CSE 431/531: Algorithm Analysis and Design (Spring 2019)

Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo

Outline

1. Syllabus
2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

- Course Webpage (contains schedule, policies, homeworks and slides):
  http://www.cse.buffalo.edu/~shil/courses/CSE531/
- Please sign up course on Piazza via link on course webpage
  - announcements, polls, asking/answering questions

- Time and location:
  - MoWeFr, 9:00-9:50am
  - Alumni 97
- Instructor:
  - Shi Li, shil@buffalo.edu
  - Office hours: TBD via poll
- TA
  - Alexander Stachnik, ajstachn@buffalo.edu
  - Office hours: TBD via poll

You should already know:

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- Mathematical Tools
  - Mathematical inductions
  - Probabilities and random variables

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  - Probabilities and random variables
- Basic data Structures
  - Stacks, queues, linked lists
- Some Programming Experience
  - C, C++, Java or Python

You Will Learn

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  - Sorting
  - Shortest paths
  - Minimum spanning tree

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  - Greedy algorithms
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  - Dynamic programming
  - Linear Programming

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- NP-completeness

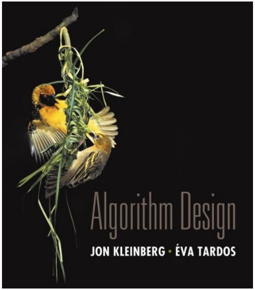
Tentative Schedule (42 Lectures)

Introduction	3 lectures
Basic Graph Algorithms	3 lectures
Greedy Algorihtms	6 lectures, 1 recitation
Divide and Conquer	6 lectures, 1 recitation
Dynamic Programming	6 lectures, 1 recitation
	1 recitation for homeworks
Mid-Term Exam	Apr 7, 2019, Mon
NP-Completeness	6 lectures, 1 recitation
Linear Programming	4 lectures
	2 recitations for homeworks
Final review, Q&A	
Final Exam	May 15 2019, Wed, 8:00am-11:00am

Textbook

Textbook (Highly Recommended):

- Algorithm Design, 1st Edition, by
  Jon Kleinberg and *Eva Tardos*



Other Reference Books

- Introduction to Algorithms, Third Edition, *Thomas Cormen, Charles Leiserson, Rondald Rivest, Clifford Stein*

Reading Before Classes

- Highly recommended: read the correspondent sections from the textbook (or reference book) before classes
- Slides and example problems for recitations will be posted online before class

Grading

- 40% for homeworks
  - 5 homeworks, 3 of which have programming tasks
- 60% for mid-term + final exams, score for two exams is

$$\max\{M \times 20\% + F \times 40\%, M \times 30\% + F \times 30\%\}$$
$$M, F \in [0, 100]$$

For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussions
  - Must write down solutions on your own, in your own words
  - Write down names of students you collaborated with

For Homeworks, You Are Not Allowed to

- Use external resources
  - Can't Google or ask questions online for solutions
  - Can't read posted solutions from other algorithm course webpages
- Copy solutions from other students

For Programming Problems

- Need to implement the algorithms by yourself
- Can not copy codes from others or the Internet
- We use Moss (`https://theory.stanford.edu/~aiken/moss/`) to detect similarity of programs

Late Policy

- You have 1 "late credit", using it allows you to turn in a homework late for three days
- With no special reasons, no other late submissions will be accepted

- Mid-Term and Final Exam will be closed-book
- Per Departmental Policy on Academia Integrity Violations, penalty for AI violation is:
  - “F” for the course
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Questions?

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- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
- Computational problem: specifies the input/output relationship.
- An algorithm solves a computational problem if it produces the correct output for any given input.

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

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- Input: 210, 270
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- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$
- $(270, 210) \to (210, 60) \to (60, 30) \to (30, 0)$

Examples

Sorting

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

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- Algorithms: insertion sort, merge sort, quicksort, ...

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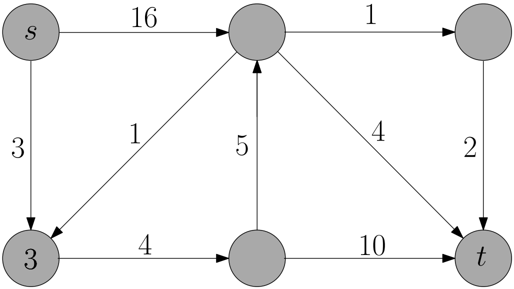

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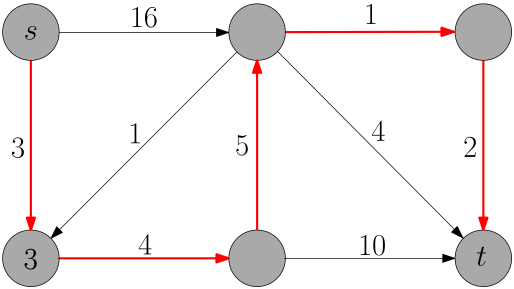

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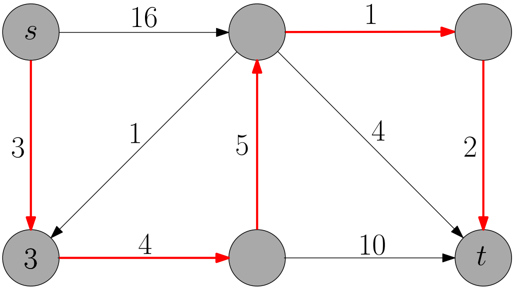

- Algorithm: Dijkstra's algorithm

Algorithm = Computer Program?

- Algorithm: "abstract", can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: "concrete", implementation of algorithm, associated with a particular programming language

Pseudo-Code

Pseudo-Code:

Euclidean(a, b)

1. while $b > 0$
2. $\quad (a, b) \leftarrow (b, a \bmod b)$
3. return a

C++ program:

```
int Euclidean(int a, int b){
    int c;
    while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```

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  4. it is fun!

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Insertion-Sort

- At the end of j-th iteration, the first j numbers are sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
- Output: $12, 15, 21, 35, 53, 59$

insertion-sort(A, n)

```
for j ← 2 to n
    key ← A[j]
    i ← j − 1
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Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

$$\text{after } j = 3 : 12, 35, 53, 21, 59, 15$$

$$\text{after } j = 4 : 12, 21, 35, 53, 59, 15$$

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$$\text{after } j = 6 : 12, 15, 21, 35, 53, 59$$

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- Q: How fast is the computer?
- Q: Programming language?
- A: Important idea: asymptotic analysis
  - Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

- Ignoring lower order terms
- Ignoring leading constant

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- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $2^{n/3+100} + 100n^{100} \Rightarrow 2^{n/3+100} \Rightarrow 2^{n/3}$

Asymptotic Analysis: O-notation

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Asymptotic Analysis: O-notation

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O-notation allows us to

- ignore architecture of computer
- ignore programming language

Asymptotic Analysis of Insertion Sort

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- Worst-case running time for iteration j in the outer loop?
  Answer: $O(j)$
- Total running time $= \sum_{j=2}^{n} O(j) = O(n^2)$ (informal)

Computation Model

- Random-Access Machine (RAM) model: read $A[j]$ takes $O(1)$ time.
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- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort, heap sort, ...

- Remember to sign up for Piazza.

Questions?

Outline

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

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- We only consider asymptotically positive functions.

O-Notation: Asymptotic Upper Bound

O-**Notation** For a function $g(n)$,

$$O(g(n)) = \{\text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\}.$$

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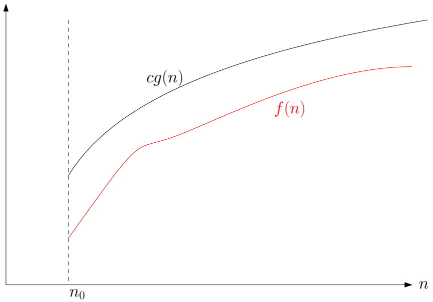

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Proof.

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$
$$= -n^2 + 40n \leq 0.$$
$$3n^2 + 2n \leq c(n^2 - 10n)$$

$\square$

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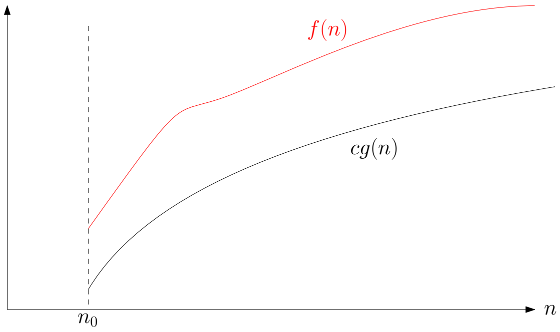

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Theorem $f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))$.

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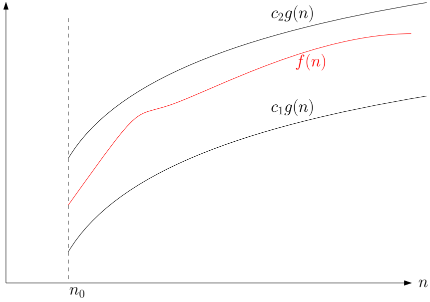

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Theorem $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Exercise

For each pair of functions f, g in the following table, indicate whether f is O, Ω or Θ of g.

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$n^3 - 100n$	$5n^2 + 3n$			
$3n - 50$	$n^2 - 7n$			
$n^2 - 100n$	$5n^2 + 30n$			
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$$f(n) = n^2$$

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2^n & \text{if } n \text{ is even} \end{cases}$$

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- $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, though weird
- $3n^2 - 10n - 5 = O(n^2)$ is simpler.

Notice that O denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.

Notice that O denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.
- We do not use Ω and Θ very often when we talk about running times.
- We say: the running time of the insertion sort algorithm is $O(n^2)$ and the bound is tight.

o and ω-Notations

o-Notation For a function $g(n)$,

$$o(g(n)) = \Big\{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0\Big\}.$$

ω-Notation For a function $g(n)$,

$$\omega(g(n)) = \Big\{\text{function } f : \forall c > 0, \exists n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0\Big\}.$$

Example:

- $3n^2 + 5n + 10 = o(n^3)$, but $3n^2 + 5n + 10 \neq o(n^2)$.
- $3n^2 + 5n + 10 = \omega(n)$, but $3n^2 + 5n + 10 \neq \omega(n^2)$.

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Questions?

Outline

1. Syllabus
2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

$O(n)$ (Linear) Running Time

Computing the sum of n numbers

sum(A, n)

```
S ← 0
for i ← 1 to n
    S ← S + A[i]
return S
```

$O(n)$ (Linear) Running Time

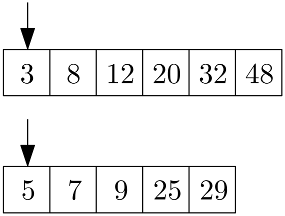
- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

$O(n)$ (Linear) Running Time

- Merge two sorted arrays



$O(n)$ (Linear) Running Time

- Merge two sorted arrays

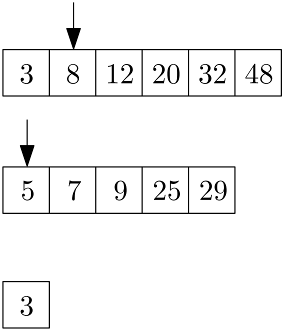
3	8	12	20	32	48

5	7	9	25	29

3

$O(n)$ (Linear) Running Time

- Merge two sorted arrays



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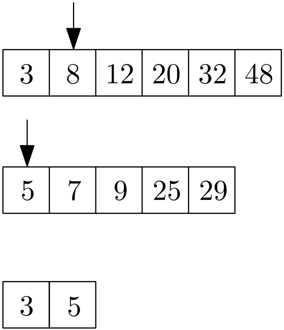

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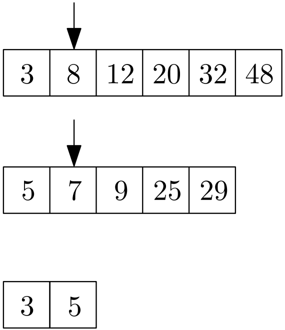

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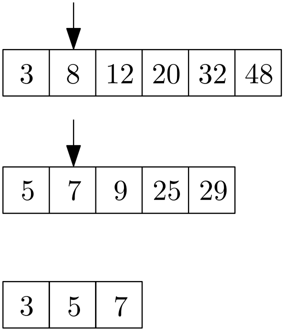


$O(n)$ (Linear) Running Time

- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

3	5	7

$O(n)$ (Linear) Running Time

- Merge two sorted arrays

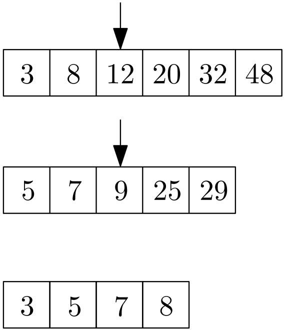
3	8	12	20	32	48

5	7	9	25	29

3	5	7	8

$O(n)$ (Linear) Running Time

- Merge two sorted arrays



- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

3	5	7	8	9	12	20	25	29

$O(n)$ (Linear) Running Time

- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

3	5	7	8	9	12	20	25	29	32	48

$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\\\ B and C are sorted, with length n_1 and n_2

```
A ← []; i ← 1; j ← 1
while i ≤ n1 and j ≤ n2
    if (B[i] ≤ C[j]) then
        append B[i] to A; i ← i + 1
    else
        append C[j] to A; j ← j + 1
if i ≤ n1 then append B[i..n1] to A
if j ≤ n2 then append C[j..n2] to A
return A
```

$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\\\ B and C are sorted, with length n_1 and n_2

$A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ and $j \leq n_2$
    if $(B[i] \leq C[j])$ then
        append $B[i]$ to A; $i \leftarrow i + 1$
    else
        append $C[j]$ to A; $j \leftarrow j + 1$
if $i \leq n_1$ then append $B[i..n_1]$ to A
if $j \leq n_2$ then append $C[j..n_2]$ to A
return A

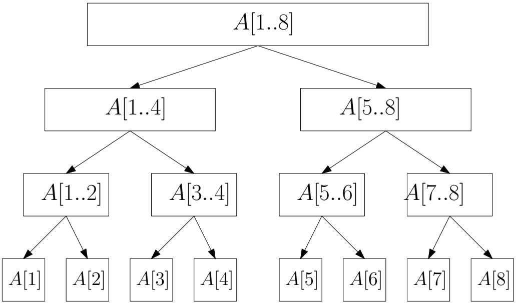
Running time $= O(n)$ where $n = n_1 + n_2$.

$O(n \lg n)$ Running Time

merge-sort(A, n)

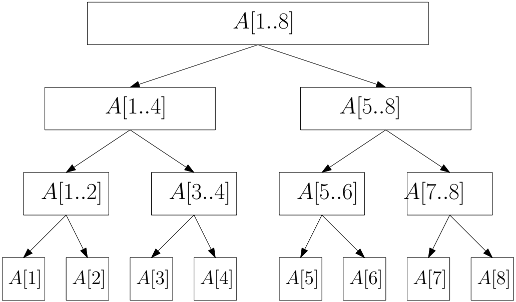
```
if n = 1 then
    return A
else
    B ← merge-sort(A[1..⌊n/2⌋], ⌊n/2⌋)
    C ← merge-sort(A[⌊n/2⌋ + 1..n], n − ⌊n/2⌋)
return merge(B, C, ⌊n/2⌋, n − ⌊n/2⌋)
```

- Merge-Sort



$O(n \lg n)$ Running Time

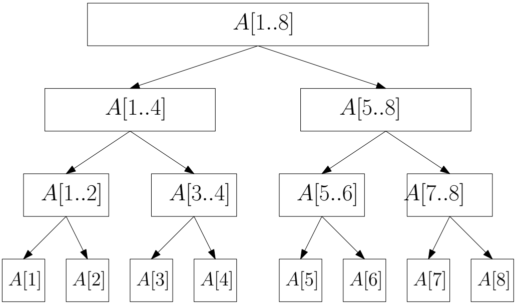
- Merge-Sort



- Each level takes running time $O(n)$

$O(n \lg n)$ Running Time

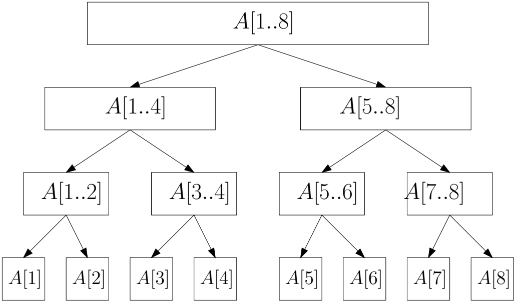
- Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels

$O(n \lg n)$ Running Time

- Merge-Sort



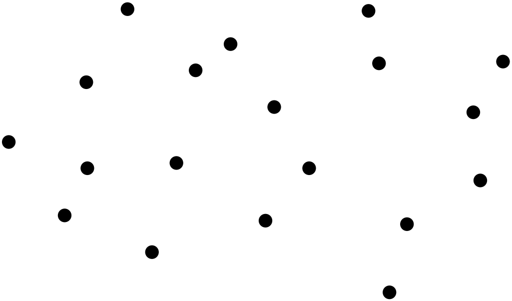
- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



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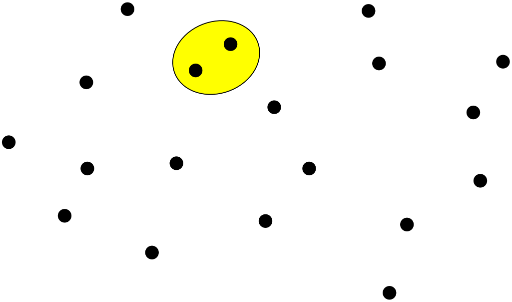

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair(x, y, n)

$bestd \leftarrow \infty$

for $i \leftarrow 1$ to $n - 1$

  for $j \leftarrow i + 1$ to n

    $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$

    if $d < bestd$ then

      $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$

return $(besti, bestj)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair(x, y, n)

```
bestd ← ∞
for i ← 1 to n − 1
    for j ← i + 1 to n
        d ← √((x[i] − x[j])² + (y[i] − y[j])²)
        if d < bestd then
            besti ← i, bestj ← j, bestd ← d
return (besti, bestj)
```

Closest pair can be solved in $O(n \lg n)$ time!

$O(n^3)$ (Cubic) Running Time

Multiply two matrices of size $n \times n$

matrix-multiplication(A, B, n)

1. $C \leftarrow$ matrix of size $n \times n$, with all entries being 0
2. for $i \leftarrow 1$ to n
3.   for $j \leftarrow 1$ to n
4.   for $k \leftarrow 1$ to n
5. $C[i, k] \leftarrow C[i, k] + A[i, j] \times B[j, k]$
6. return C

$O(n^k)$ Running Time for Integer $k \geq 4$

Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

$O(n^k)$ Running Time for Integer $k \geq 4$

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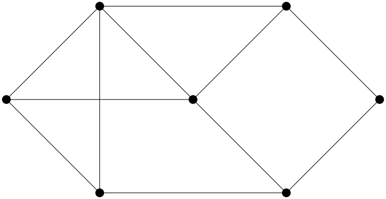

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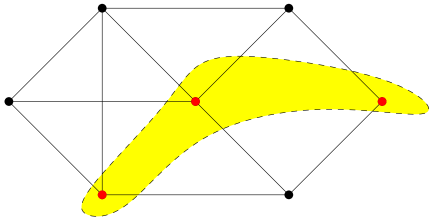

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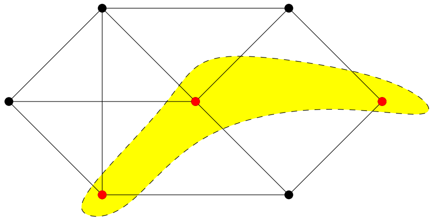


Independent set of size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

$O(n^k)$ Running Time for Integer $k \geq 4$

Independent Set of Size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

independent-set($G = (V, E)$)

1. for every set $S \subseteq V$ of size k
2. $\quad b \leftarrow$ true
3. $\quad$ for every $u, v \in S$
4. $\quad\quad$ if $(u, v) \in E$ then $b \leftarrow$ false
5. $\quad$ if b return true
6. return false

Running time $= O(\frac{n^k}{k!} \times k^2) = O(n^k)$ (assume k is a constant)

Beyond Polynomial Time: $O(2^n)$

Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the maximum independent set of G

max-independent-set($G = (V, E)$)

1. $R \leftarrow \emptyset$
2. for every set $S \subseteq V$
3. $\quad b \leftarrow$ true
4. $\quad$ for every $u, v \in S$
5. $\quad\quad$ if $(u, v) \in E$ then $b \leftarrow$ false
6. $\quad$ if b and $|S| > |R|$ then $R \leftarrow S$
7. return R

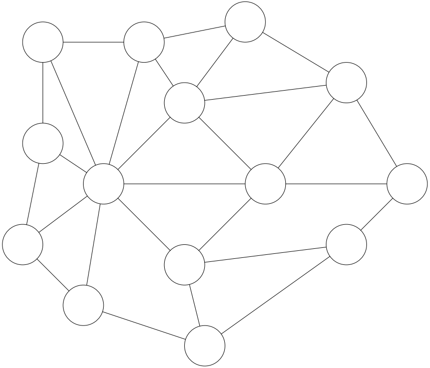
Running time $= O(2^n n^2)$.

Beyond Polynomial Time: $O(n!)$

Hamiltonian Cycle Problem

Input: a graph with n vertices

Output: a cycle that visits each node exactly once, or say no such cycle exists



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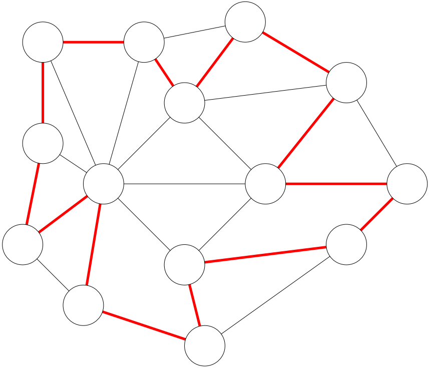

Beyond Polynomial Time: $n!$

Hamiltonian($G = (V, E)$)

1. for every permutation $(p_1, p_2, \cdots, p_n)$ of V
2. $\quad b \leftarrow$ true
3. $\quad$ for $i \leftarrow 1$ to $n - 1$
4. $\quad\quad$ if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
5. $\quad$ if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
6. $\quad$ if b then return $(p_1, p_2, \cdots, p_n)$
7. return "No Hamiltonian Cycle"

Running time $= O(n! \times n)$

$O(\lg n)$ (Logarithmic) Running Time

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- Binary search
  - Input: sorted array A of size n, an integer t;
  - Output: whether t appears in A.

$O(\lg n)$ (Logarithmic) Running Time

- Binary search
  - Input: sorted array A of size n, an integer t;
  - Output: whether t appears in A.
- E.g, search 35 in the following array:

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

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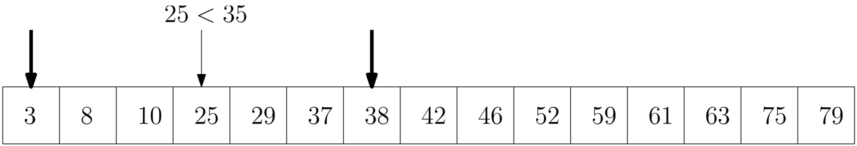

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$O(\lg n)$ (Logarithmic) Running Time

Binary search

- Input: sorted array A of size n, an integer t;
- Output: whether t appears in A.

binary-search(A, n, t)

1. $i \leftarrow 1, j \leftarrow n$
2. while $i \leq j$ do
3. $\quad k \leftarrow \lfloor (i+j)/2 \rfloor$
4. $\quad$ if $A[k] = t$ return true
5. $\quad$ if $A[k] < t$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6. return false

$O(\lg n)$ (Logarithmic) Running Time

Binary search

- Input: sorted array A of size n, an integer t;
- Output: whether t appears in A.

binary-search(A, n, t)

```
i ← 1, j ← n
while i ≤ j do
    k ← ⌊(i + j)/2⌋
    if A[k] = t return true
    if A[k] < t then j ← k − 1 else i ← k + 1
return false
```

Running time $= O(\lg n)$

Compare the Orders

- Sort the functions from smallest to largest asymptotically
  $n^{\sqrt{n}}, \ \lg n, \ n, \ n^2, \ n \lg n, \ n!, \ 2^n, \ e^n, \ \lg(n!), \ n^n$

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  $n^{\sqrt{n}}, \ \lg n, \ n, \ n^2, \ n \lg n, \ n!, \ 2^n, \ e^n, \ \lg(n!), \ n^n$
- $f < g$ stands for $f = o(g)$, $f = g$ stands for $f = \Theta(g)$!
- $\lg n < n^{\sqrt{n}}$

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- $\lg n < n < n^{\sqrt{n}}$
- $\lg n < n < n^2 < n^{\sqrt{n}}$

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- $\lg n < n < n^{\sqrt{n}}$
- $\lg n < n < n^2 < n^{\sqrt{n}}$
- $\lg n < n < n\lg n < n^2 < n^{\sqrt{n}}$
- $\lg n < n < n\lg n < n^2 < n^{\sqrt{n}} < n!$

Compare the Orders

- Sort the functions from smallest to largest asymptotically
  $n^{\sqrt{n}},\ \lg n,\ n,\ n^2,\ n\lg n,\ n!,\ 2^n,\ e^n,\ \lg(n!),\ n^n$
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Compare the Orders

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- $\lg n < n < n \lg n < n^2 < n^{\sqrt{n}} < 2^n < e^n < n!$
- $\lg n < n < n \lg n = \lg(n!) < n^2 < n^{\sqrt{n}} < 2^n < e^n < n!$

Compare the Orders

- Sort the functions from smallest to largest asymptotically
  $n^{\sqrt{n}},\ \lg n,\ n,\ n^2,\ n\lg n,\ n!,\ 2^n,\ e^n,\ \lg(n!),\ n^n$
- $f < g$ stands for $f = o(g)$, $f = g$ stands for $f = \Theta(g)$!
- $\lg n < n^{\sqrt{n}}$
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Terminologies

When we talk about upper bounds:

- Logarithmic time: $O(\lg n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

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- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

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- However, when n is big enough, $1000n < 0.1n^2$
- For "natural" algorithms, constants are not so big!
- So, for reasonable n, algorithm with lower order running time beats algorithm with higher order running time.