CSE 431/531: Algorithm Analysis and Design (Spring 2019)
Greedy Algorithms

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Main Goal of Algorithm Design

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Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming

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- At each step, make an irrevocable decision using a "reasonable" strategy

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- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Outline

Toy Problem 1: Bill Changing

Input: Integer $A \geq 0$

Currency denominations: $\$1, \$2, \$5, \$10, \$20$

Output: A way to pay A dollars using fewest number of bills

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Cashier's Algorithm

1. while $A \geq 0$ do
2. $\quad a \leftarrow \max\{t \in \{1, 2, 5, 10, 20\} : t \leq A\}$
3. $\quad$ pay a \$$a$ bill
4. $\quad A \leftarrow A - a$

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- strategy: choose the largest bill that does not exceed A
- the strategy is "reasonable": choosing a larger bill help us in minimizing the number of bills
- The decision is irrevocable : once we choose a \$$a$ bill, we let $A \leftarrow A - a$ and proceed to the next

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- $n_1 < 2$ $\quad$ $2 \leq A < 5$: pay a \$2 bill

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- Trivial: in residual problem, we need to pay $A' = A - a$ dollars, using the fewest number of bills

Toy Example 2: Box Packing

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

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Example:

- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: $45 \rightarrow 60, 20 \rightarrow 40, 19 \rightarrow 25$

Box Packing: Design a Safe Strategy

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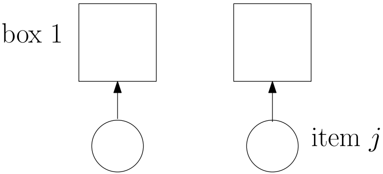

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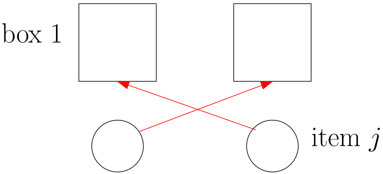

- Residual task: solve the instance obtained by removing box 1 and item j

Greedy Algorithm for Box Packing

1. $T \leftarrow \{1, 2, 3, \cdots, m\}$
2. for $i \leftarrow 1$ to n do
3.   if some item in T can be put into box i, then
4. $j \leftarrow$ the largest item in T that can be put into box i
5.   print("put item j in box i")
6. $T \leftarrow T \setminus \{j\}$

Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
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Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

Exchanging argument: let S be an arbitrary optimum solution. If S is consistent with the greedy choice, we are done. Otherwise, modify it to another optimum solution S' such that S' is consistent with the greedy choice.

Generic Greedy Algorithm

1. **while** the instance is non-trivial
2. make the choice using the greedy strategy
3. reduce the instance

Algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
- Greedy stretegy is not safe: we will miss the optimum solution for some instance, since the choices we made are irrevocable.

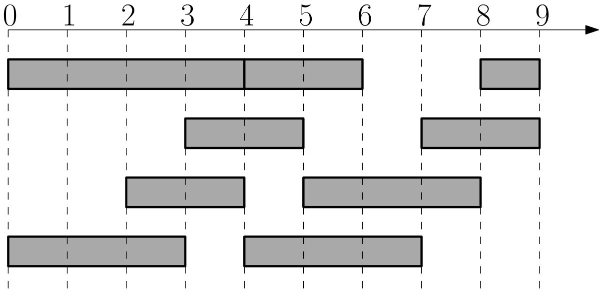
Outline

Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs



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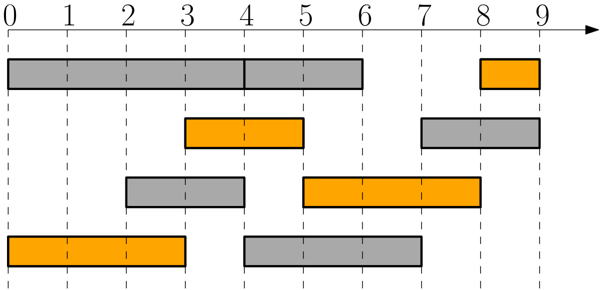

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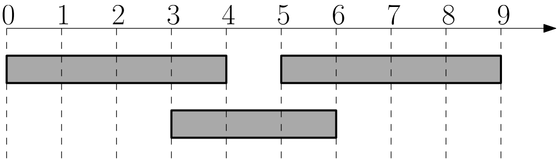

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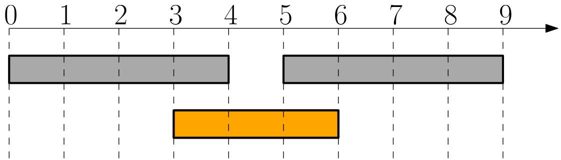

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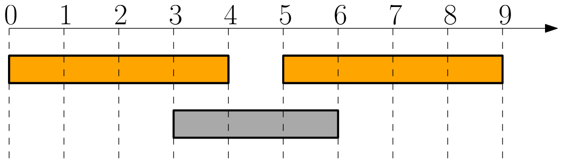

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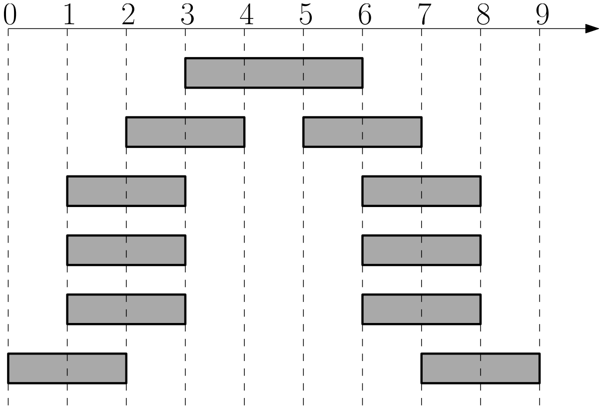

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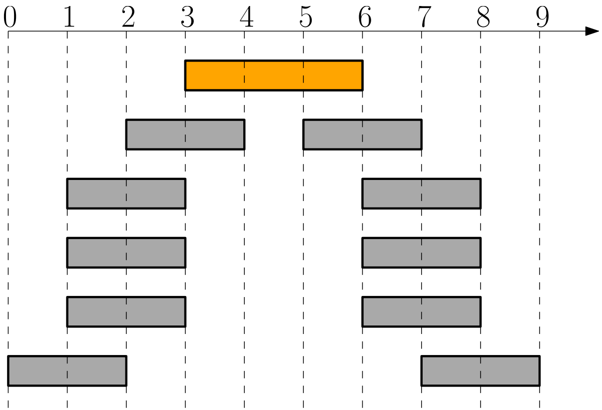

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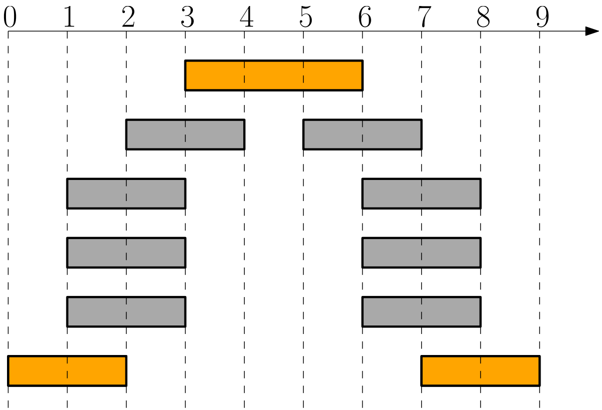

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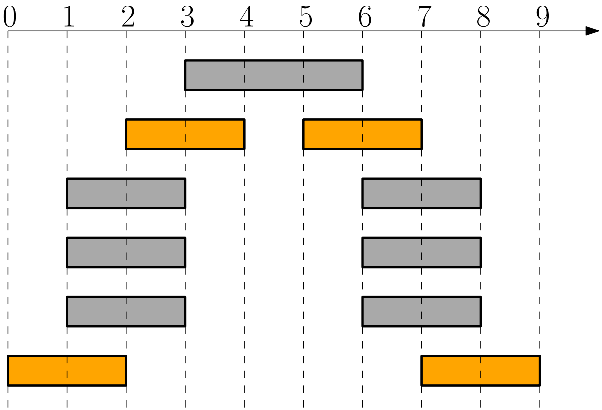

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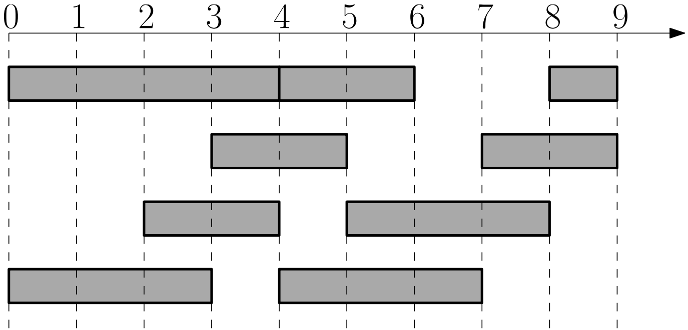

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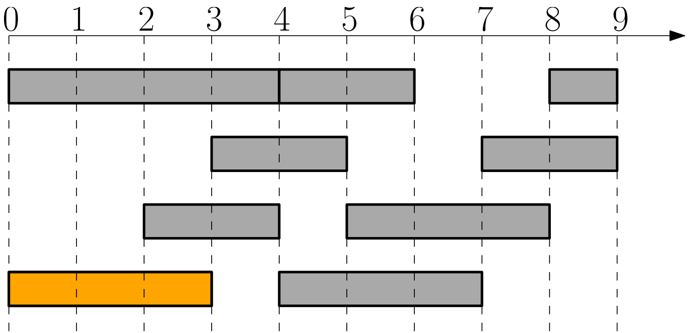

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Lemma It is safe to schedule the job j with the earliest finish time: there is an optimum solution where j is scheduled.

Proof.

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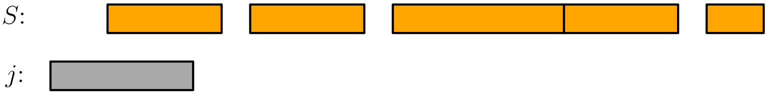

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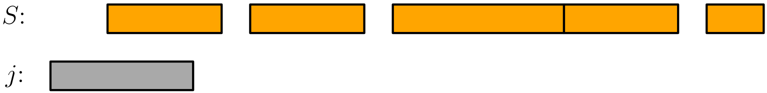

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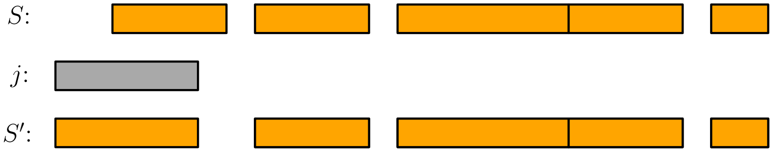

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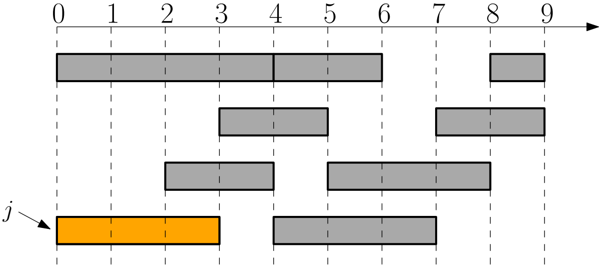
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem?



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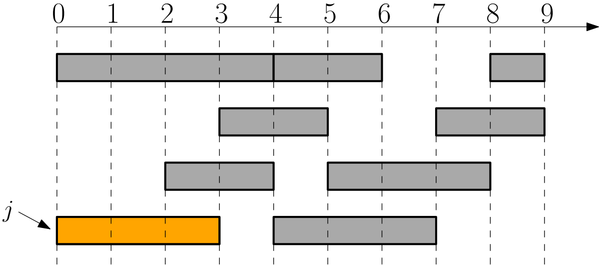

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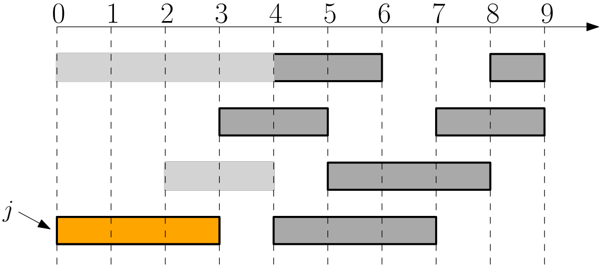

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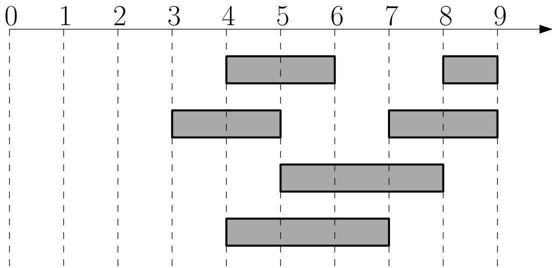

Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1. $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2. while $A \neq \emptyset$
3. $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
4. $\quad S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5. return S

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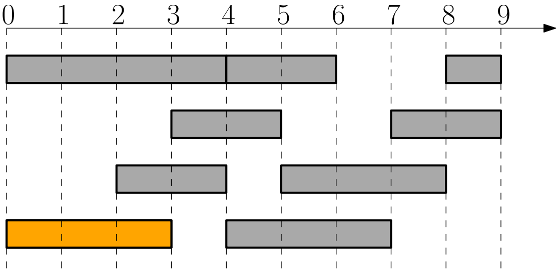

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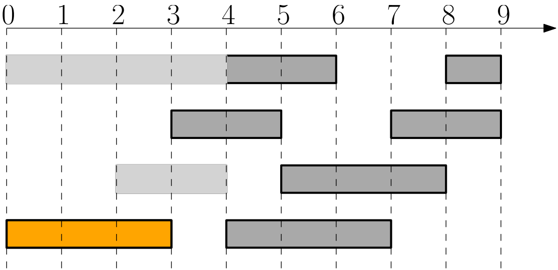

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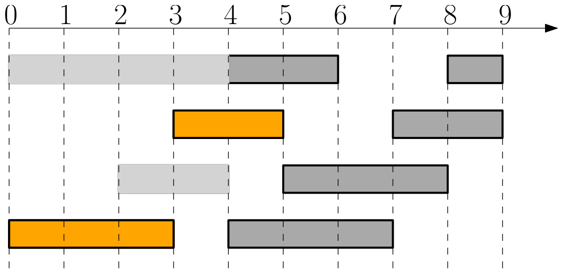

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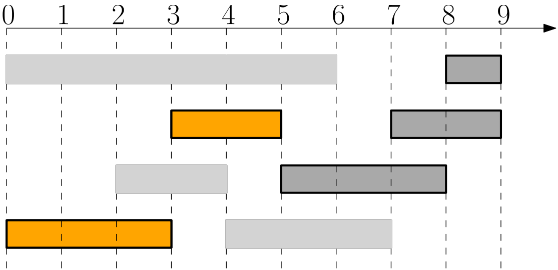

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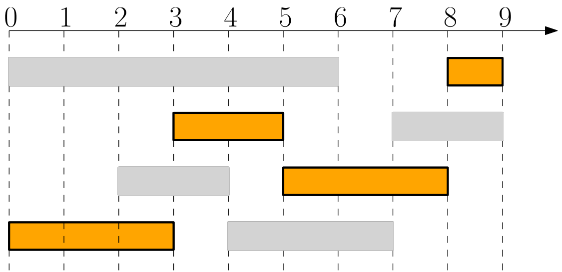

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- Naive implementation: $O(n^2)$ time

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Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

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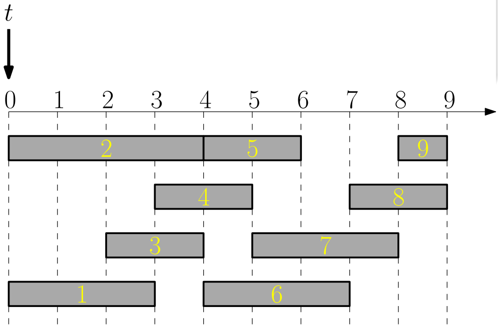

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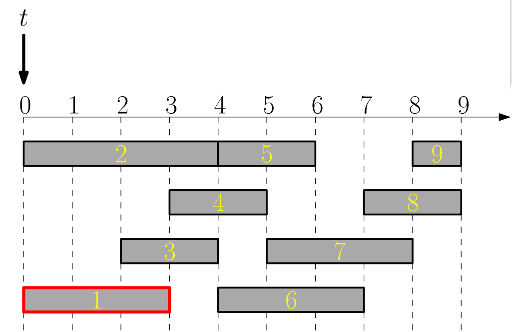

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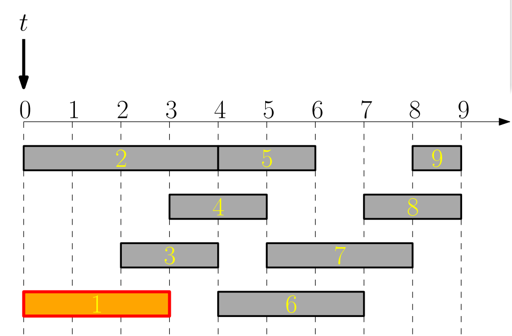

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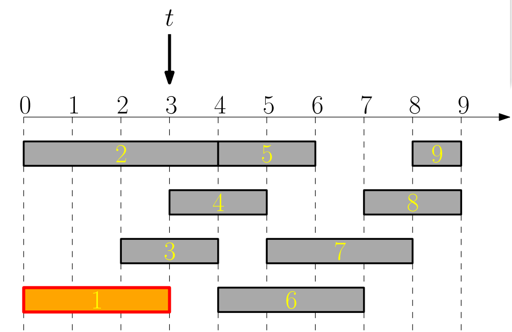

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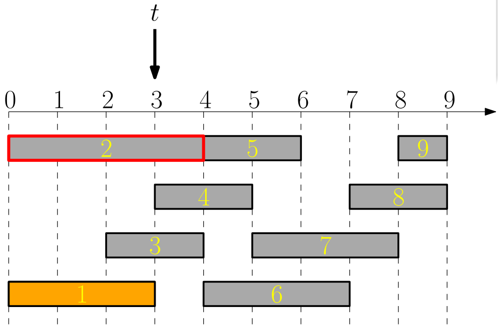

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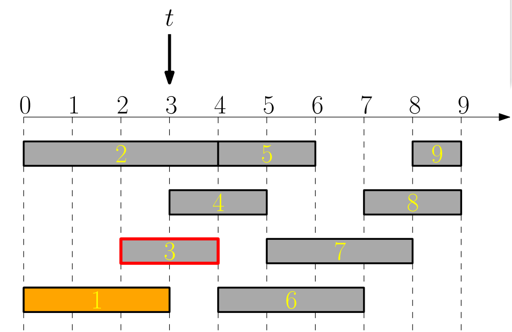

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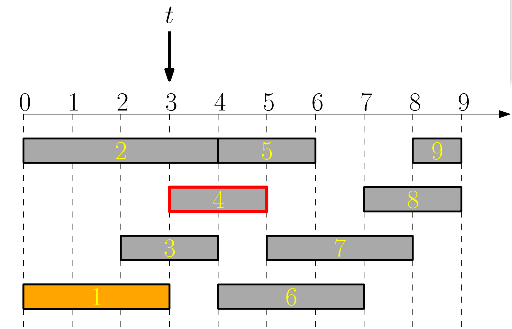

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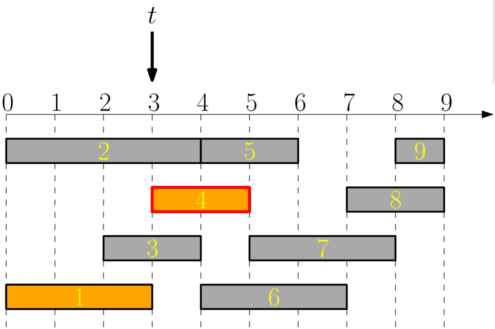

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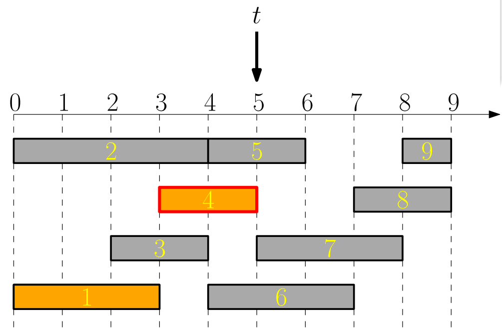

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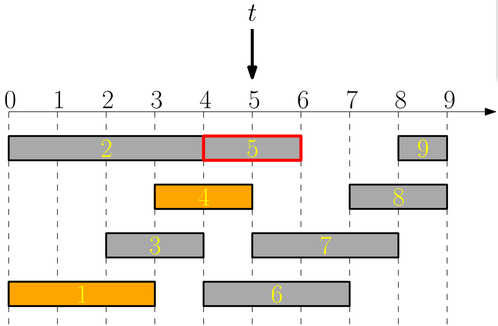

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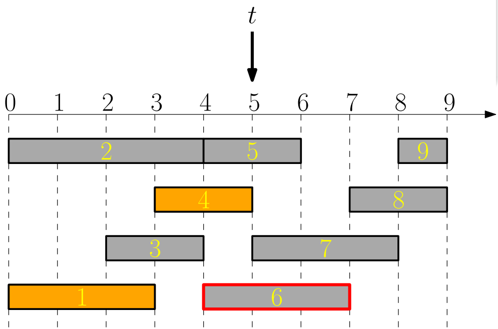

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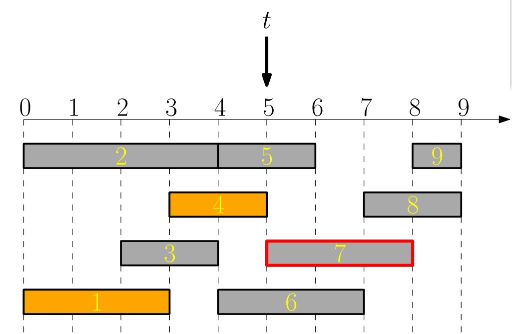

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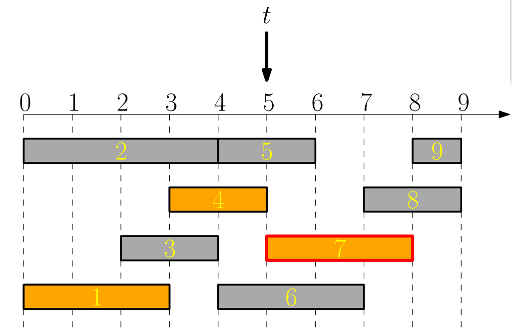

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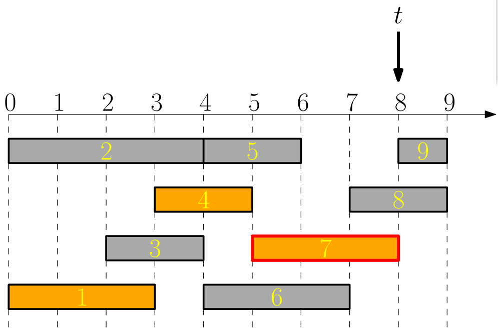

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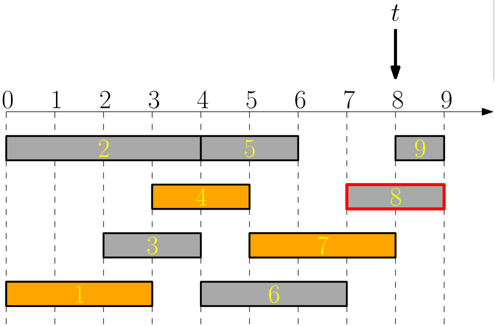

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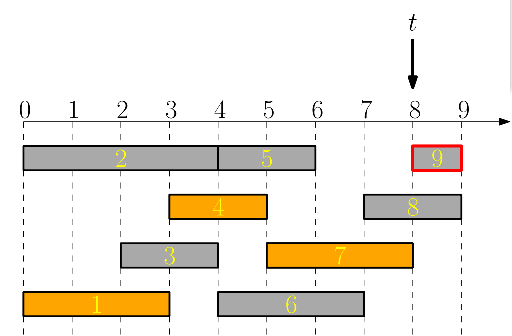

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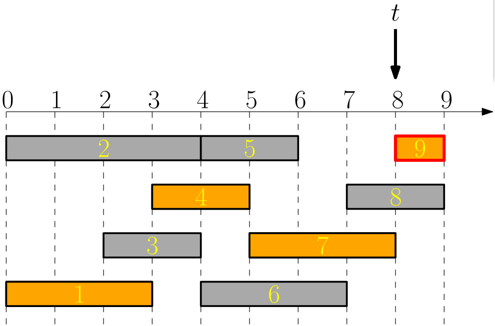

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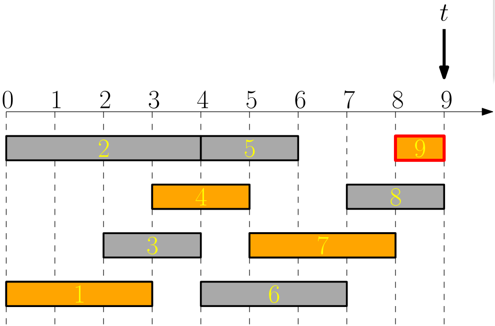
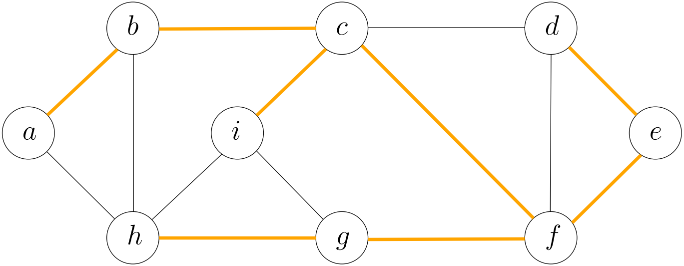
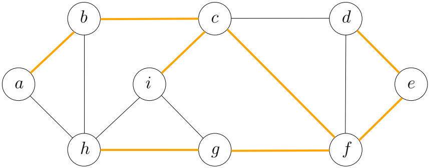


Outline

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight

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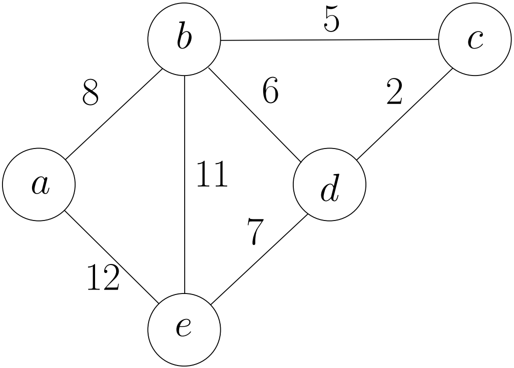

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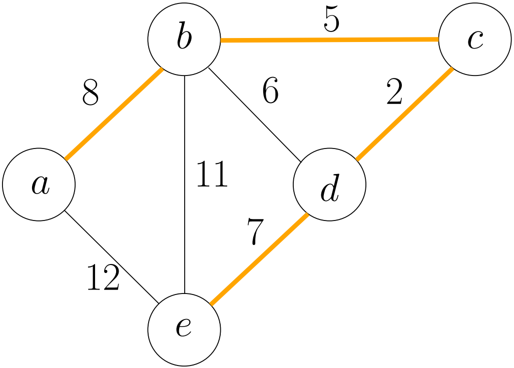

Recall: Steps of Designing A Greedy Algorithm

- Design a "reasonable" strategy
- Prove that the reasonable strategy is "safe" (key, usually done by "exchanging argument")
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

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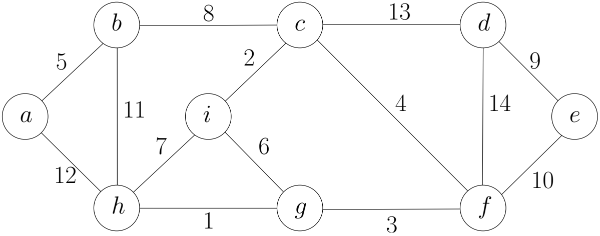
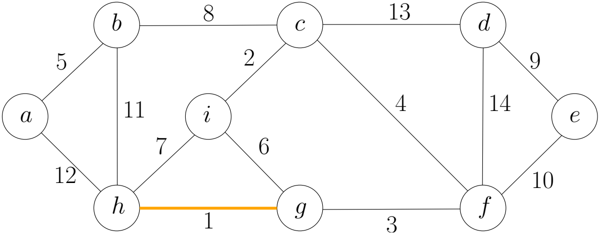
Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

Outline



Q: Which edge can be safely included in the MST?



Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

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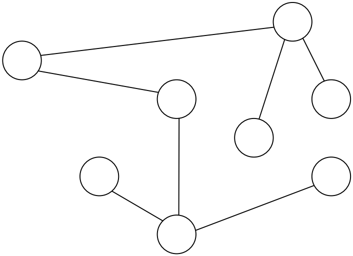
Proof.

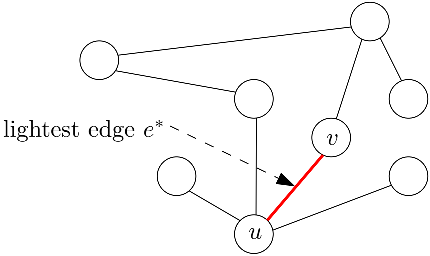
- Take a minimum spanning tree T



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T



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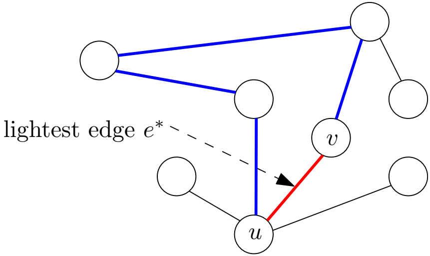

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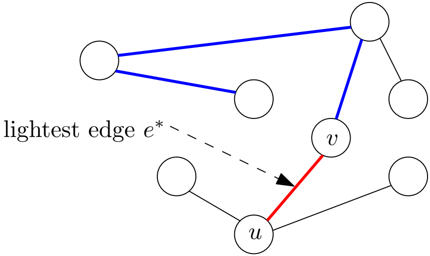
- Remove any edge e in the path to obtain tree T'



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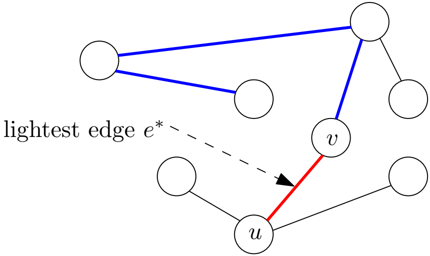
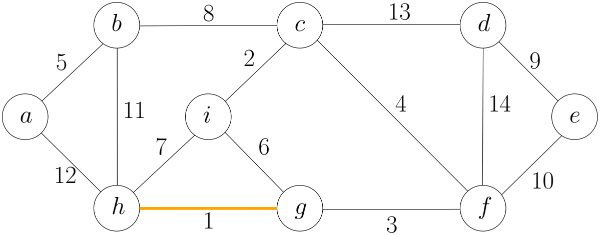
- $w(e^*) \le w(e) \implies w(T') \le w(T)$: T' is also a MST □



Is the Residual Problem Still a MST Problem?



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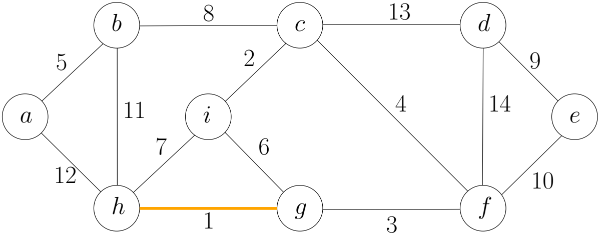

- Residual problem: find the minimum spanning tree that contains edge (g, h)

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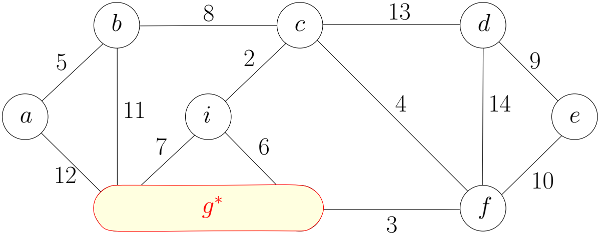

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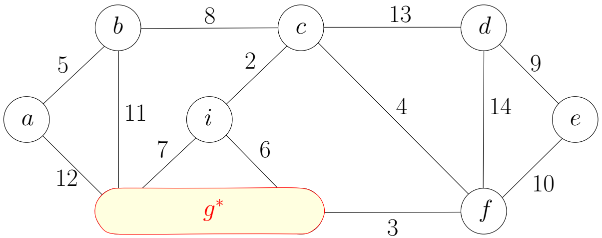
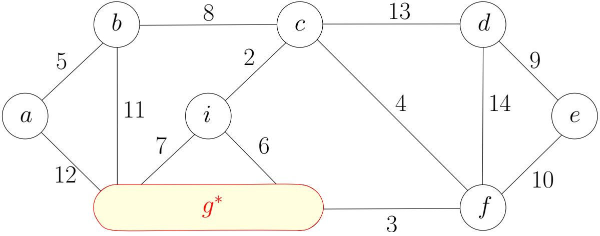
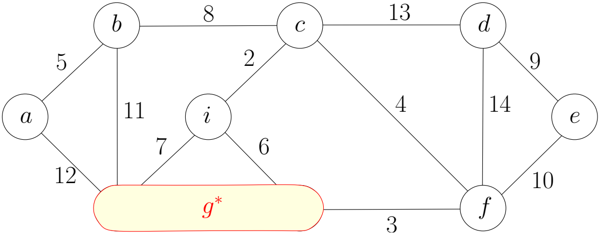


- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*

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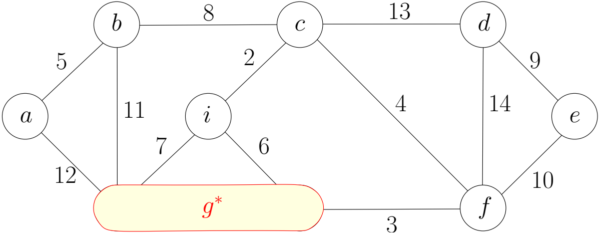

- Remove u and v from the graph, and add a new vertex u^*
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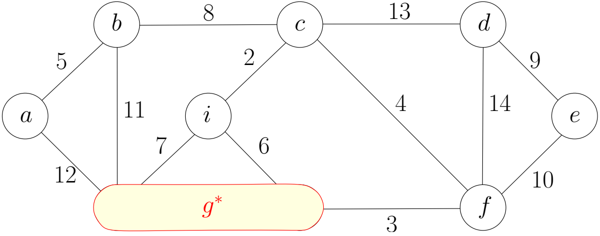

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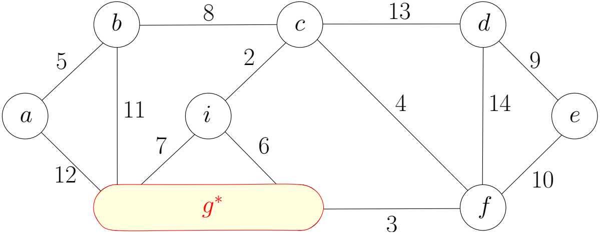

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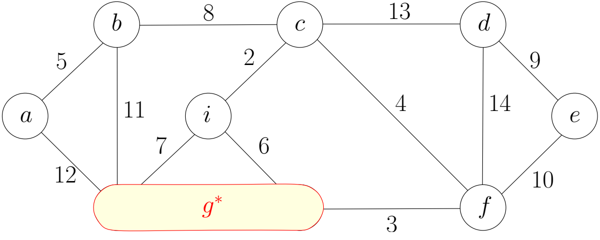

- Remove u and v from the graph, and add a new vertex u^*
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- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

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1. Choose the lightest edge e^*, add e^* to the spanning tree
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Q: What edges are removed due to contractions?

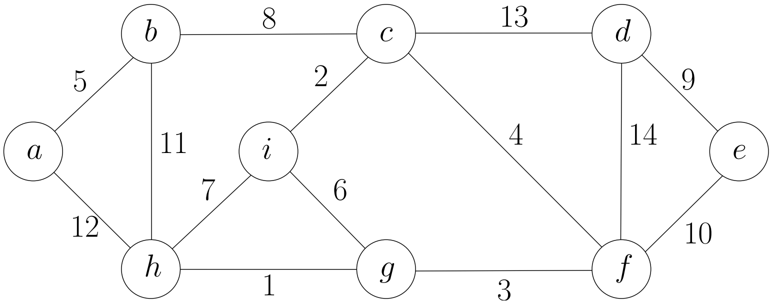
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

1. $F = \emptyset$
2. sort edges in E in non-decreasing order of weights w
3. for each edge (u, v) in the order
4.   if u and v are not connected by a path of edges in F
5. $F = F \cup \{(u, v)\}$
6. return (V, F)

Kruskal's Algorithm: Example



Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

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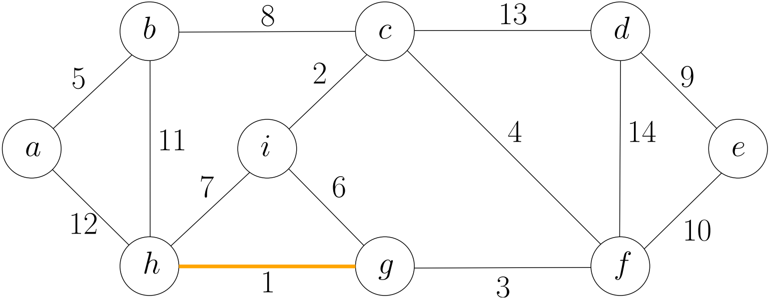

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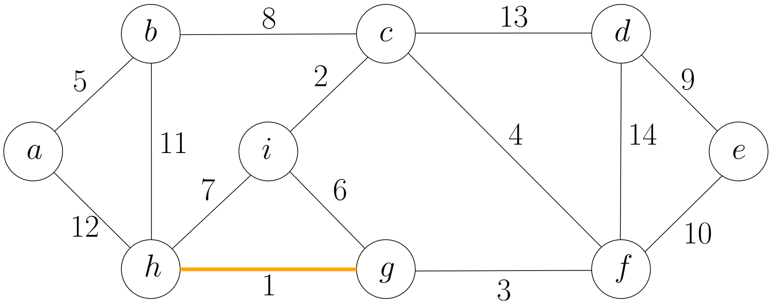

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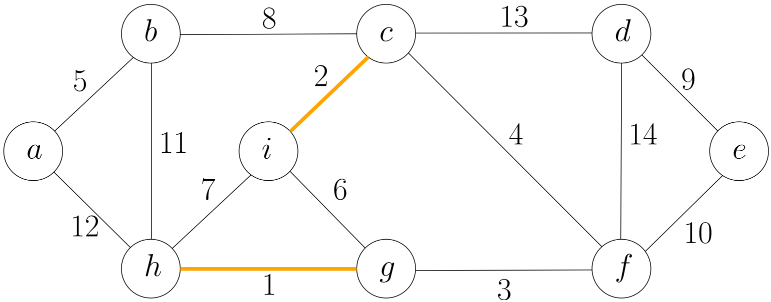

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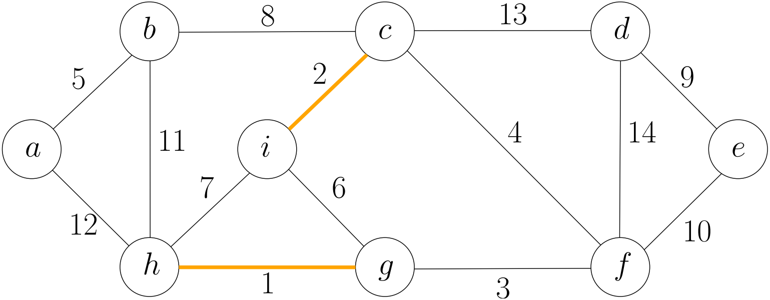

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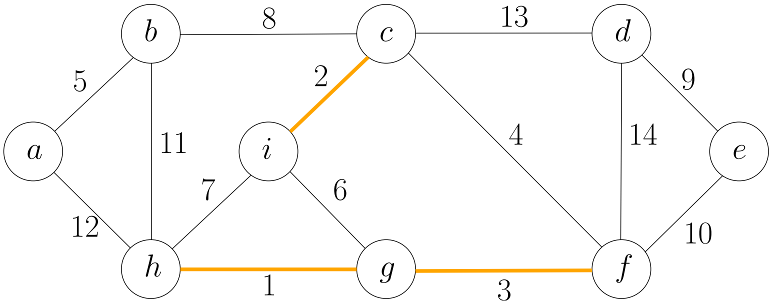

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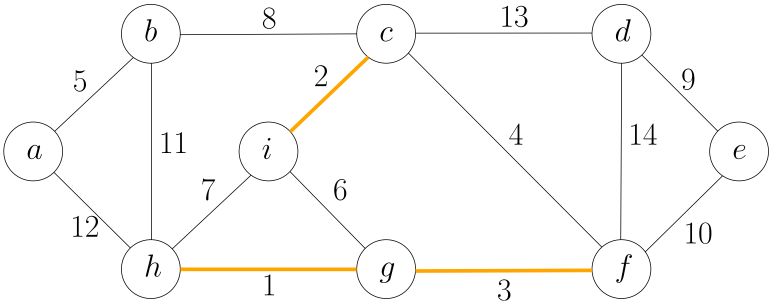

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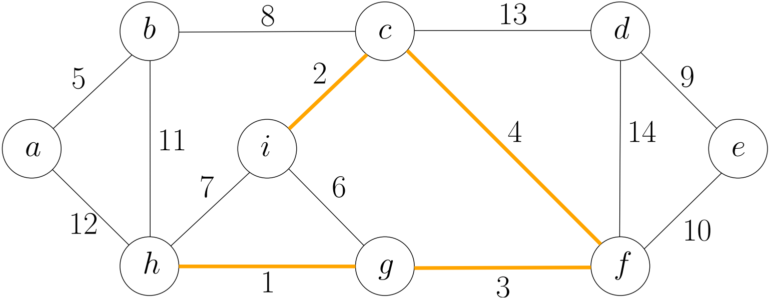

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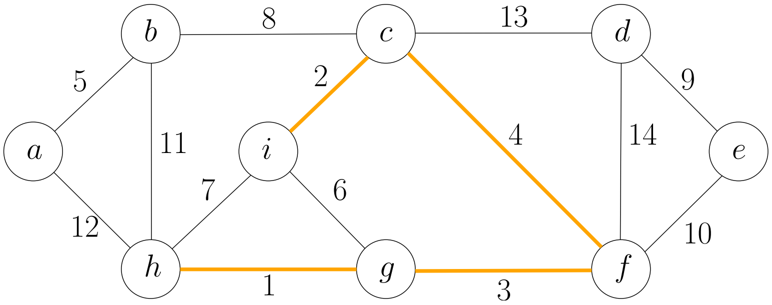

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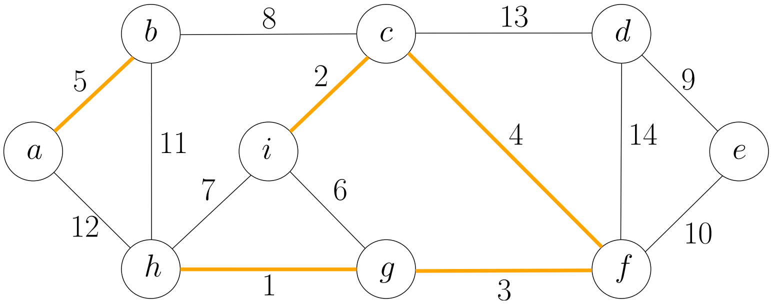

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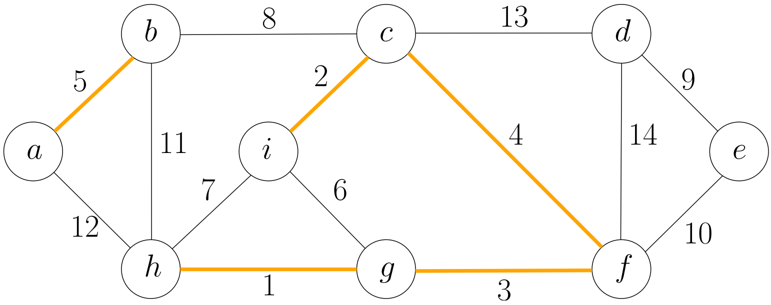

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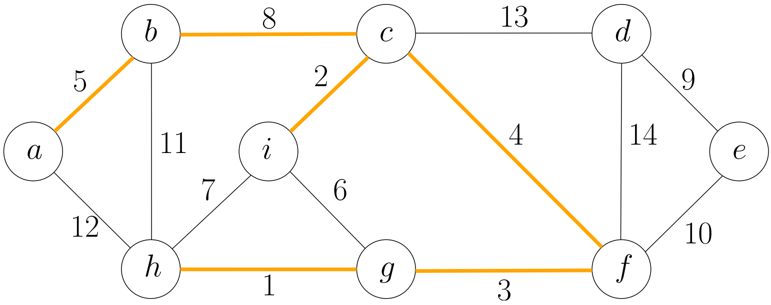

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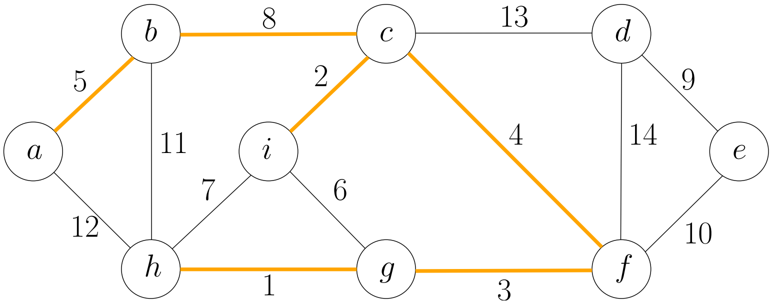

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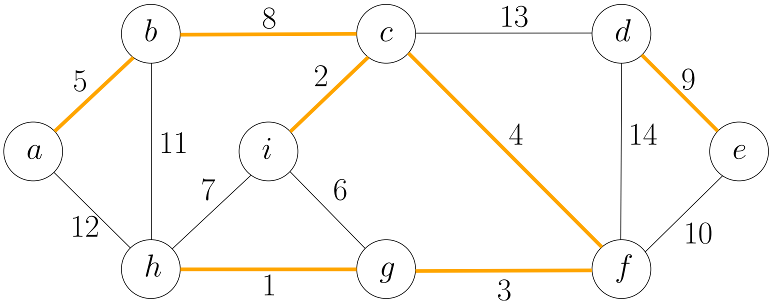

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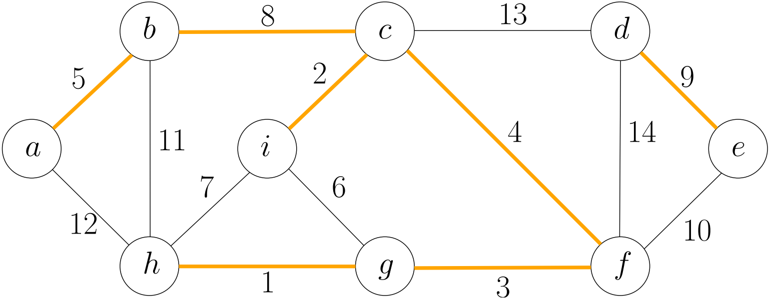

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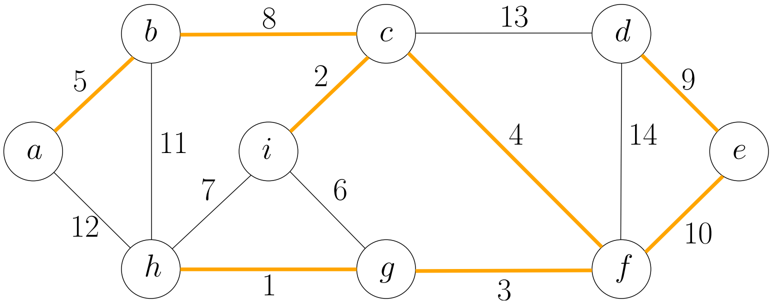

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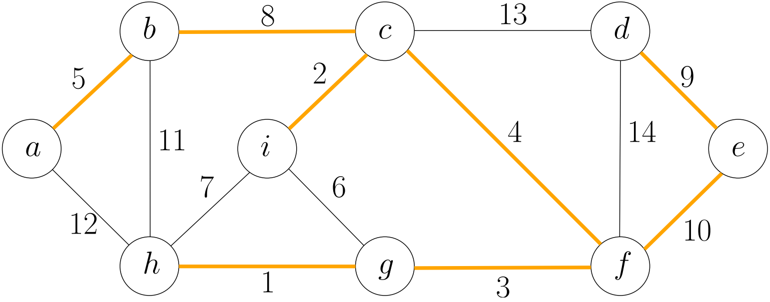


Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

```
F ← ∅
𝒮 ← {{v} : v ∈ V}
sort the edges of E in non-decreasing order of weights w
for each edge (u, v) ∈ E in the order
    S_u ← the set in 𝒮 containing u
    S_v ← the set in 𝒮 containing v
    if S_u ≠ S_v
        F ← F ∪ {(u, v)}
        𝒮 ← 𝒮 \ {S_u} \ {S_v} ∪ {S_u ∪ S_v}
return (V, F)
```

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

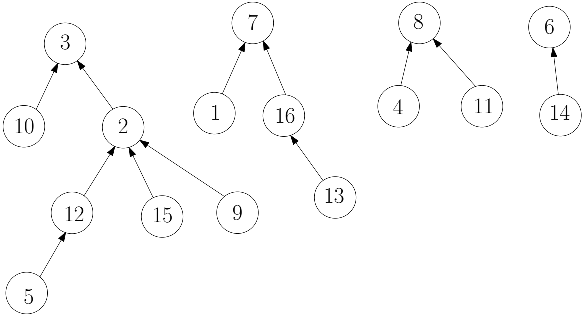
1. $F \leftarrow \emptyset$
2. $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3. sort the edges of E in non-decreasing order of weights w
4. for each edge $(u, v) \in E$ in the order
5. $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6. $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7.   if $S_u \neq S_v$
8. $F \leftarrow F \cup \{(u, v)\}$
9. $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10. return (V, F)

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

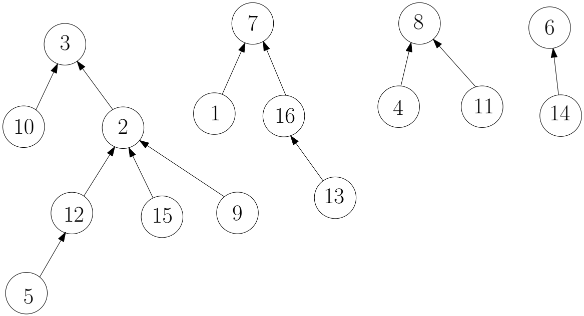
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

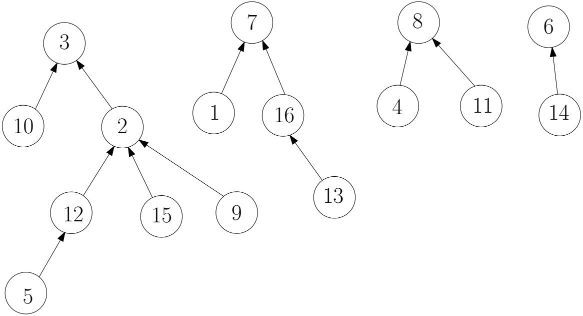
- $V = \{1, 2, 3, \cdots, 16\}$
- Partition:
  $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] =$ nil if i is a root).

Union-Find Data Structure





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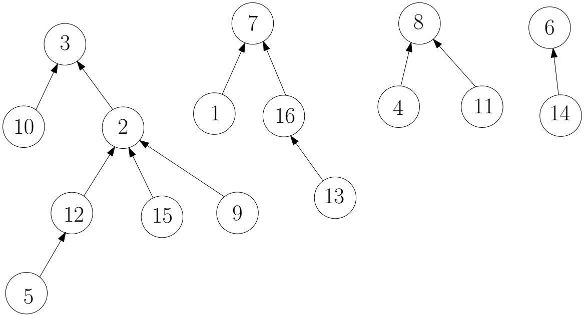

- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).

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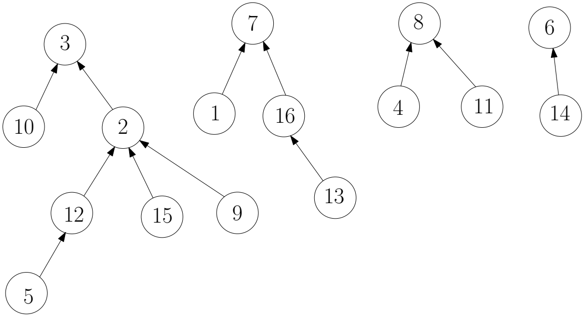

- Q: how can we check if u and v are in the same set?
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- $\text{root}(u)$: the root of the tree containing u

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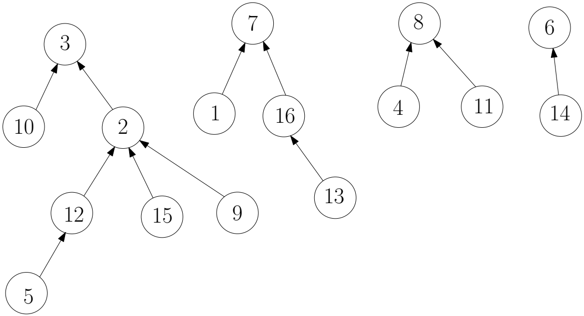

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- Merge the trees with root r and r': $par[r] \leftarrow r'$.

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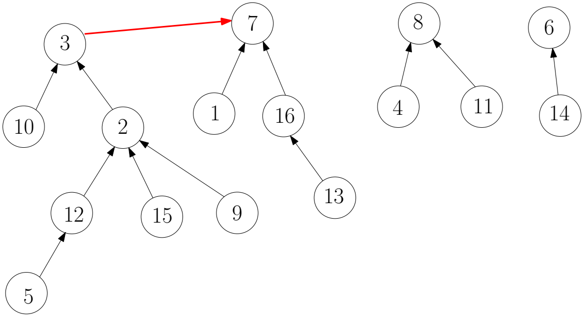

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if par[v] = nil then
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    return root(par[v])
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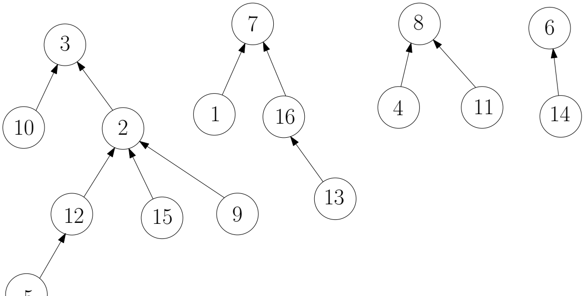

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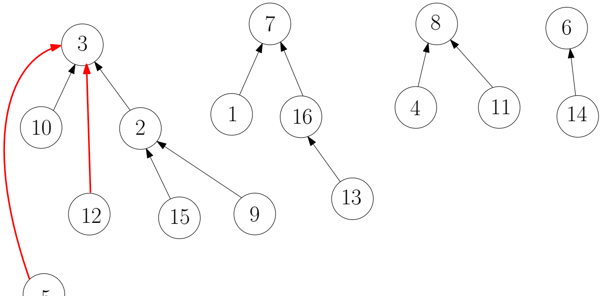

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MST-Kruskal(G, w)

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4. for each edge $(u, v) \in E$ in the order
5. $u' \leftarrow$ root(u)
6. $v' \leftarrow$ root(v)
7.   if $u' \neq v'$
8. $F \leftarrow F \cup \{(u, v)\}$
9. $par[u'] \leftarrow v'$
10. return (V, F)

- 2, 5, 6, 7, 9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

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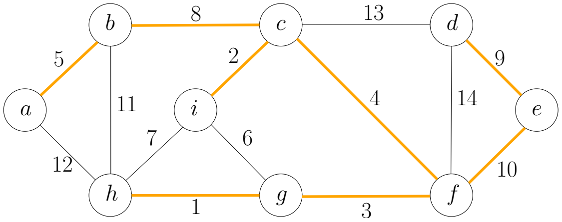
- Running time $=$ time for ③ $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



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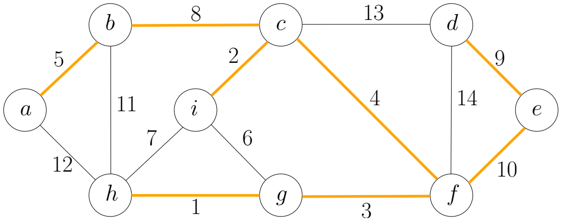

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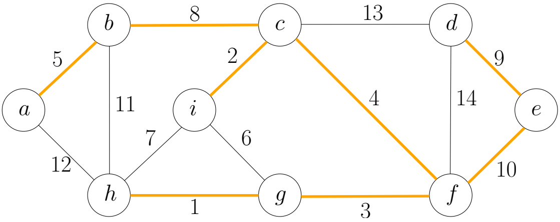

- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree

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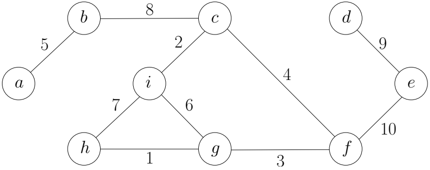

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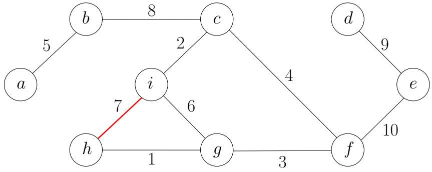

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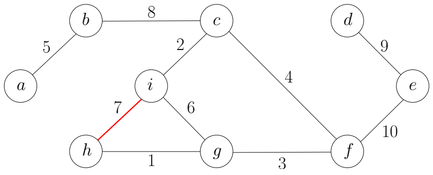


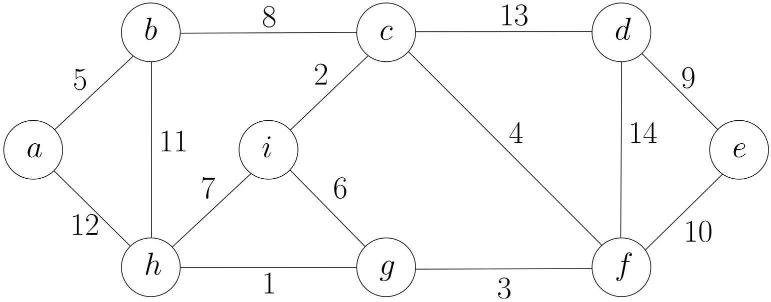
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

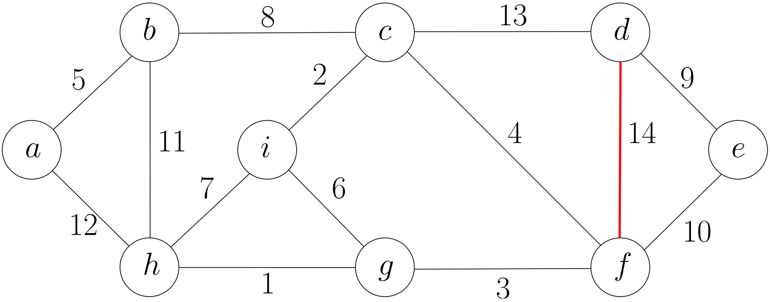
MST-Greedy(G, w)

1. $F \leftarrow E$
2. sort E in non-increasing order of weights
3. for every e in this order
4.   if $(V, F \setminus \{e\})$ is connected then
5. $F \leftarrow F \setminus \{e\}$
6. return (V, F)

Reverse Kruskal's Algorithm: Example



Reverse Kruskal's Algorithm: Example



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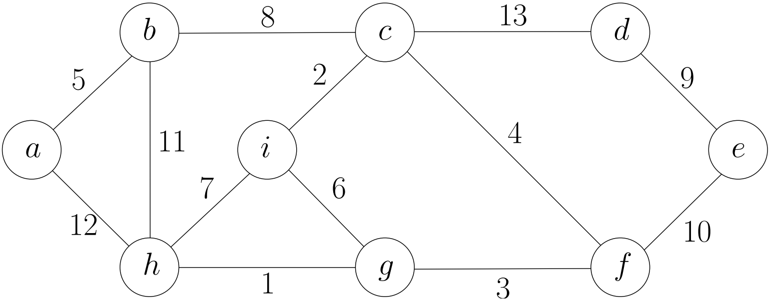

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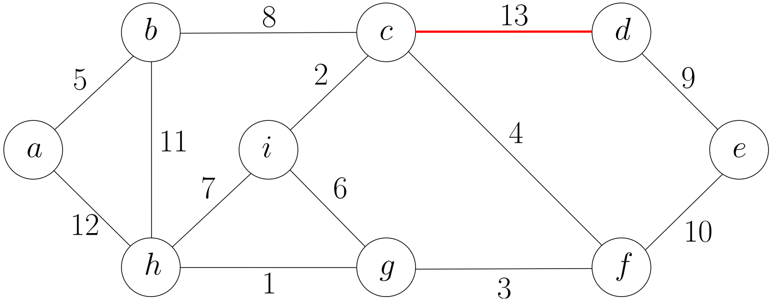

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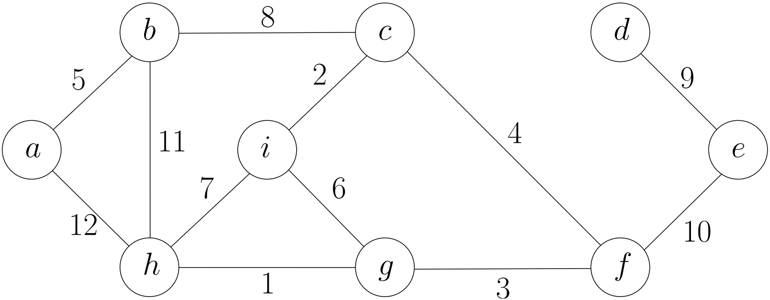

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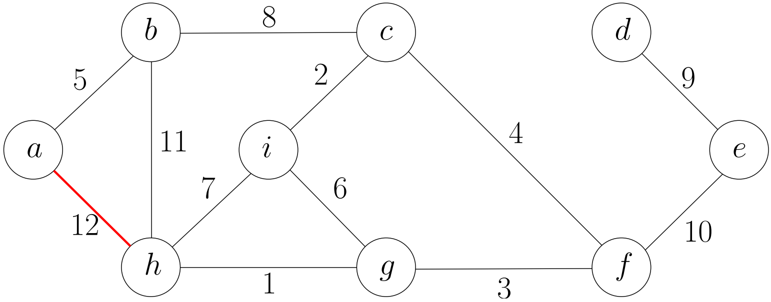

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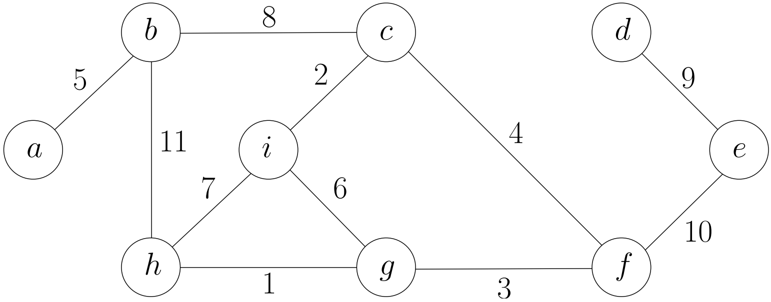

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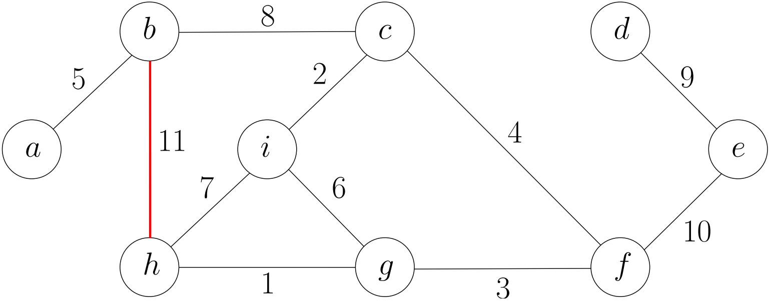

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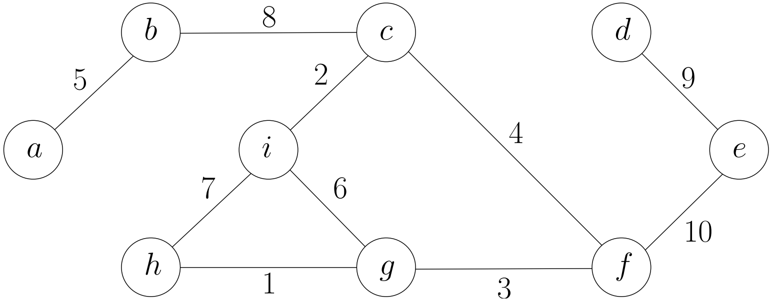

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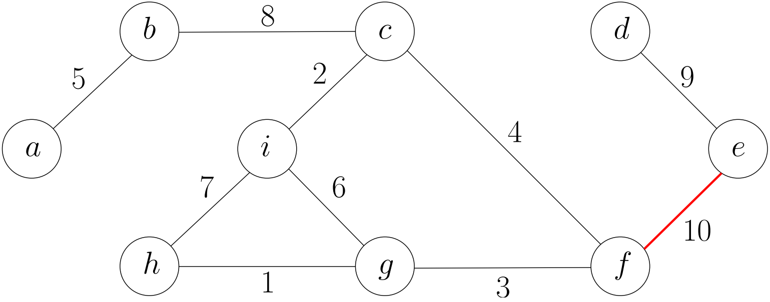

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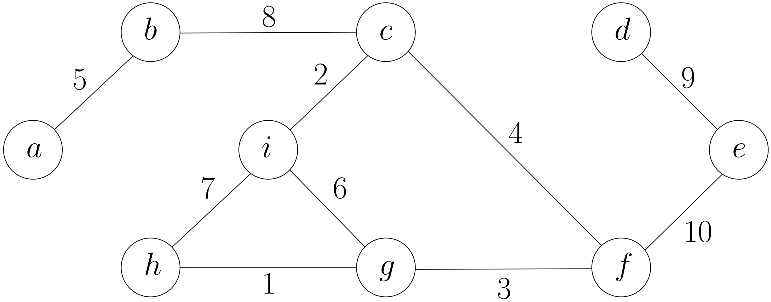

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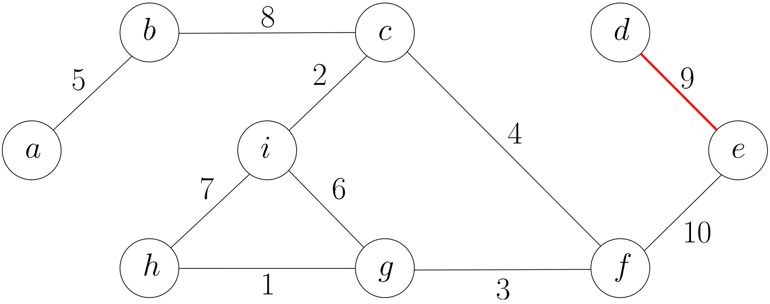

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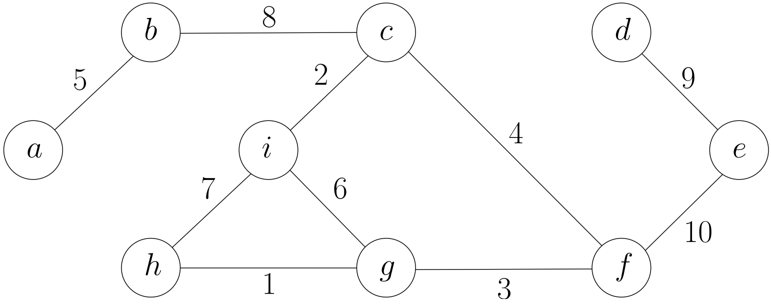

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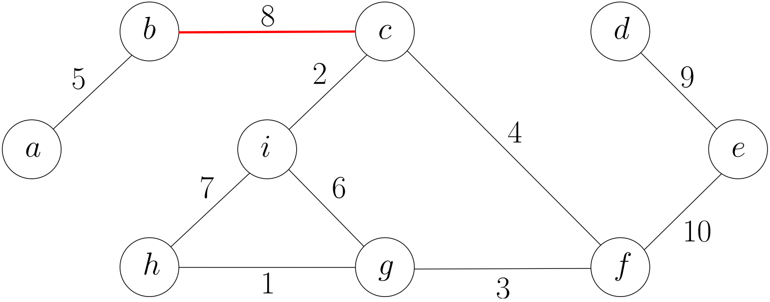

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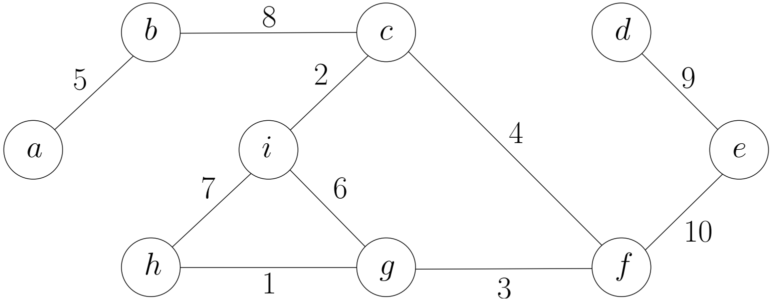

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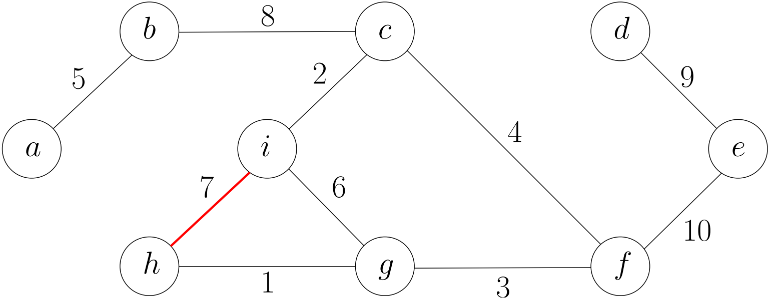

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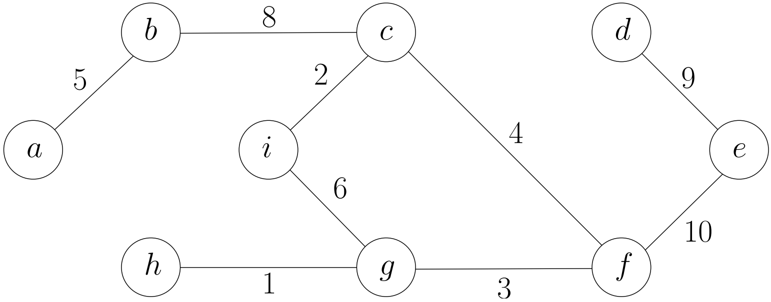

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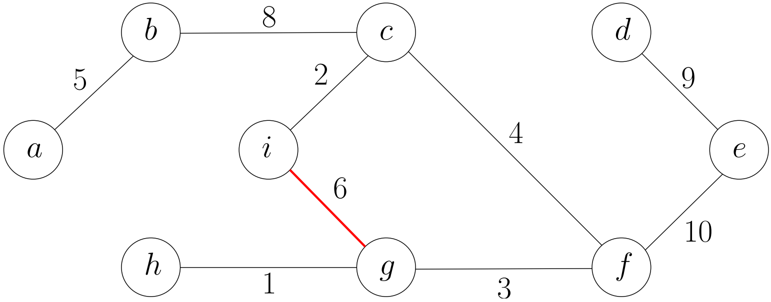

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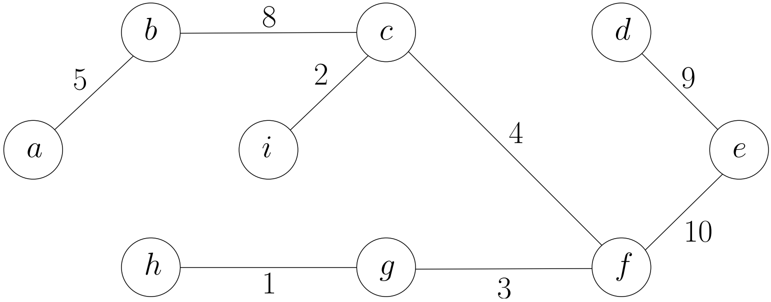
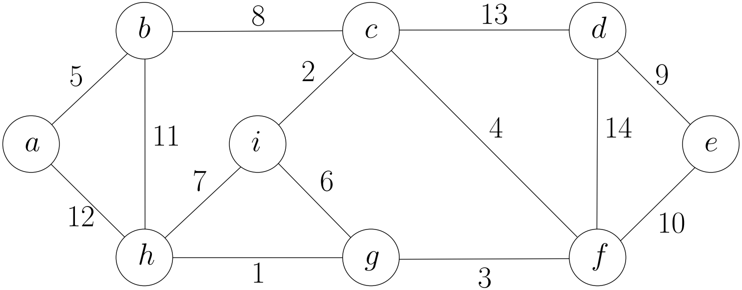


Outline

Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



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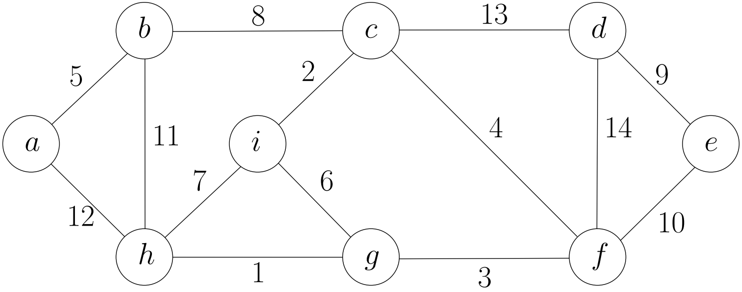

- Greedy strategy for Prim's algorithm: choose the lightest edge incident to a.

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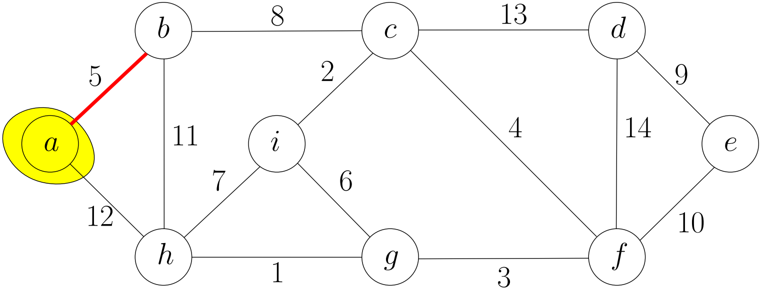

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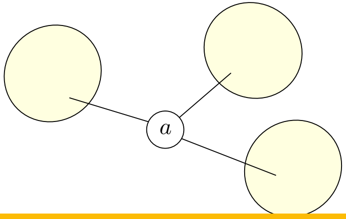
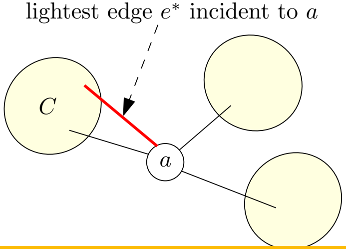


Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

Lemma It is safe to include the lightest edge incident to a.



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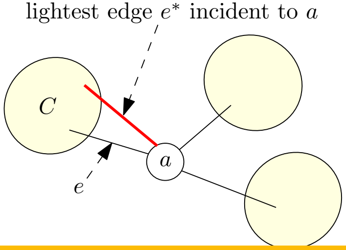

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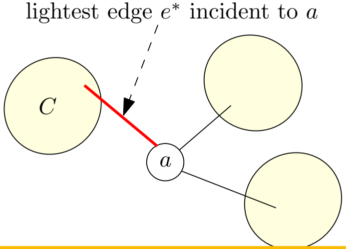

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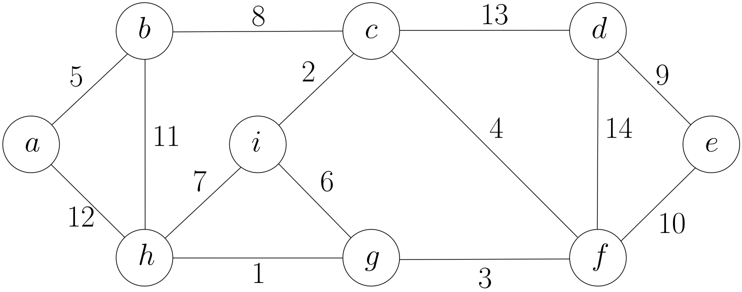
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus e \cup \{e^*\}$ is a spanning tree with $w(T') \le w(T)$ $\square$

Prim's Algorithm: Example



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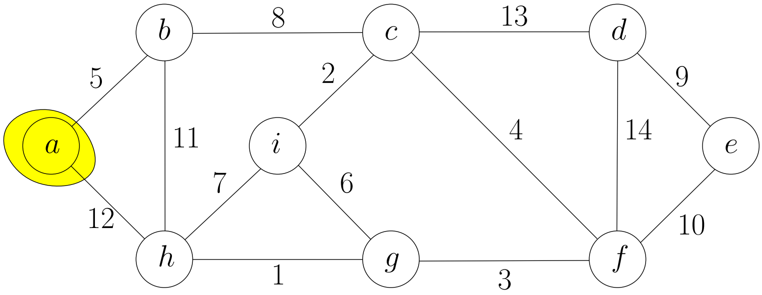

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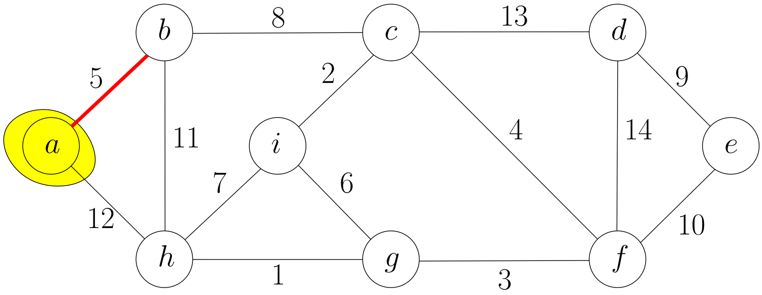

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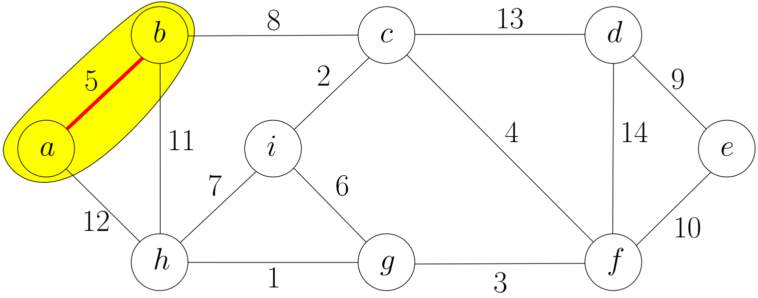

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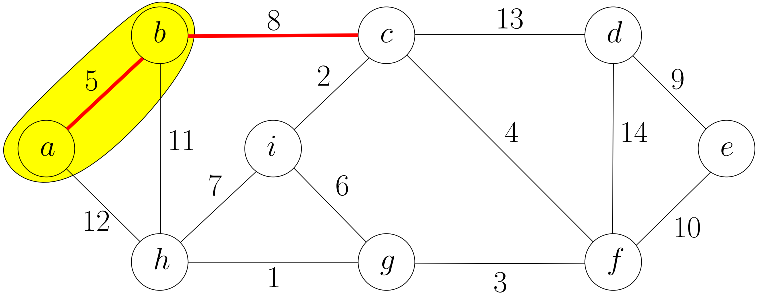

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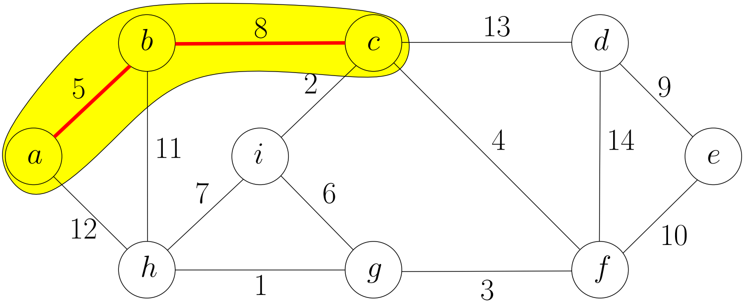

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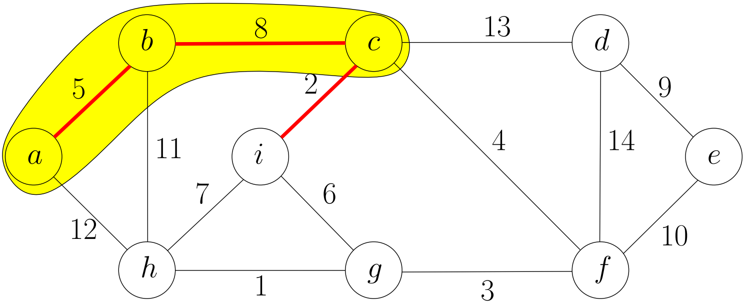

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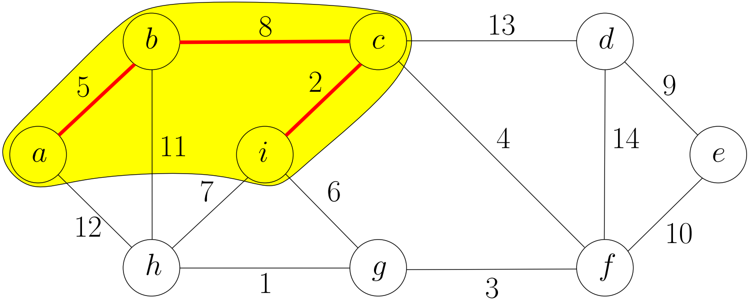

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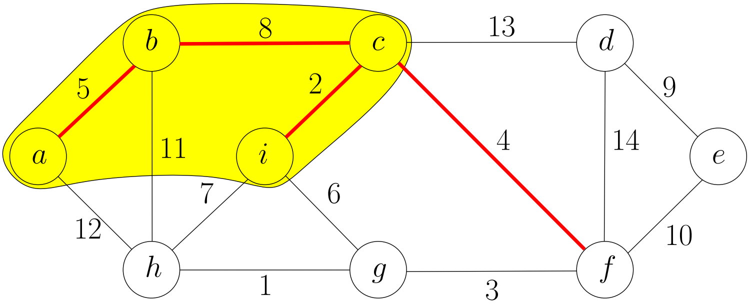

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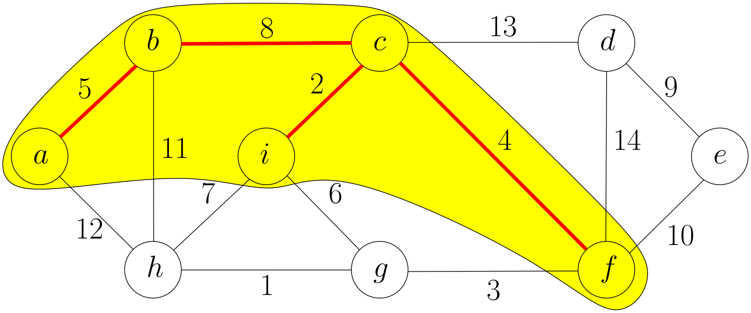

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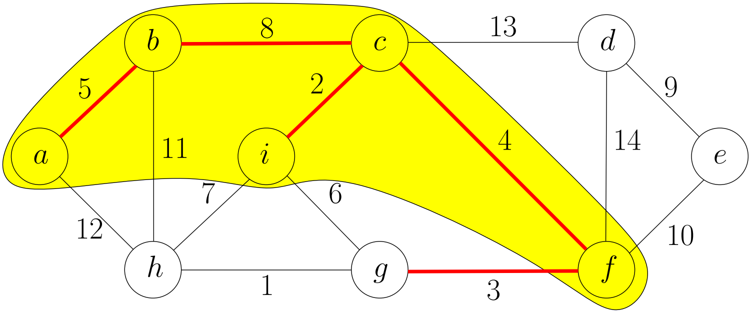

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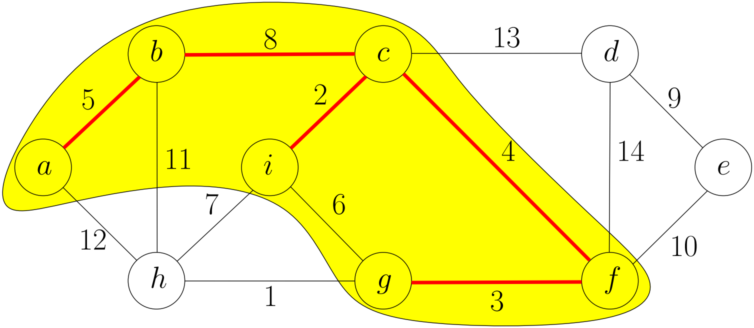

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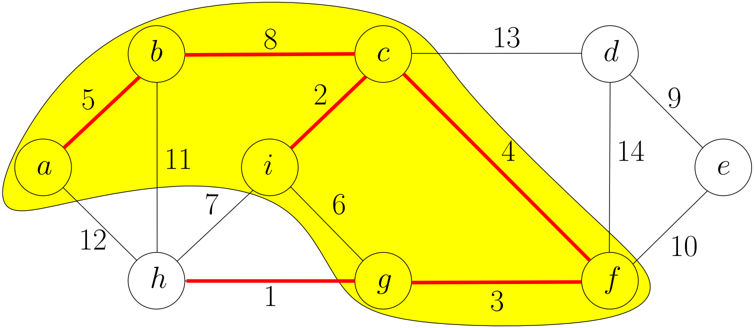

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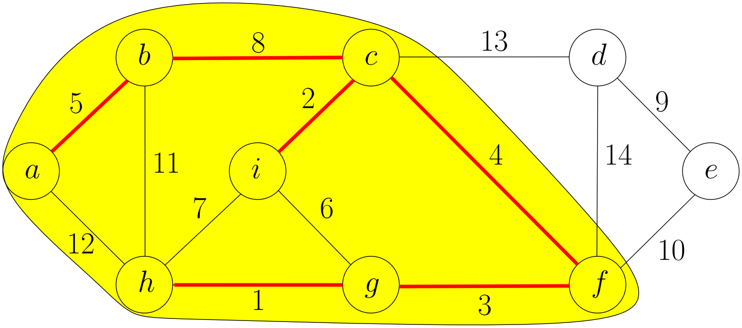

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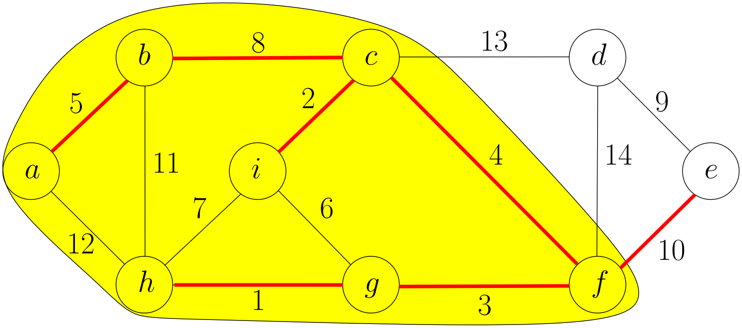

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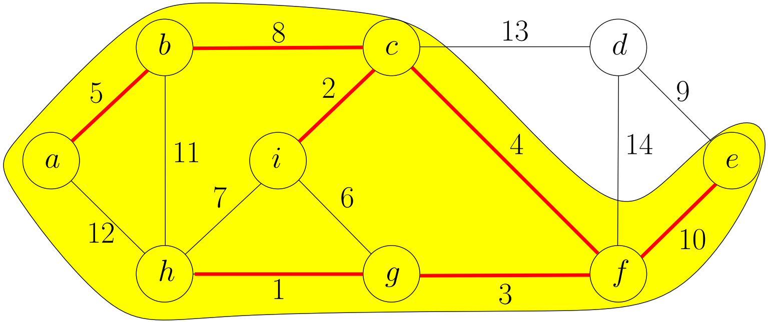

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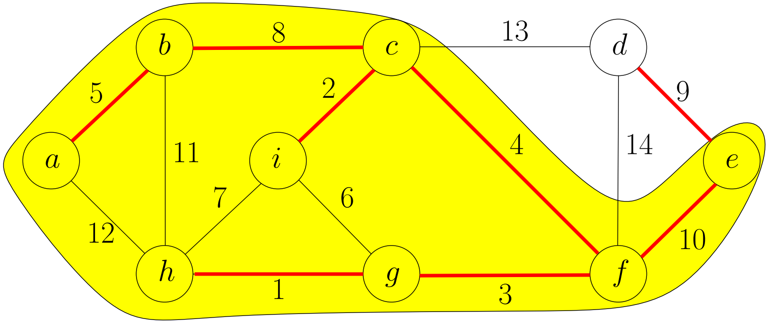

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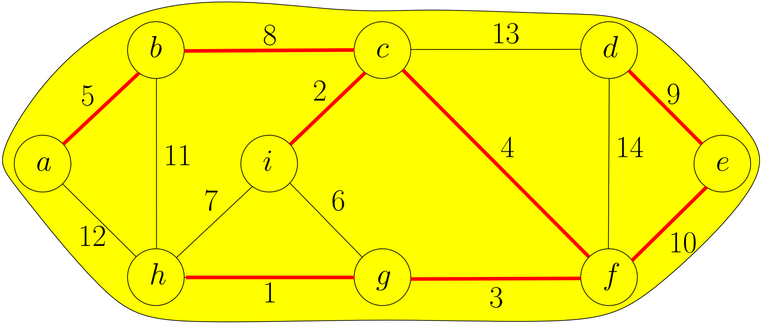

Greedy Algorithm

MST-Greedy1(G, w)

1. $S \leftarrow \{s\}$, where s is arbitrary vertex in V
2. $F \leftarrow \emptyset$
3. while $S \neq V$
4. $\quad (u, v) \leftarrow$ lightest edge between S and $V \setminus S$, where $u \in S$ and $v \in V \setminus S$
5. $\quad S \leftarrow S \cup \{v\}$
6. $\quad F \leftarrow F \cup \{(u, v)\}$
7. return (V, F)

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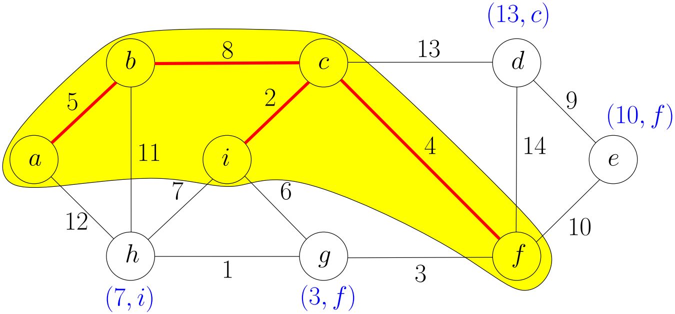
return (V, F)

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S



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  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

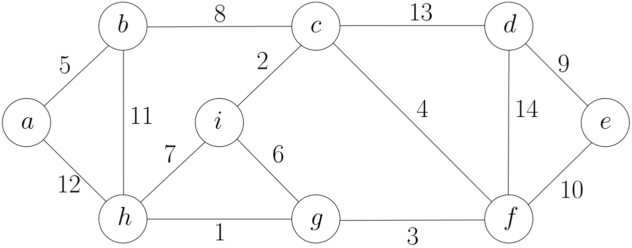
- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$, do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$
            $\pi(v) \leftarrow u$
return $\big\{(u, \pi(u)) | u \in V \setminus \{s\}\big\}$

Example



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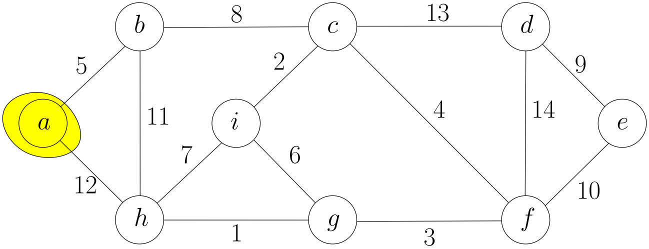

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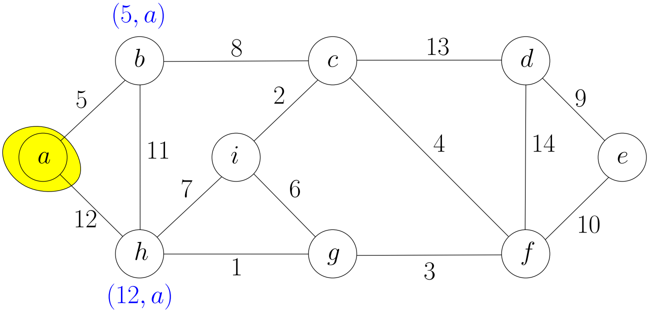

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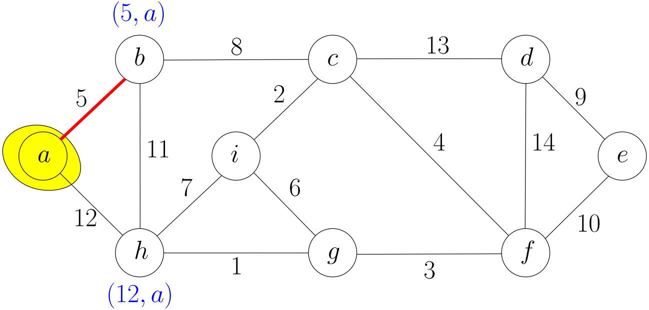

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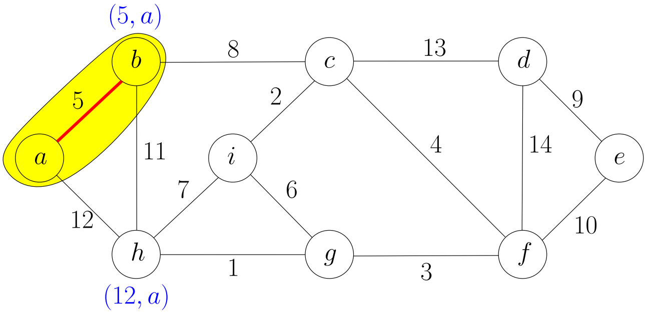

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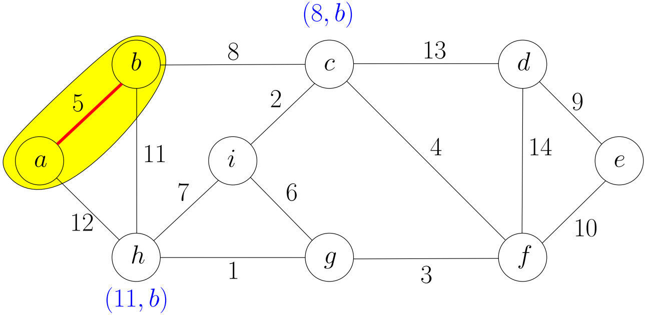

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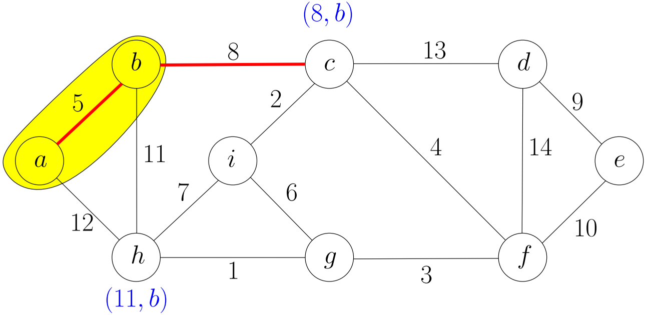

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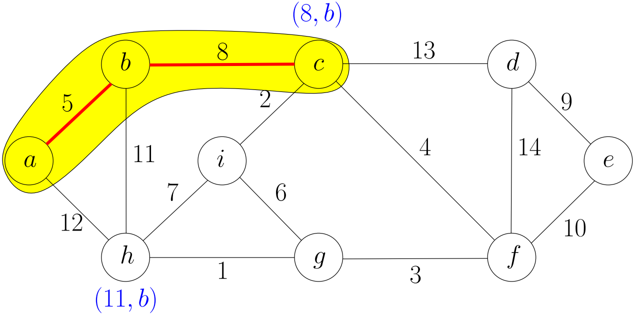

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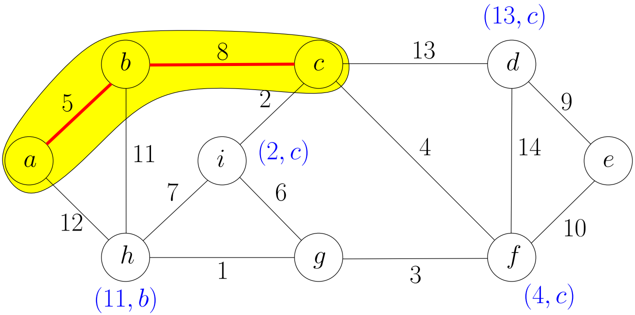

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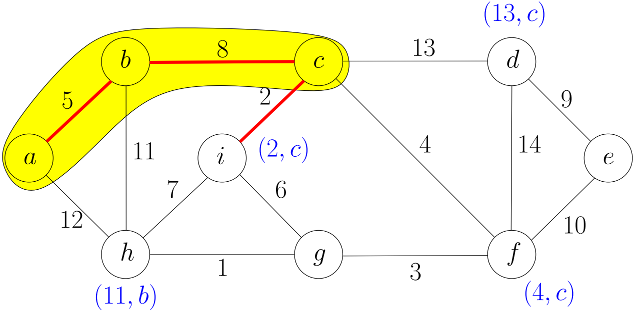

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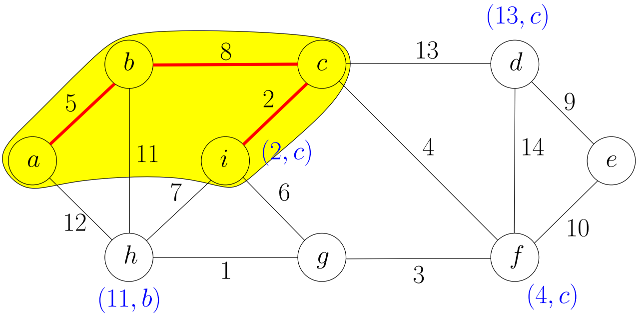

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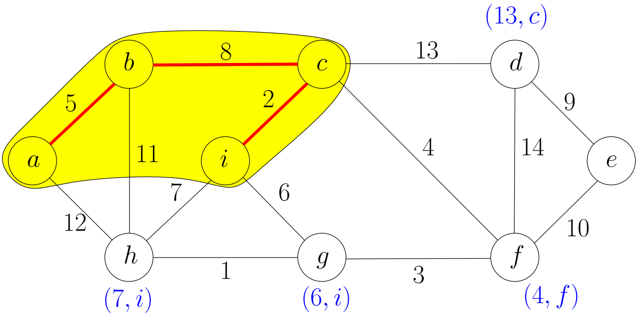

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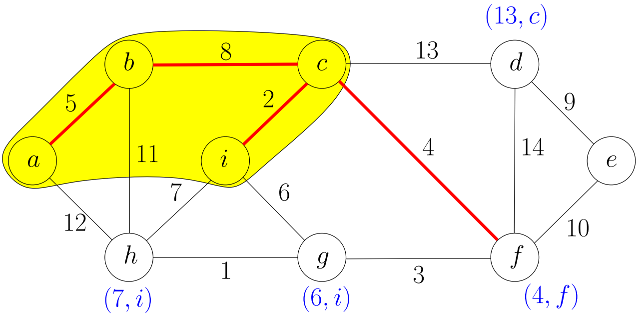

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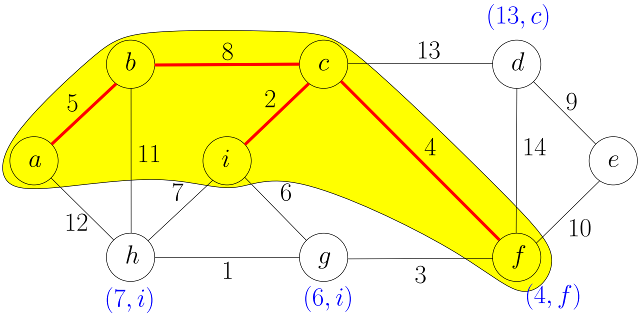

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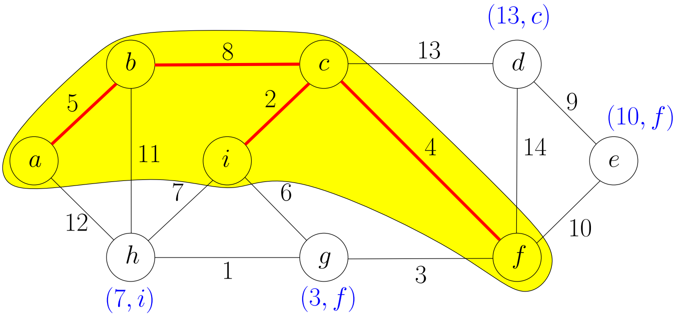

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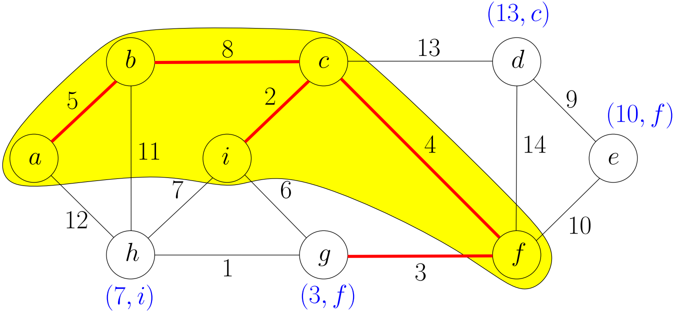

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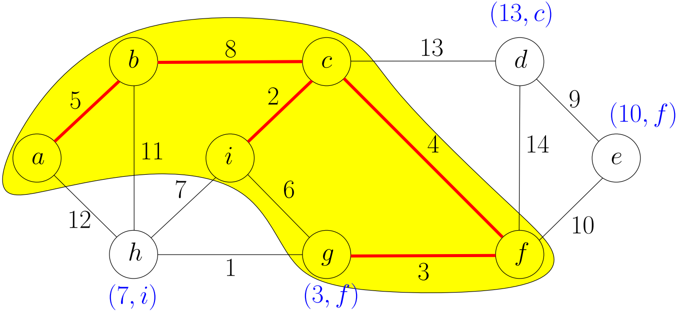

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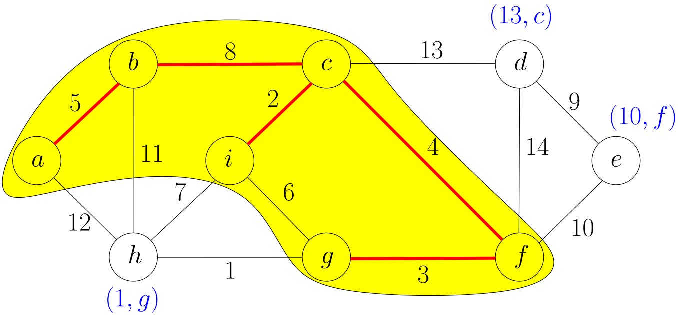
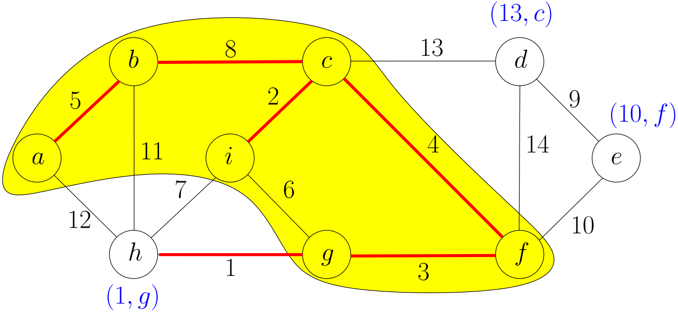


Example



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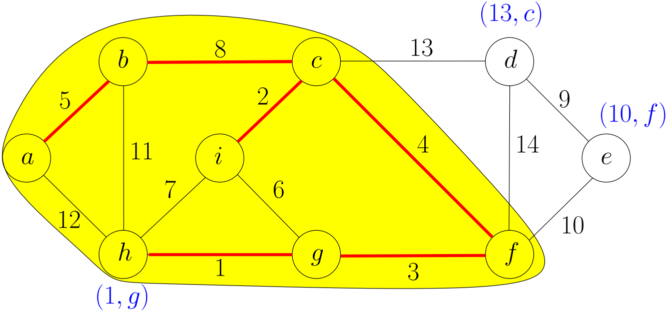
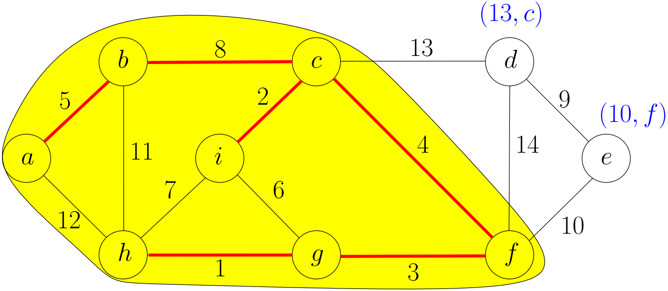


Example



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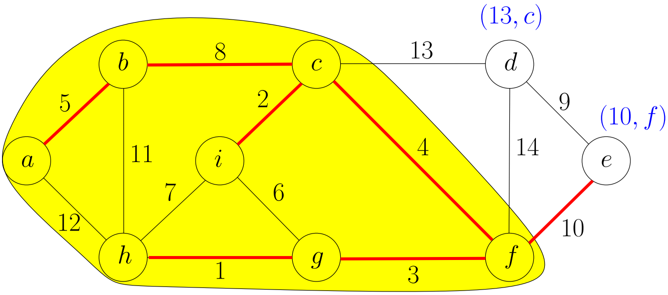

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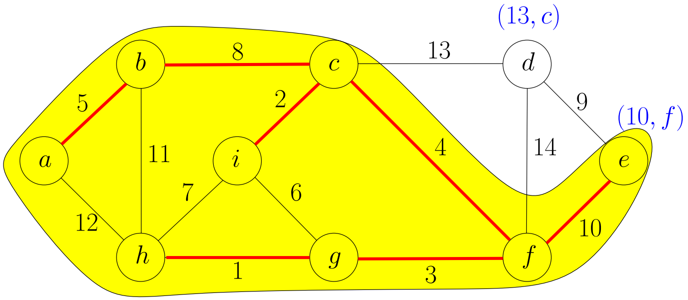

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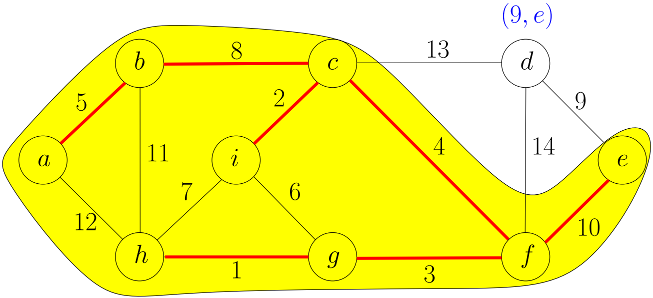

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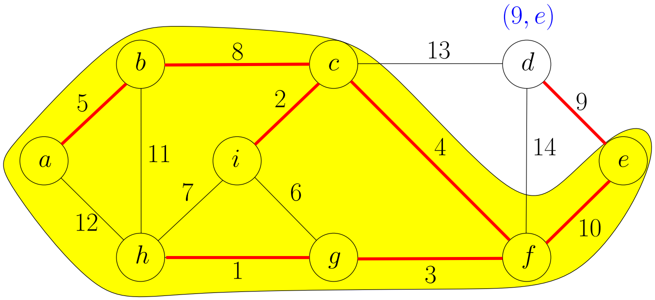
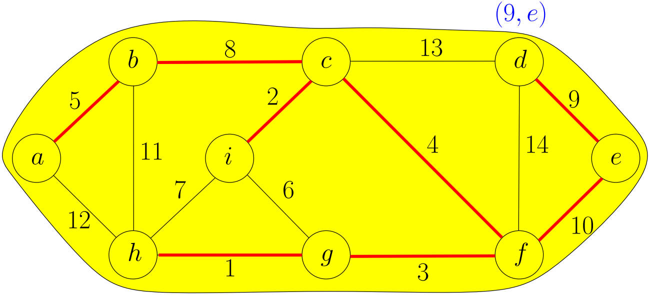


Example



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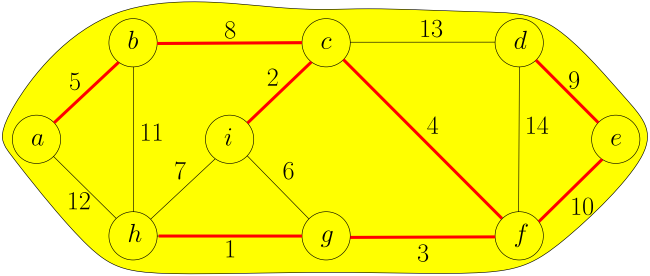

Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to F
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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value — extract_min
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values. — decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min(): return and remove the element in queue with the smallest key value
- $\cdots$

MST-Prim(G, w)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d(v) ← ∞ for every v ∈ V \ {s}
while S ≠ V, do
    u ← vertex in V \ S with the minimum d(u)
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    for each v ∈ V \ S such that (u, v) ∈ E
        if w(u, v) < d(v) then
            d(v) ← w(u, v)
            π(v) ← u
return {(u, π(u))|u ∈ V \ {s}}
```

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+\ O(m)\times$ (time for decrease_key)

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concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
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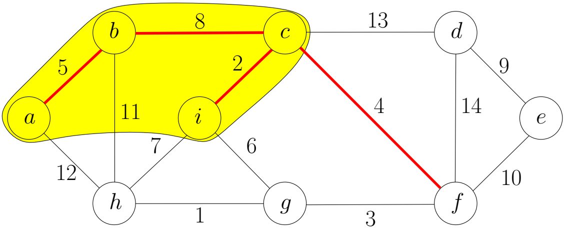

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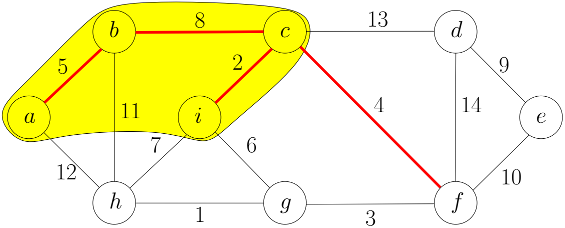

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

"Evidence" for $e \in$ MST or $e \notin$ MST

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- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
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Thus, the minimum spanning tree is unique with assumption.

Outline

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

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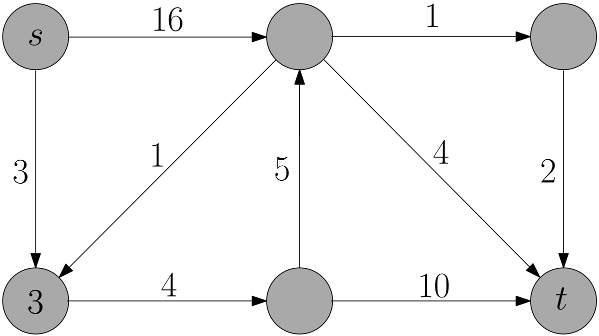

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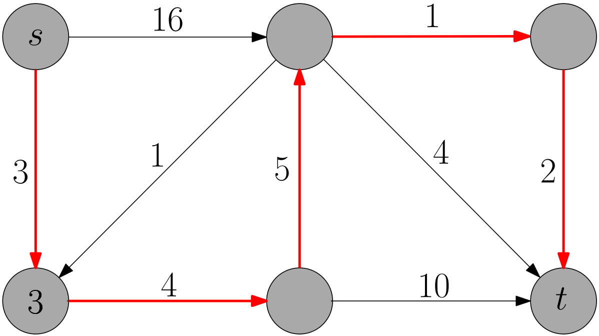

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Input: (directed or undirected) graph $G = (V, E)$, $s \in V$

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- Not acceptable if graph is sparse

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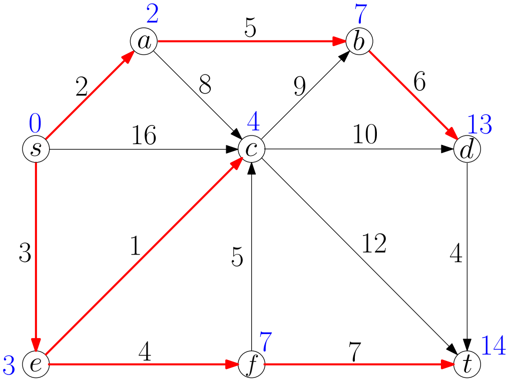

Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi(v), v \in V \setminus s$: the parent of v

$d(v), v \in V \setminus s$: the length of shortest path from s to v

Q: How to compute shortest paths from s when all edges have weight 1?

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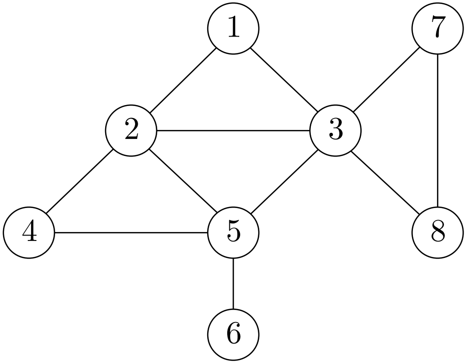

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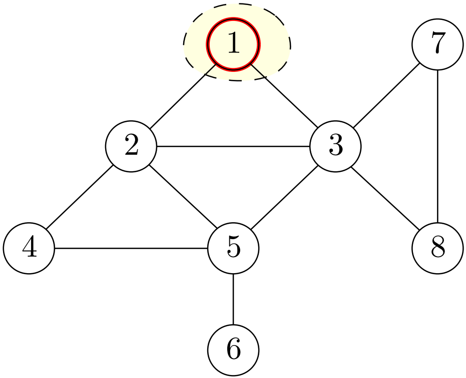

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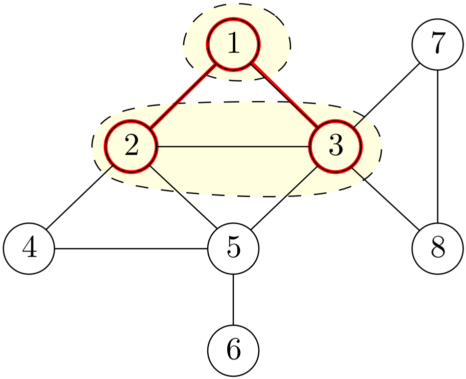

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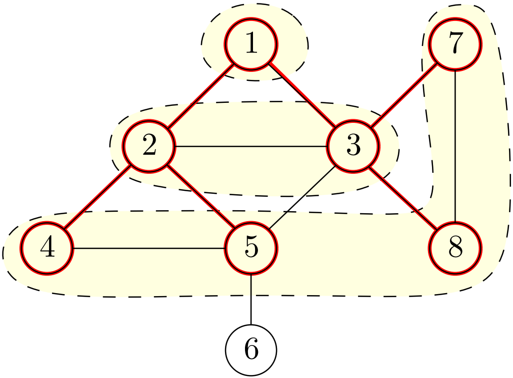

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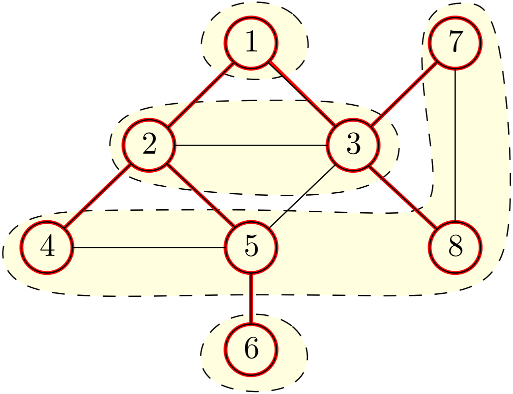

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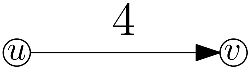

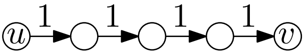

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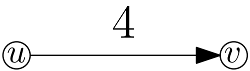

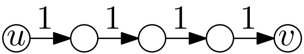

Shortest Path Algorithm by Running BFS

1. replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2. run BFS
3. $\pi(v) =$ vertex from which v is visited
4. $d(v) =$ index of the level containing v

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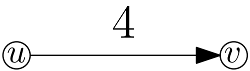
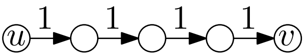




Shortest Path Algorithm by Running BFS

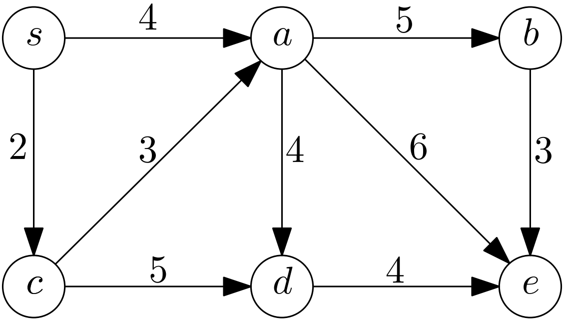
1. replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2. run BFS virtually
3. $\pi(v)$ = vertex from which v is visited
4. $d(v)$ = index of the level containing v

- Problem: $w(u,v)$ may be too large!

Shortest Path Algorithm by Running BFS Virtually

1. $S \leftarrow \{s\}, d(s) \leftarrow 0$
2. while $|S| \leq n$
3. find a $v \notin S$ that minimizes $\min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$
4. $S \leftarrow S \cup \{v\}$
5. $d(v) \leftarrow \min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$

Virtual BFS: Example



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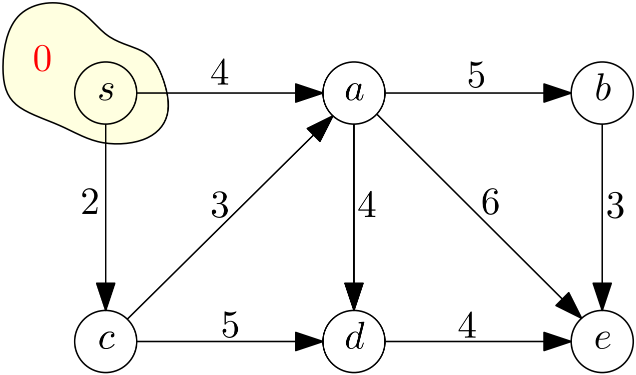

Time 0

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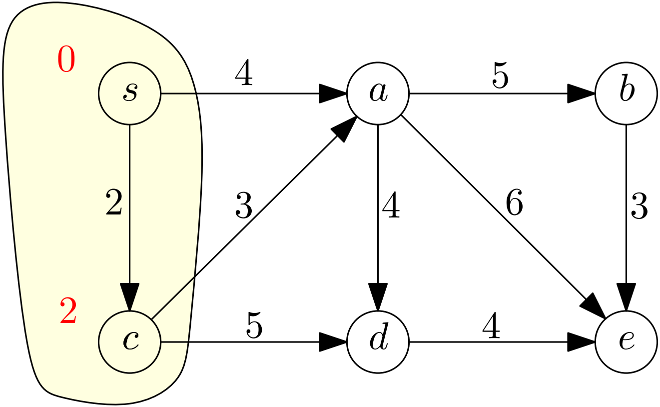
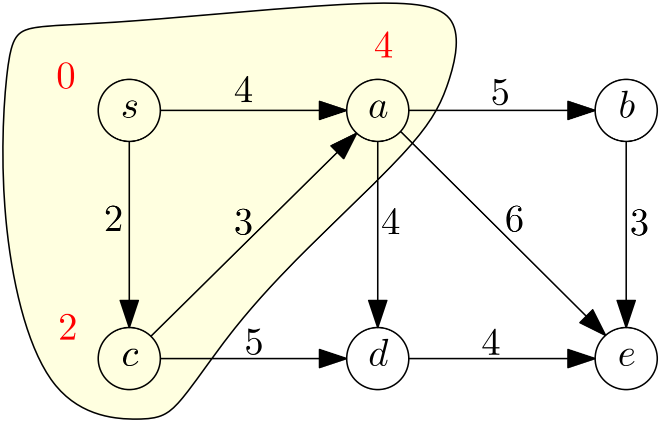


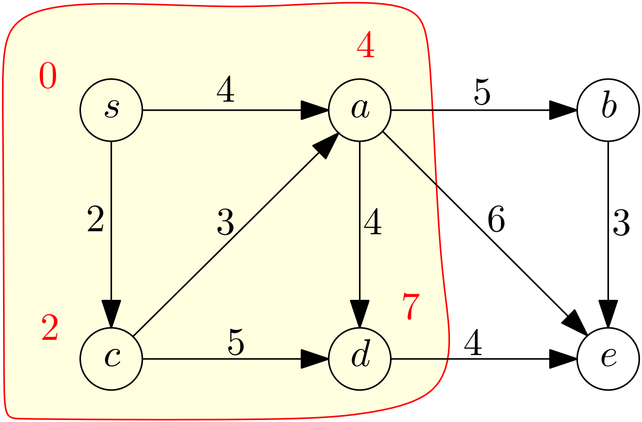
Time 2

Virtual BFS: Example



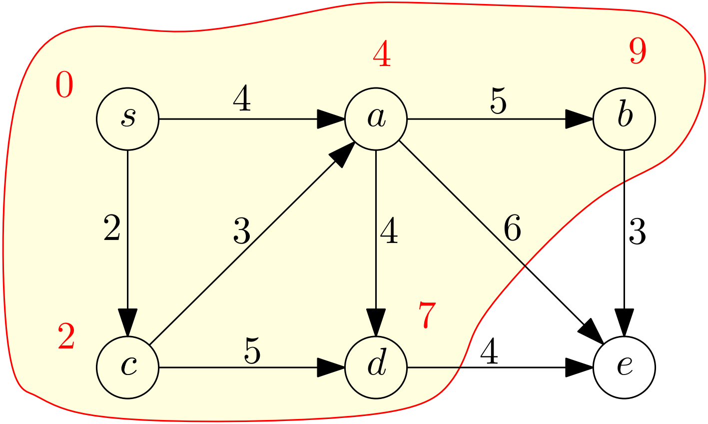
Time 4

Virtual BFS: Example



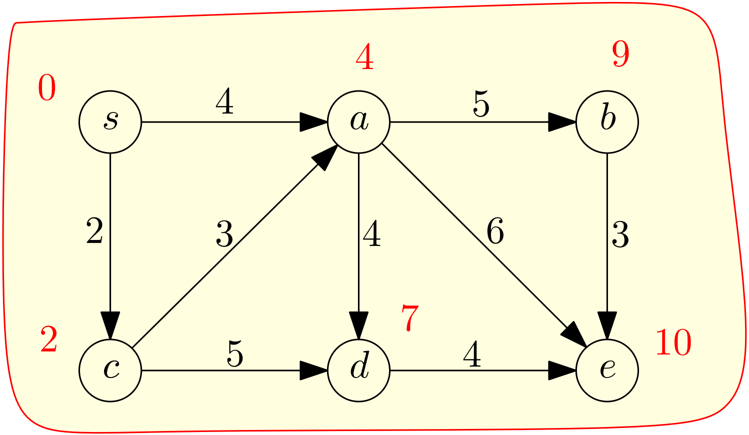
Time 7

Virtual BFS: Example



Time 9

Virtual BFS: Example



Time 10

Outline

Dijkstra's Algorithm

Dijkstra(G, w, s)

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    add u to S
    for each v ∈ V \ S such that (u, v) ∈ E
        if d(u) + w(u, v) < d(v) then
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return (d, π)
```

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while $S \neq V$ do

    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$

    add u to S

    for each $v \in V \setminus S$ such that $(u, v) \in E$

        if $d(u) + w(u, v) < d(v)$ then

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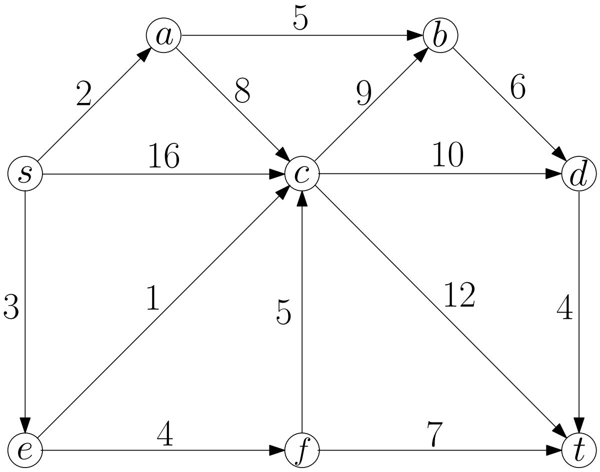
return (d, π)

- Running time $= O(n^2)$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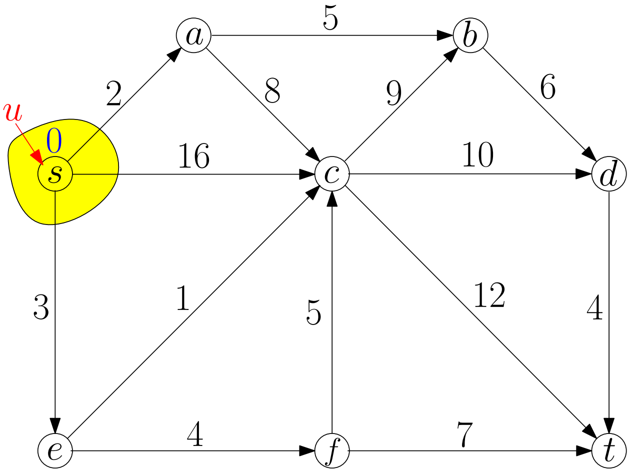

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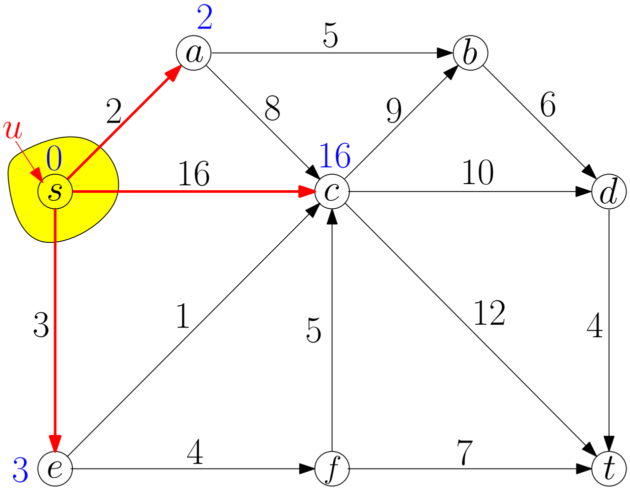
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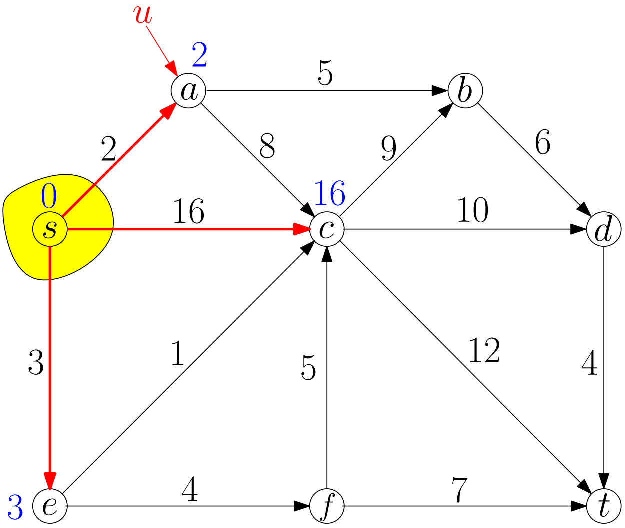

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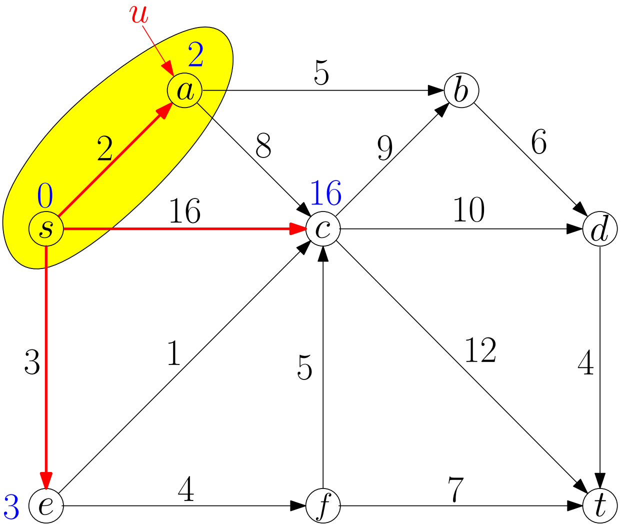

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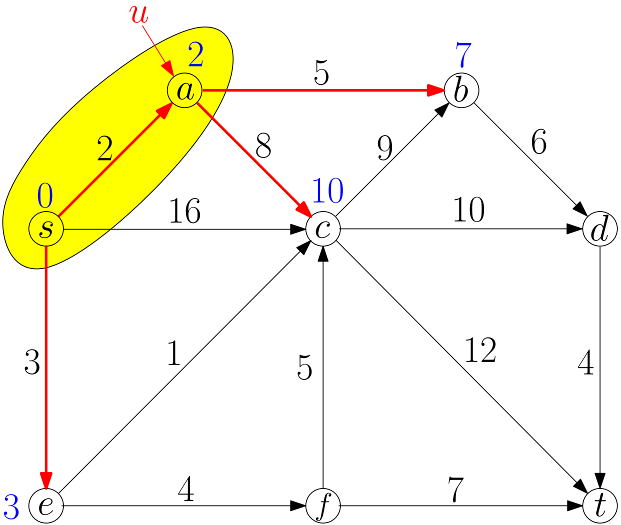
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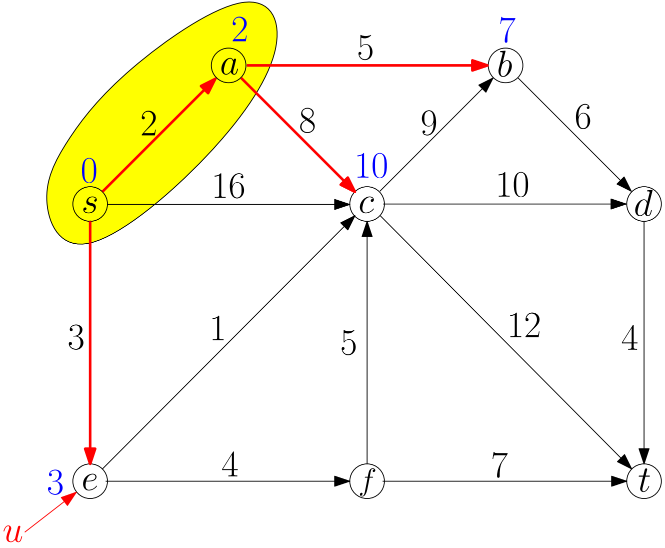
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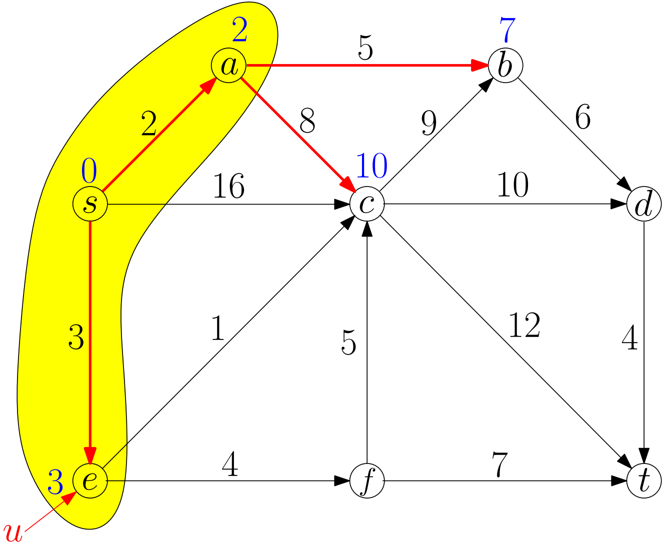
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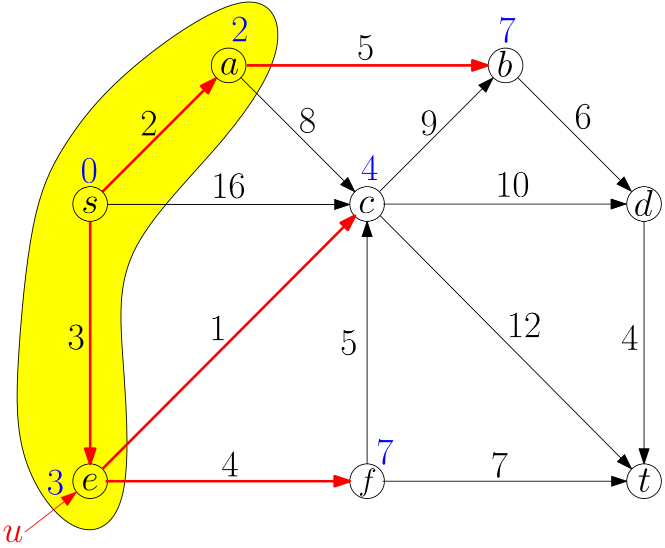

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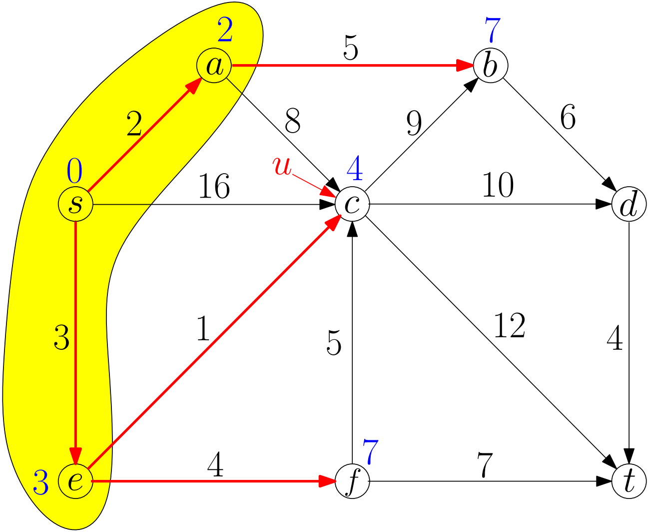
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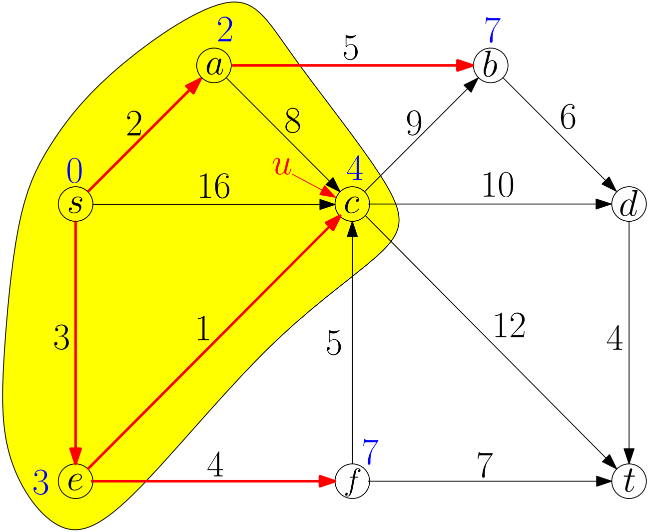
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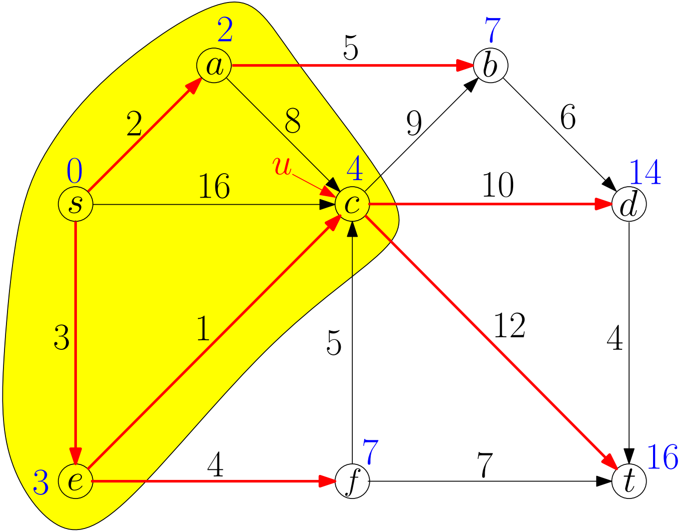
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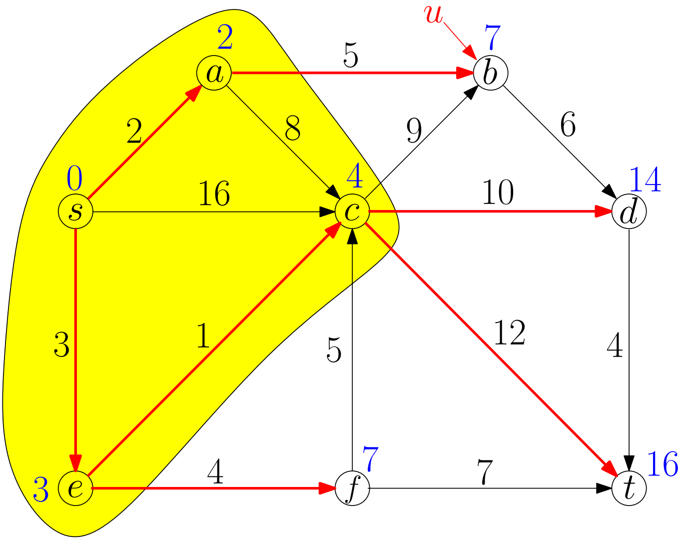
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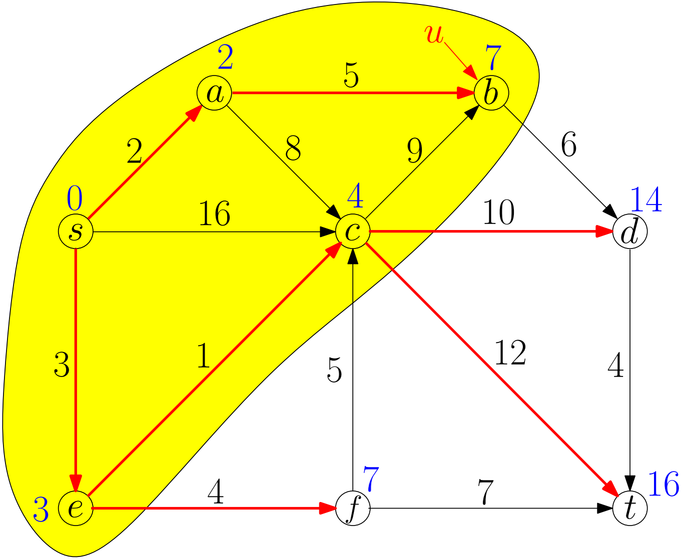
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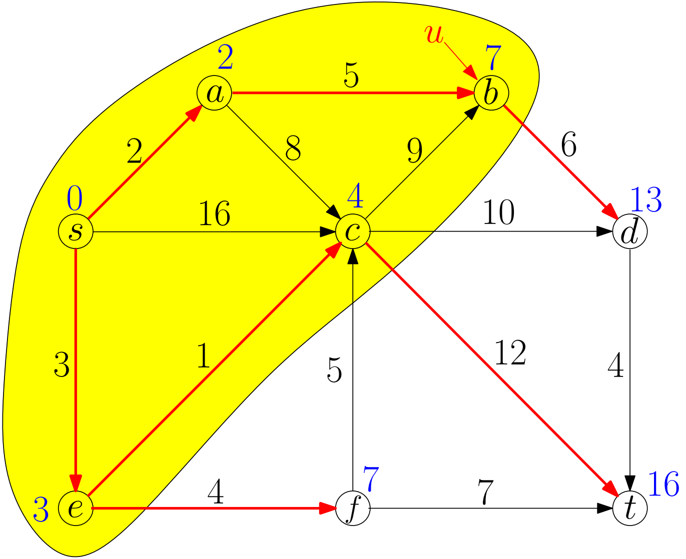
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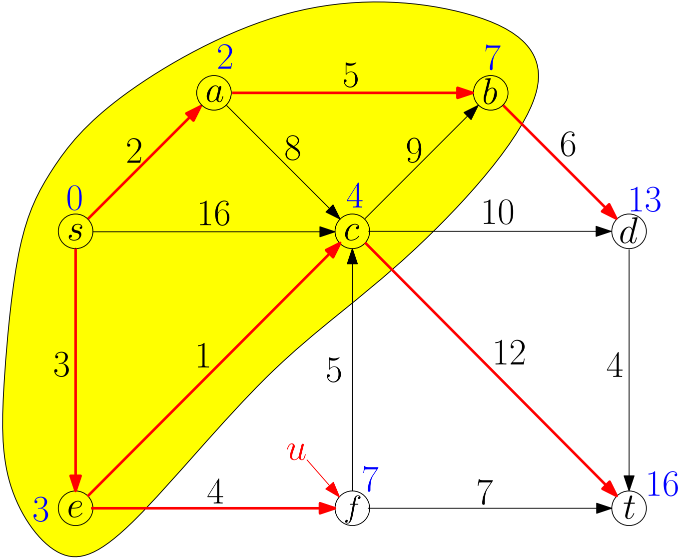

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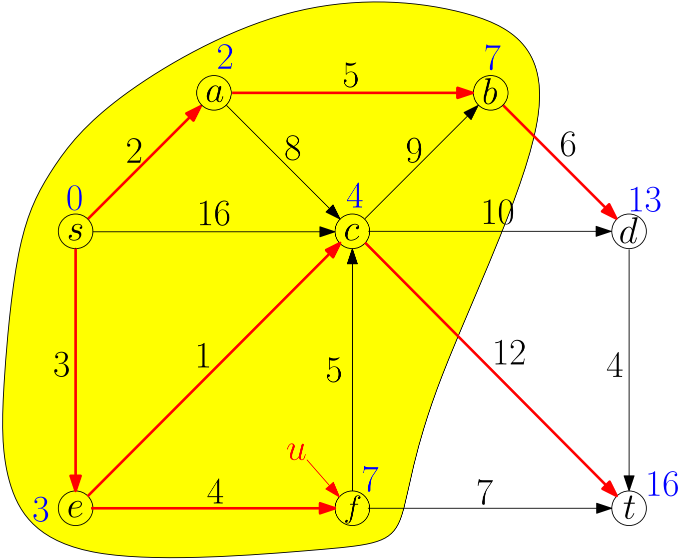
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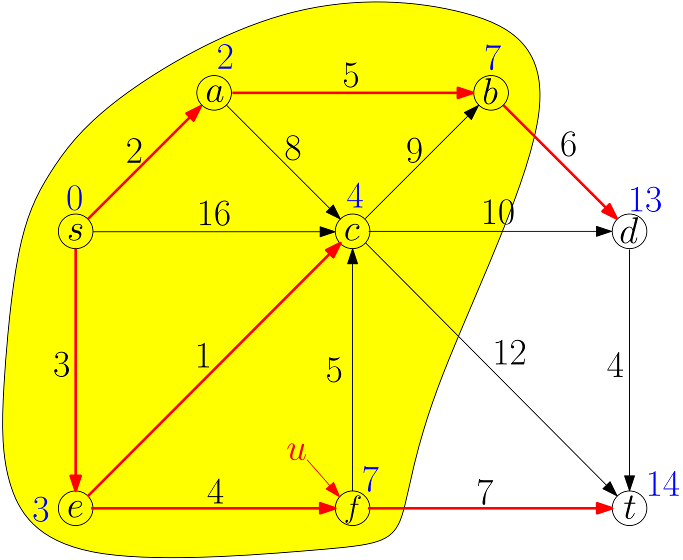

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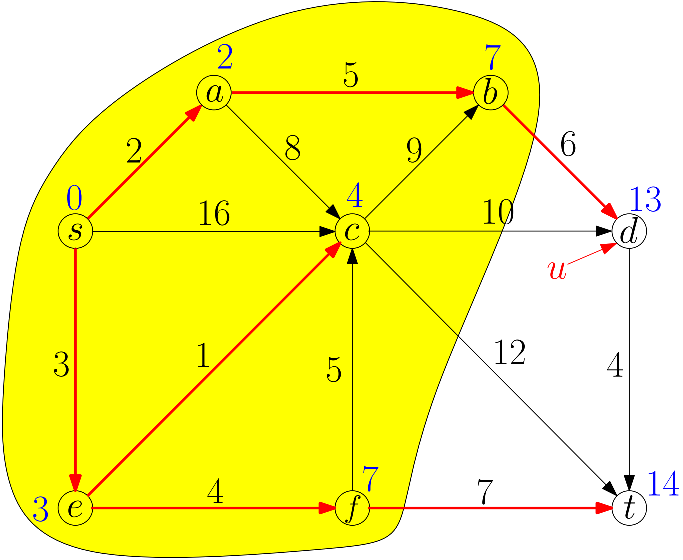

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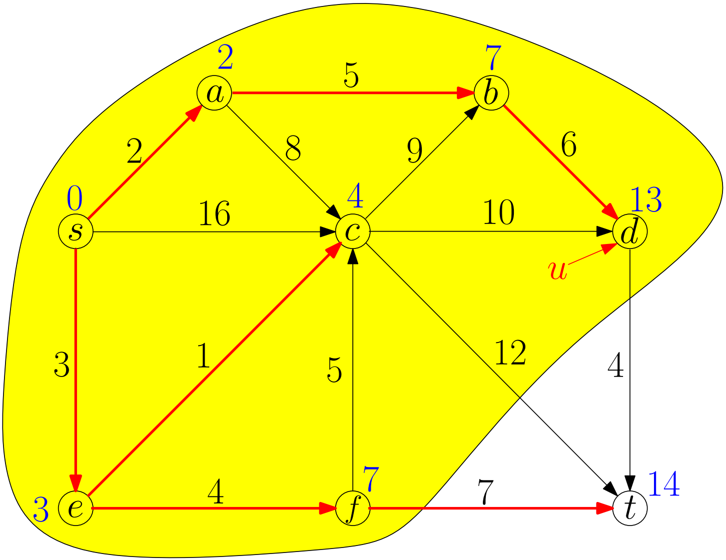
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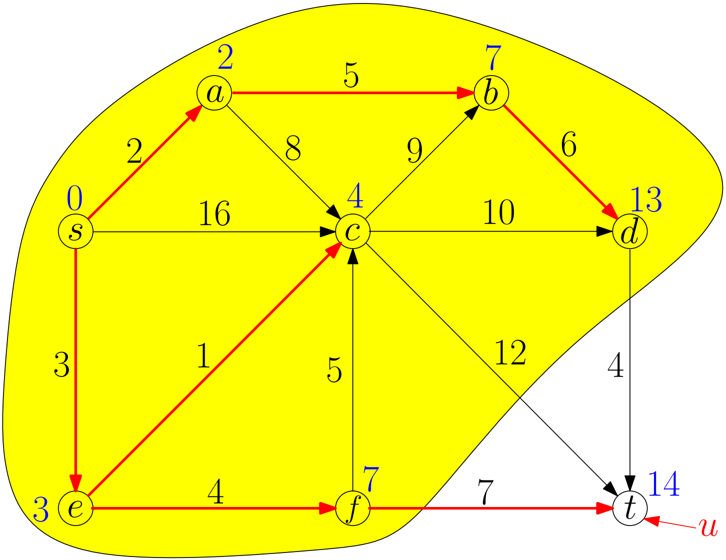

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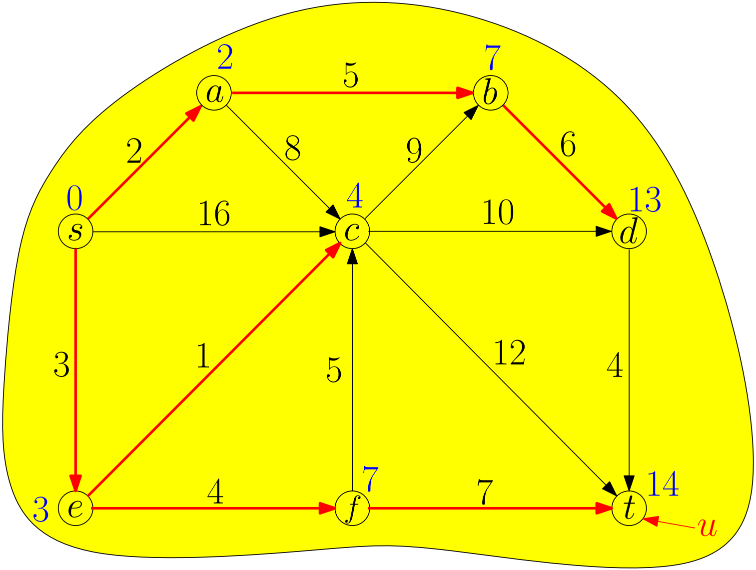

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Improved Running Time using Priority Queue

Dijkstra(G, w, s)

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$

$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$

while $S \neq V$, do

    $u \leftarrow Q$.extract_min()

    $S \leftarrow S \cup \{u\}$

    for each $v \in V \setminus S$ such that $(u, v) \in E$

        if $d(u) + w(u, v) < d(v)$ then

            $d(v) \leftarrow d(u) + w(u, v)$, Q.decrease_key$(v, d(v))$

            $\pi(v) \leftarrow u$

return (π, d)

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$

Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

Encoding Symbols Using Bits

- assume: 8 symbols a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per symbol

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \to 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per symbol

Q: What if some symbols appear more frequently than the others in expectation?

Q: If some symbols appear more frequently than the others in expectation, can we have a better encoding scheme?

A: Maybe. Using variable-length encoding scheme.

Idea

- using fewer bits for symbols that are more frequently used, and more bits for symbols that are less frequently used.

Need to use prefix codes to guarantee a unique decoding.

Prefix Codes

Def. A prefix code for a set S of symbols is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

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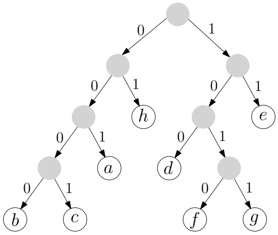
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



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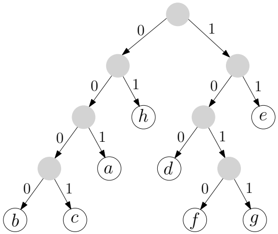

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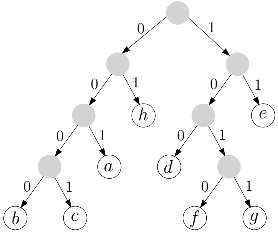

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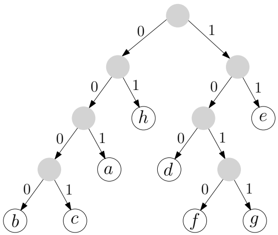

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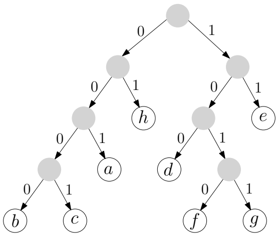

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e	f	g	h
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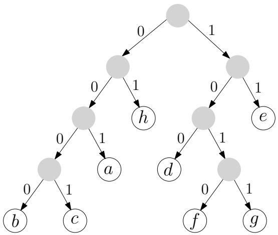

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a	b	c	d
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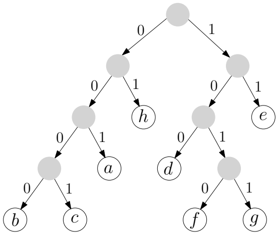
e	f	g	h
11	1010	1011	01



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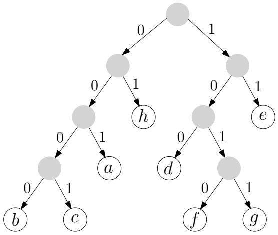

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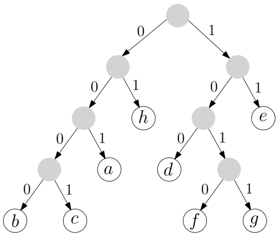

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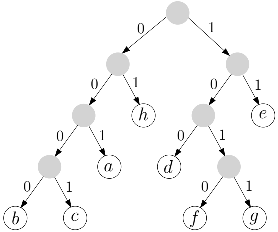

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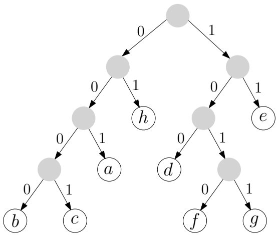

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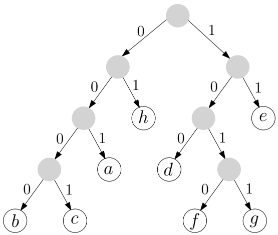

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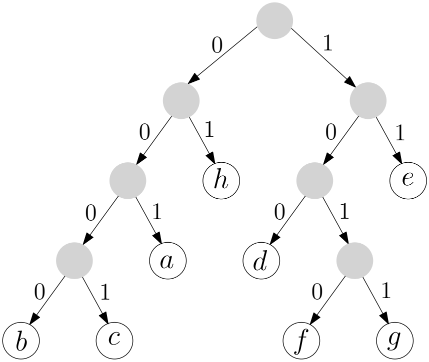
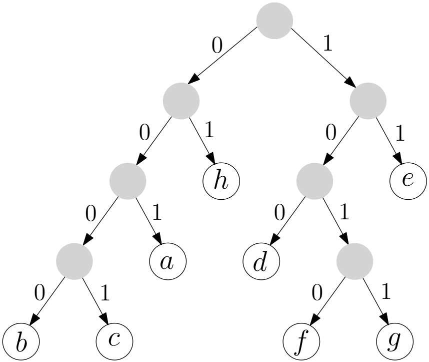
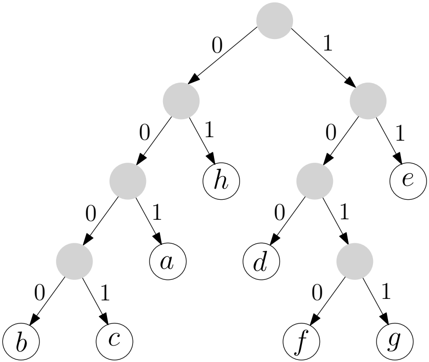


Properties of Encoding Tree



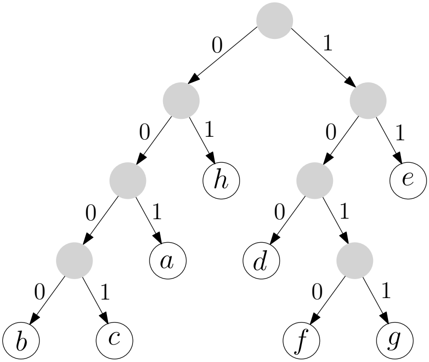
Properties of Encoding Tree

- Rooted binary tree



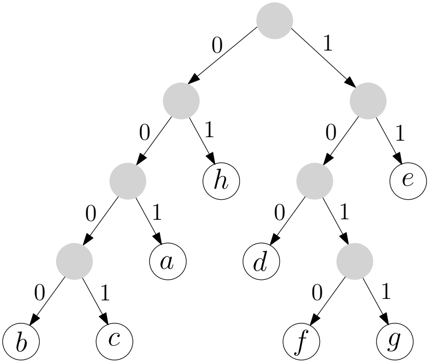
Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1



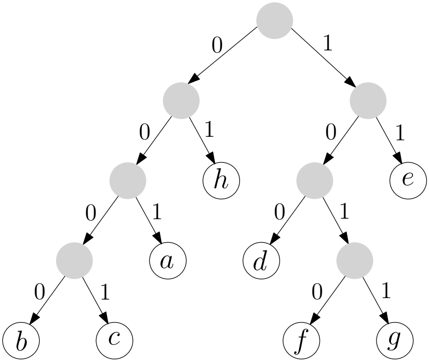
Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some symbol



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some symbol
- If coding scheme is not wasteful: a non-leaf has exactly two children



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some symbol
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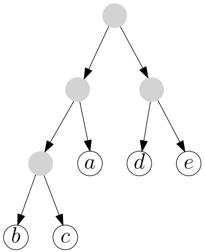
Best Prefix Codes

Input: frequencies of letters in a message

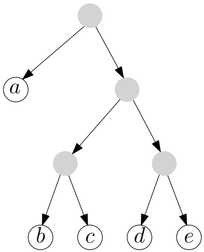
Output: prefix coding scheme giving the shortest encoding for the message

example

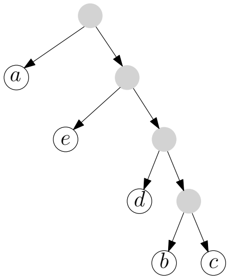
symbols	a	b	c	d	e	
frequencies	18	3	4	6	10	



scheme 1



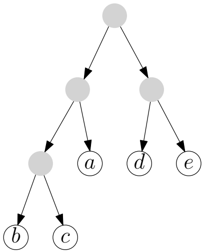
scheme 2



scheme 3

example

symbols	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total = 89
scheme 2 length	1	3	3	3	3	total = 87
scheme 3 length	1	4	4	3	2	total = 84



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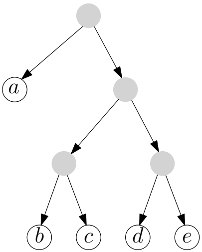

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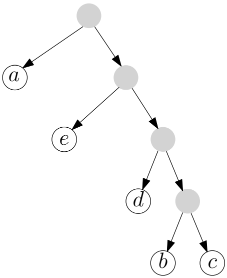

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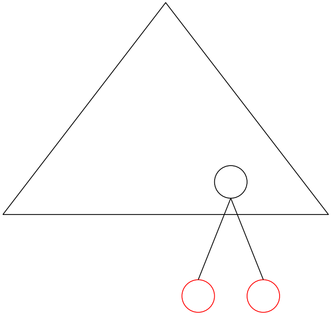
A: Choose two letters and make them brothers in the tree.

Which Two symbols Can Be Safely Put Together As Brothers?

- Focus a tree structure, without leaf labeling



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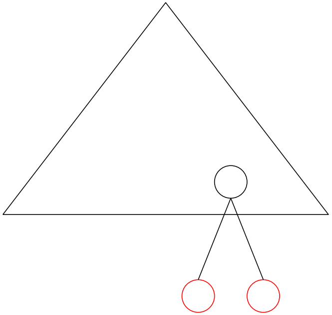

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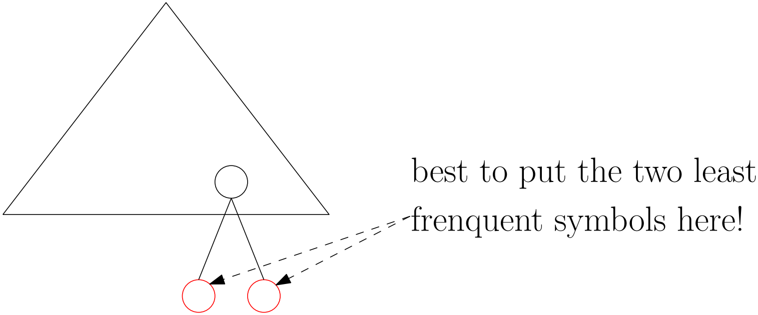

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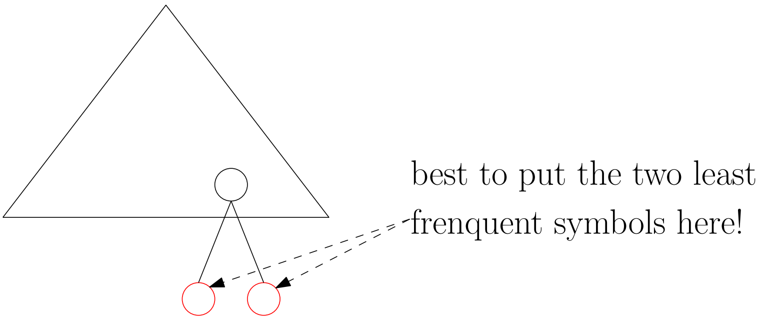
- Focus a tree structure, without leaf labeling
- There are two deepest leaves that are brothers
- It is safe to make the two least frequent symbols brothers!



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Q: Is the residual problem an instance of the best prefix codes problem?

A: Yes, although the answer is not immediate.

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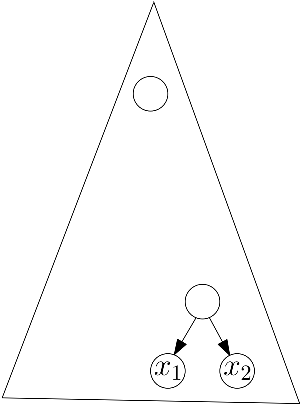
- x_1 and x_2: the two symbols we decided to put together.
- d_x the depth of symbol x in our output encoding tree.



$$\begin{aligned}&\sum_{x \in S} f_x d_x \\ =& \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\ =& \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}\end{aligned}$$

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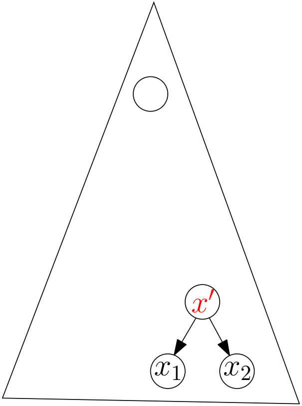

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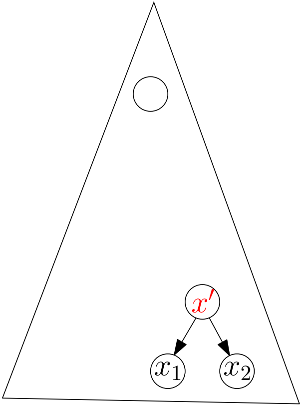

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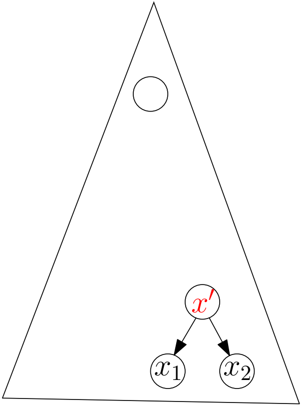

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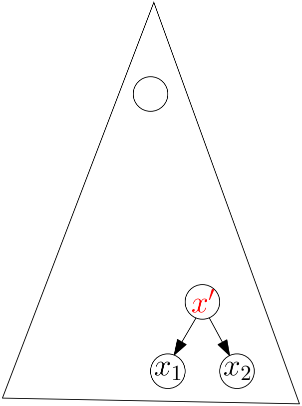

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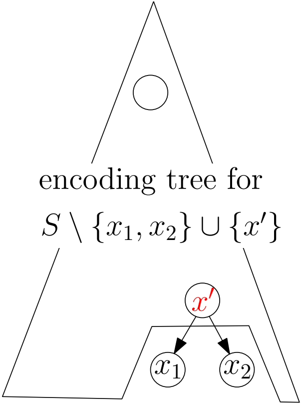

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$$

In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with symbols $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

Huffman(S, f)

if $|S| > 1$ **then**
    let x_1, x_2 be the two symbols with the smallest f values
    introduce a new symbol x' and let $f_{x'} = f_{x_1} + f_{x_2}$
    $S' \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
    call Huffman$(S', f|_{S'})$ to build an encoding tree T'
    let T be obtained from T' by adding x_1, x_2 as two children of x'
    return T
else
    let x be the symbol in S
    return a tree with a single node labeled x

Huffman codes: Iterative Algorithm

Huffman(S, f)

while $|S| > 1$ **do**
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    let x_1 and x_2 be the two children of x'
    $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
return the tree constructed

Example

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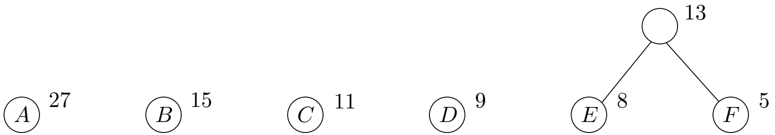

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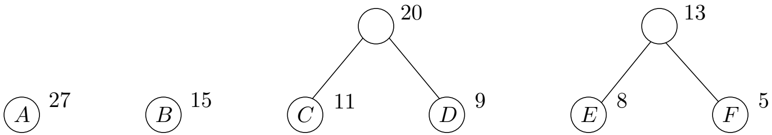

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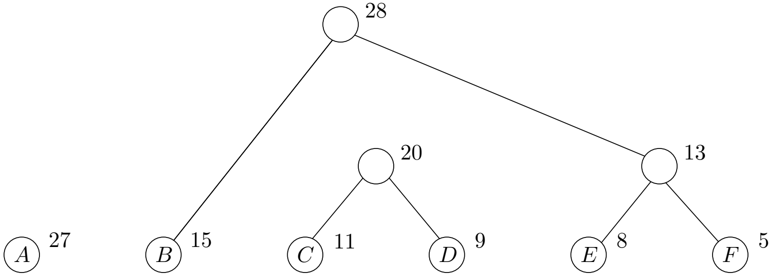

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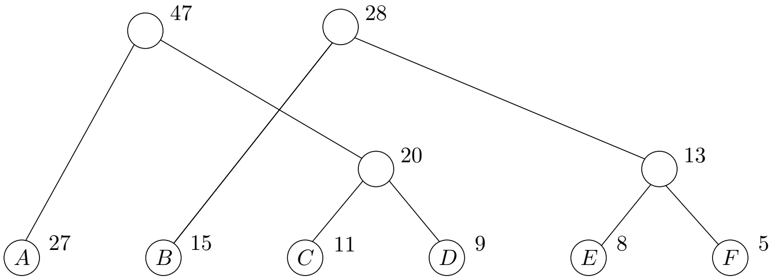

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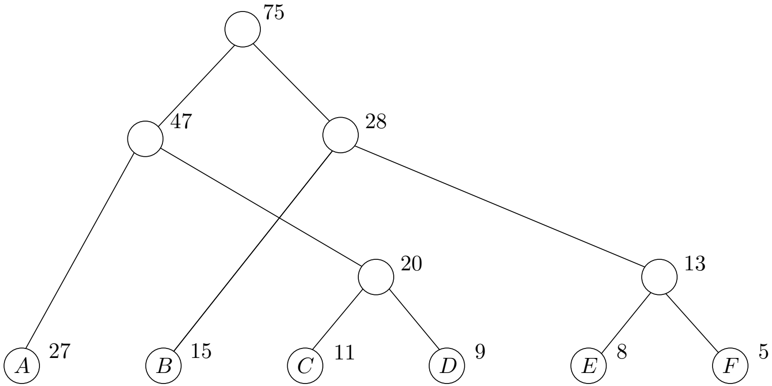

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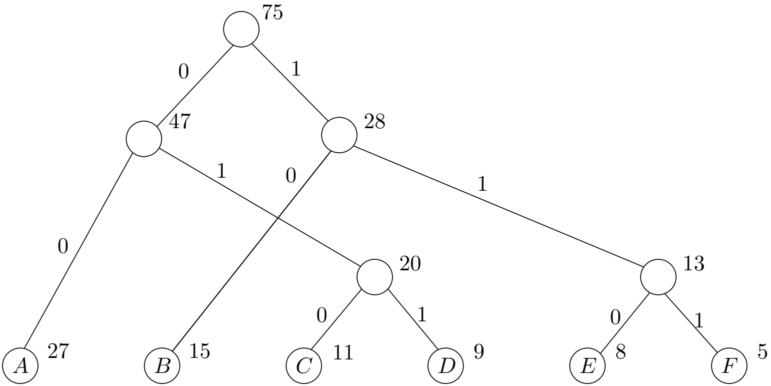

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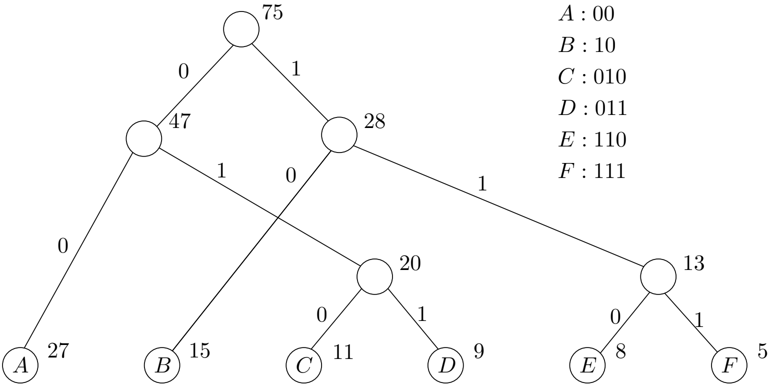

A : 00
B : 10
C : 010
D : 011
E : 110
F : 111

Algorithm using Priority Queue

Huffman(S, f)

```
Q ← build-priority-queue(S)
while Q.size > 1 do
    x_1 ← Q.extract-min()
    x_2 ← Q.extract-min()
    introduce a new symbol x' and let f_{x'} = f_{x_1} + f_{x_2}
    let x_1 and x_2 be the two children of x'
    Q.insert(x')
return the tree constructed
```

Outline

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- Dijkstra's algorithm is very similar to Prim's algorithm for MST