# CSE 431/531: Algorithm Analysis and Design (Spring 2020)

# Divide-and-Conquer

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# Outline


1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
   - Quicksort
   - Lower Bound for Comparison-Based Sorting Algorithms
   - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
6. Solving Recurrences
7. Computing $n$-th Fibonacci Number

### Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

### Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

# Divide-and-Conquer

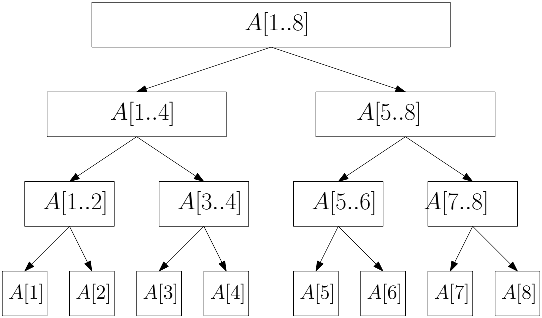
- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

**merge-sort$(A, n)$**

1. if $n = 1$ then
2.     return $A$
3. else
4.     $B \leftarrow$ merge-sort$\Big(A\big[1..\lfloor n/2\rfloor\big], \lfloor n/2\rfloor\Big)$
5.     $C \leftarrow$ merge-sort$\Big(A\big[\lfloor n/2\rfloor + 1..n\big], \lceil n/2\rceil\Big)$
6. return merge$(B, C, \lfloor n/2\rfloor, \lceil n/2\rceil)$

- Divide: trivial
- Conquer: 4, 5
- Combine: 6

# Running Time for Merge-Sort




- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$
- Better than insertion sort

# Running Time for Merge-Sort Using Recurrence

- $T(n)$ = running time for sorting $n$ numbers,then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

- With some tolerance of informality:

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2T(n/2) + O(n) & \text{if } n \geq 2 \end{cases}$$

- Even simpler: $T(n) = 2T(n/2) + O(n)$. (Implicit assumption: $T(n) = O(1)$ if $n$ is at most some constant.)
- Solving this recurrence, we have $T(n) = O(n \lg n)$ (we shall show how later)

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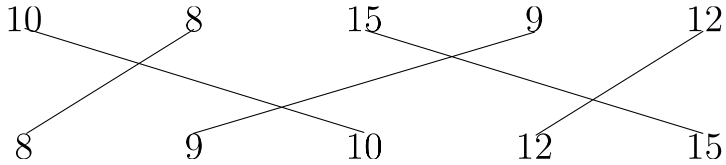
**Def.** Given an array $A$ of $n$ integers, an inversion in $A$ is a pair $(i, j)$ of indices such that $i < j$ and $A[i] > A[j]$.

## Counting Inversions

**Input:** an sequence $A$ of $n$ numbers

**Output:** number of inversions in $A$

### Example:



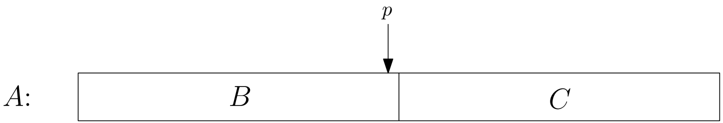

- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

# Naive Algorithm for Counting Inversions

count-inversions$(A, n)$

1. $c \leftarrow 0$
2. for every $i \leftarrow 1$ to $n - 1$
3. for every $j \leftarrow i + 1$ to $n$
4. if $A[i] > A[j]$ then $c \leftarrow c + 1$
5. return $c$

# Divide-and-Conquer



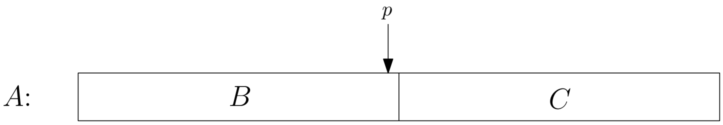

- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$
  $$m = \left|\{(i,j) : B[i] > C[j]\}\right|$$

**Q:** How fast can we compute $m$, via trivial algorithm?

**A:** $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

# Divide-and-Conquer

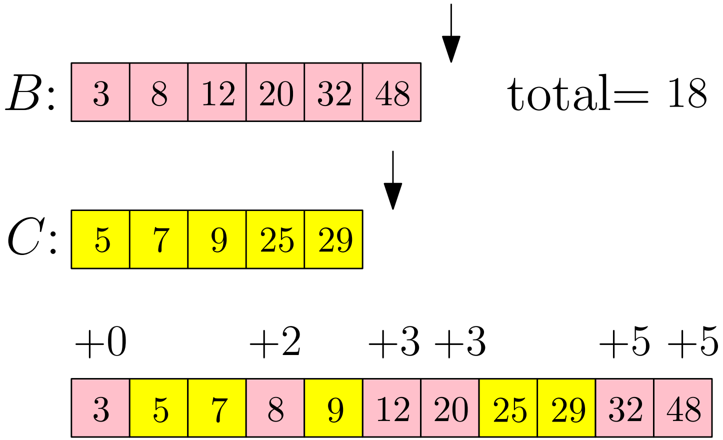



- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\mathsf{invs}(A) = \#\mathsf{invs}(B) + \#\mathsf{invs}(C) + m$$
  $$m = \Big|\big\{(i,j) : B[i] > C[j]\big\}\Big|$$

**Lemma** If both $B$ and $C$ are sorted, then we can compute $m$ in $O(n)$ time!

# Counting Inversions between $B$ and $C$

Count pairs $i, j$ such that $B[i] > C[j]$:



# Count Inversions between $B$ and $C$

- Procedure that merges $B$ and $C$ and counts inversions between $B$ and $C$ at the same time

**merge-and-count$(B, C, n_1, n_2)$**

$count \leftarrow 0$;
$A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ or $j \leq n_2$
    if $j > n_2$ or $(i \leq n_1$ and $B[i] \leq C[j])$ then
        append $B[i]$ to $A$; $i \leftarrow i + 1$
        $count \leftarrow count + (j - 1)$
    else
        append $C[j]$ to $A$; $j \leftarrow j + 1$
return $(A, count)$

# Sort and Count Inversions in $A$

- A procedure that returns the sorted array of $A$ and counts the number of inversions in $A$:

**sort-and-count$(A, n)$**

1. if $n = 1$ then
2.     return $(A, 0)$
3. else
4.     $(B, m_1) \leftarrow$ sort-and-count$\Big(A\big[1..\lfloor n/2\rfloor\big], \lfloor n/2\rfloor\Big)$
5.     $(C, m_2) \leftarrow$ sort-and-count$\Big(A\big[\lfloor n/2\rfloor + 1..n\big], \lceil n/2\rceil\Big)$
6.     $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2\rfloor, \lceil n/2\rceil)$
7.     return $(A, m_1 + m_2 + m_3)$

- Divide: trivial
- Conquer: 4, 5
- Combine: 6, 7

**sort-and-count$(A, n)$**

1. if $n = 1$ then
2. return $(A, 0)$
3. else
4. $(B, m_1) \leftarrow$ sort-and-count$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
5. $(C, m_2) \leftarrow$ sort-and-count$\Big(A\big[\lfloor n/2 \rfloor + 1..n\big], \lceil n/2 \rceil\Big)$
6. $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
7. return $(A, m_1 + m_2 + m_3)$

- Recurrence for the running time: $T(n) = 2T(n/2) + O(n)$
- Running time $= O(n \lg n)$

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# Quicksort vs Merge-Sort

| | **Merge Sort** | **Quicksort** |
|---|---|---|
| Divide | Trivial | Separate small and big numbers |
| Conquer | Recurse | Recurse |
| Combine | Merge 2 sorted arrays | Trivial |

# Quicksort Example

**Assumption** We can choose median of an array of size $n$ in $O(n)$ time.

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 29 | 82 | 75 | 64 | 38 | 45 | 94 | 69 | 25 | 76 | 15 | 92 | 37 | 17 | 85 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 29 | 38 | 45 | 25 | 15 | 37 | 17 | 64 | 82 | 75 | 94 | 92 | 69 | 76 | 85 |

| | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 25 | 15 | 17 | 29 | 38 | 45 | 37 | 64 | 82 | 75 | 94 | 92 | 69 | 76 | 85 |

# Quicksort

**quicksort$(A, n)$**

1. if $n \leq 1$ then return $A$
2. $x \leftarrow$ lower median of $A$
3. $A_L \leftarrow$ elements in $A$ that are less than $x$ \\\\ Divide
4. $A_R \leftarrow$ elements in $A$ that are greater than $x$ \\\\ Divide
5. $B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\\\ Conquer
6. $B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\\\ Conquer
7. $t \leftarrow$ number of times $x$ appear $A$
8. return the array obtained by concatenating $B_L$, the array containing $t$ copies of $x$, and $B_R$

- Recurrence $T(n) \leq 2T(n/2) + O(n)$
- Running time $= O(n \lg n)$

**Assumption** We can choose median of an array of size $n$ in $O(n)$ time.

**Q:** How to remove this assumption?

**A:**

1. There is an algorithm to find median in $O(n)$ time, using divide-and-conquer (we shall not talk about it; it is complicated and not practical)
2. Choose a pivot randomly and pretend it is the median (it is practical)

# Quicksort Using A Random Pivot

quicksort$(A, n)$

1. if $n \leq 1$ then return $A$
2. $x \leftarrow$ a random element of $A$ ($x$ is called a pivot)
3. $A_L \leftarrow$ elements in $A$ that are less than $x$ \\ Divide
4. $A_R \leftarrow$ elements in $A$ that are greater than $x$ \\ Divide
5. $B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
6. $B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
7. $t \leftarrow$ number of times $x$ appear $A$
8. return the array obtained by concatenating $B_L$, the array containing $t$ copies of $x$, and $B_R$

# Randomized Algorithm Model

**Assumption** There is a procedure to produce a random real number in $[0, 1]$.

**Q:** Can computers really produce random numbers?

**A:** No! The execution of a computer programs is deterministic!

- In practice: use pseudo-random-generator, a deterministic algorithm returning numbers that "look like" random
- In theory: assume they can.

# Quicksort Using A Random Pivot

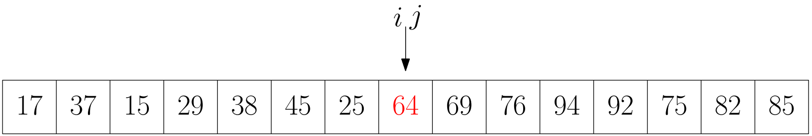
**quicksort$(A, n)$**

1. if $n \leq 1$ then return $A$
2. $x \leftarrow$ a random element of $A$ ($x$ is called a pivot)
3. $A_L \leftarrow$ elements in $A$ that are less than $x$ \\ Divide
4. $A_R \leftarrow$ elements in $A$ that are greater than $x$ \\ Divide
5. $B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$ \\ Conquer
6. $B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$ \\ Conquer
7. $t \leftarrow$ number of times $x$ appear $A$
8. return the array obtained by concatenating $B_L$, the array containing $t$ copies of $x$, and $B_R$

**Lemma** The expected running time of the algorithm is $O(n \lg n)$.

# Quicksort Can Be Implemented as an "In-Place" Sorting Algorithm

- In-Place Sorting Algorithm: an algorithm that only uses "small" extra space.




- To partition the array into two parts, we only need $O(1)$ extra space.

**partition$(A, \ell, r)$**

$p \leftarrow$ random integer between $\ell$ and $r$, swap $A[p]$ and $A[\ell]$

$i \leftarrow \ell, j \leftarrow r$

while $\mathbf{true}$ do

  while $i < j$ and $A[i] < A[j]$ do $j \leftarrow j - 1$

  **if** $i = j$ **then** break

  swap $A[i]$ and $A[j]$; $i \leftarrow i + 1$

  while $i < j$ and $A[i] < A[j]$ do $i \leftarrow i + 1$

  **if** $i = j$ **then** break

  swap $A[i]$ and $A[j]$; $j \leftarrow j - 1$

return $i$

# In-Place Implementation of Quick-Sort

**quicksort$(A, \ell, r)$**

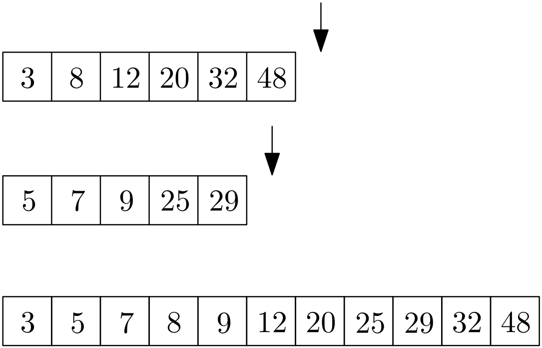
1. **if** $\ell \geq r$ **then return**
2. $m \leftarrow$ patition$(A, \ell, r)$
3. quicksort$(A, \ell, m-1)$
4. quicksort$(A, m+1, r)$

- To sort an array $A$ of size $n$, call quicksort$(A, 1, n)$.

**Note:** We pass the array $A$ by reference, instead of by copying.

# Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays



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# Comparison-Based Sorting Algorithms

**Q:** Can we do better than $O(n \log n)$ for sorting?

**A:** No, for comparison-based sorting algorithms.

**Comparison-Based Sorting Algorithms**

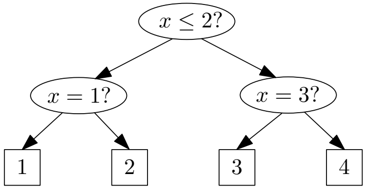
- To sort, we are only allowed to compare two elements
- We can not use "internal structures" of the elements

**Lemma** The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$.

- Bob has one number $x$ in his hand, $x \in \{1, 2, 3, \cdots, N\}$.
- You can ask Bob "yes/no" questions about $x$.

**Q:** How many questions do you need to ask Bob in order to know $x$?

**A:** $\lceil \log_2 N \rceil$.



# Comparison-Based Sorting Algorithms

**Q:** Can we do better than $O(n \log n)$ for sorting?

**A:** No, for comparison-based sorting algorithms.

- Bob has a permutation $\pi$ over $\{1, 2, 3, \cdots, n\}$ in his hand.
- You can ask Bob "yes/no" questions about $\pi$.

**Q:** How many questions do you need to ask in order to get the permutation $\pi$?

**A:** $\log_2 n! = \Theta(n \lg n)$

# Comparison-Based Sorting Algorithms

**Q:** Can we do better than $O(n \log n)$ for sorting?

**A:** No, for comparison-based sorting algorithms.

- Bob has a permutation $\pi$ over $\{1, 2, 3, \cdots, n\}$ in his hand.
- You can ask Bob questions of the form "does $i$ appear before $j$ in $\pi$?"

**Q:** How many questions do you need to ask in order to get the permutation $\pi$?

**A:** At least $\log_2 n! = \Theta(n \lg n)$

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## Selection Problem

**Input:** a set $A$ of $n$ numbers, and $1 \leq i \leq n$

**Output:** the $i$-th smallest number in $A$

- Sorting solves the problem in time $O(n \lg n)$.
- Our goal: $O(n)$ running time

# Recall: Quicksort with Median Finder

**quicksort$(A, n)$**

1. if $n \leq 1$ then return $A$
2. $x \leftarrow$ lower median of $A$
3. $A_L \leftarrow$ elements in $A$ that are less than $x$ \\ Divide
4. $A_R \leftarrow$ elements in $A$ that are greater than $x$ \\ Divide
5. $B_L \leftarrow$ quicksort$(A_L, A_L.\text{size})$ \\ Conquer
6. $B_R \leftarrow$ quicksort$(A_R, A_R.\text{size})$ \\ Conquer
7. $t \leftarrow$ number of times $x$ appear $A$
8. return the array obtained by concatenating $B_L$, the array containing $t$ copies of $x$, and $B_R$

# Selection Algorithm with Median Finder

**selection$(A, n, i)$**

1. if $n = 1$ then return $A$
2. $x \leftarrow$ lower median of $A$
3. $A_L \leftarrow$ elements in $A$ that are less than $x$ \\ Divide
4. $A_R \leftarrow$ elements in $A$ that are greater than $x$ \\ Divide
5. if $i \leq A_L$.size then
6.     return selection$(A_L, A_L.\text{size}, i)$ \\ Conquer
7. elseif $i > n - A_R$.size then
8.     return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ \\ Conquer
9. else return $x$

- Recurrence for selection: $T(n) = T(n/2) + O(n)$
- Solving recurrence: $T(n) = O(n)$

# Randomized Selection Algorithm

**selection$(A, n, i)$**

1. if $n = 1$ then return $A$
2. $x \leftarrow$ random element of $A$ (called pivot)
3. $A_L \leftarrow$ elements in $A$ that are less than $x$ \\\\ Divide
4. $A_R \leftarrow$ elements in $A$ that are greater than $x$ \\\\ Divide
5. if $i \leq A_L$.size then
6.     return selection$(A_L, A_L.\text{size}, i)$ \\\\ Conquer
7. elseif $i > n - A_R$.size then
8.     return selection$(A_R, A_R.\text{size}, i - (n - A_R.\text{size}))$ \\\\ Conquer
9. else return $x$

- expected running time $= O(n)$

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## Polynomial Multiplication

**Input:** two polynomials of degree $n-1$

**Output:** product of two polynomials

## Example:

$$\begin{aligned}
&(3x^3 + 2x^2 - 5x + 4) \times (2x^3 - 3x^2 + 6x - 5) \\
=\ & 6x^6 - 9x^5 + 18x^4 - 15x^3 \\
& + 4x^5 - 6x^4 + 12x^3 - 10x^2 \\
& - 10x^4 + 15x^3 - 30x^2 + 25x \\
& + 8x^3 - 12x^2 + 24x - 20 \\
=\ & 6x^6 - 5x^5 + 2x^4 + 20x^3 - 52x^2 + 49x - 20
\end{aligned}$$

- **Input**: $(4, -5, 2, 3), (-5, 6, -3, 2)$
- **Output**: $(-20, 49, -52, 20, 2, -5, 6)$

# Naïve Algorithm

polynomial-multiplication$(A, B, n)$

1. let $C[k] = 0$ for every $k = 0, 1, 2, \cdots, 2n - 2$
2. for $i \leftarrow 0$ to $n - 1$
3.     for $j \leftarrow 0$ to $n - 1$
4.         $C[i + j] \leftarrow C[i + j] + A[i] \times B[j]$
5. return $C$

Running time: $O(n^2)$

# Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$
$$q(x) = 2x^3 - 3x^2 + 6x - 5 = (2x - 3)x^2 + (6x - 5)$$

- $p(x)$: degree of $n - 1$ (assume $n$ is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
- $p_H(x), p_L(x)$: polynomials of degree $n/2 - 1$.

$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right)x^{n/2} + p_L q_L \end{aligned}$$

# Divide-and-Conquer for Polynomial Multiplication

$$
\begin{aligned}
pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\
&= p_H q_H x^n + \left(p_H q_L + p_L q_H\right) x^{n/2} + p_L q_L
\end{aligned}
$$

$$
\begin{aligned}
\text{multiply}(p, q) = \; & \text{multiply}(p_H, q_H) \times x^n \\
& + \left(\text{multiply}(p_H, q_L) + \text{multiply}(p_L, q_H)\right) \times x^{n/2} \\
& + \text{multiply}(p_L, q_L)
\end{aligned}
$$

- Recurrence: $T(n) = 4T(n/2) + O(n)$
- $T(n) = O(n^2)$

# Reduce Number from 4 to 3

$$\begin{aligned} pq &= \left(p_H x^{n/2} + p_L\right)\left(q_H x^{n/2} + q_L\right) \\ &= p_H q_H x^n + \left(p_H q_L + p_L q_H\right) x^{n/2} + p_L q_L \end{aligned}$$

- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

# Divide-and-Conquer for Polynomial Multiplication

$$r_H = \text{multiply}(p_H, q_H)$$
$$r_L = \text{multiply}(p_L, q_L)$$

$$\begin{aligned}\text{multiply}(p, q) = {} & r_H \times x^n \\ & + \big(\text{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L\big) \times x^{n/2} \\ & + r_L\end{aligned}$$

- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

**Assumption** $n$ is a power of $2$. Arrays are $0$-indexed.

**multiply**$(A, B, n)$

if $n = 1$ then return $(A[0]B[0])$

$A_L \leftarrow A[0 .. n/2-1], A_H \leftarrow A[n/2 .. n-1]$

$B_L \leftarrow B[0 .. n/2-1], B_H \leftarrow B[n/2 .. n-1]$

$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$

$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$

$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$

$C \leftarrow$ array of $(2n-1)$ 0's

for $i \leftarrow 0$ to $n-2$ do

    $C[i] \leftarrow C[i] + C_L[i]$

    $C[i+n] \leftarrow C[i+n] + C_H[i]$

    $C[i+n/2] \leftarrow C[i+n/2] + C_M[i] - C_L[i] - C_H[i]$

return $C$

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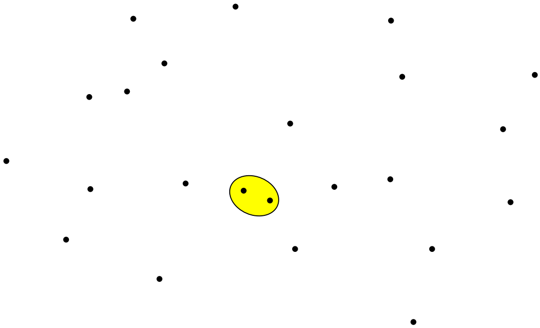
7. Computing $n$-th Fibonacci Number

- Closest pair
- Convex hull
- Matrix multiplication
- FFT(Fast Fourier Transform): polynomial multiplication in $O(n \lg n)$ time

## Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

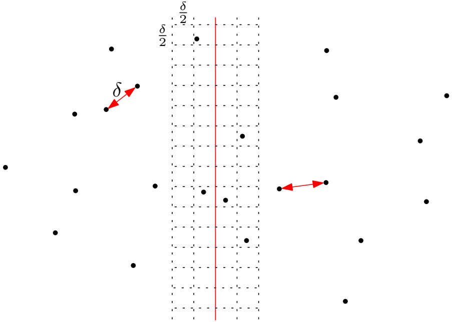
**Output:** the pair of points that are closest



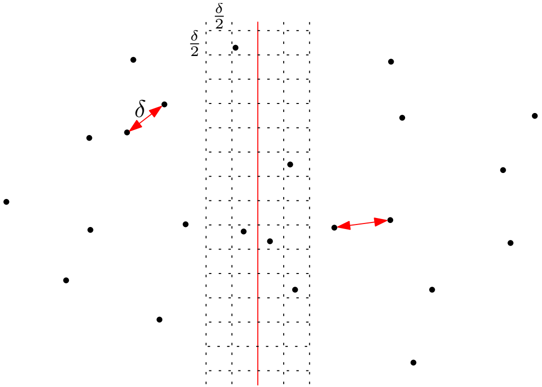
- Trivial algorithm: $O(n^2)$ running time

# Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively
- **Combine**: Check if there is a closer pair between left-half and right-half

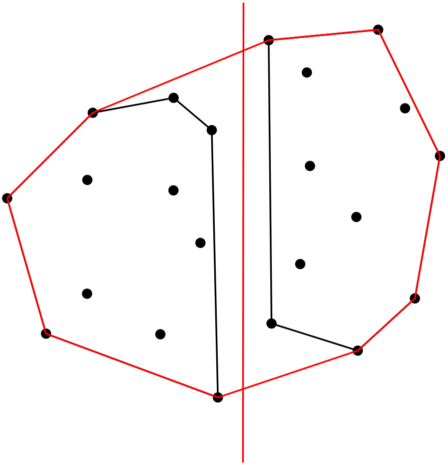


# Divide-and-Conquer Algorithm for Closest Pair




- Each box contains at most one pair
- For each point, only need to consider $O(1)$ boxes nearby
- time for combine $= O(n)$ (many technicalities omitted)
- Recurrence: $T(n) = 2T(n/2) + O(n)$
- Running time: $O(n \lg n)$

# $O(n \lg n)$-Time Algorithm for Convex Hull



# Strassen's Algorithm for Matrix Multiplication

**Matrix Multiplication**

**Input:** two $n \times n$ matrices $A$ and $B$

**Output:** $C = AB$

**Naive Algorithm: matrix-multiplication$(A, B, n)$**

for $i \leftarrow 1$ to $n$
    for $j \leftarrow 1$ to $n$
        $C[i, j] \leftarrow 0$
        for $k \leftarrow 1$ to $n$
            $C[i, j] \leftarrow C[i, j] + A[i, k] \times B[k, j]$
return $C$

- running time $= O(n^3)$

# Try to Use Divide-and-Conquer

$$A = \begin{array}{|c|c|} \hline A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline \end{array} \qquad B = \begin{array}{|c|c|} \hline B_{11} & B_{12} \\ \hline B_{21} & B_{22} \\ \hline \end{array}$$

- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- matrix_multiplication$(A, B)$ recursively calls matrix_multiplication$(A_{11}, B_{11})$, matrix_multiplication$(A_{12}, B_{21})$, $\cdots$
- Recurrence for running time: $T(n) = 8T(n/2) + O(n^2)$
- $T(n) = O(n^3)$

# Strassen's Algorithm

- $T(n) = 8T(n/2) + O(n^2)$
- Strassen's Algorithm: improve the number of multiplications from $8$ to $7$!
- New recurrence: $T(n) = 7T(n/2) + O(n^2)$
- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

# Outline

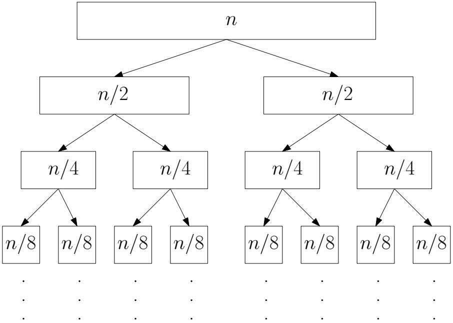

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
   - Quicksort
   - Lower Bound for Comparison-Based Sorting Algorithms
   - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
6. Solving Recurrences
7. Computing $n$-th Fibonacci Number

# Methods for Solving Recurrences

- The recursion-tree method
- The master theorem

# Recursion-Tree Method

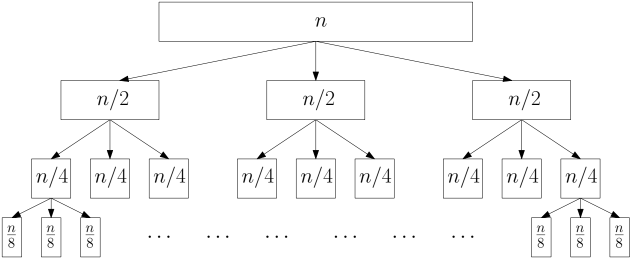
- $T(n) = 2T(n/2) + O(n)$




- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$

# Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n)$

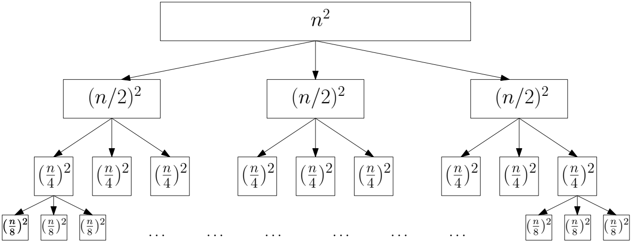



- Total running time at level $i$? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{2}\right)^i n = O\left(n\left(\frac{3}{2}\right)^{\lg_2 n}\right) = O(3^{\lg_2 n}) = O(n^{\lg_2 3}).$$

# Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$




- Total running time at level $i$? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$
- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{4}\right)^i n^2 = O(n^2).$$

# Master Theorem

| Recurrences | $a$ | $b$ | $c$ | time |
|---|---|---|---|---|
| $T(n) = 2T(n/2) + O(n)$ | 2 | 2 | 1 | $O(n \lg n)$ |
| $T(n) = 3T(n/2) + O(n)$ | 3 | 2 | 1 | $O(n^{\lg_2 3})$ |
| $T(n) = 3T(n/2) + O(n^2)$ | 3 | 2 | 2 | $O(n^2)$ |

**Theorem** $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} O(n^{\lg_b a}) & \text{if } c < \lg_b a \\ O(n^c \lg n) & \text{if } c = \lg_b a \\ O(n^c) & \text{if } c > \lg_b a \end{cases}$$

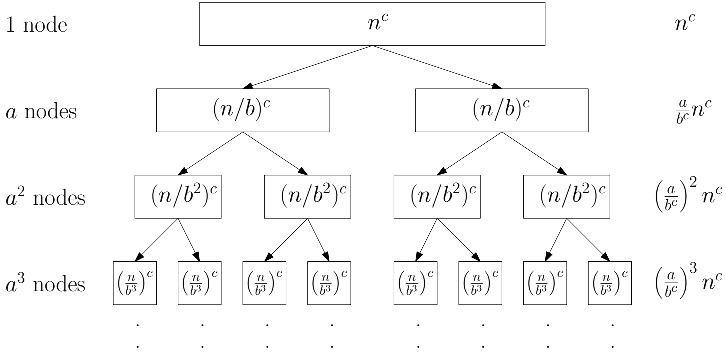
**Theorem** $T(n) = aT(n/b) + O(n^c)$, where $a \geq 1, b > 1, c \geq 0$ are constants. Then,

$$T(n) = \begin{cases} O(n^{\lg_b a}) & \text{if } c < \lg_b a \\ O(n^c \lg n) & \text{if } c = \lg_b a \\ O(n^c) & \text{if } c > \lg_b a \end{cases}$$

- Ex: $T(n) = 4T(n/2) + O(n^2)$. Case 2. $T(n) = O(n^2 \lg n)$
- Ex: $T(n) = 3T(n/2) + O(n)$. Case 1. $T(n) = O(n^{\lg_2 3})$
- Ex: $T(n) = T(n/2) + O(1)$. Case 2. $T(n) = O(\lg n)$
- Ex: $T(n) = 2T(n/2) + O(n^2)$. Case 3. $T(n) = O(n^2)$

# Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$




- $c < \lg_b a$ : bottom-level dominates: $\left(\frac{a}{b^c}\right)^{\lg_b n} n^c = n^{\lg_b a}$
- $c = \lg_b a$ : all levels have same time: $n^c \lg_b n = O(n^c \lg n)$
- $c > \lg_b a$ : top-level dominates: $O(n^c)$

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# Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

**$n$-th Fibonacci Number**

**Input:** integer $n > 0$

**Output:** $F_n$

# Computing $F_n$ : Stupid Divide-and-Conquer Algorithm

**Fib($n$)**

1. if $n = 0$ return 0
2. if $n = 1$ return 1
3. return Fib($n-1$) $+$ Fib($n-2$)

**Q:** Is the running time of the algorithm polynomial or exponential in $n$?

**A:** Exponential

- Running time is at least $\Omega(F_n)$
- $F_n$ is exponential in $n$

# Computing $F_n$: Reasonable Algorithm

**Fib($n$)**

1. $F[0] \leftarrow 0$
2. $F[1] \leftarrow 1$
3. for $i \leftarrow 2$ to $n$ do
4. $\quad F[i] \leftarrow F[i-1] + F[i-2]$
5. return $F[n]$

- Dynamic Programming
- Running time $= O(n)$

# Computing $F_n$: Even Better Algorithm

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\cdots$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

### power($n$)

1. if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
2. $R \leftarrow$ power($\lfloor n/2 \rfloor$)
3. $R \leftarrow R \times R$
4. if $n$ is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
5. return $R$

### Fib($n$)

1. if $n = 0$ then return $0$
2. $M \leftarrow$ power($n - 1$)
3. return $M[1][1]$

- Recurrence for running time? $T(n) = T(n/2) + O(1)$
- $T(n) = O(\lg n)$

# Running time = $O(\lg n)$: We Cheated!

**Q:** How many bits do we need to represent $F(n)$?

**A:** $\Theta(n)$

- We can not add (or multiply) two integers of $\Theta(n)$ bits in $O(1)$ time
- Even printing $F(n)$ requires time much larger than $O(\lg n)$

**Fixing the Problem**

To compute $F_n$, we need $O(\lg n)$ basic arithmetic operations on integers

# Summary: Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

- Write down recurrence for running time
- Solve recurrence using master theorem

# Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, $\cdots$:
  $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \lg n)$
- Integer Multiplication:
  $T(n) = 3T(n/2) + O(n) \Rightarrow T(n) = O(n^{\lg_2 3})$
- Matrix Multiplication:
  $T(n) = 7T(n/2) + O(n^2) \Rightarrow T(n) = O(n^{\lg_2 7})$

- Usually, designing better algorithm for "combine" step is key to improve running time