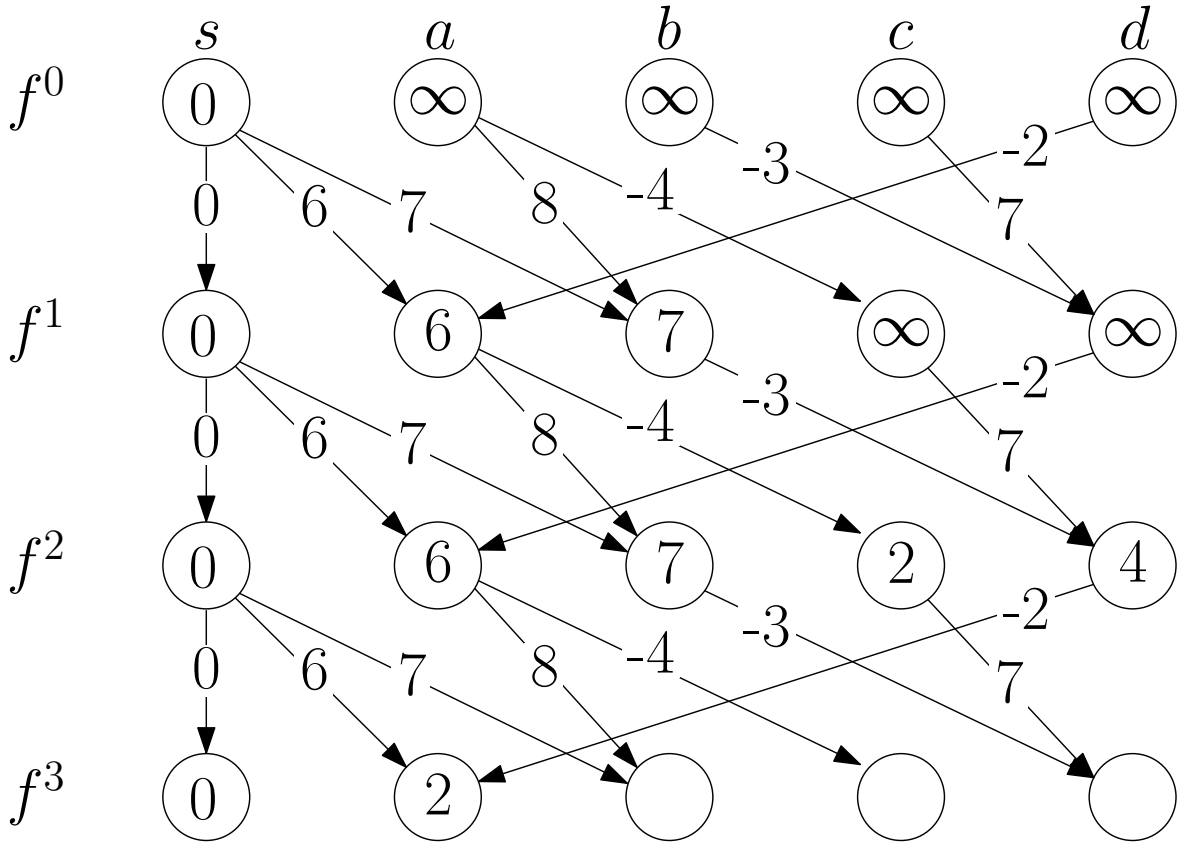

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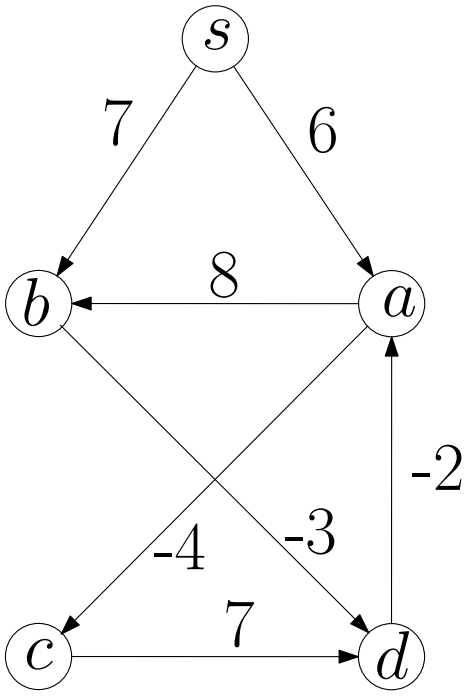

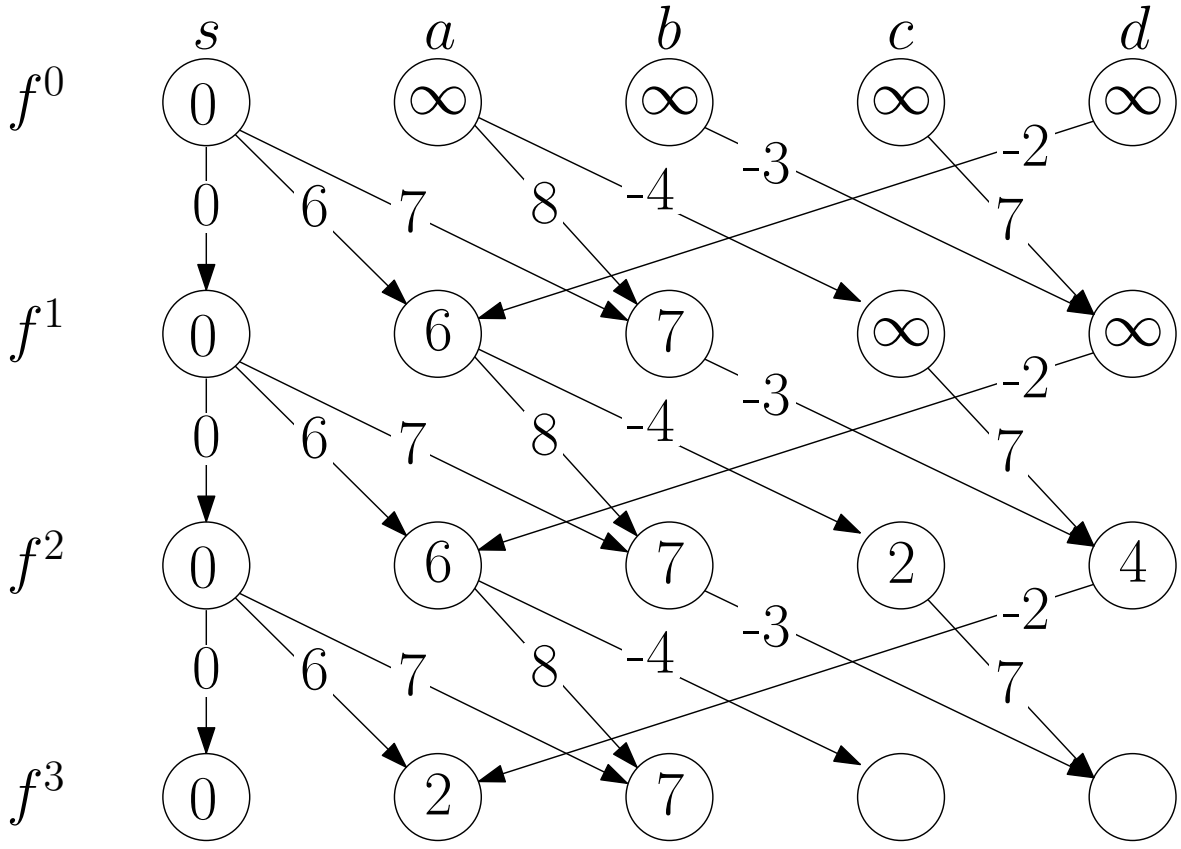

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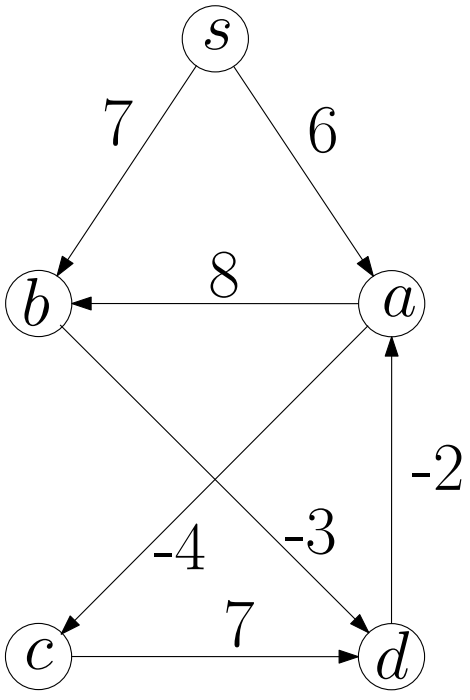

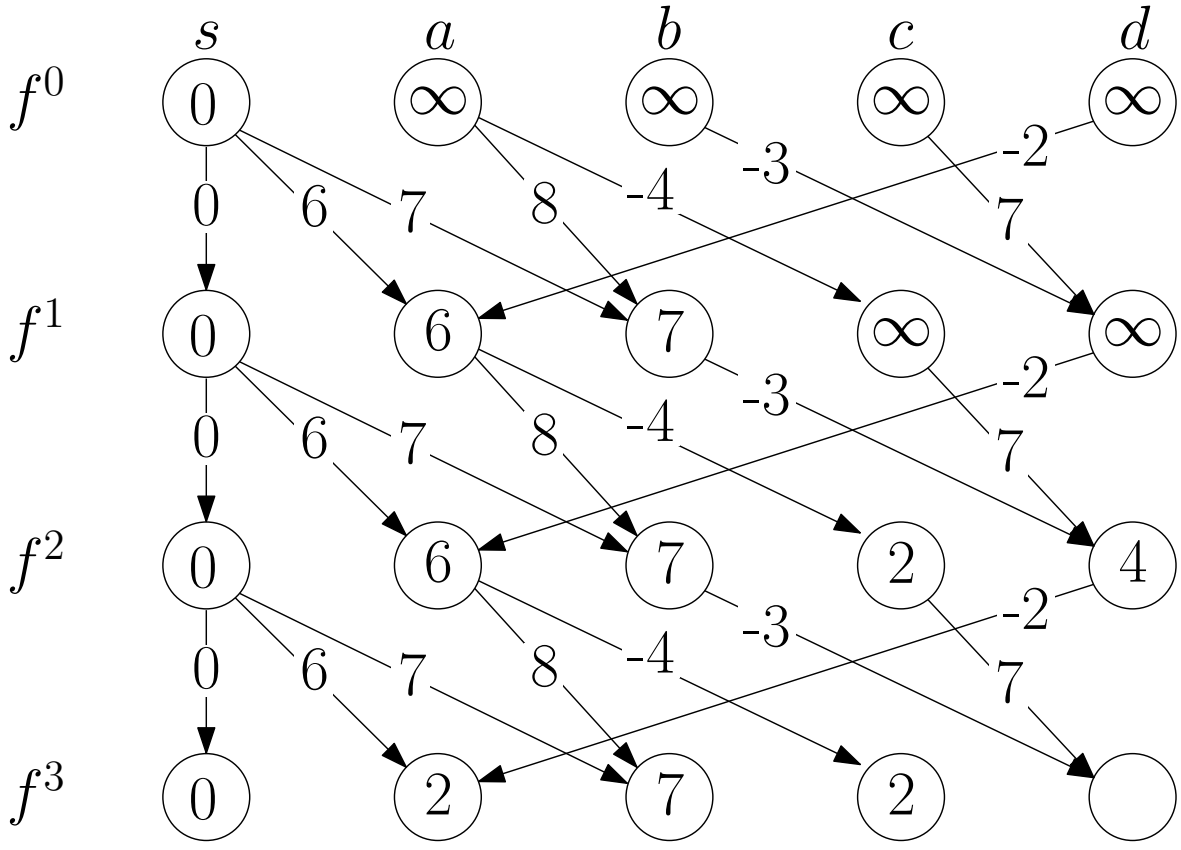

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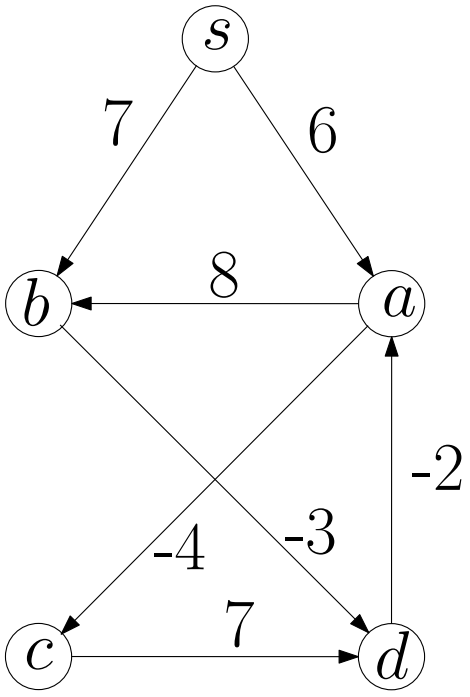

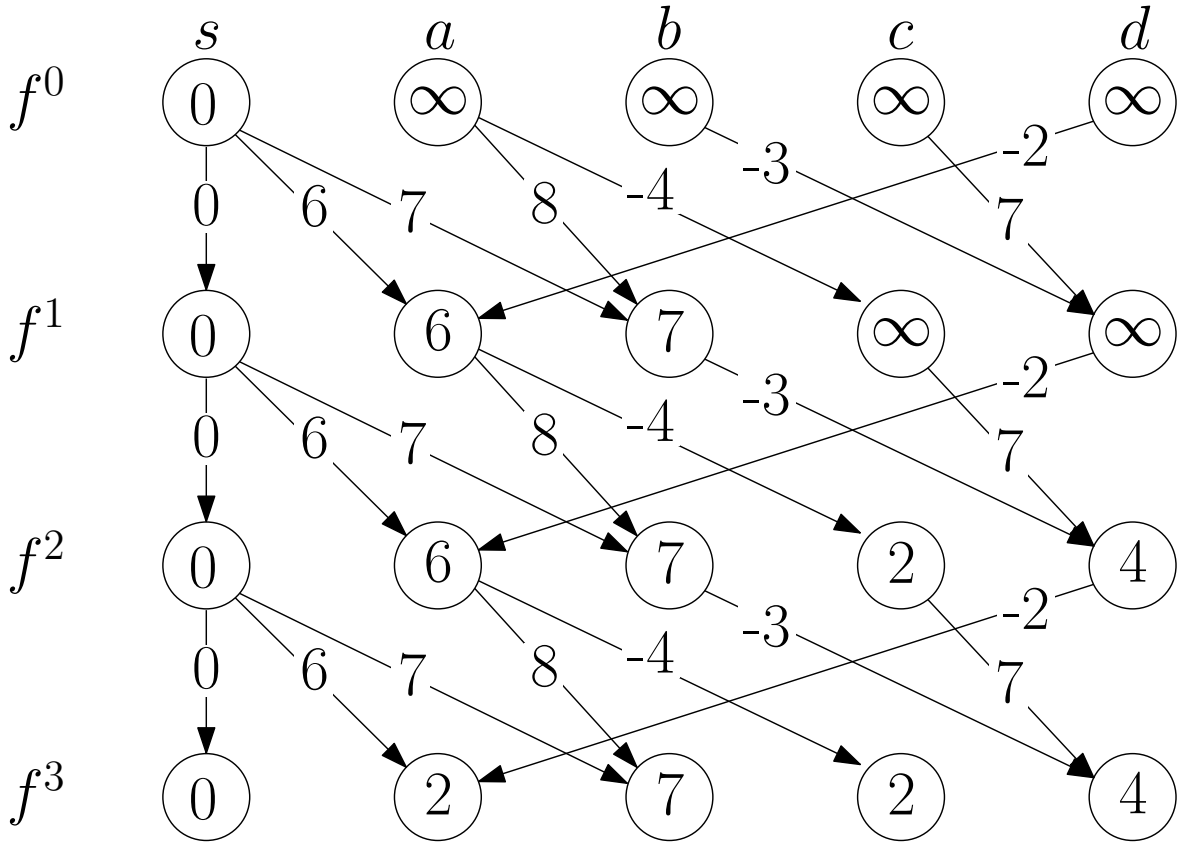

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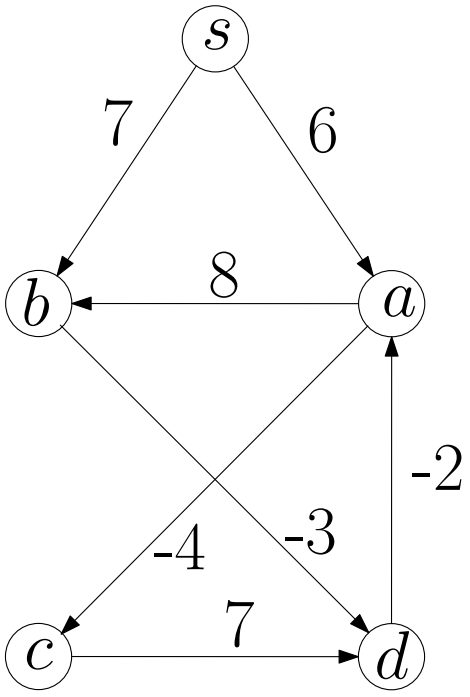

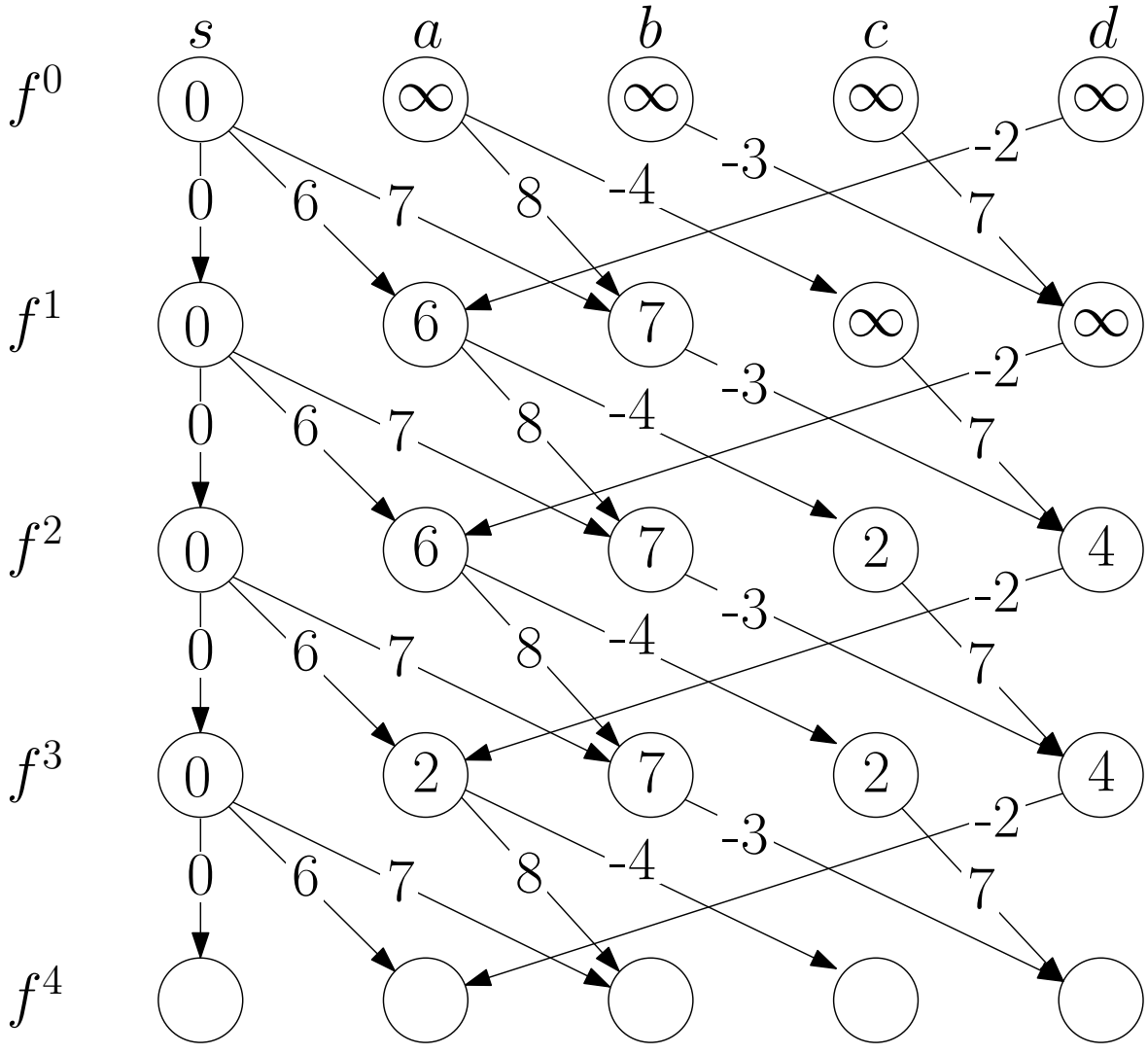

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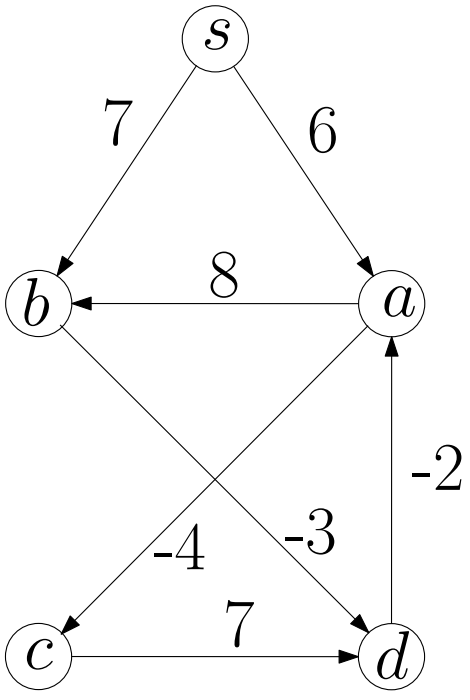

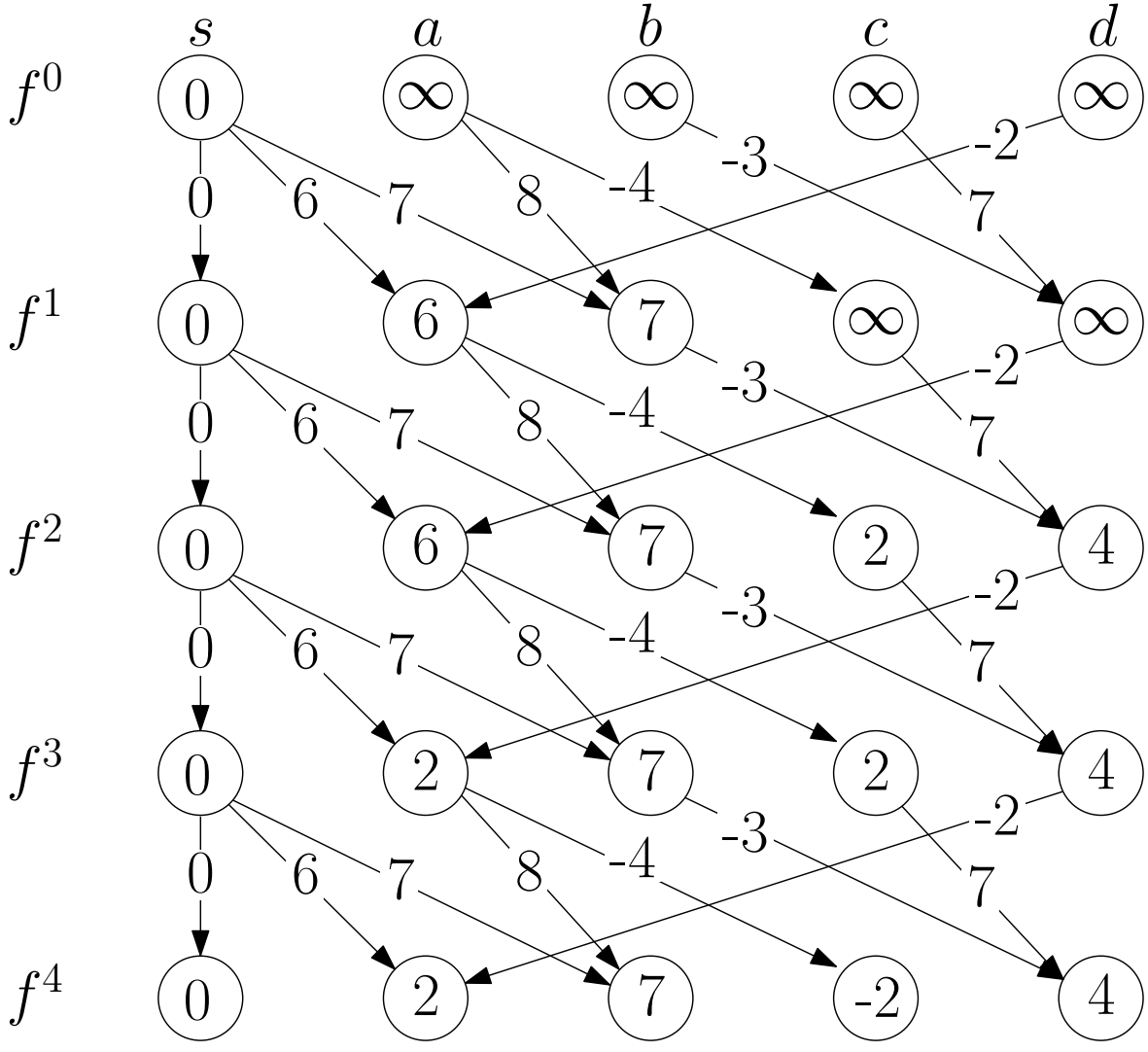

dynamic-programming(G, w, s)

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n-1$ do
3. copy $f^{\ell-1} \rightarrow f^{\ell}$
4. for each $(u, v) \in E$
5. if $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$
6. $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
7. return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

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Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Q: What if there are negative cycles?

Dynamic Programming With Negative Cycle Detection

dynamic-programming(G, w, s)

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
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4. for each $(u, v) \in E$
5. if $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$
6. $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
7. for each $(u, v) \in E$
8. if $f^{n-1}[u] + w(u, v) < f^{n-1}[v]$
9. report "negative cycle exists" and exit
10. return $(f^{n-1}[v])_{v \in V}$

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

1. $f^{\text{old}}[s] \leftarrow 0$ and $f^{\text{old}}[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n - 1$ do
3. copy $f^{\text{old}} \rightarrow f^{\text{new}}$
4. for each $(u, v) \in E$
5. if $f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v]$
6. $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)$
7. copy $f^{\text{new}} \rightarrow f^{\text{old}}$
8. return f^{old}

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors

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6. $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)$
7. copy $f^{\text{new}} \rightarrow f^{\text{old}}$
8. return f^{old}

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors
- only need 1 vector!

Dynamic Programming with Better Space Usage

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- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges

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- This is OK: it can only "accelerate" the process!
- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v

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5. $f[v] \leftarrow f[u] + w(u, v)$
6. return f

- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v
- Assuming there are no negative cycles, after iteration $n - 1$, $f[v] =$ length of shortest path from s to v

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

1. $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to n do
3. $\quad updated \leftarrow$ false
4. $\quad$ for each $(u, v) \in E$
5. $\quad\quad$ if $f[u] + w(u, v) < f[v]$
6. $\quad\quad\quad f[v] \leftarrow f[u] + w(u, v)$
7. $\quad\quad\quad updated \leftarrow$ true
8. $\quad$ if not $updated$, then return f
9. output "negative cycle exists"

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

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6. $\quad\quad\quad f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
7. $\quad\quad\quad updated \leftarrow$ true
8. $\quad$ if not $updated$, then return f
9. output "negative cycle exists"

- $\pi[v]$: the parent of v in the shortest path tree

Bellman-Ford Algorithm

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- $\pi[v]$: the parent of v in the shortest path tree
- Running time $= O(nm)$

Outline

Matrix Chain Multiplication

Matrix Chain Multiplication

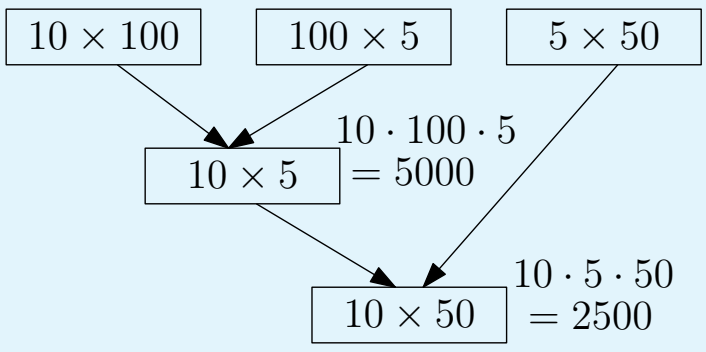
Input: n matrices $A_1, A_2, \cdots, A_n$ of sizes $r_1 \times c_1, r_2 \times c_2, \cdots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \cdots, n-1$.

Output: the order of computing $A_1 A_2 \cdots A_n$ with the minimum number of multiplications

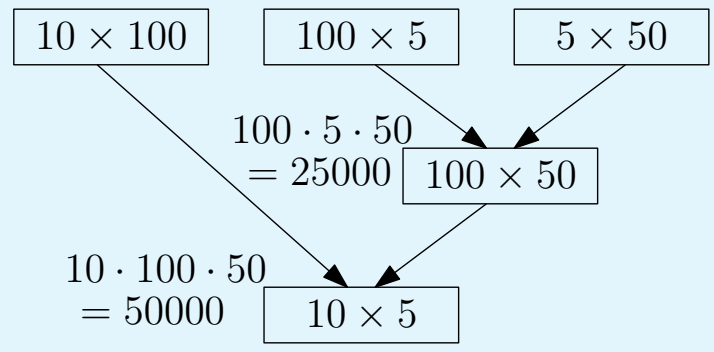
Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$



cost $= 5000 + 2500 = 7500$

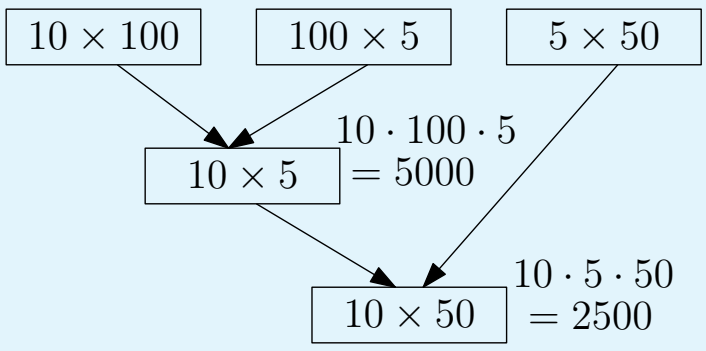


cost $= 25000 + 50000 = 75000$

- $(A_1A_2)A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1(A_2A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$



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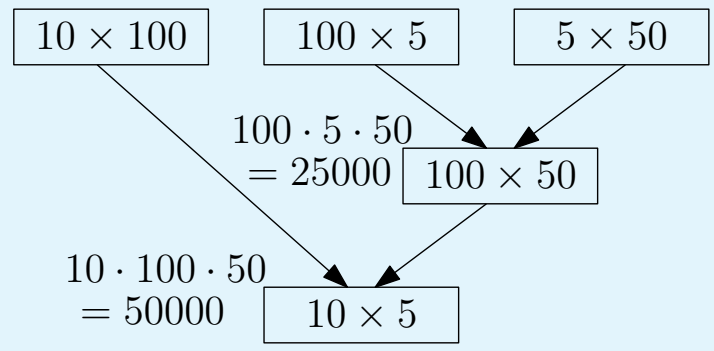

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$$opt[i,j] = \begin{cases} 0 & i = j \\ \min_{k:i \le k < j} \left(opt[i,k] + opt[k+1,j] + r_i c_k c_j \right) & i < j \end{cases}$$

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

1. let $opt[i,i] \leftarrow 0$ for every $i = 1, 2, \cdots, n$
2. for $\ell \leftarrow 2$ to n
3. for $i \leftarrow 1$ to $n - \ell + 1$
4. $j \leftarrow i + \ell - 1$
5. $opt[i,j] \leftarrow \infty$
6. for $k \leftarrow i$ to $j - 1$
7. if $opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j]$
8. $opt[i,j] \leftarrow opt[i,k] + opt[k+1,j] + r_i c_k c_j$
9. return $opt[1,n]$

Outline

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
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- Matrix chain multiplication: $opt[i, j]$ = value of instances defined by matrices i to j
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Bellman-Ford: $f^{\ell}[v]$ = length of shortest path from s to v that uses at most ℓ edges