CSE 431/531: Algorithm Analysis and Design (Spring 2019)

Graph Basics

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Outline

1. Graphs
2. Connectivity and Graph Traversal
   - Testing Bipartiteness
3. Topological Ordering

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

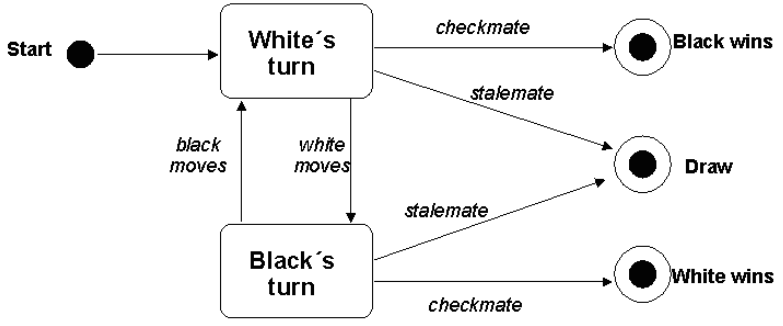
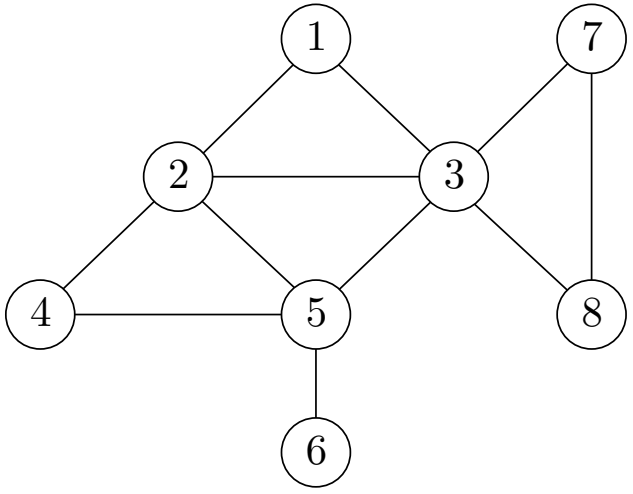


Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



- V: set of vertices (nodes);
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

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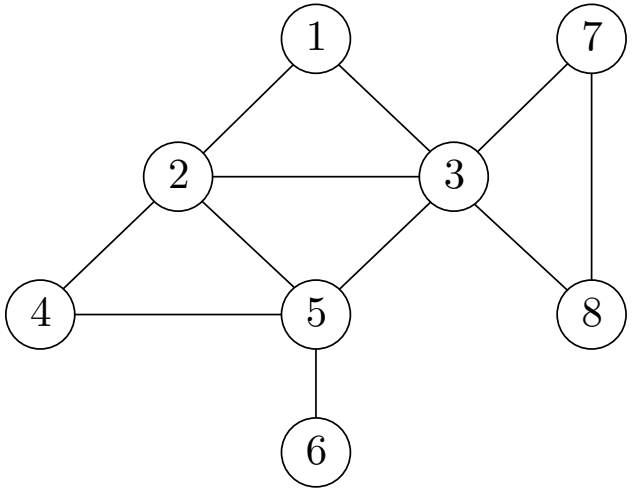

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  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
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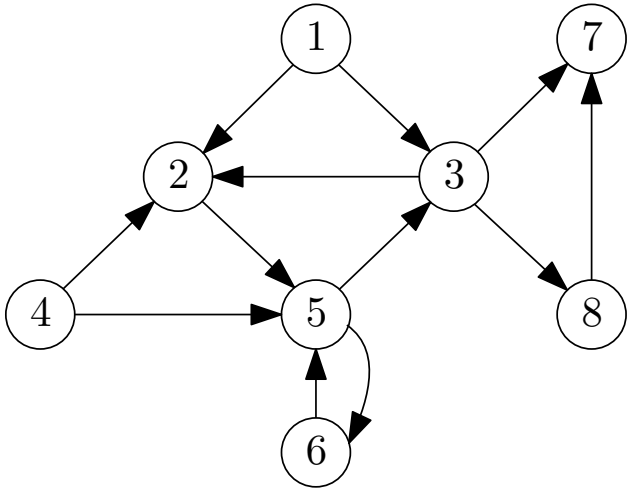
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
  - $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$

Directed Graph $G = (V, E)$



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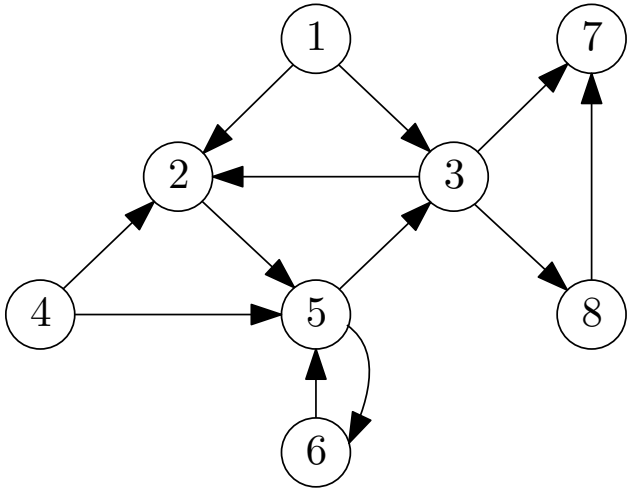

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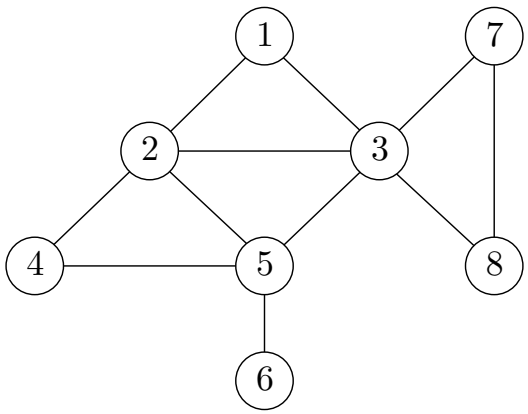
Abuse of Notations

- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



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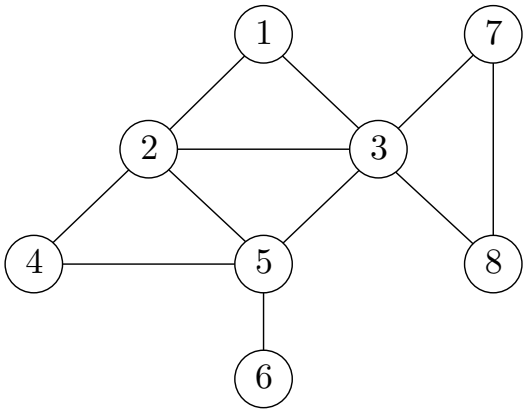
- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

Representation of Graphs



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

- Adjacency matrix
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected

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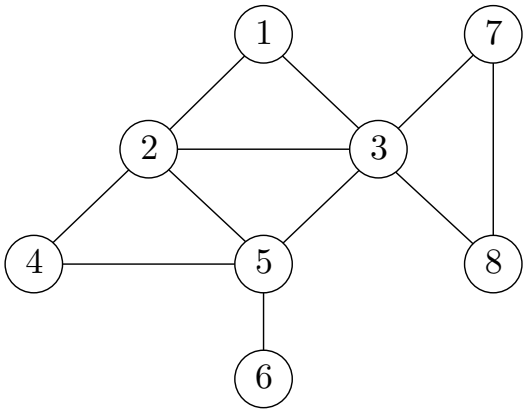

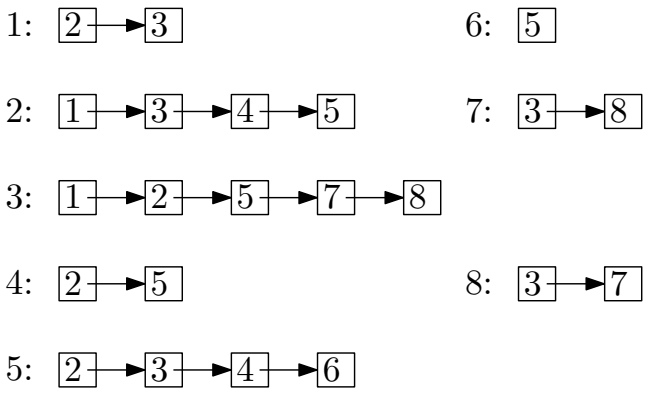

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  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbours of v.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n - 1 \leq m \leq n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u, v) \in E$		
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time to check $(u, v) \in E$	$O(1)$	
time to list all neighbours of v		

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time to list all neighbours of v	$O(n)$	

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Input: graph $G = (V, E)$, (using linked lists)

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  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j

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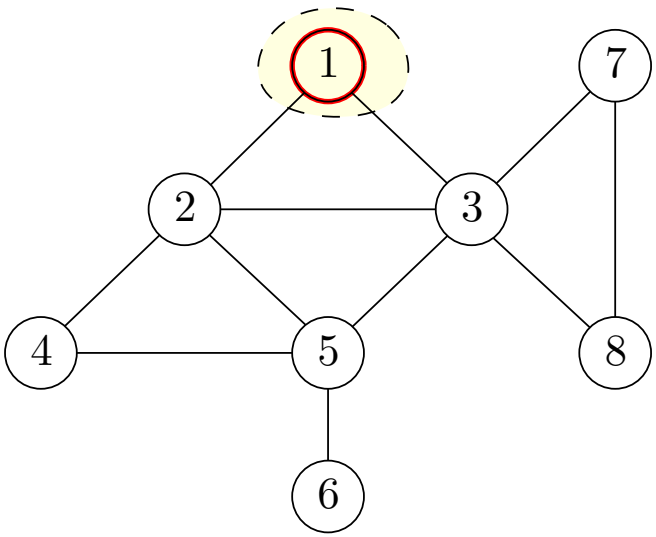

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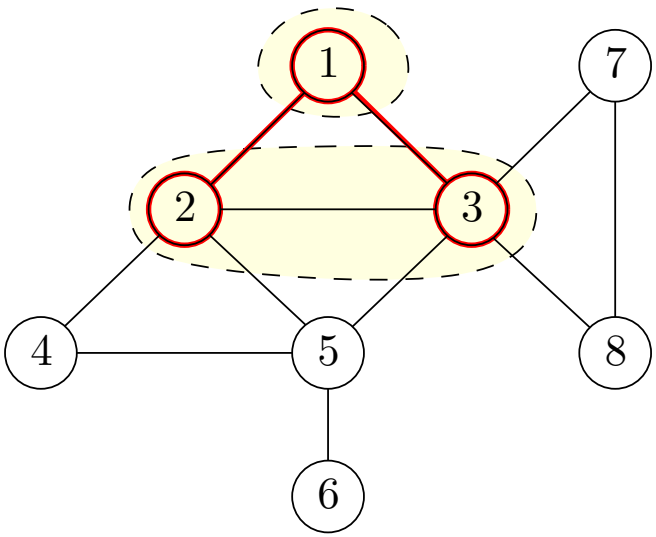

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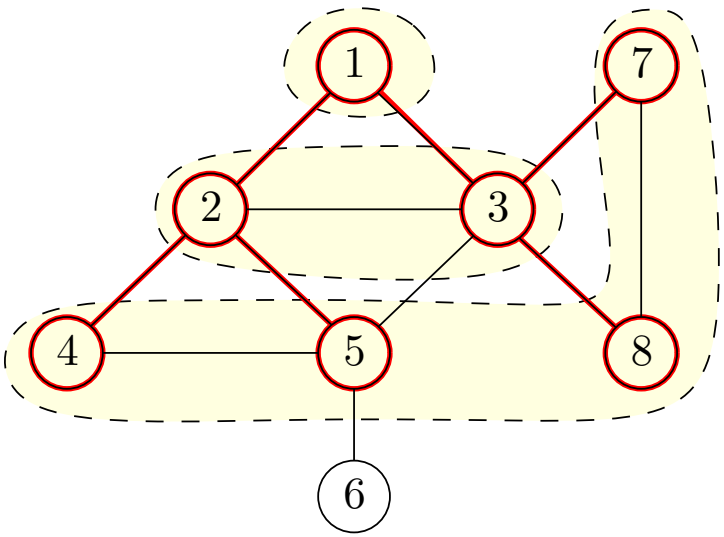

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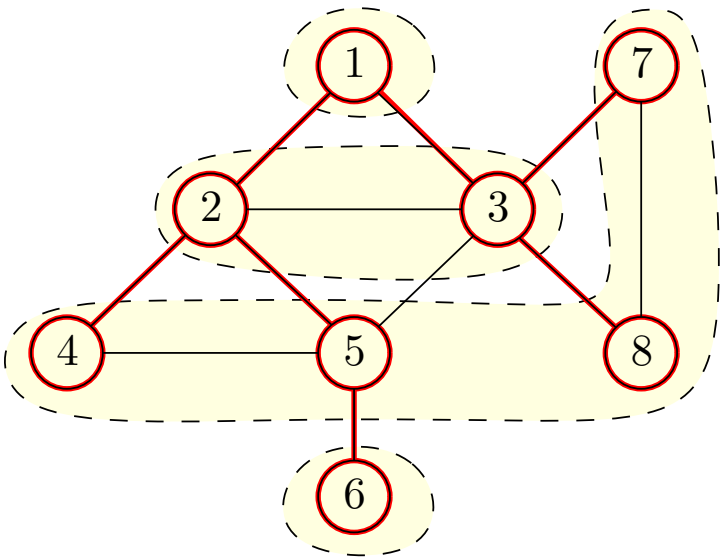


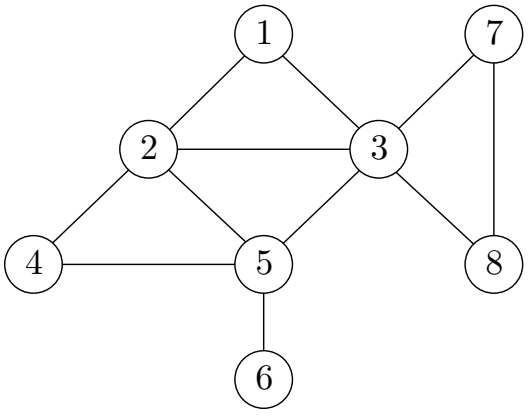
Implementing BFS using a Queue

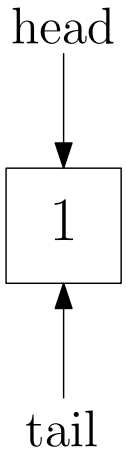
BFS(s)

1. $head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
2. mark s as "visited" and all other vertices as "unvisited"
3. while head $\geq$ tail
4. $\quad v \leftarrow queue[tail], tail \leftarrow tail + 1$
5. $\quad$ for all neighbours u of v
6. $\quad\quad$ if u is "unvisited" then
7. $\quad\quad\quad head \leftarrow head + 1, queue[head] = u$
8. $\quad\quad\quad$ mark u as "visited"

- Running time: $O(n + m)$.

Example of BFS via Queue





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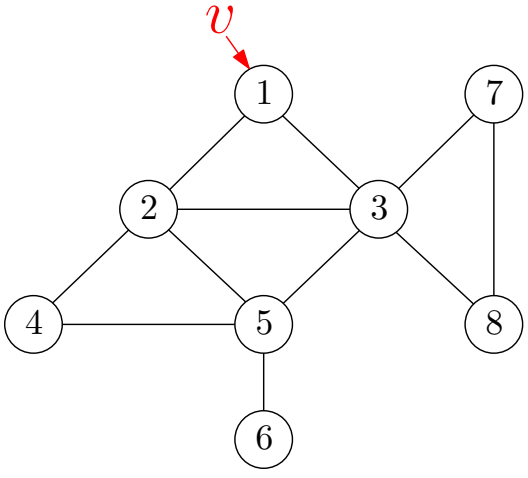

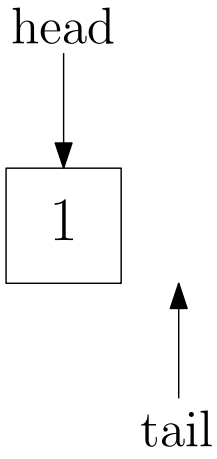

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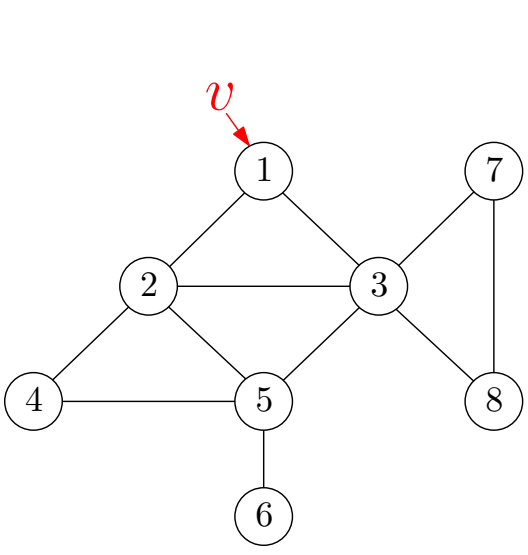

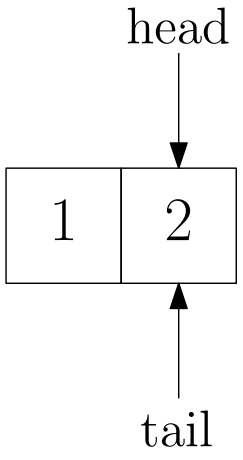

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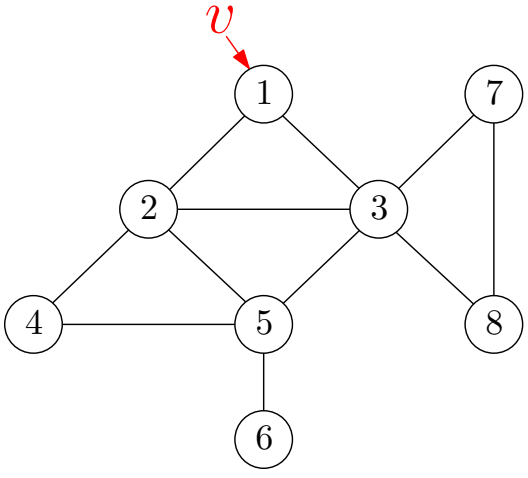

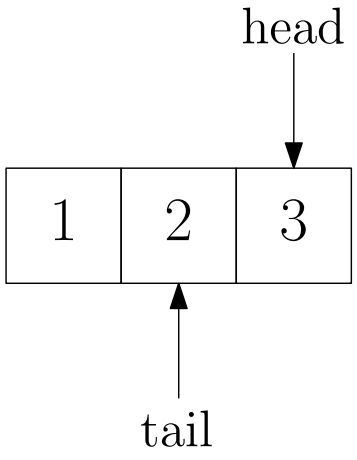

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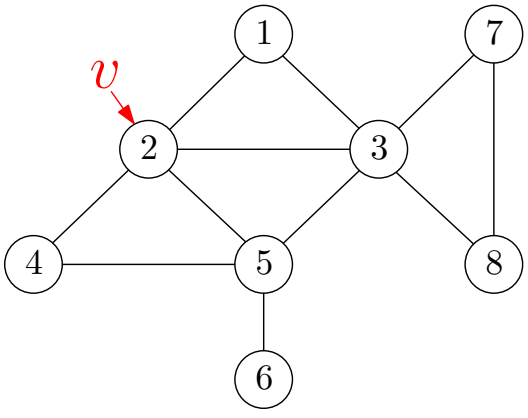

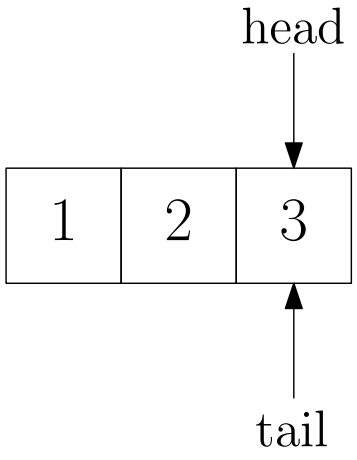

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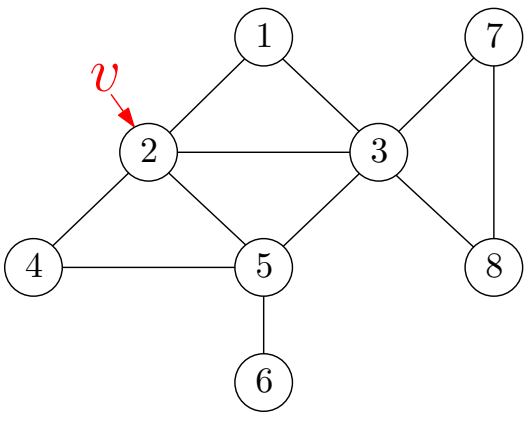

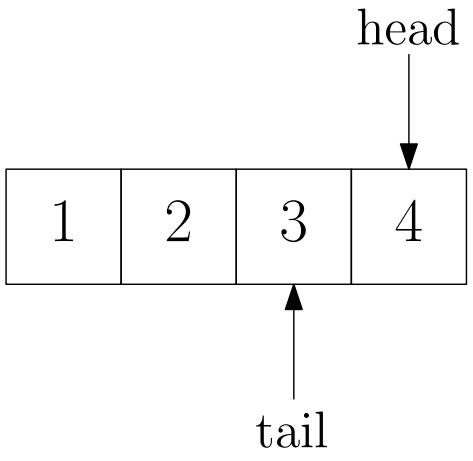

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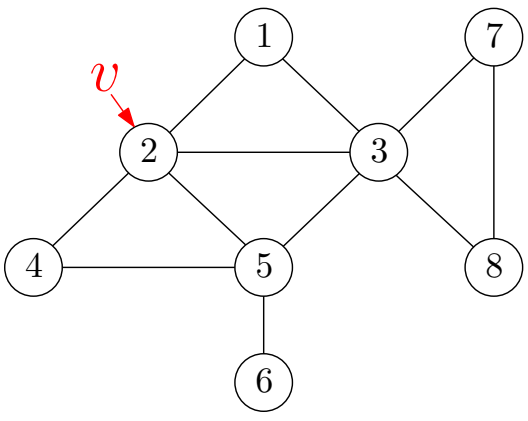

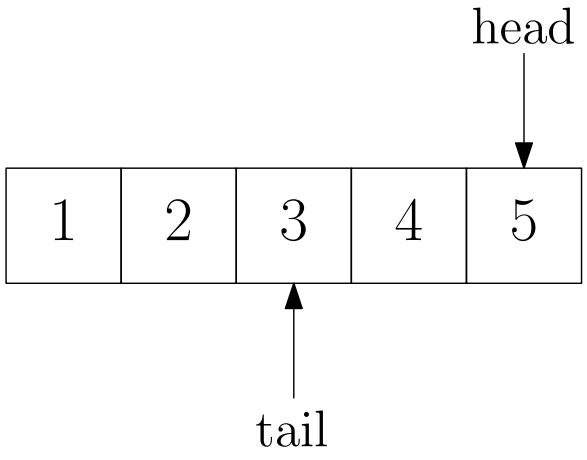

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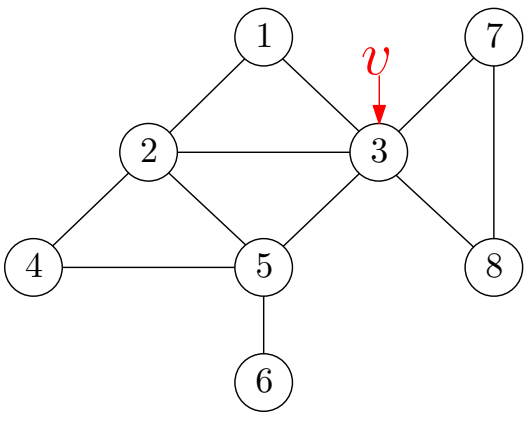

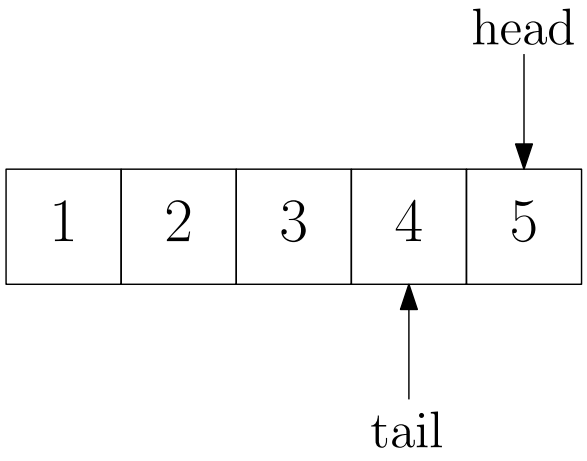

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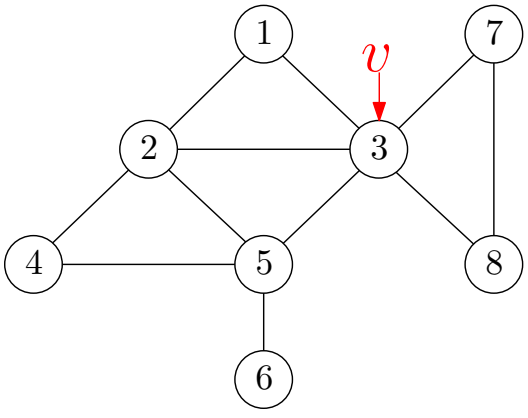

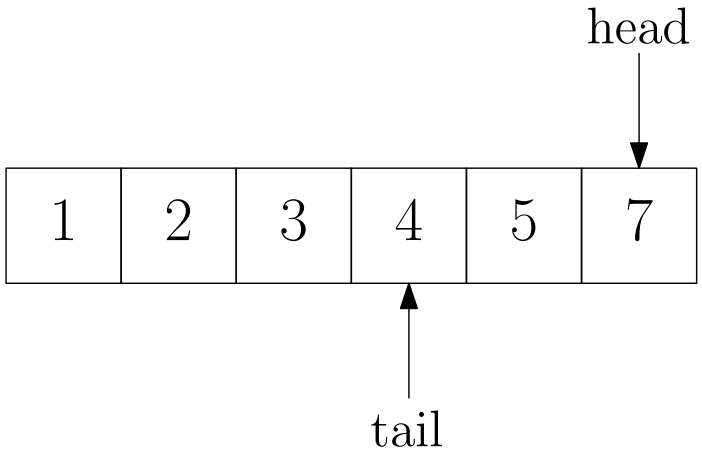

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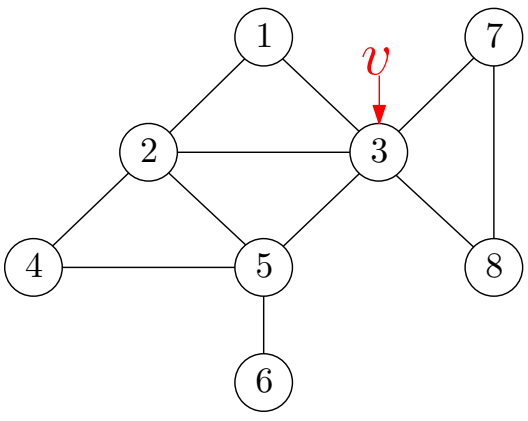

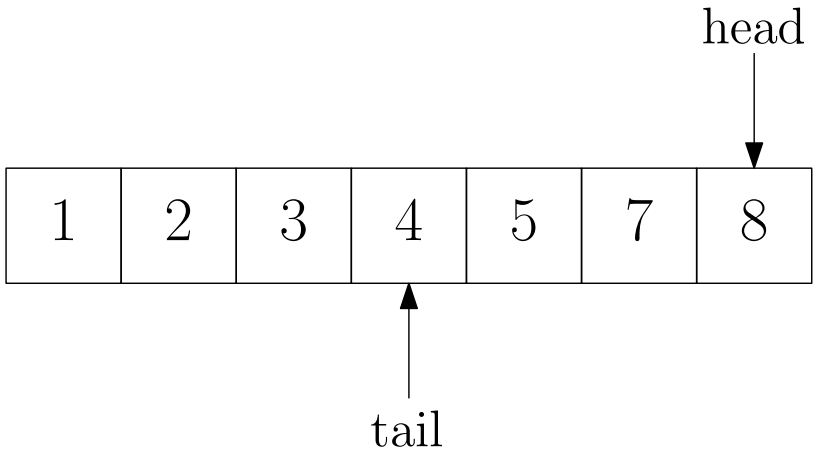

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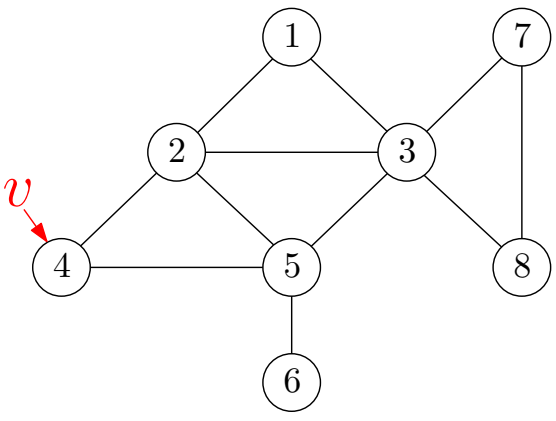

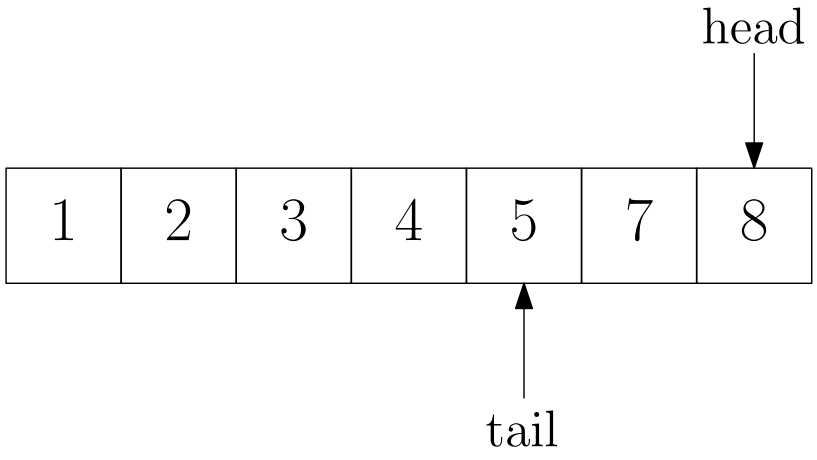

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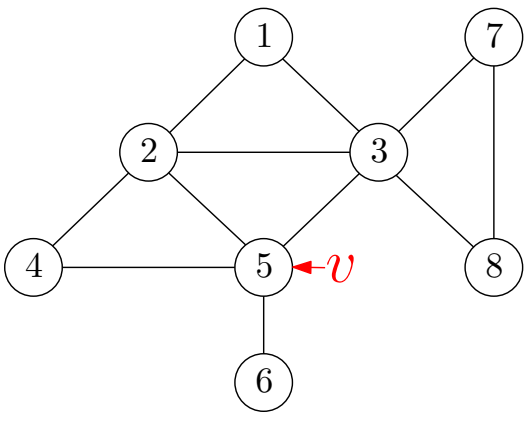
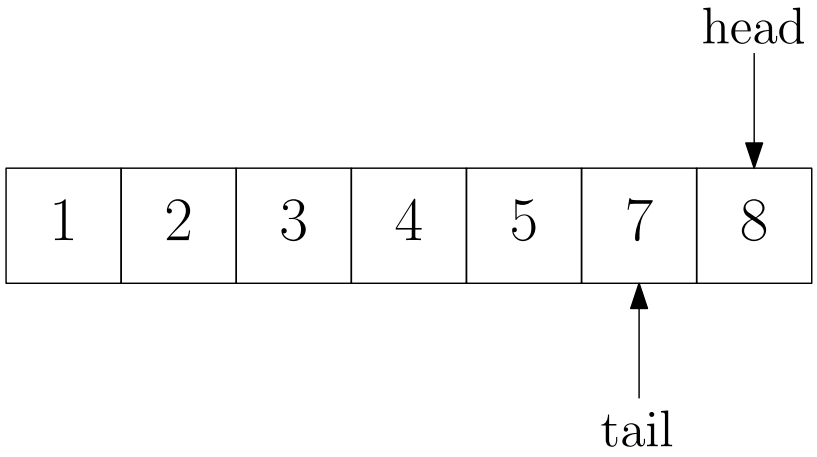
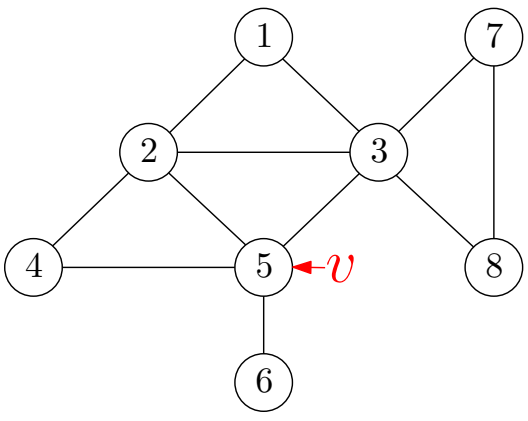
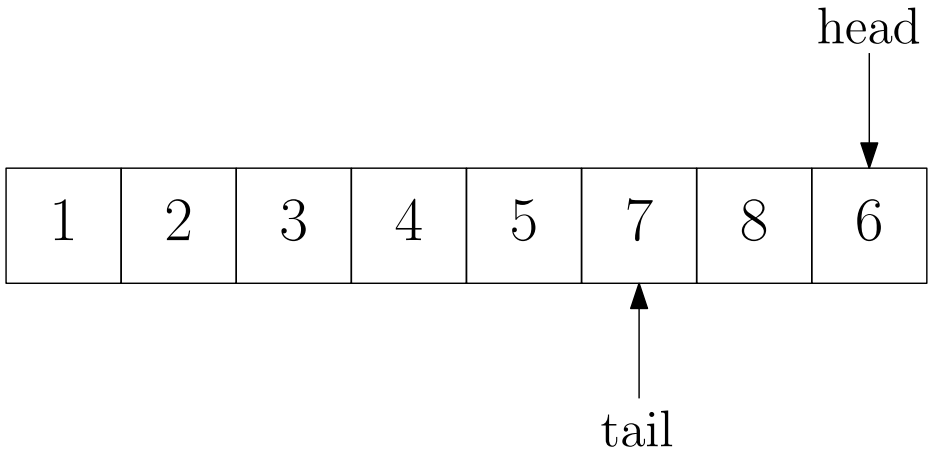




Example of BFS via Queue





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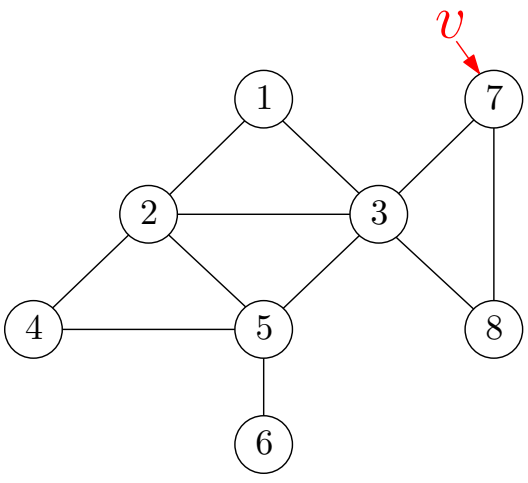

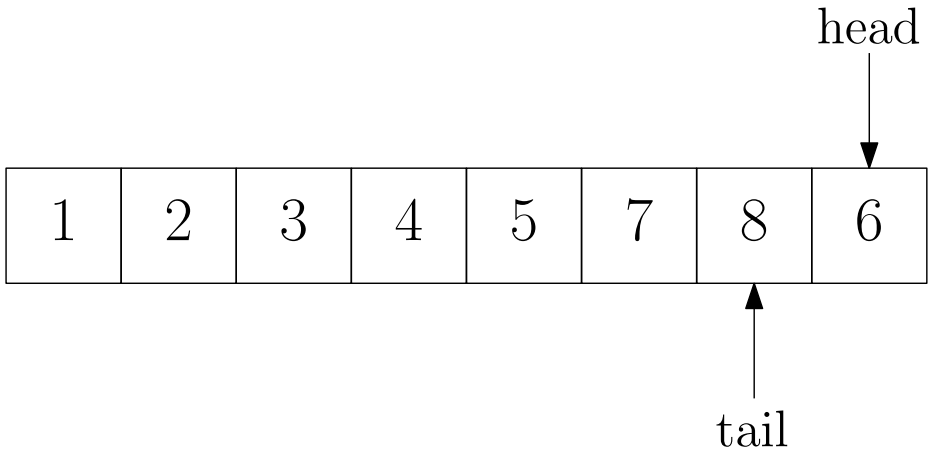

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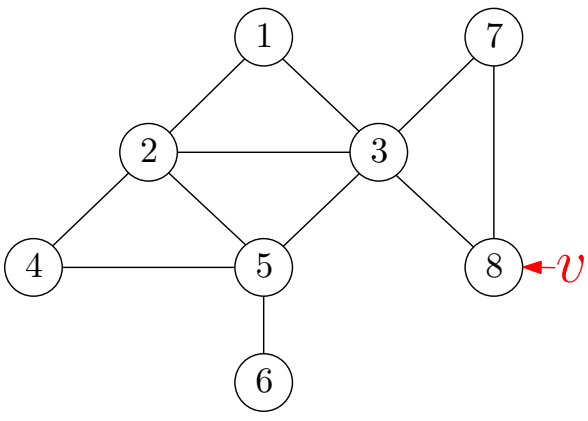

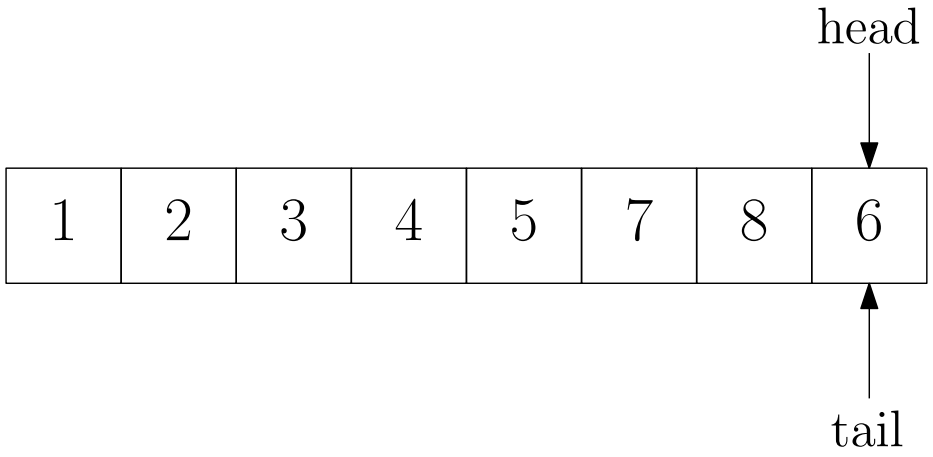

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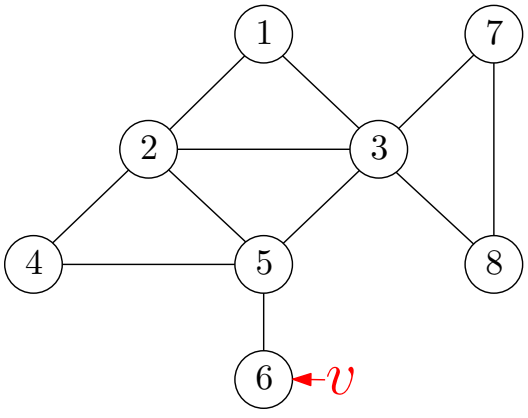

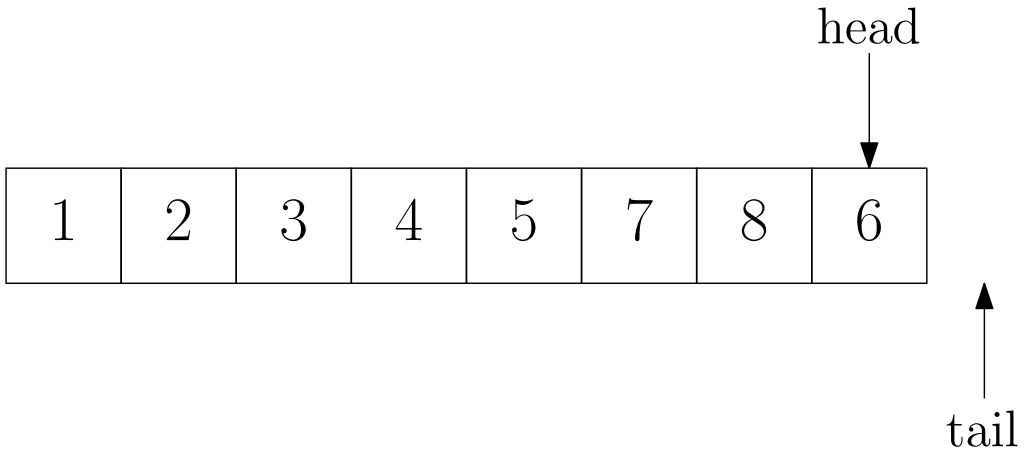

Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
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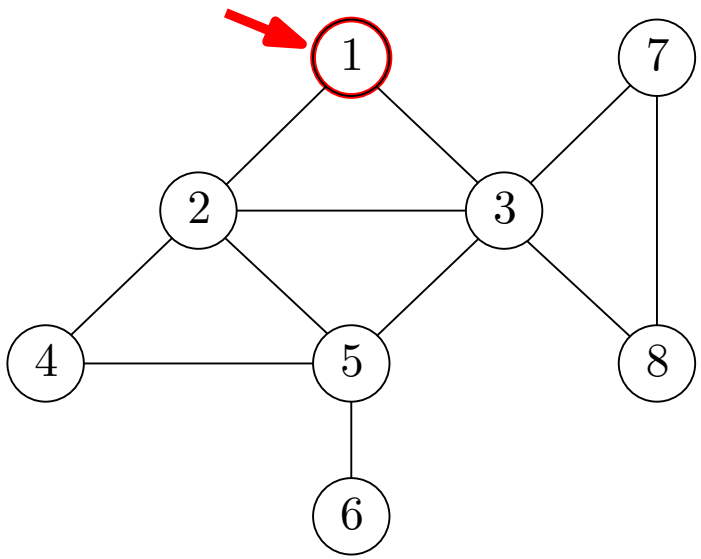

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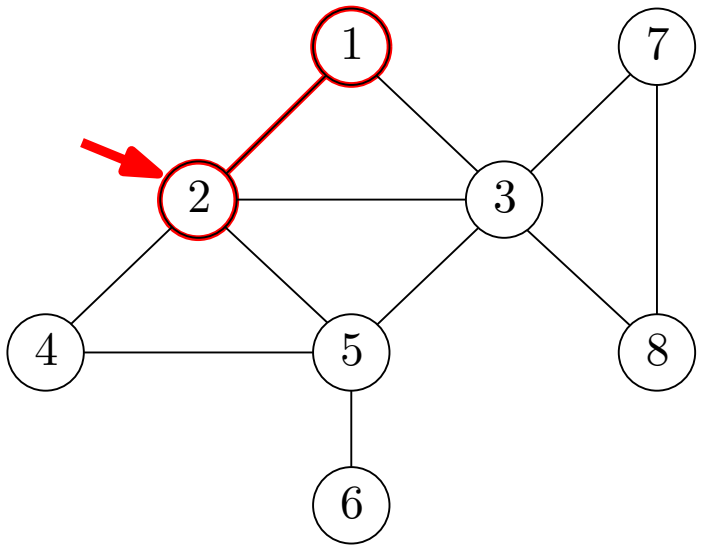

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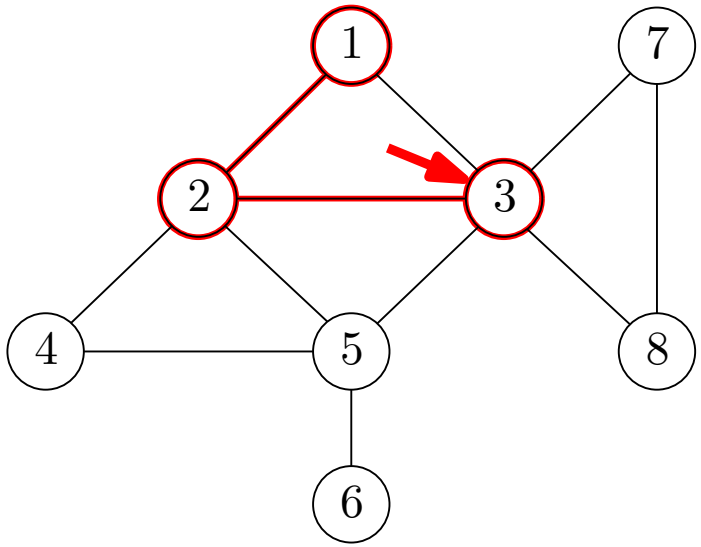

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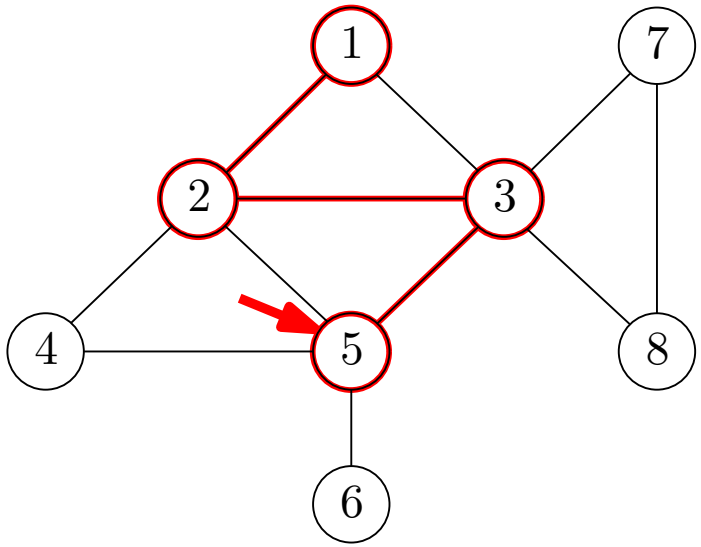

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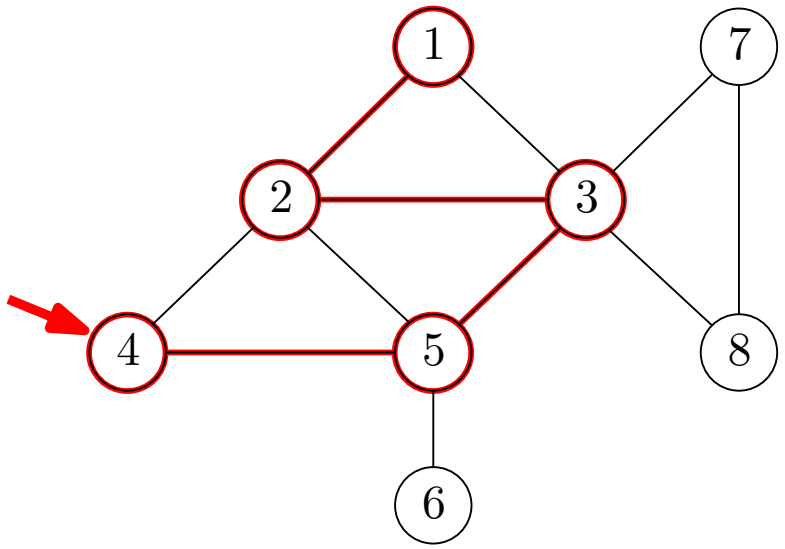

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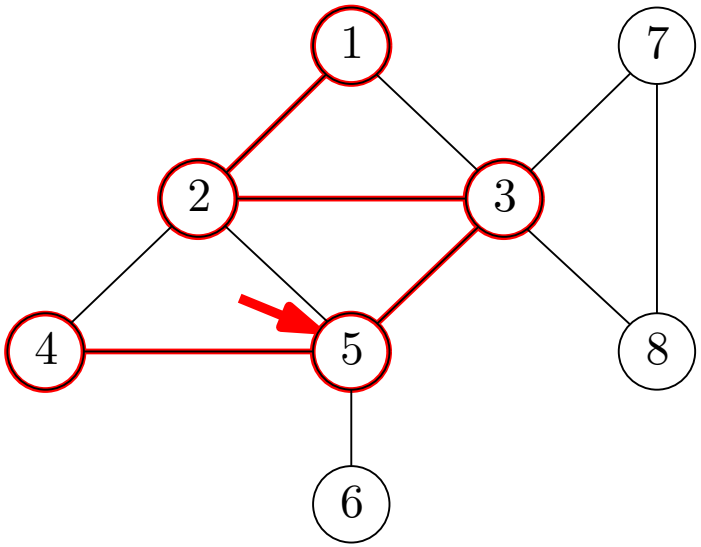

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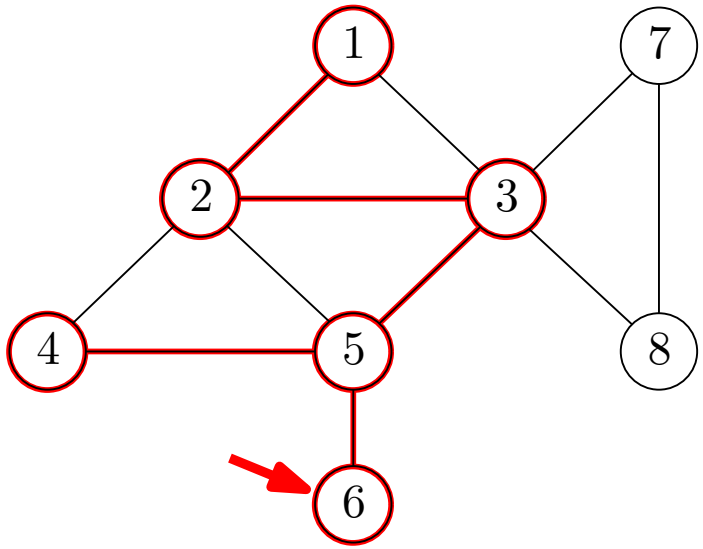

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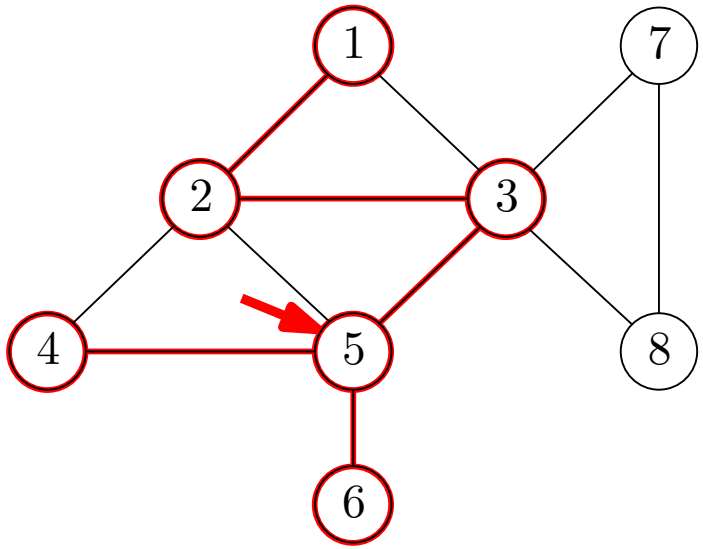

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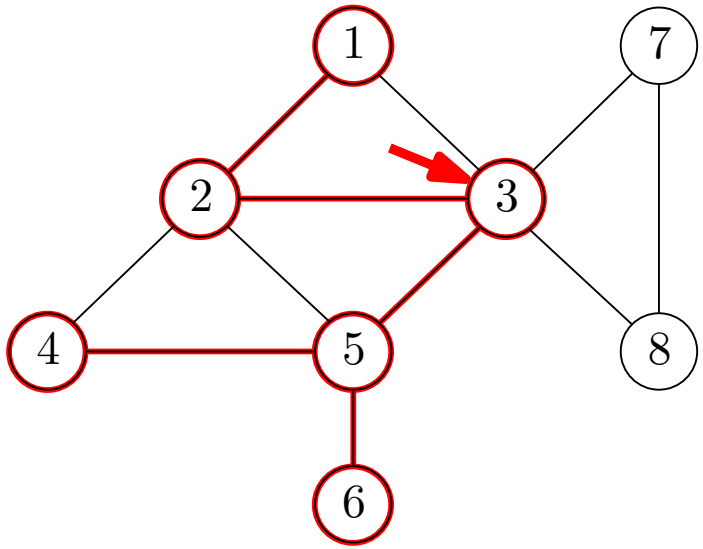

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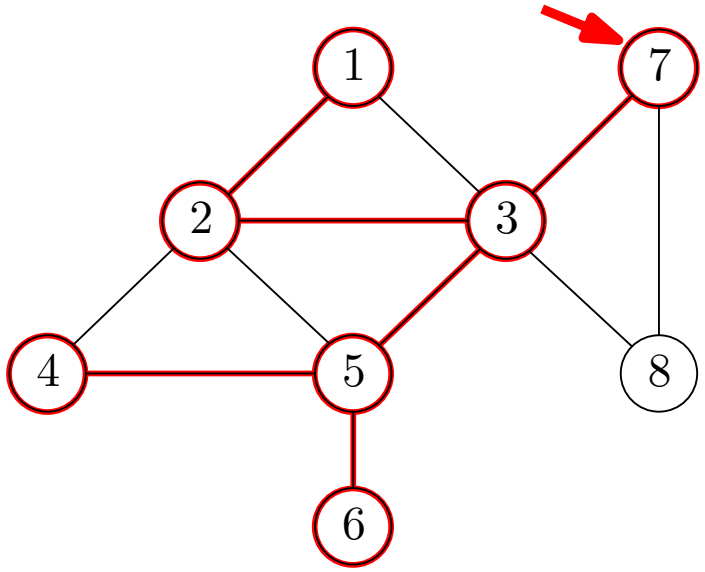

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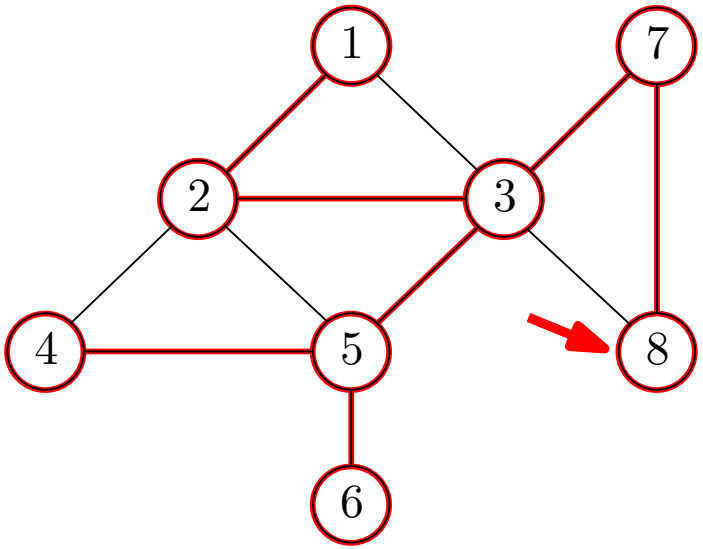


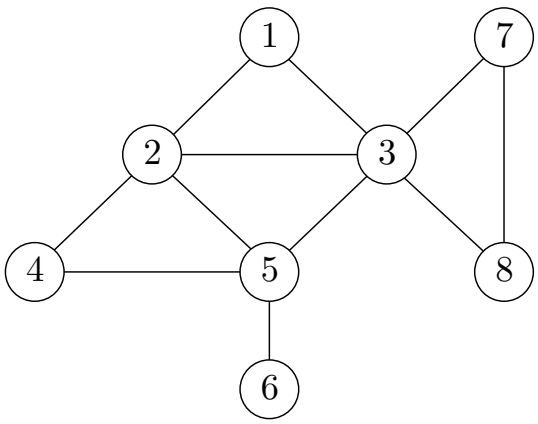
Implementing DFS using a Stack

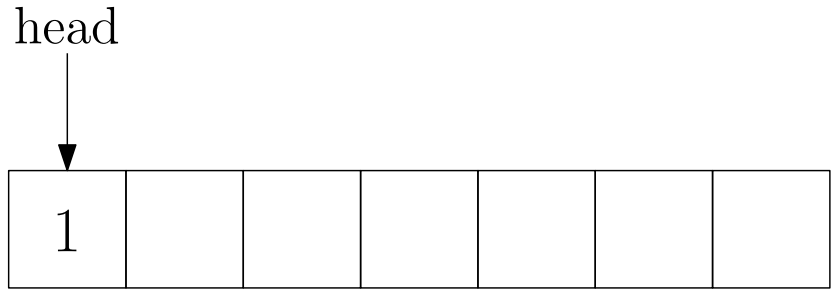
DFS(s)

1. $head \leftarrow 1, stack[1] \leftarrow s$
2. mark all vertices as "unexplored"
3. while head ≥ 1
4. $\quad v \leftarrow stack[head], head \leftarrow head - 1$
5. $\quad$ if v is unexplored then
6. $\quad\quad$ mark v as "explored"
7. $\quad\quad$ for all neighbours u of v
8. $\quad\quad\quad$ if u is not explored then
9. $\quad\quad\quad\quad head \leftarrow head + 1, stack[head] = u$

- Running time: $O(n + m)$.

Example of DFS using Stack





explored vertices:

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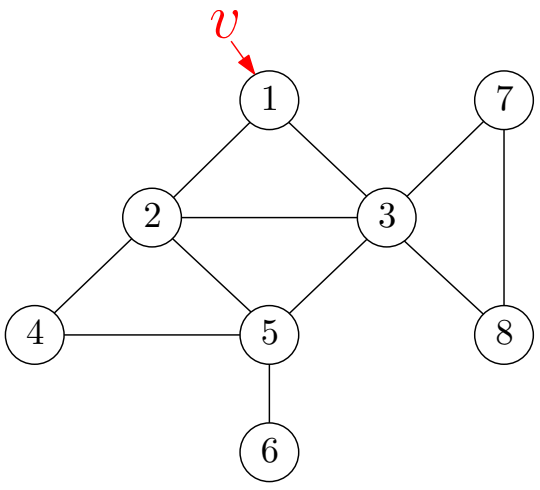

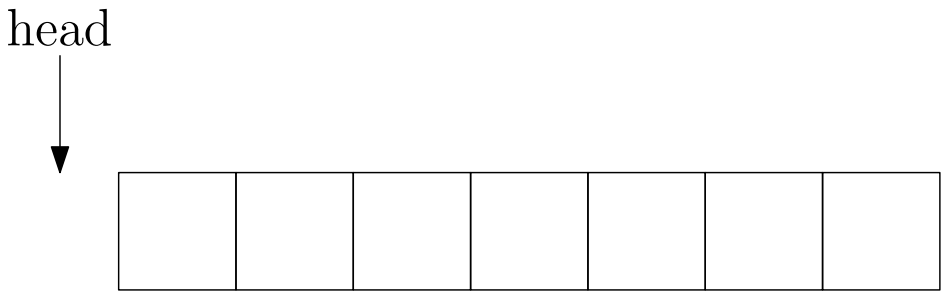

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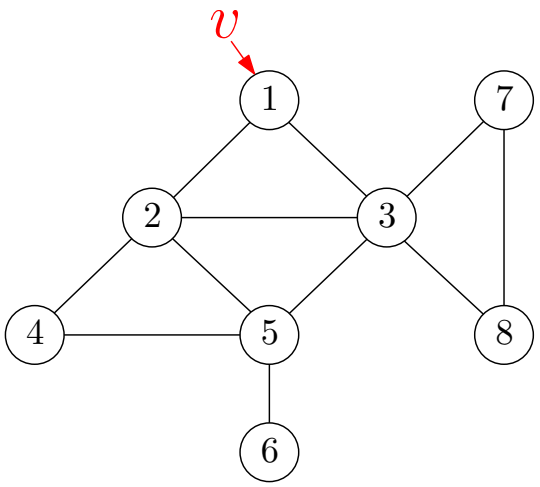

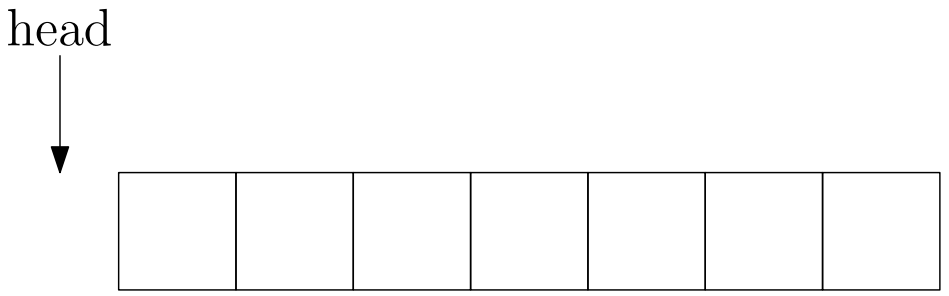

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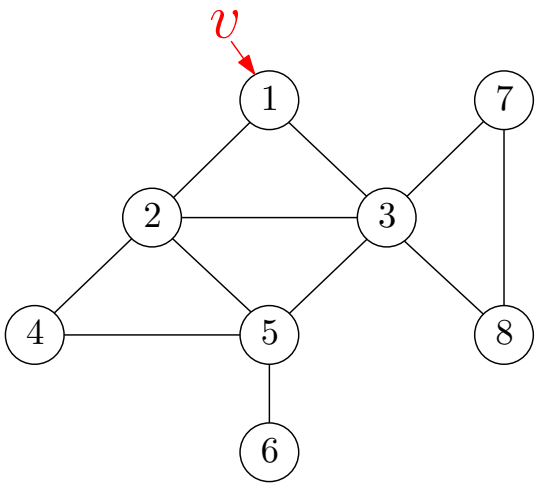

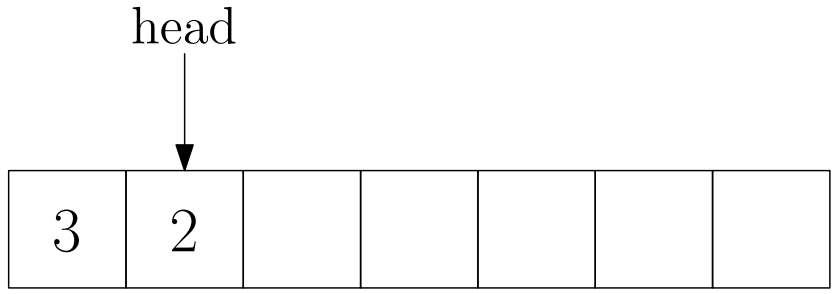

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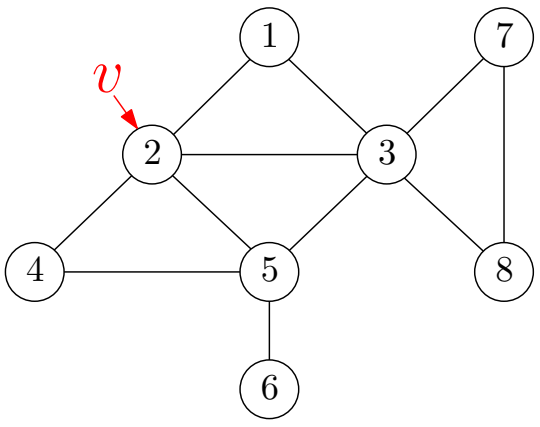

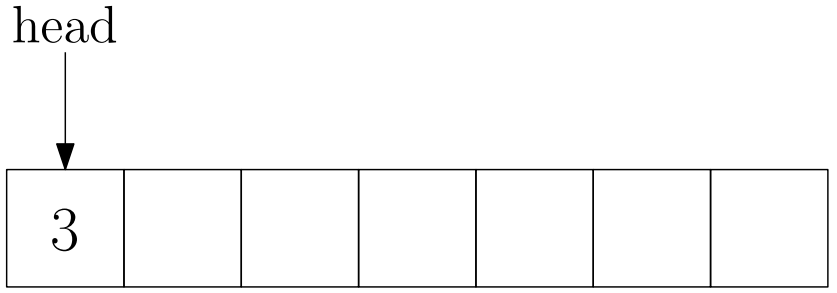

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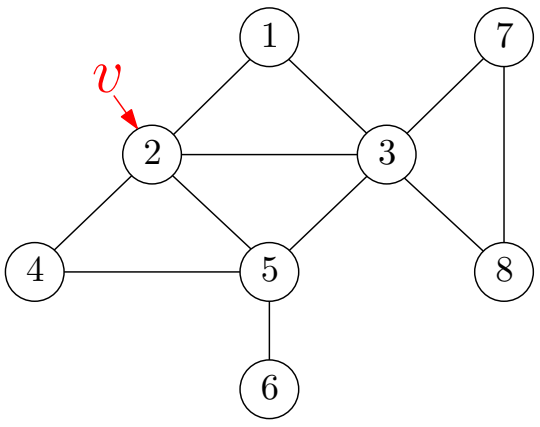

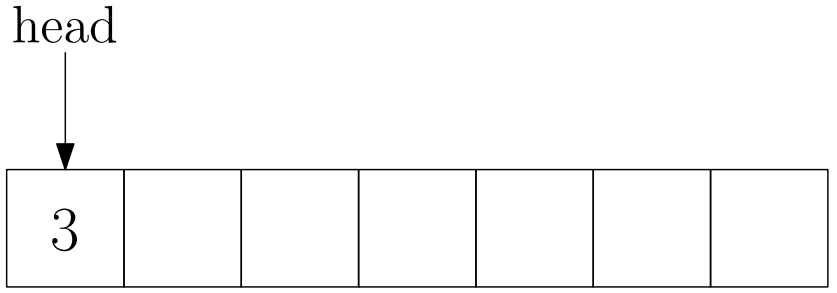

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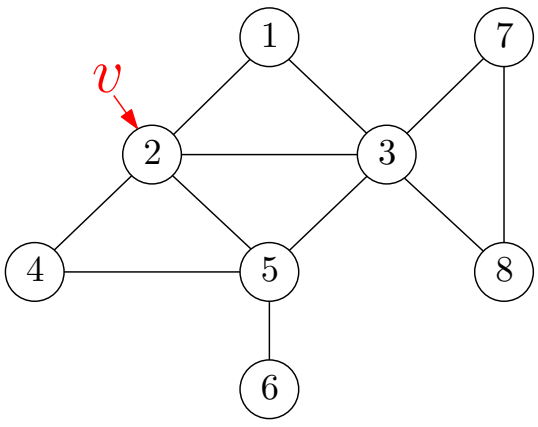

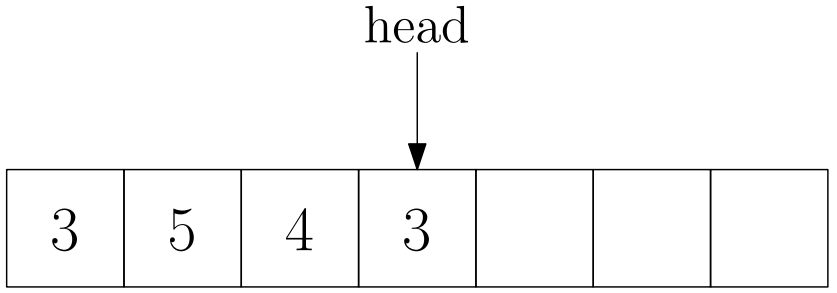

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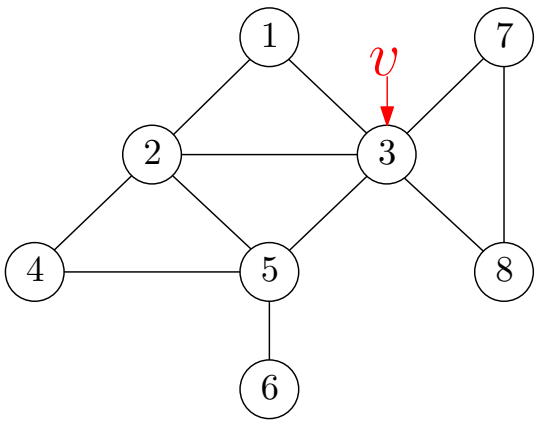

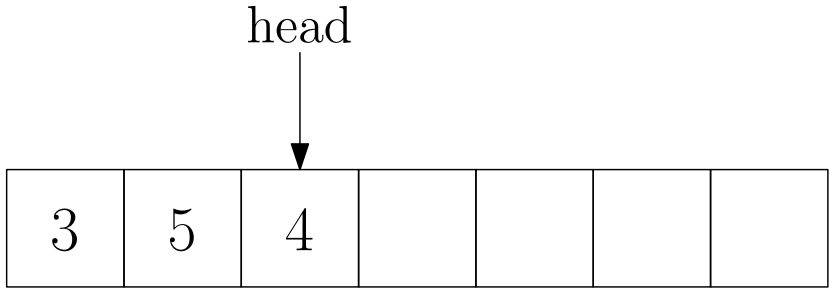

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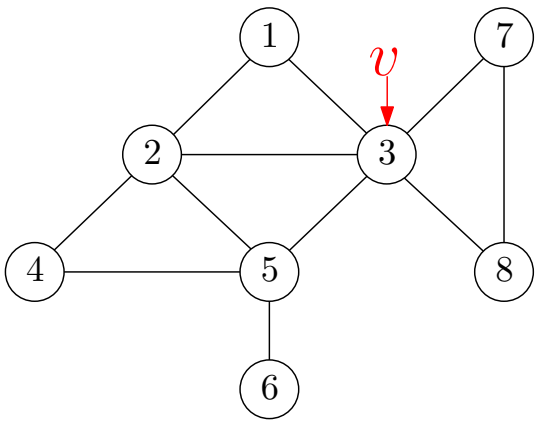

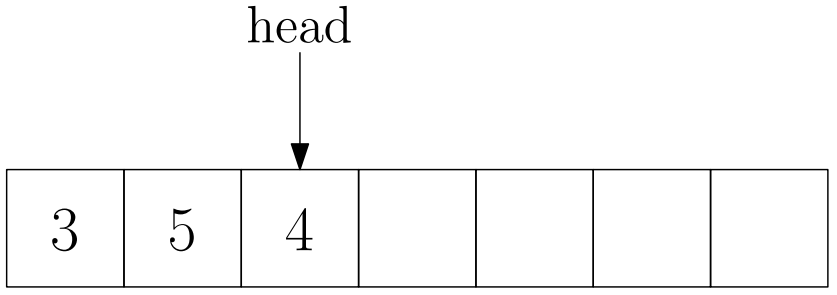

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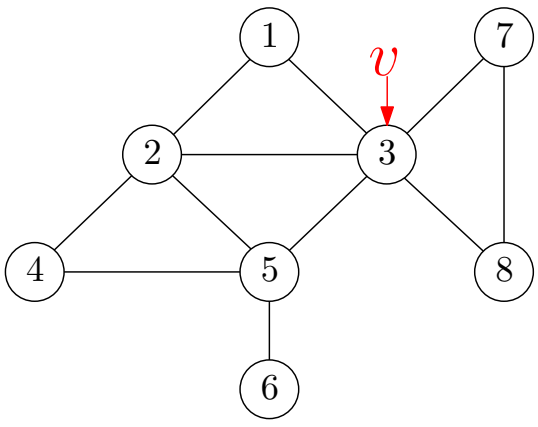

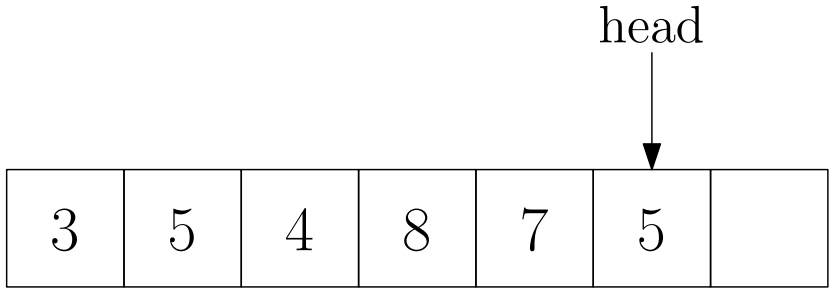

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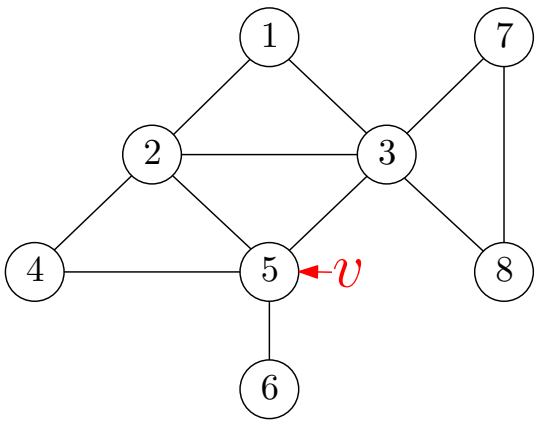

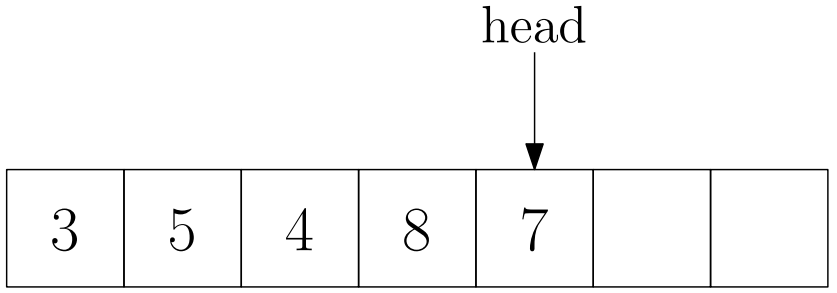

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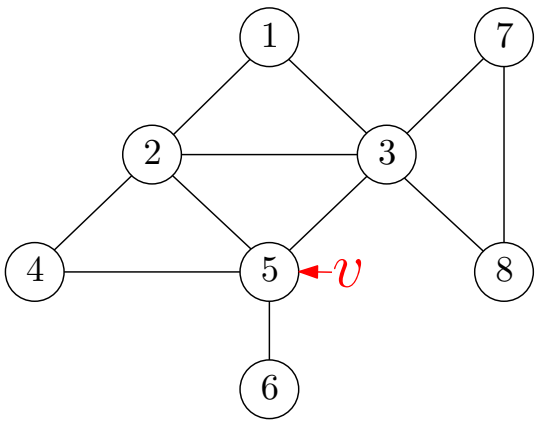

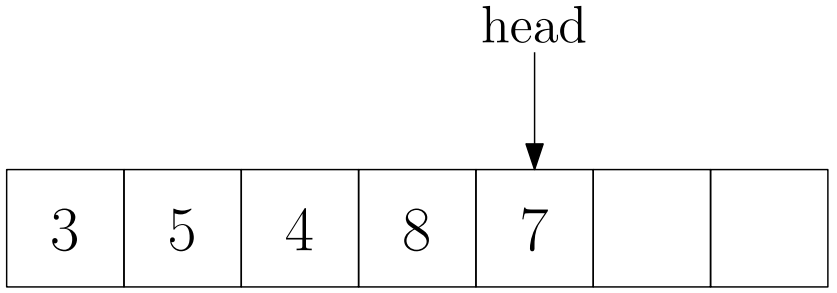

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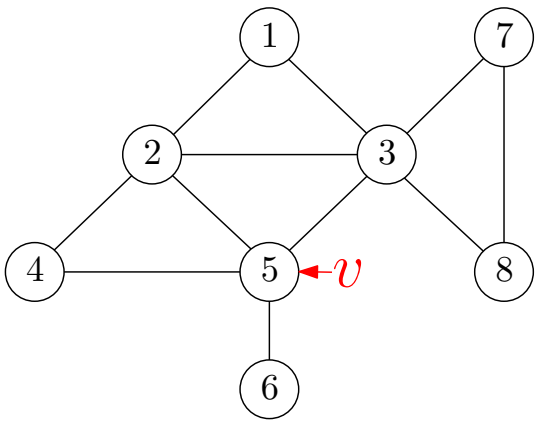

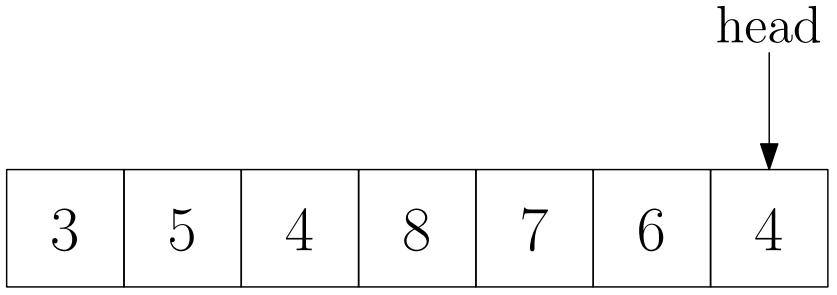

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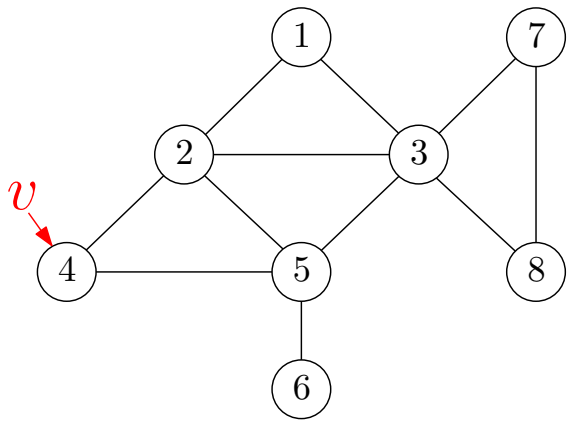

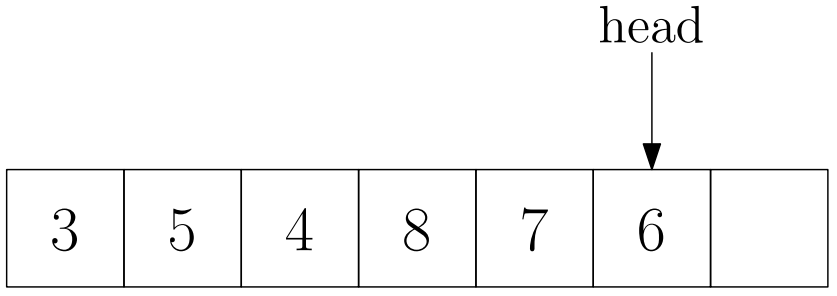

explored vertices: 1 2 3 5

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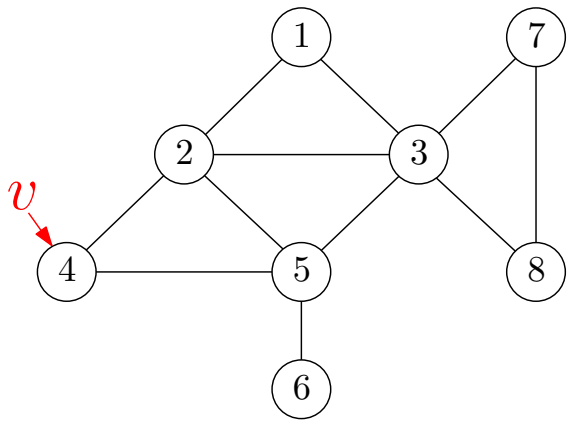

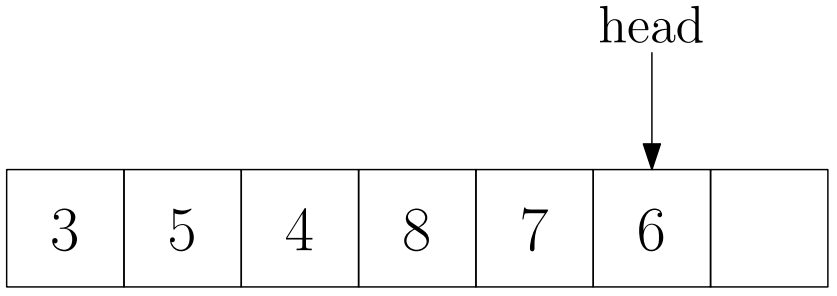

explored vertices: 1 2 3 5 4

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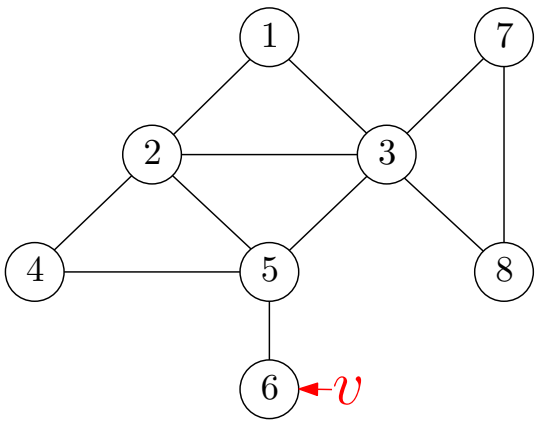

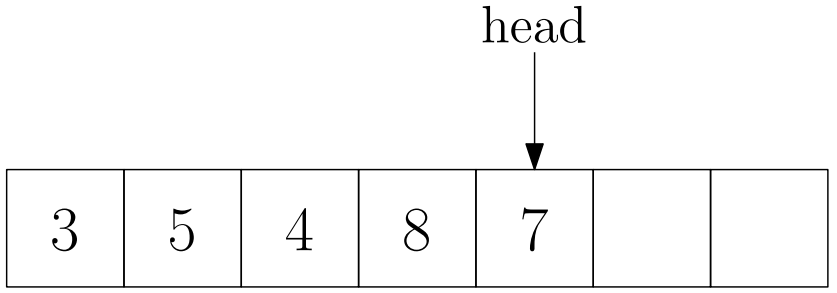

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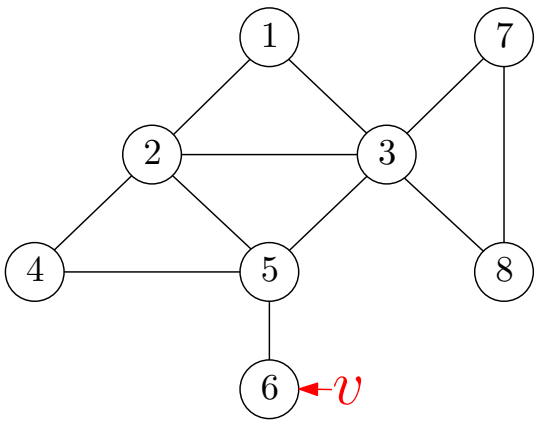

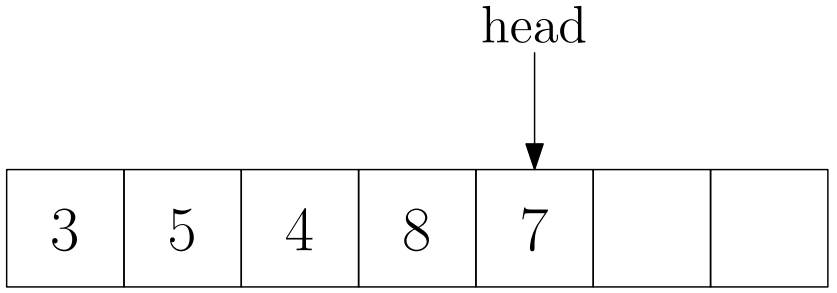

explored vertices: 1 2 3 5 4 6

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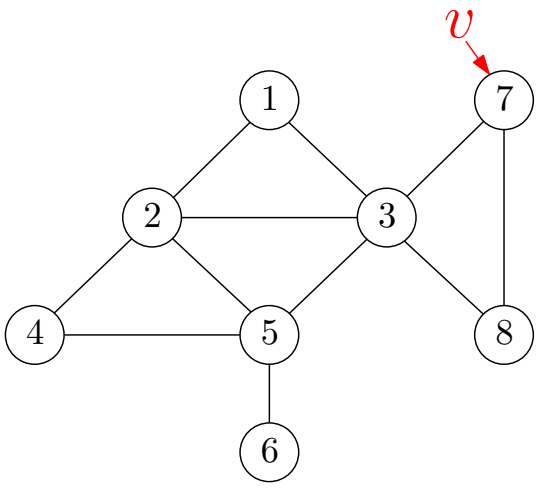

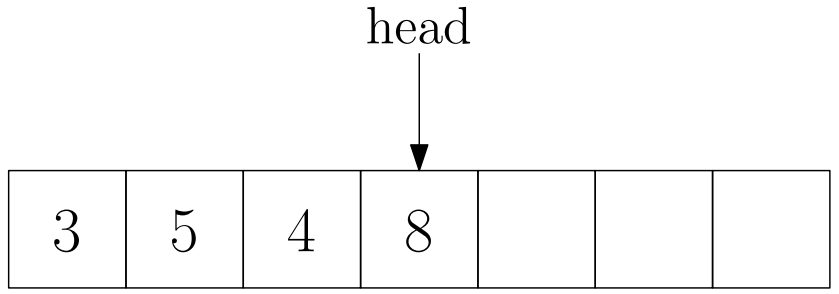

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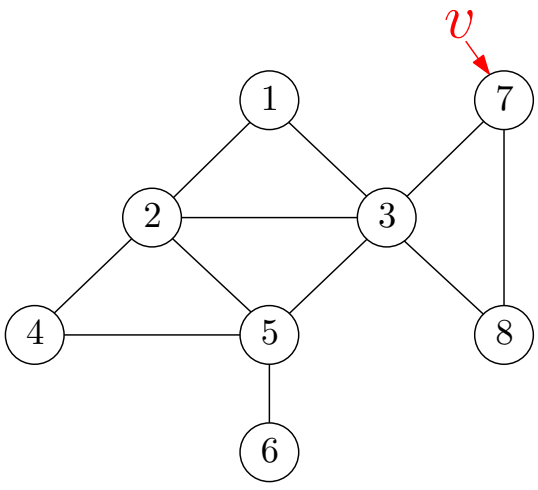

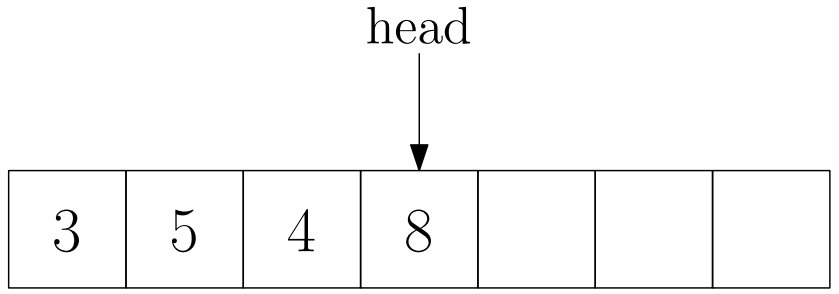

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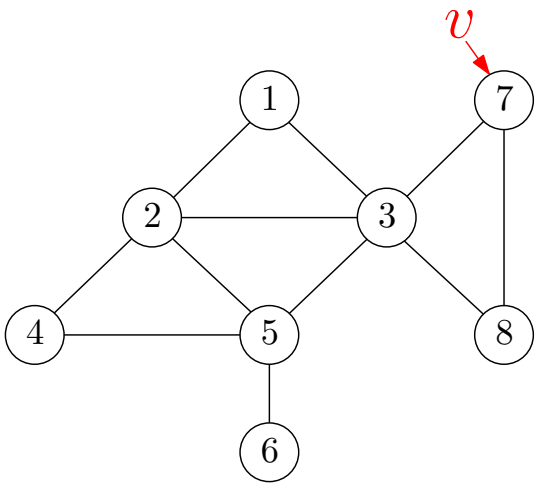

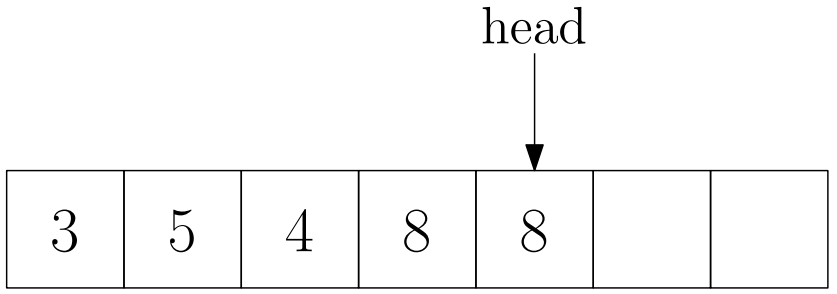

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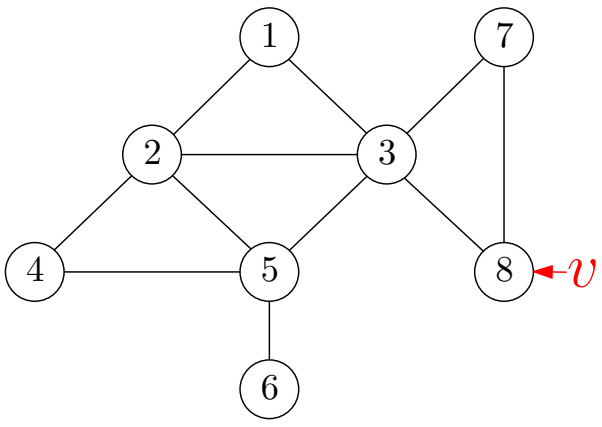

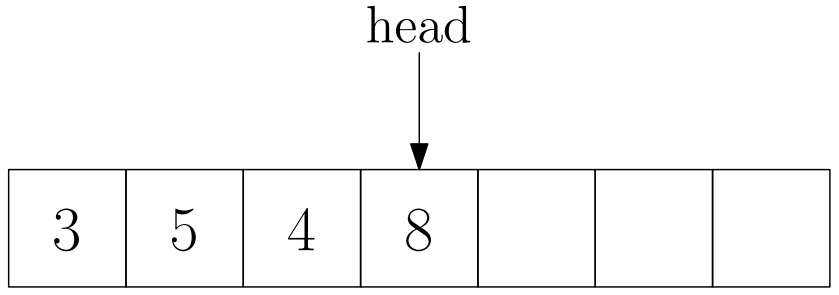

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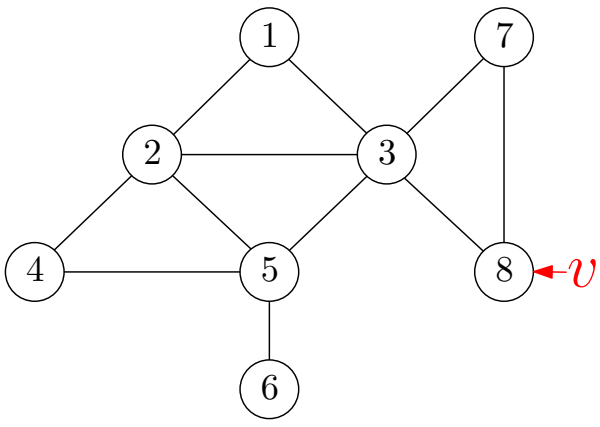

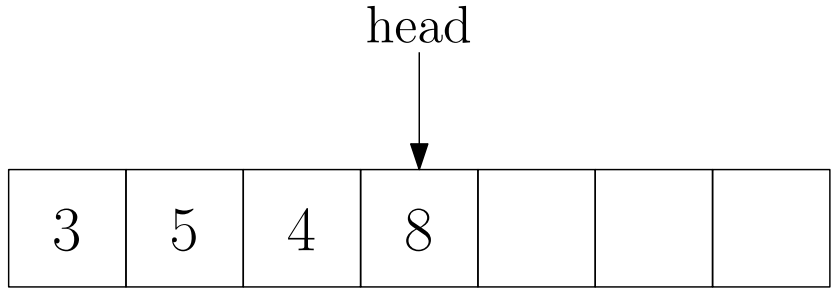

explored vertices: 1 2 3 5 4 6 7 8

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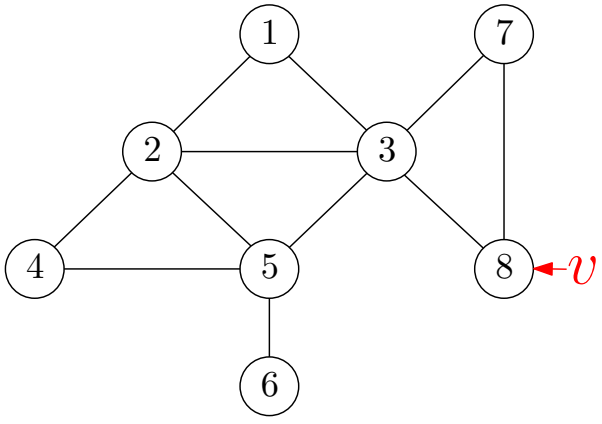

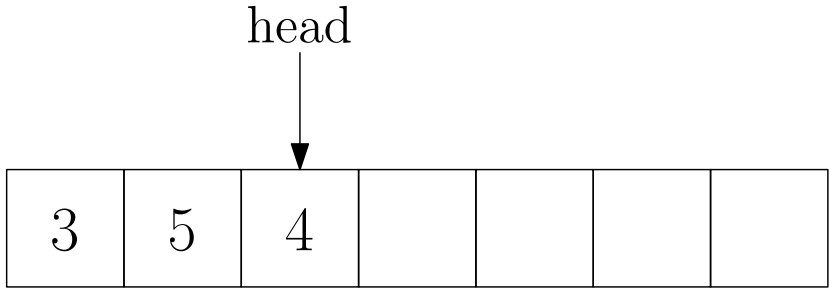

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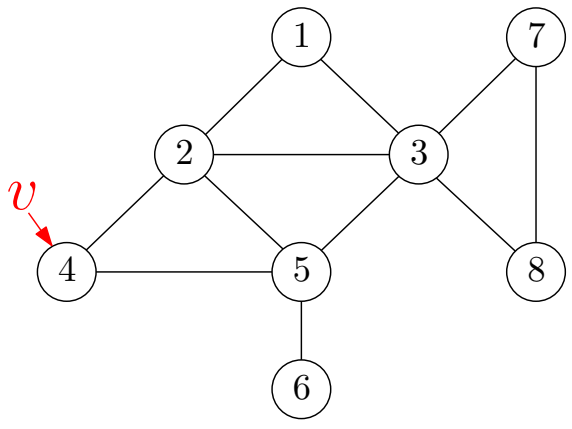

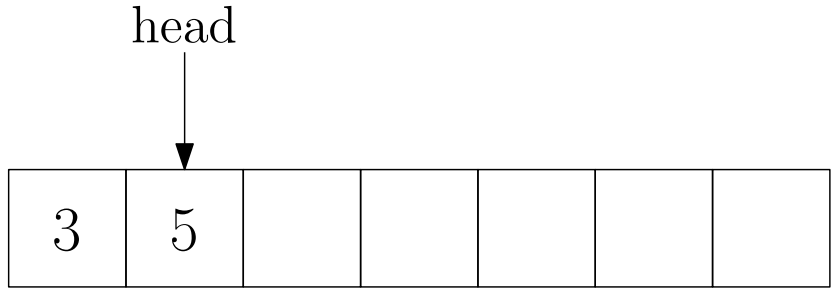

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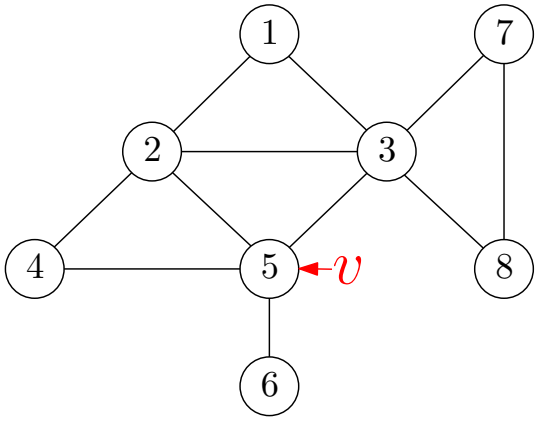

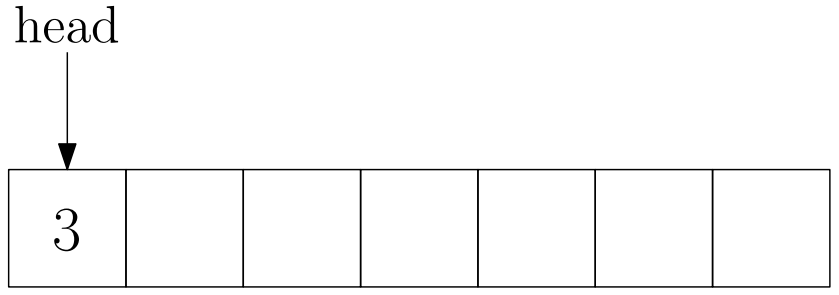

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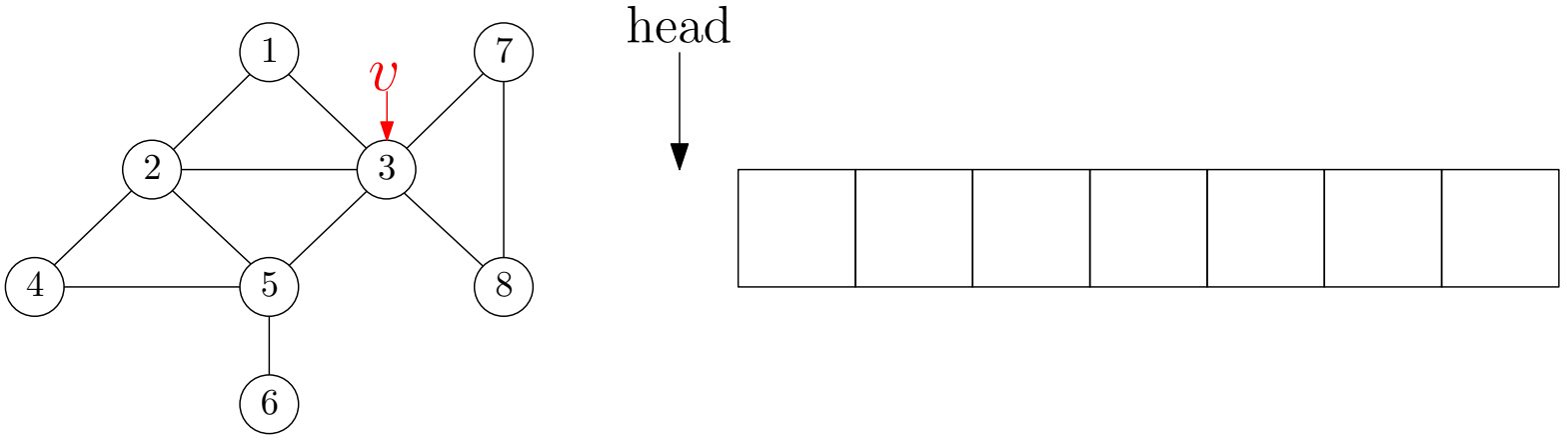


explored vertices: 1 2 3 5 4 6 7 8

Implementing DFS using Recurrsion

DFS(s)

1. mark all vertices as "unexplored"
2. recursive-DFS(s)

recursive-DFS(v)

1. if v is explored then return
2. mark v as "explored"
3. for all neighbours u of v
4. recursive-DFS(u)

Outline

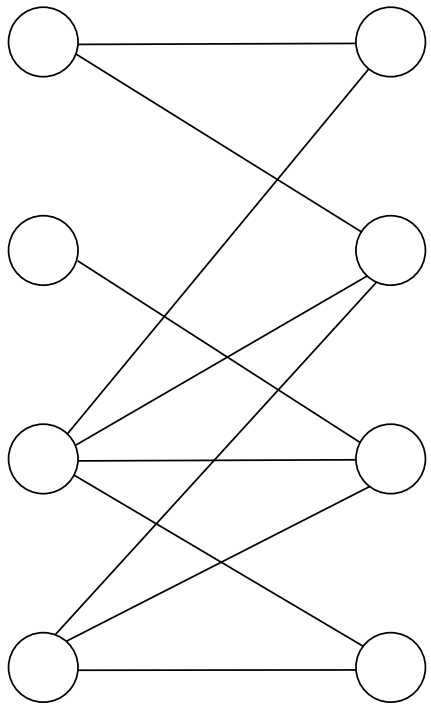
1 Graphs

2 Connectivity and Graph Traversal
- Testing Bipartiteness

3 Topological Ordering

Testing Bipartiteness: Applications of BFS

Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, we have either $u \in L, v \in R$ or $v \in L, u \in R$.



Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$

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- Assuming $s \in L$ w.l.o.g
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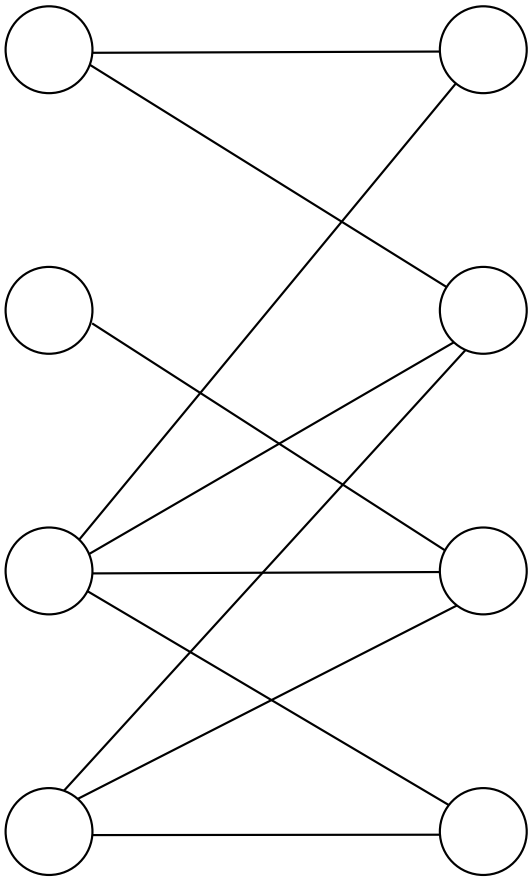
- ...
- Report "not a bipartite graph" if contradiction was found

Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L
- ...
- Report "not a bipartite graph" if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

Test Bipartiteness



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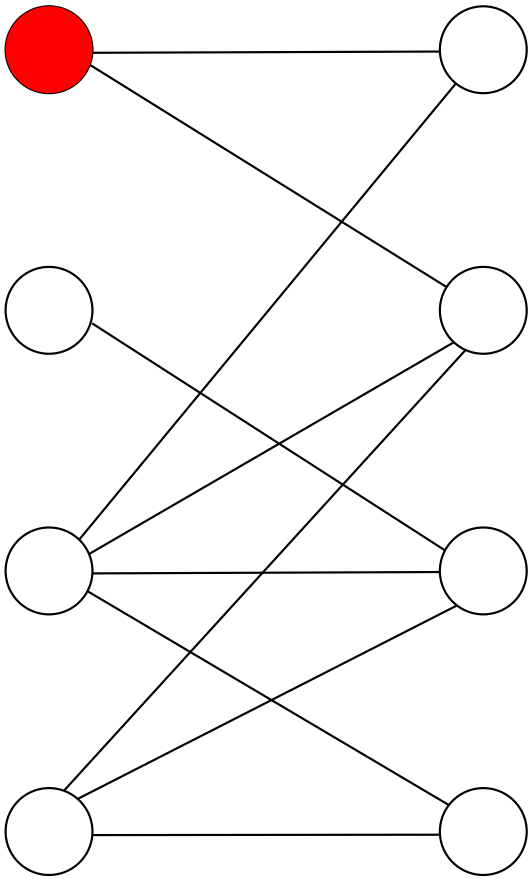

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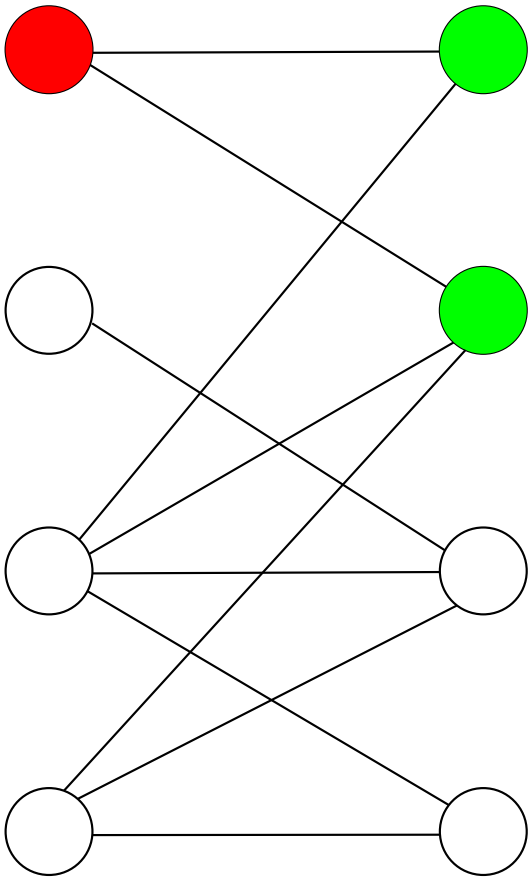

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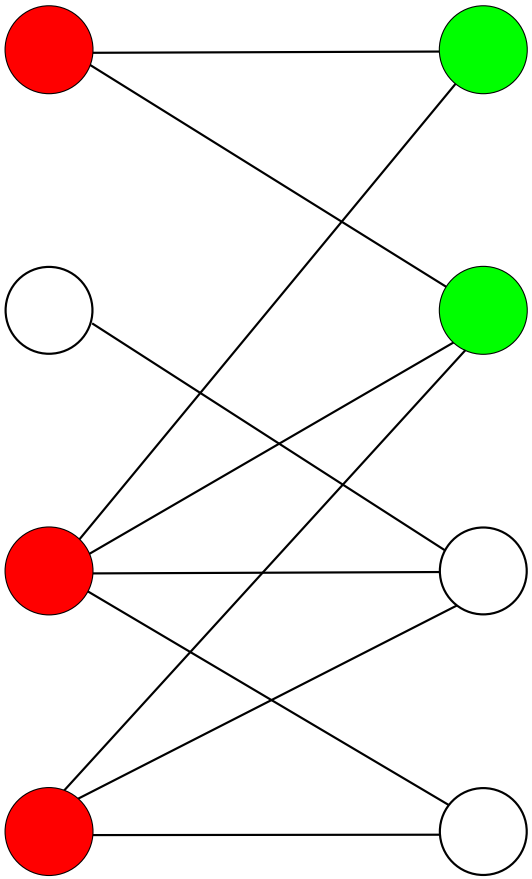

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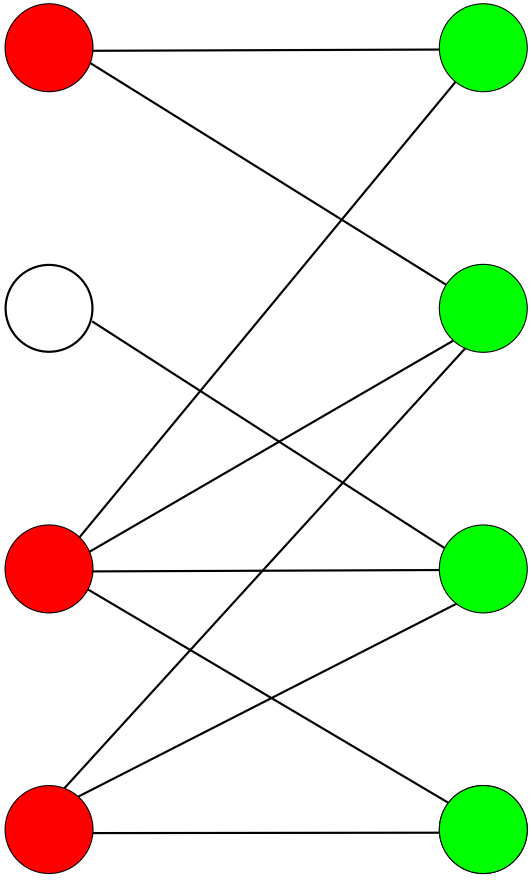

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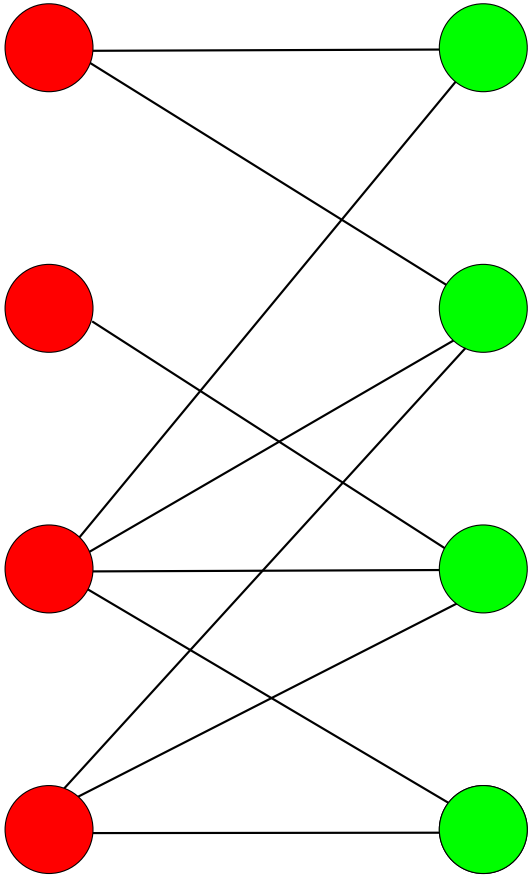

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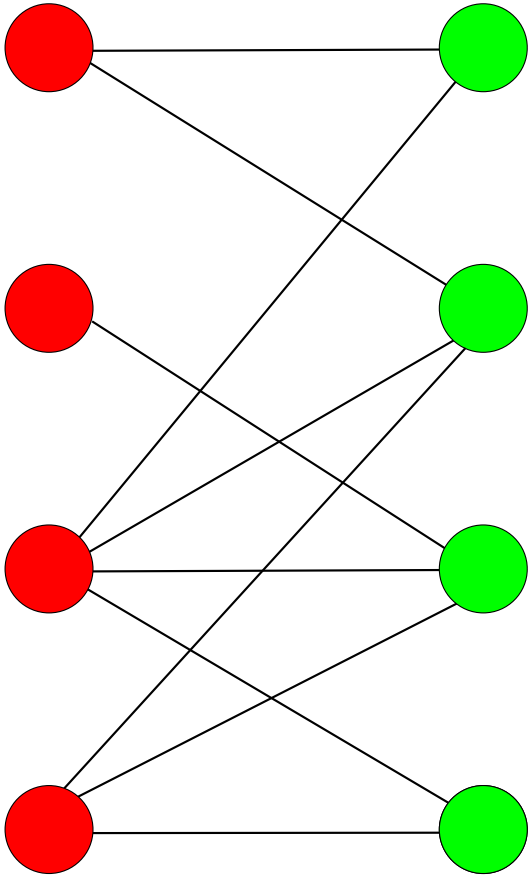

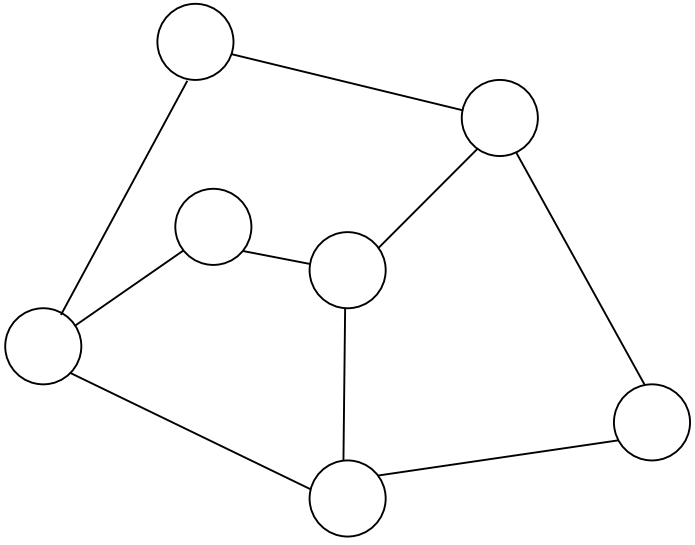

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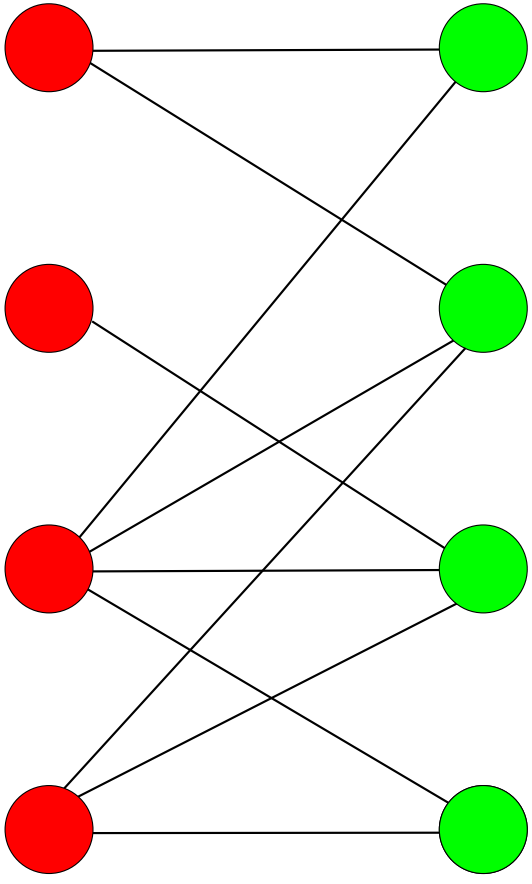

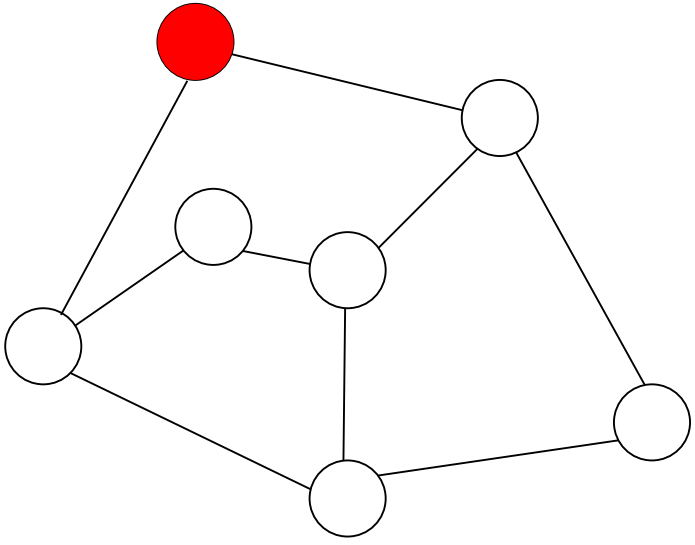

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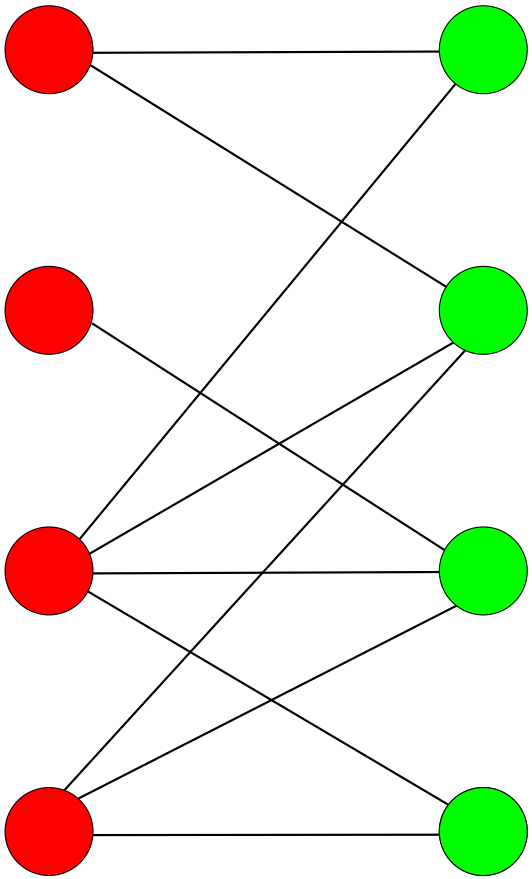

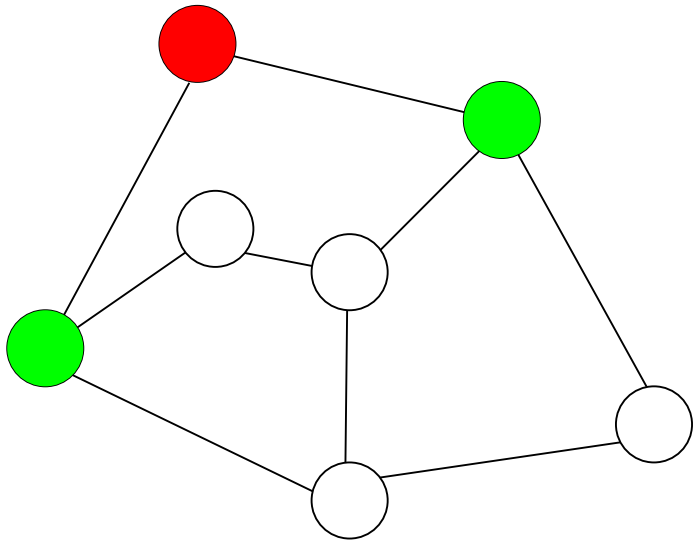

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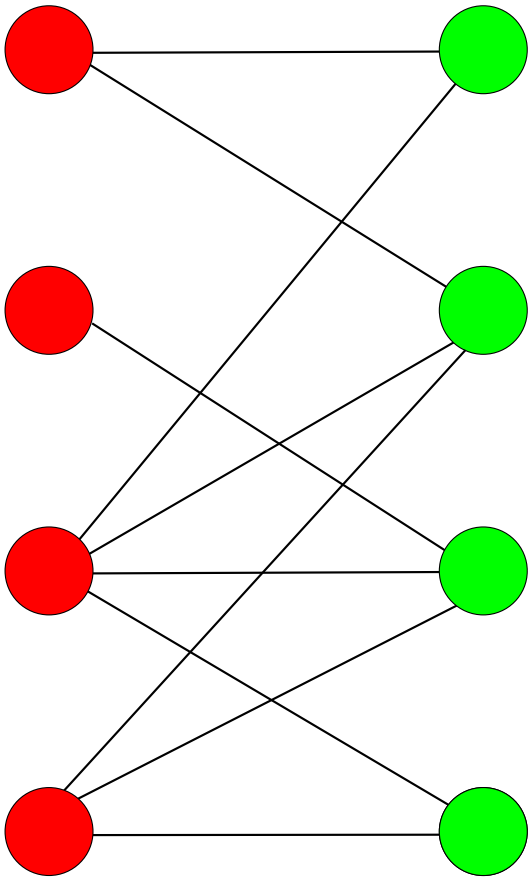

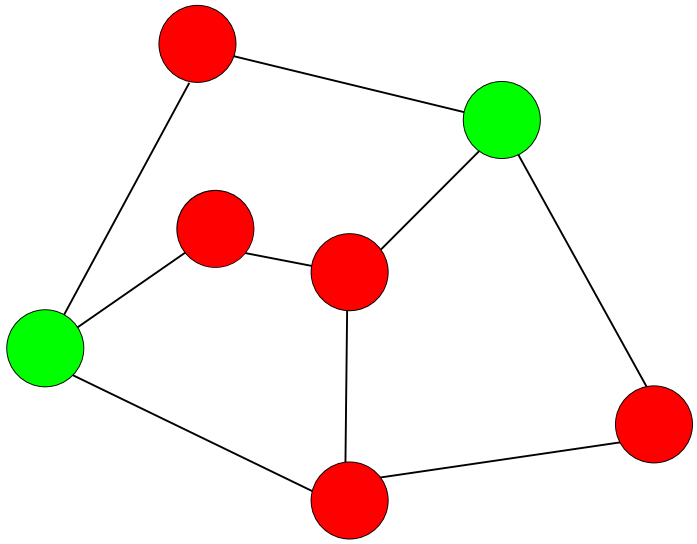

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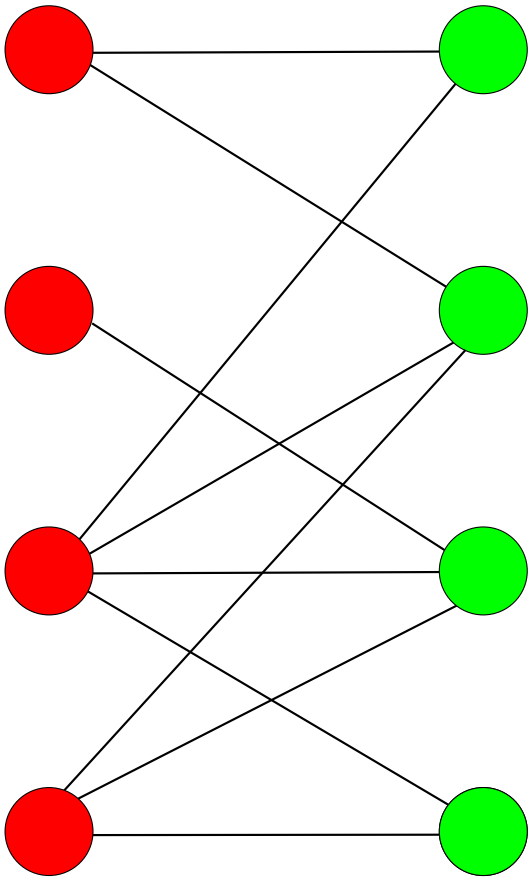

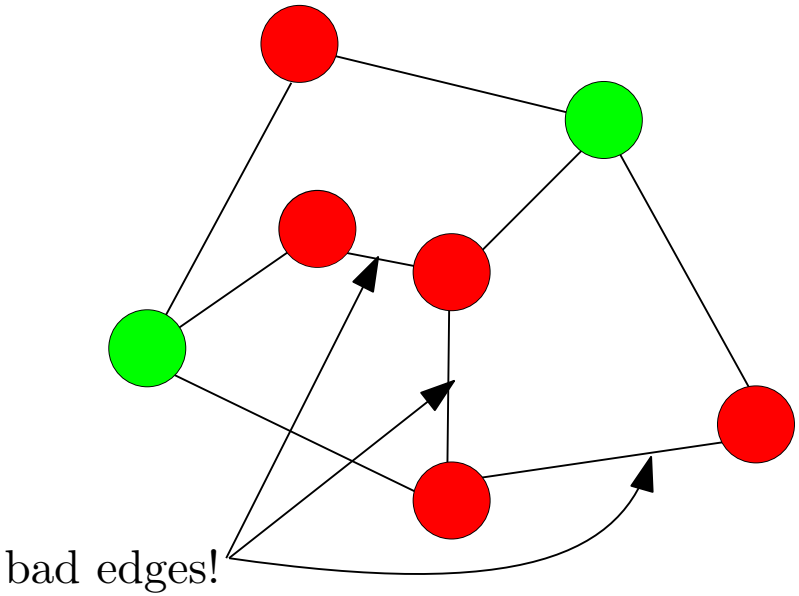

Testing Bipartiteness using BFS

BFS(s)

1. $head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
2. mark s as "visited" and all other vertices as "unvisited"
3. while head $\geq$ tail
4. $\quad v \leftarrow queue[tail], tail \leftarrow tail + 1$
5. $\quad$ for all neighbours u of v
6. $\quad\quad$ if u is "unvisited" then
7. $\quad\quad\quad head \leftarrow head + 1, queue[head] = u$
8. $\quad\quad\quad$ mark u as "visited"

Testing Bipartiteness using BFS

test-bipartiteness(s)

1. $head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
2. mark s as "visited" and all other vertices as "unvisited"
3. $color[s] \leftarrow 0$
4. while head $\geq$ tail
5. $\quad v \leftarrow queue[tail], tail \leftarrow tail + 1$
6. $\quad$ for all neighbours u of v
7. $\quad\quad$ if u is "unvisited" then
8. $\quad\quad\quad head \leftarrow head + 1, queue[head] = u$
9. $\quad\quad\quad$ mark u as "visited"
10. $\quad\quad\quad color[u] \leftarrow 1 - color[v]$
11. $\quad\quad$ elseif $color[u] = color[v]$ then
12. $\quad\quad\quad$ print("G is not bipartite") and exit

Testing Bipartiteness using BFS

1. mark all vertices as "unvisited"
2. for each vertex $v \in V$
3. if v is "unvisited" then
4. test-bipartiteness(v)
5. print("G is bipartite")

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Obs. Running time of algorithm $= O(n + m)$

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Homework problem: using DFS to implement test-bipartiteness.

Outline

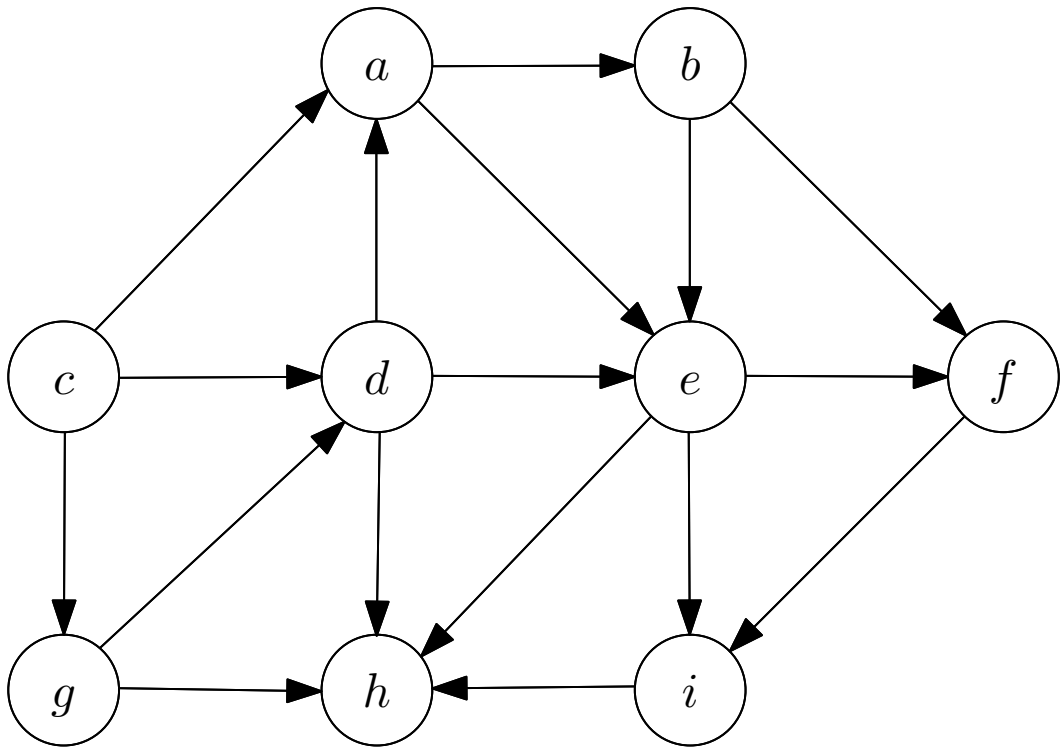
1. Graphs
2. Connectivity and Graph Traversal
   - Testing Bipartiteness
3. Topological Ordering

Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

Output: 1-to-1 function $\pi : V \to \{1, 2, 3 \cdots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



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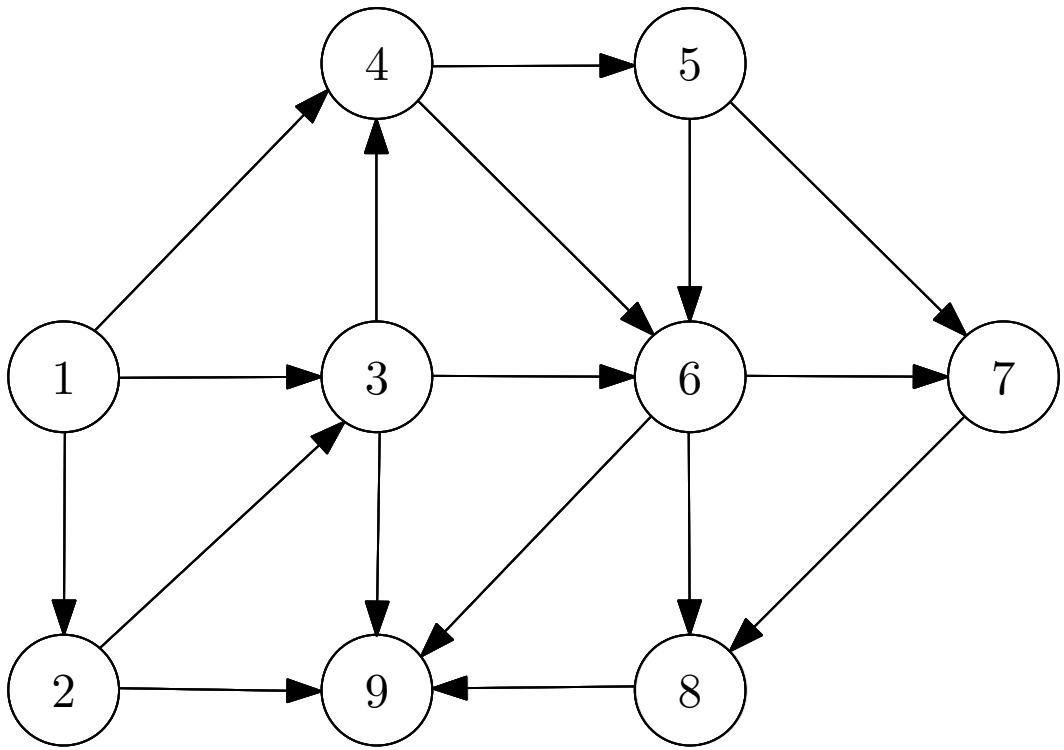
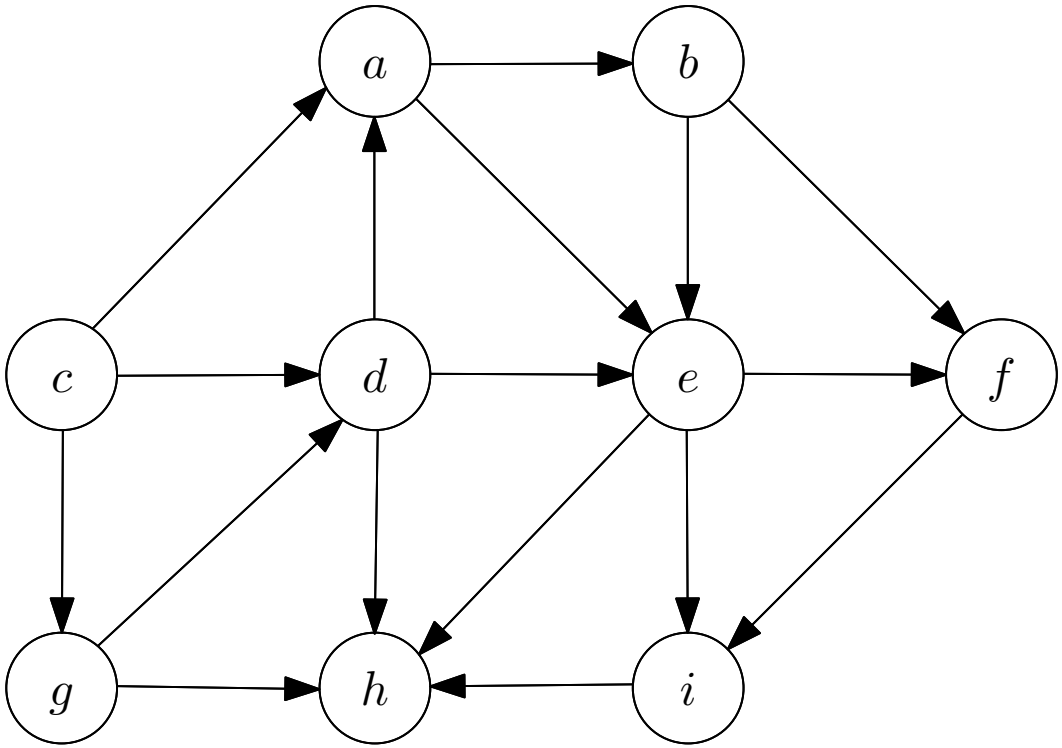


Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



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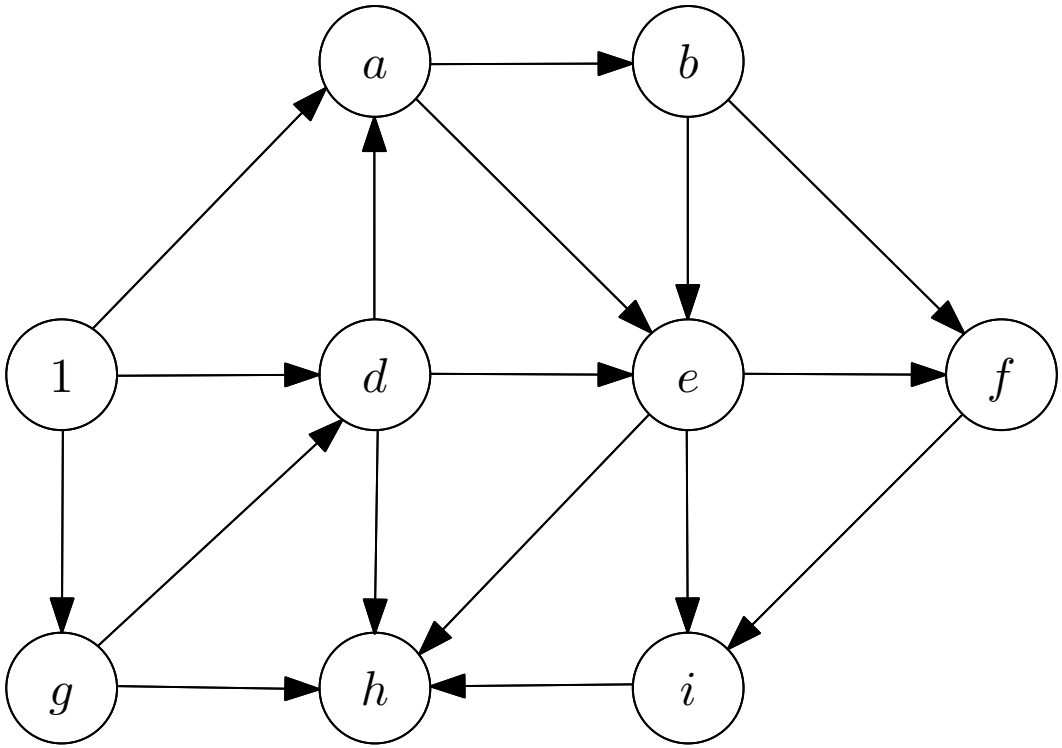

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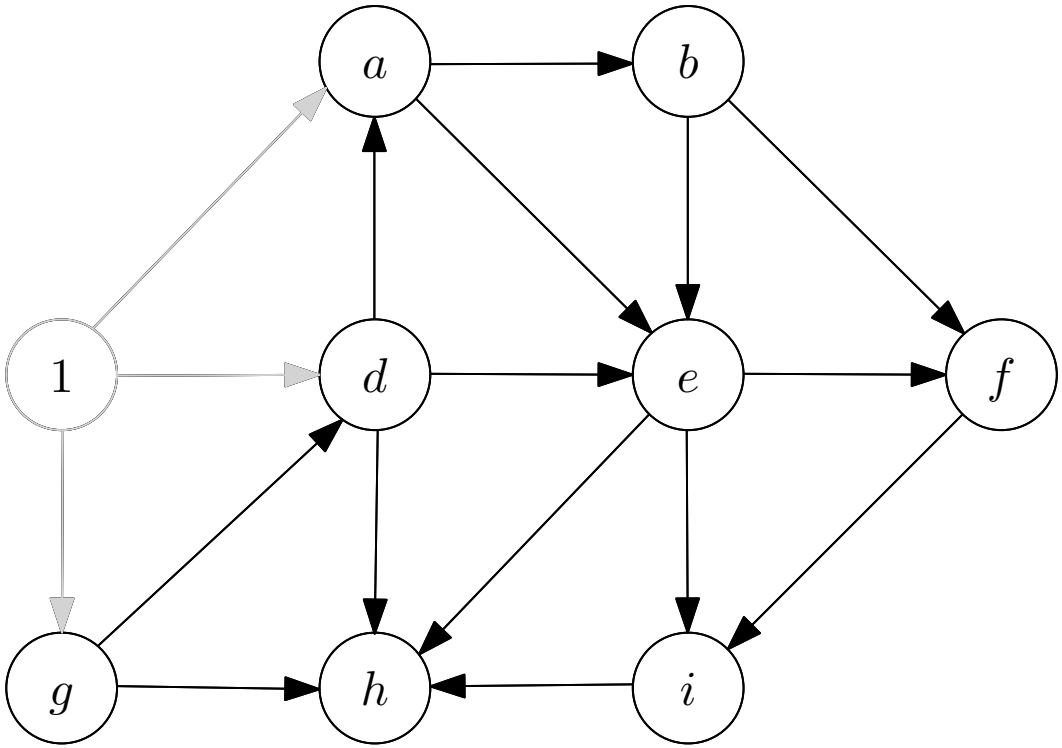

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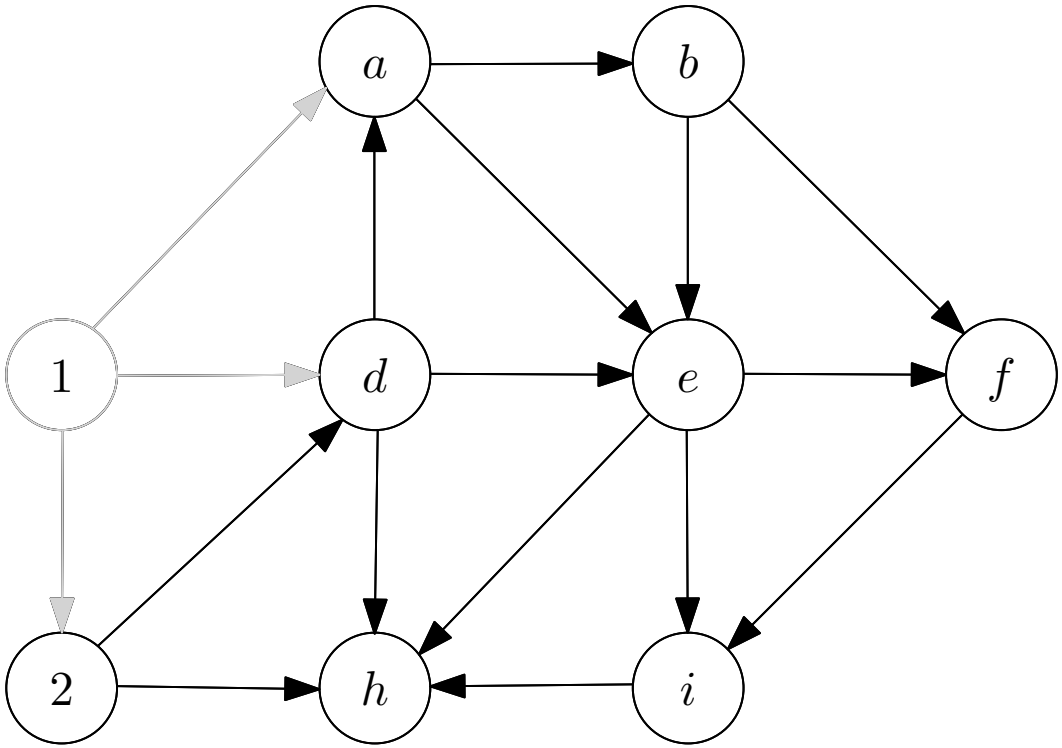

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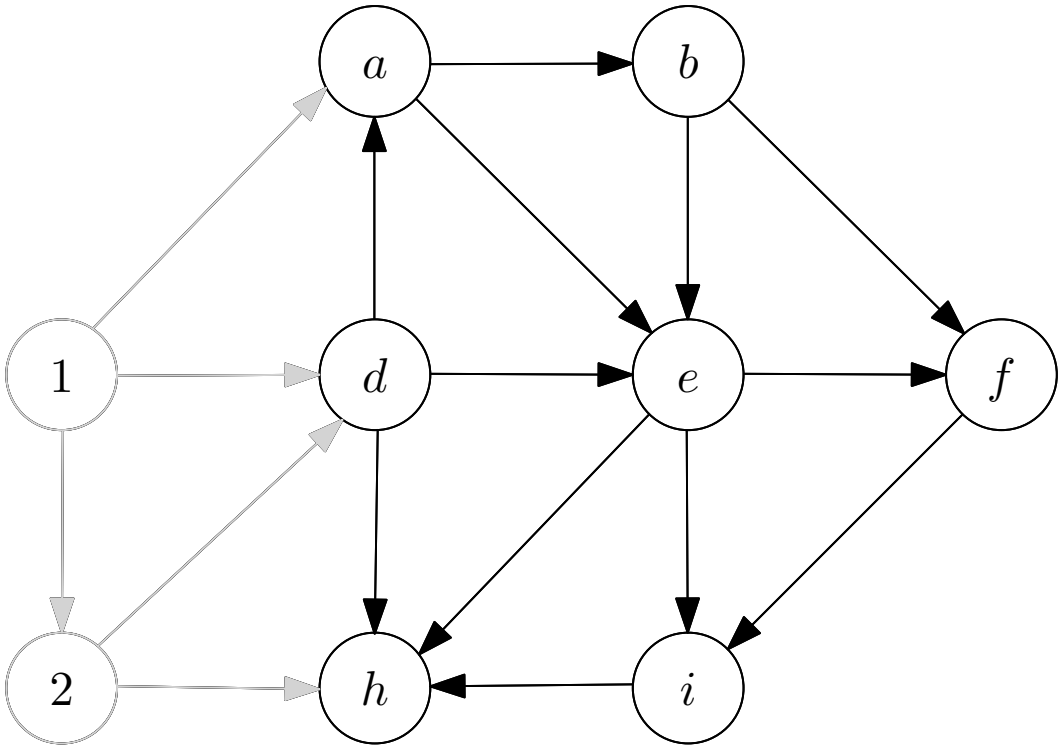

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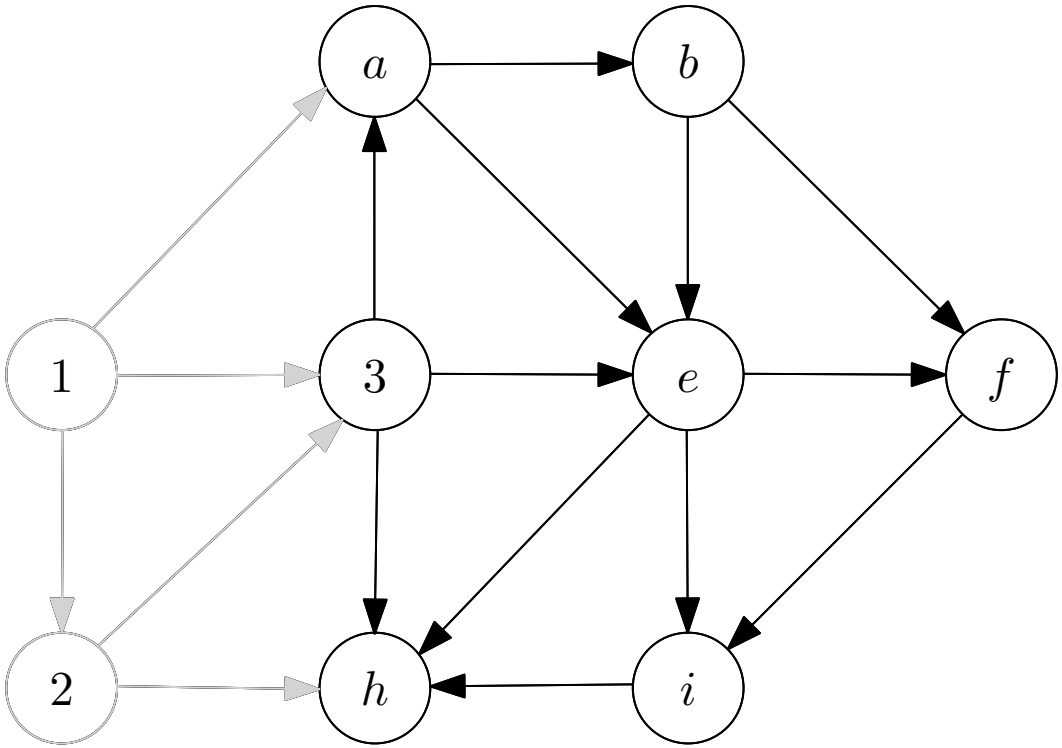

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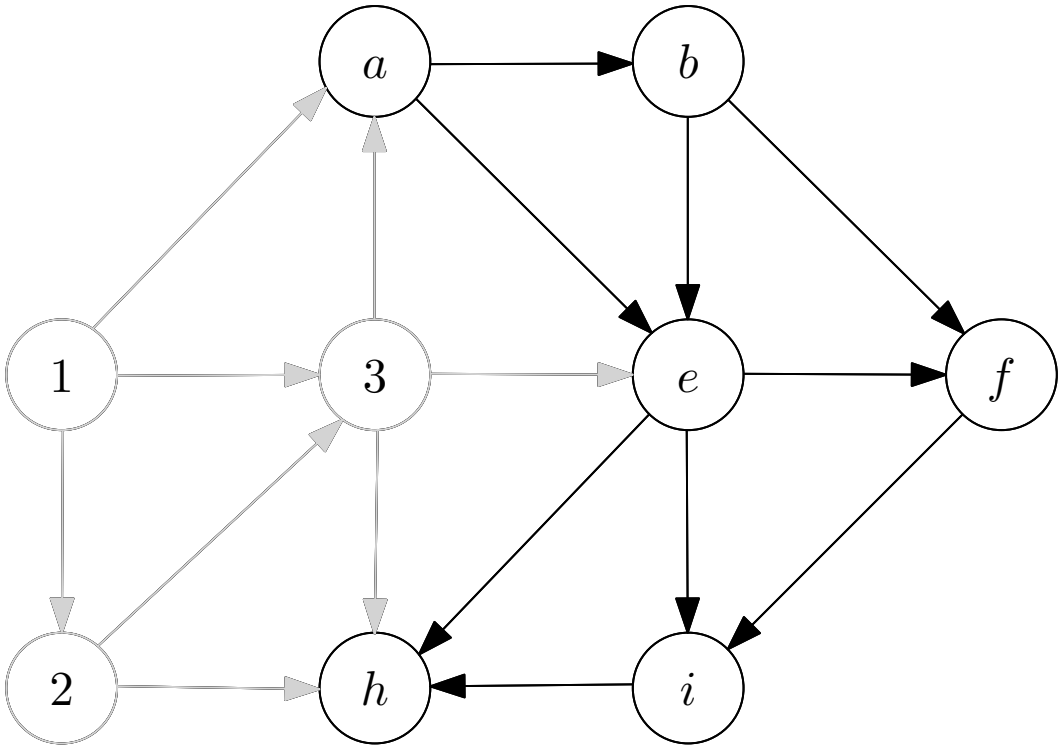

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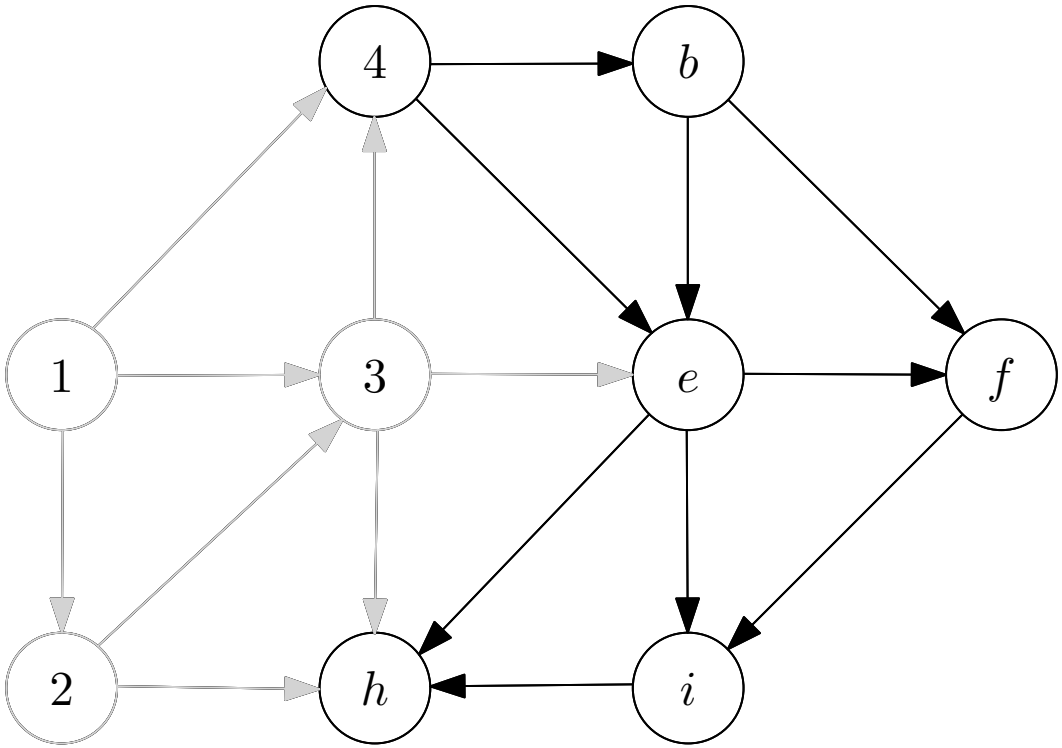

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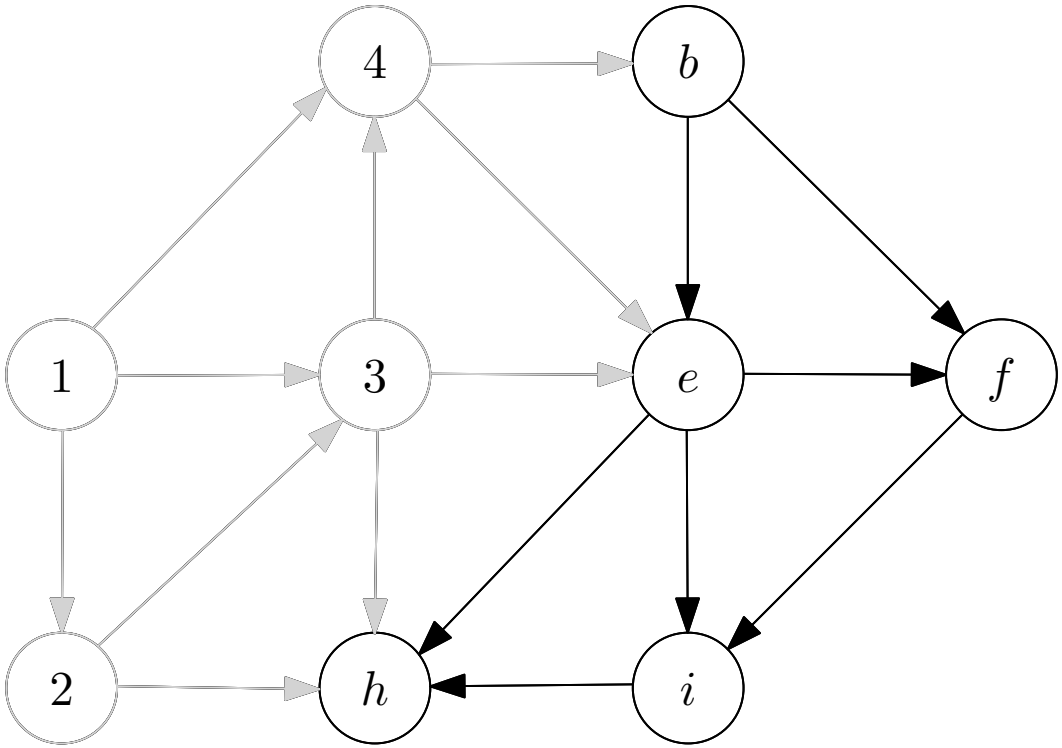

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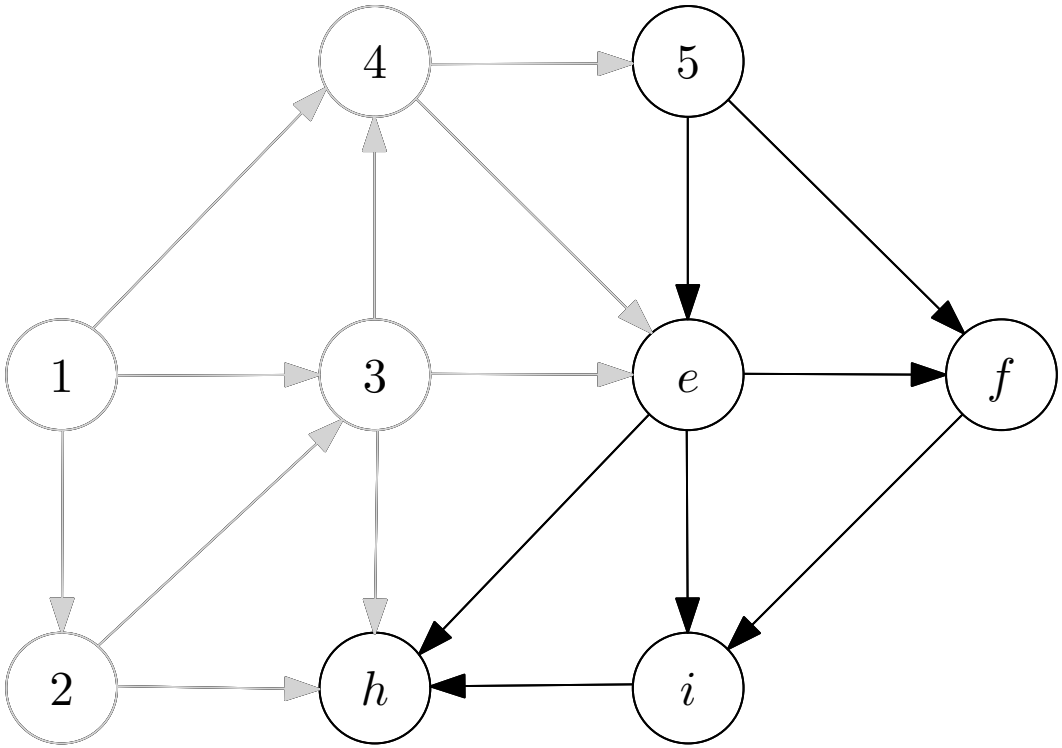

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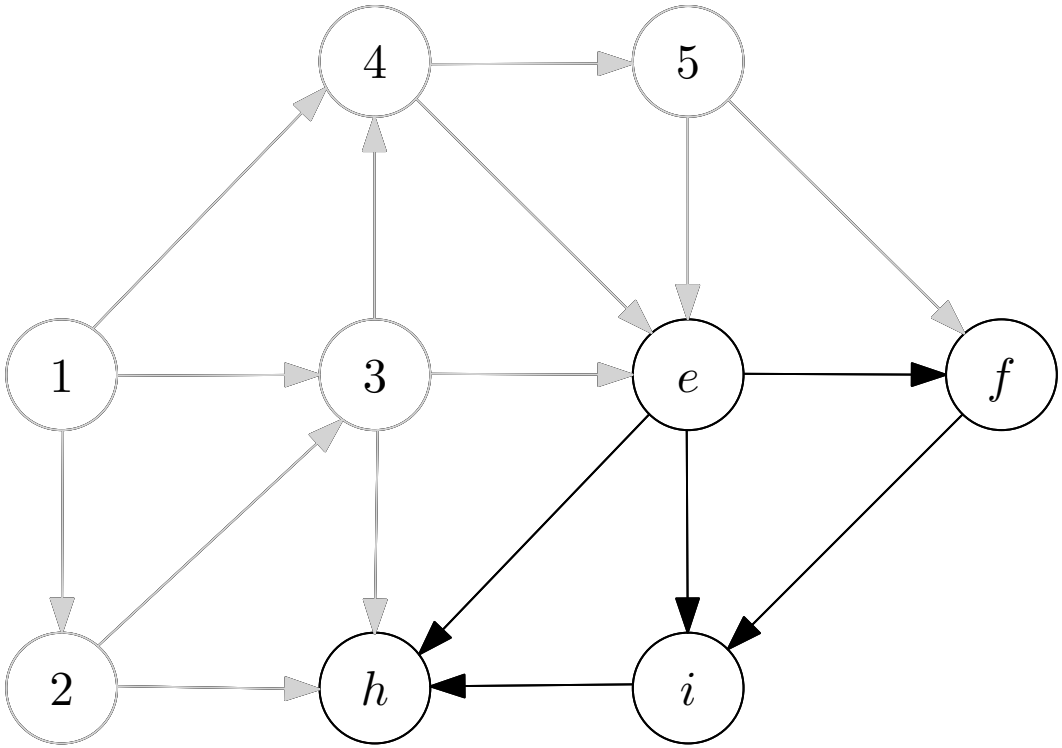

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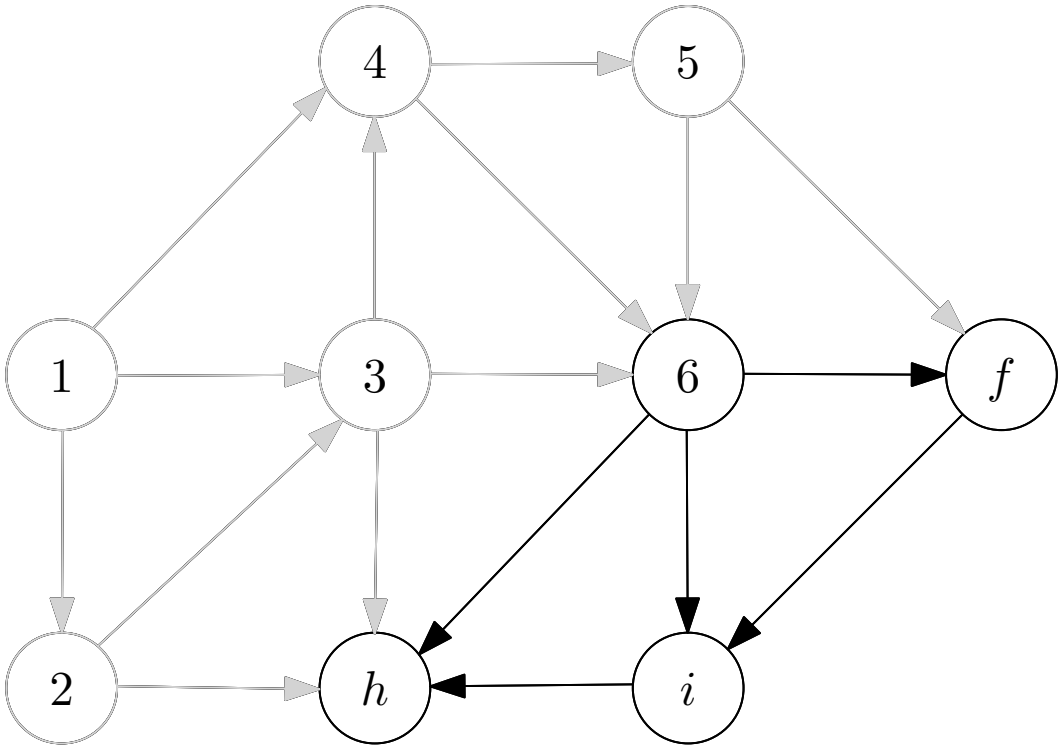

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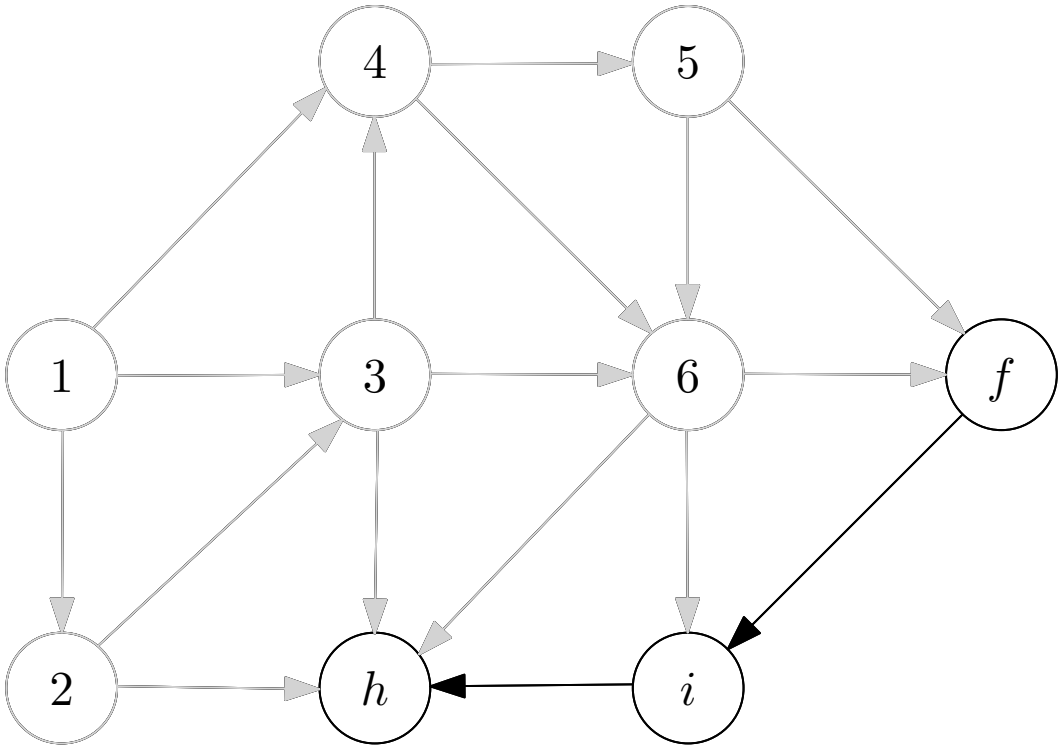

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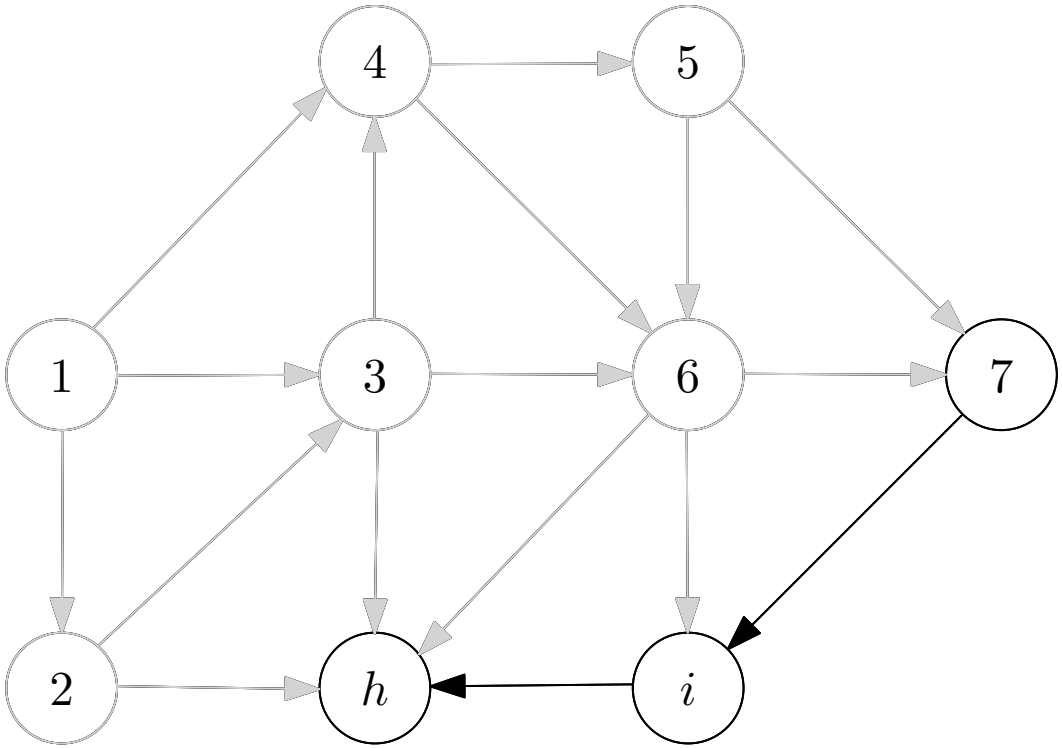

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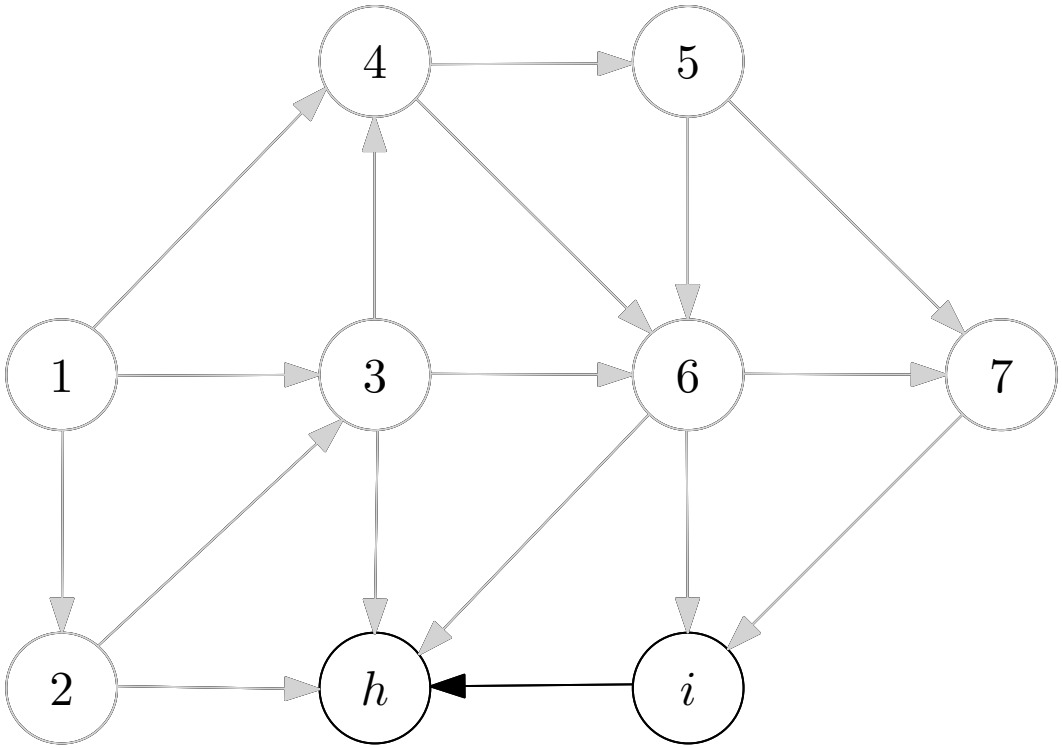

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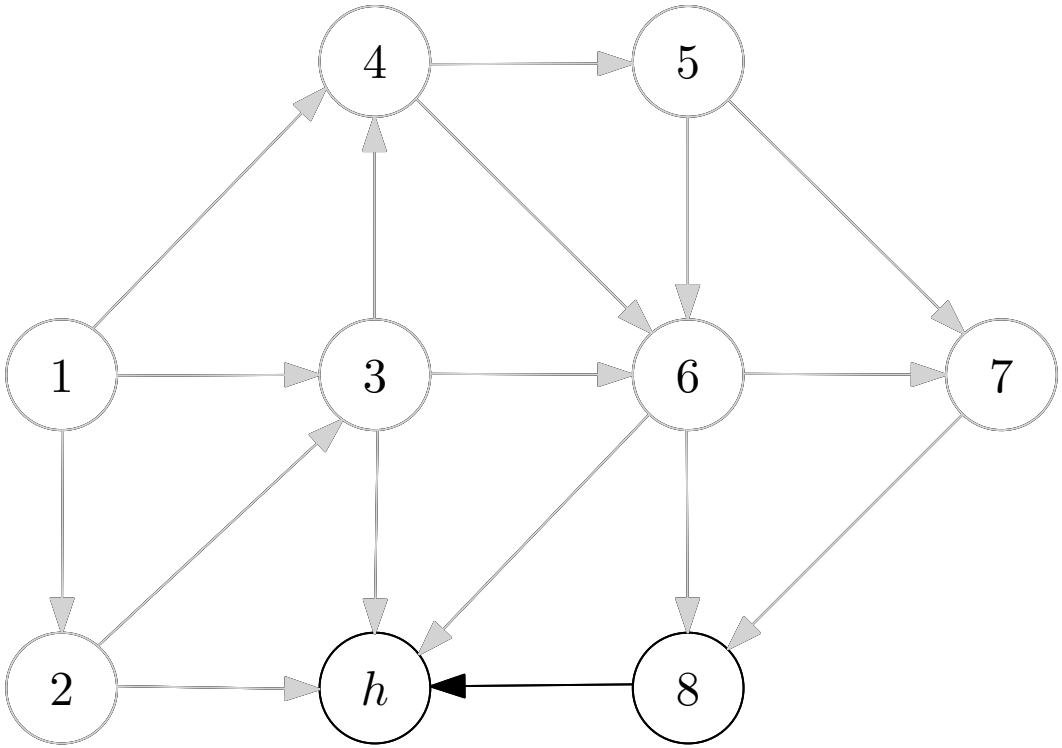

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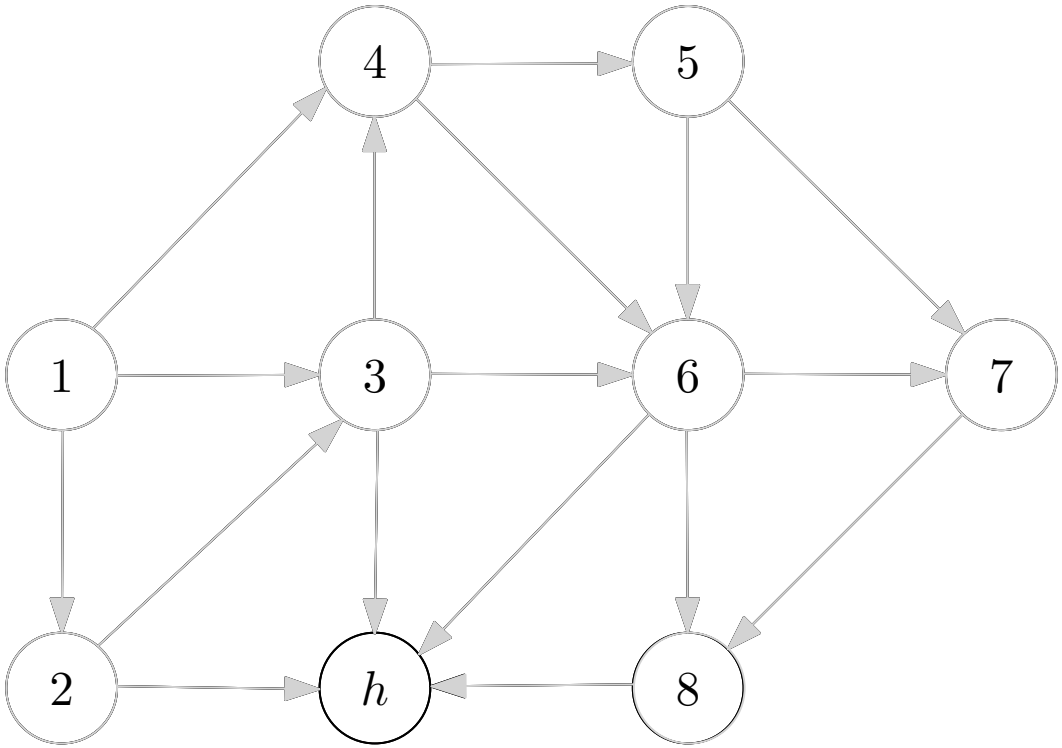

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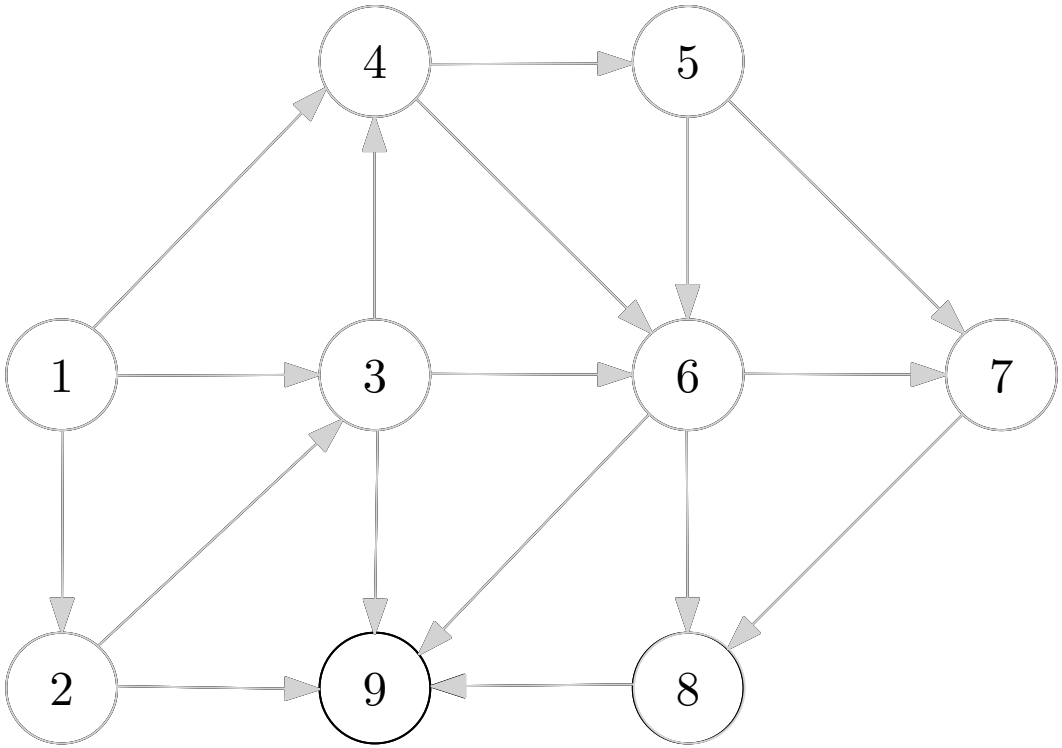

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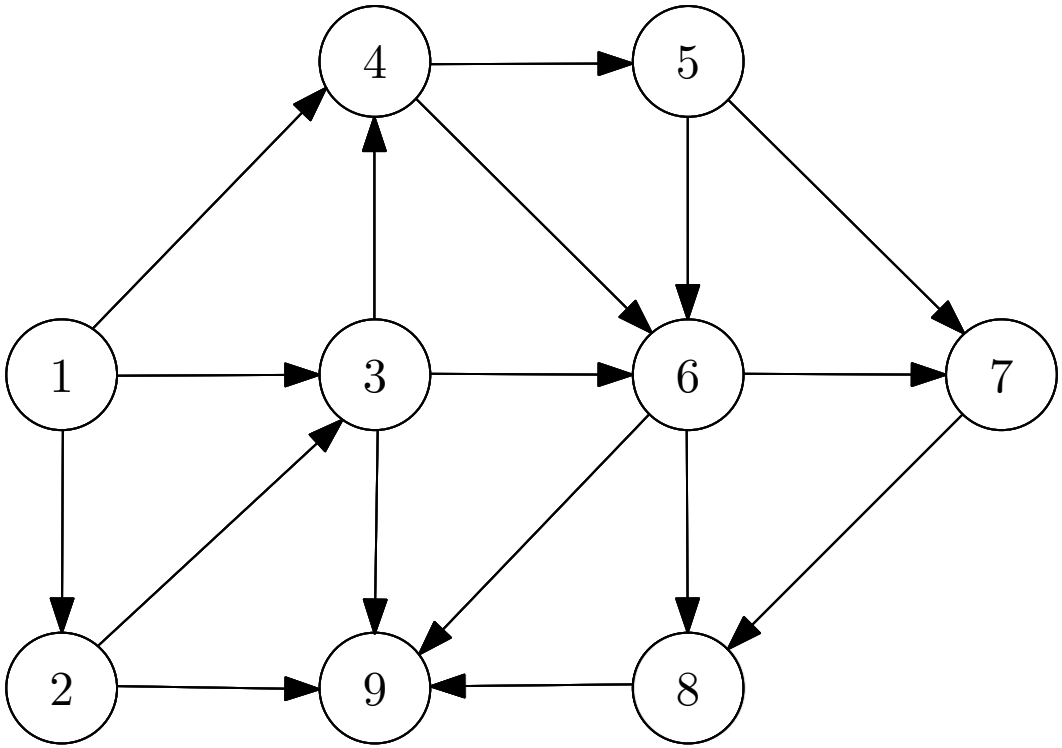

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Q: How to make the algorithm as efficient as possible?

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Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

1. let $d_v \leftarrow 0$ for every $v \in V$
2. for every $v \in V$
3. for every u such that $(v, u) \in E$
4. $d_u \leftarrow d_u + 1$
5. $S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
6. while $S \neq \emptyset$
7. $v \leftarrow$ arbitrary vertex in S, $S \leftarrow S \setminus \{v\}$
8. $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
9. for every u such that $(v, u) \in E$
10. $d_u \leftarrow d_u - 1$
11. if $d_u = 0$ then add u to S
12. if $i < n$ then output "not a DAG"

- S can be represented using a queue or a stack
- Running time $= O(n + m)$