CSE 431/531: Algorithm Analysis and Design (Spring 2020)

Graph Algorithms

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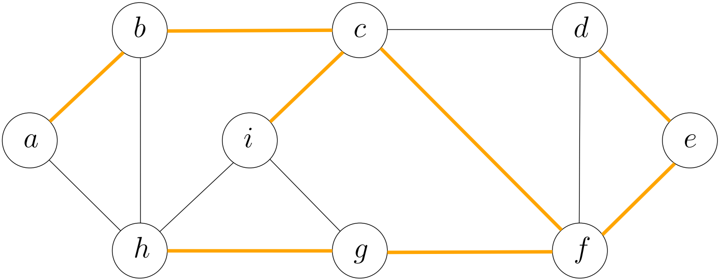
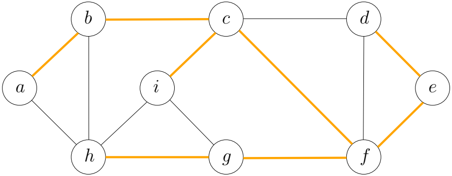
Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
   - Bellman-Ford Algorithm
4. All-Pair Shortest Paths and Floyd-Warshall

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





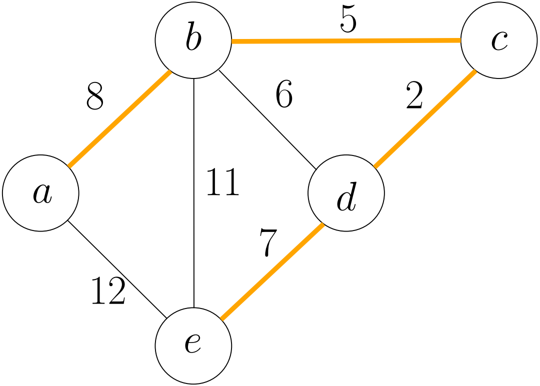
Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight



Recall: Steps of Designing A Greedy Algorithm

- Design a "reasonable" strategy
- Prove that the reasonable strategy is "safe" (key, usually done by "exchanging argument")
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

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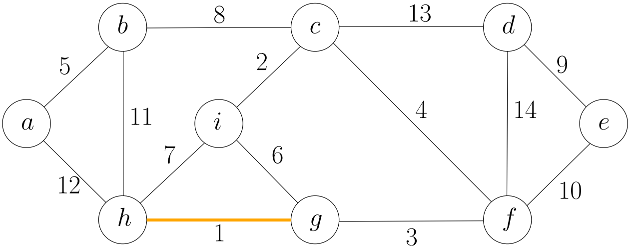


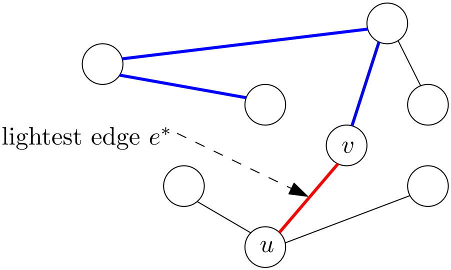
Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

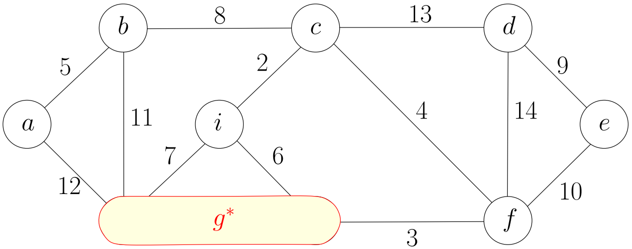
Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v
- Remove any edge e in the path to obtain tree T'
- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: T' is also a MST $\square$

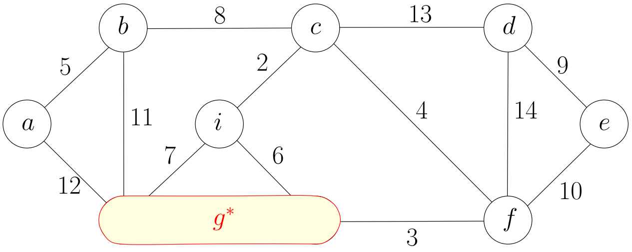


Is the Residual Problem Still a MST Problem?



- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

Q: What edges are removed due to contractions?

A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

$F = \emptyset$

sort edges in E in non-decreasing order of weights w

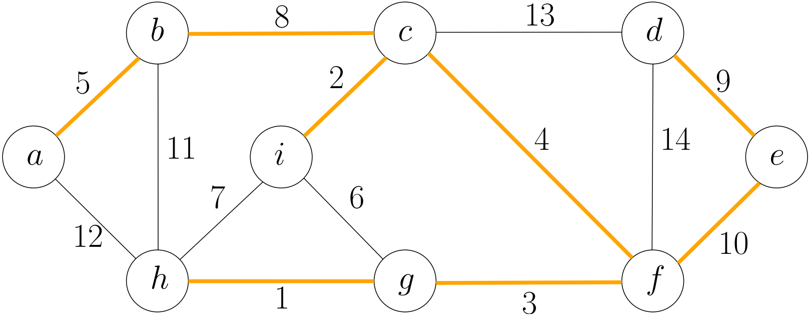
for each edge (u, v) in the order

    if u and v are not connected by a path of edges in F

        $F = F \cup \{(u, v)\}$

return (V, F)

Kruskal's Algorithm: Example



Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

$F \leftarrow \emptyset$

$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$

sort the edges of E in non-decreasing order of weights w

for each edge $(u, v) \in E$ in the order

    $S_u \leftarrow$ the set in $\mathcal{S}$ containing u

    $S_v \leftarrow$ the set in $\mathcal{S}$ containing v

    if $S_u \neq S_v$

        $F \leftarrow F \cup \{(u, v)\}$

        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$

return (V, F)

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

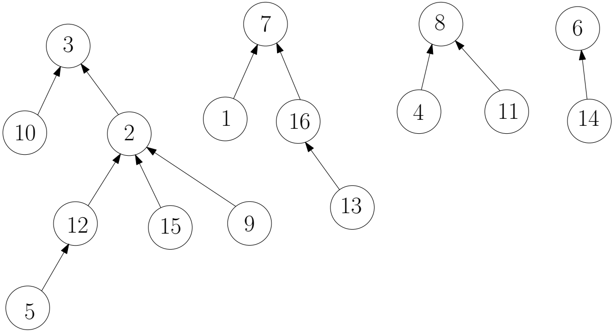
1. $F \leftarrow \emptyset$
2. $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3. sort the edges of E in non-decreasing order of weights w
4. for each edge $(u, v) \in E$ in the order
5. $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6. $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7.   if $S_u \neq S_v$
8. $F \leftarrow F \cup \{(u, v)\}$
9. $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10. return (V, F)

Use union-find data structure to support 2, 5, 6, 7, 9.

Union-Find Data Structure

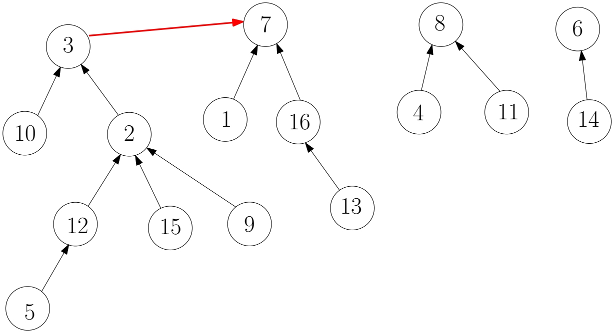
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] =$ nil if i is a root).

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).
- root(u): the root of the tree containing u
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure

root(v)

if $par[v]$ = nil then
    return v
else
    return root($par[v]$)

root(v)

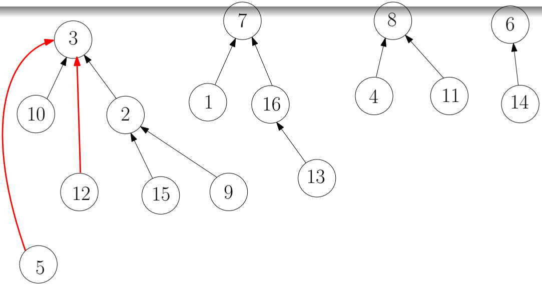
if par$[v]$ = nil then
    return v
else
    $par[v] \leftarrow$ root($par[v]$)
    return $par[v]$

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.

Union-Find Data Structure

root(v)

if $par[v] = \text{nil}$ then
    return v
else
    $par[v] \leftarrow \text{root}(par[v])$
    return $par[v]$



MST-Kruskal(G, w)

$F \leftarrow \emptyset$

$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$

sort the edges of E in non-decreasing order of weights w

for each edge $(u, v) \in E$ in the order

  $S_u \leftarrow$ the set in $\mathcal{S}$ containing u

  $S_v \leftarrow$ the set in $\mathcal{S}$ containing v

  if $S_u \neq S_v$

    $F \leftarrow F \cup \{(u, v)\}$

    $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$

return (V, F)

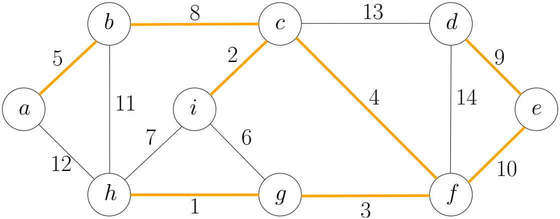
MST-Kruskal(G, w)

```
1  F ← ∅
2  for every v ∈ V: let par[v] ← nil
3  sort the edges of E in non-decreasing order of weights w
4  for each edge (u, v) ∈ E in the order
5      u' ← root(u)
6      v' ← root(v)
7      if u' ≠ v'
8          F ← F ∪ {(u, v)}
9          par[u'] ← v'
10 return (V, F)
```

- 2, 5, 6, 7, 9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $=$ time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.

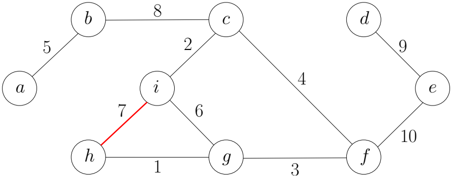


- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from F one by one until we obtain a spanning tree



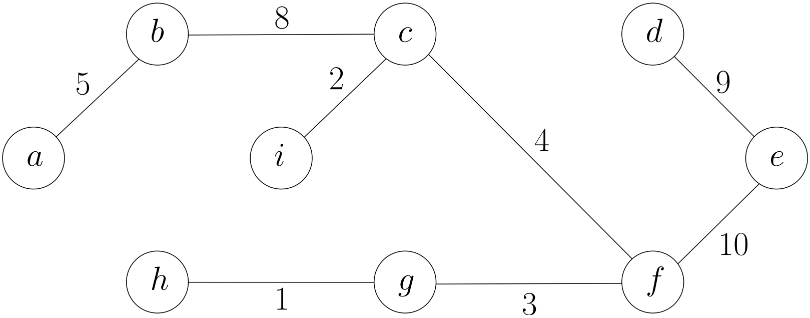
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

1. $F \leftarrow E$
2. sort E in non-increasing order of weights
3. for every e in this order
4. if $(V, F \setminus \{e\})$ is connected then
5. $F \leftarrow F \setminus \{e\}$
6. return (V, F)

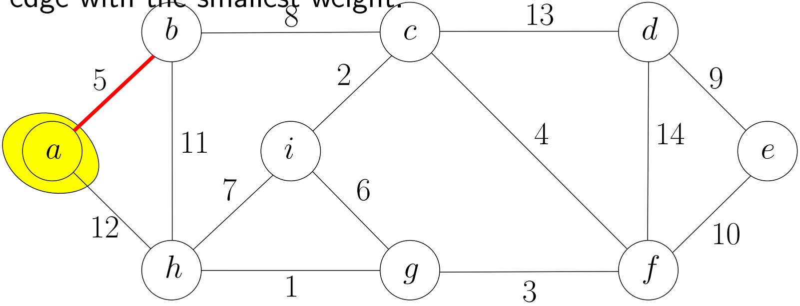
Reverse Kruskal's Algorithm: Example



Outline

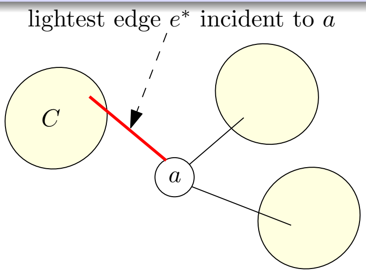
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



- Greedy strategy for Prim's algorithm: choose the lightest edge incident to a.

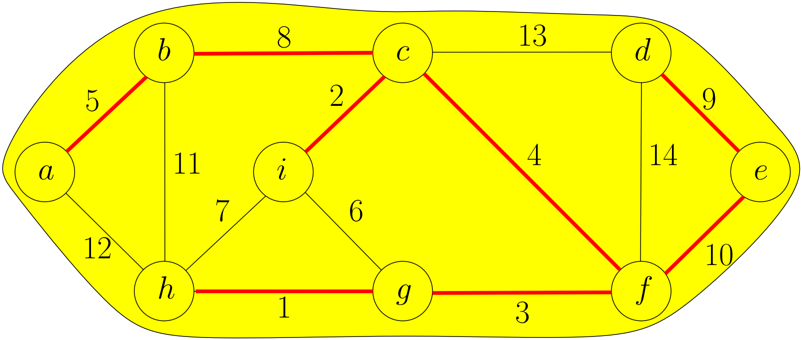
Lemma It is safe to include the lightest edge incident to a.



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus e \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ $\square$

Prim's Algorithm: Example



Greedy Algorithm

MST-Greedy1(G, w)

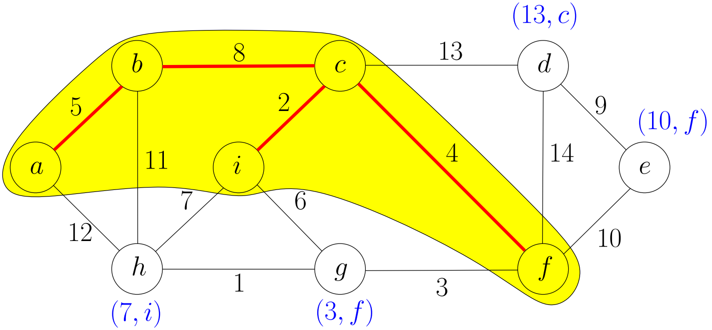
1. $S \leftarrow \{s\}$, where s is arbitrary vertex in V
2. $F \leftarrow \emptyset$
3. while $S \neq V$
4. $\quad (u, v) \leftarrow$ lightest edge between S and $V \setminus S$, where $u \in S$ and $v \in V \setminus S$
5. $\quad S \leftarrow S \cup \{v\}$
6. $\quad F \leftarrow F \cup \{(u, v)\}$
7. return (V, F)

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S



Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

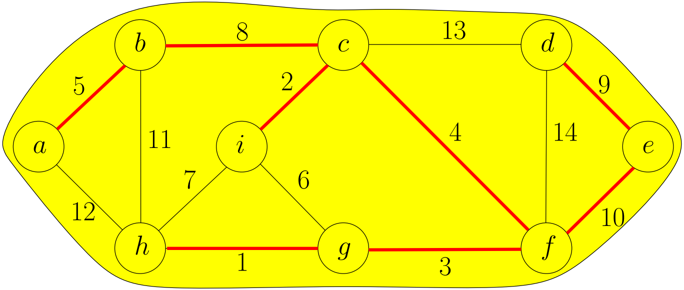
- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$, do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$
            $\pi(v) \leftarrow u$
return $\big\{(u, \pi(u)) | u \in V \setminus \{s\}\big\}$

Example



Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S:(u,v)\in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi(v) = \arg\min_{u \in S:(u,v)\in E} w(u,v)$:
  $(\pi(v), v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value extract_min
- Add $(\pi(u), u)$ to F
- Add u to S, update d and π values. decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$

while $S \neq V$, do
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $w(u, v) < d(v)$ then
            $d(v) \leftarrow w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$

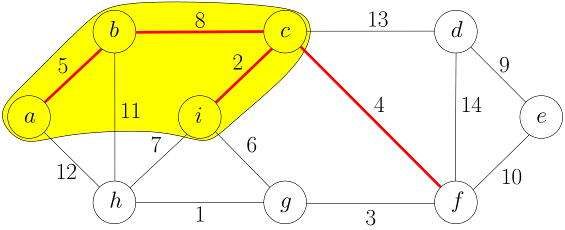
Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+$ $O(m)\times$ (time for decrease_key)

concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.



- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

"Evidence" for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

Exactly one of the following is true:

- There is a cut in which e is the lightest edge
- There is a cycle in which e is the heaviest edge

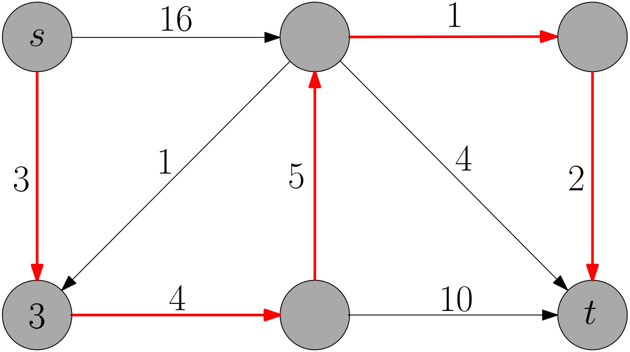
Thus, the minimum spanning tree is unique with assumption.

Outline

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$
$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest path from s to t



Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

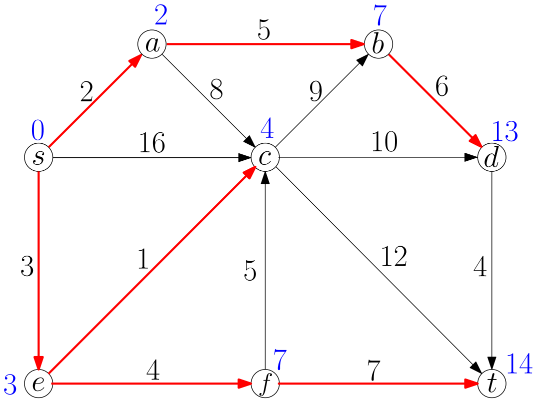
- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

- Shortest path from s to v may contain $\Omega(n)$ edges
- There are $\Omega(n)$ different vertices v
- Thus, printing out all shortest paths may take time $\Omega(n^2)$
- Not acceptable if graph is sparse

Shortest Path Tree

- $O(n)$-size data structure to represent all shortest paths
- For every vertex v, we only need to remember the parent of v: second-to-last vertex in the shortest path from s to v (why?)



Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

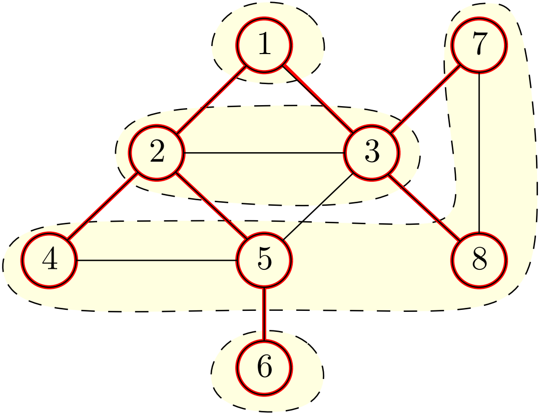
$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi(v), v \in V \setminus s$: the parent of v

$d(v), v \in V \setminus s$: the length of shortest path from s to v

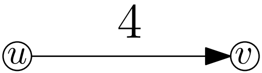
Q: How to compute shortest paths from s when all edges have weight 1?

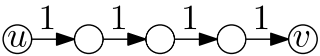
A: Breadth first search (BFS) from source s



Assumption Weights $w(u,v)$ are integers (w.l.o.g).

- An edge of weight $w(u,v)$ is equivalent to a pah of $w(u,v)$ unit-weight edges





Shortest Path Algorithm by Running BFS

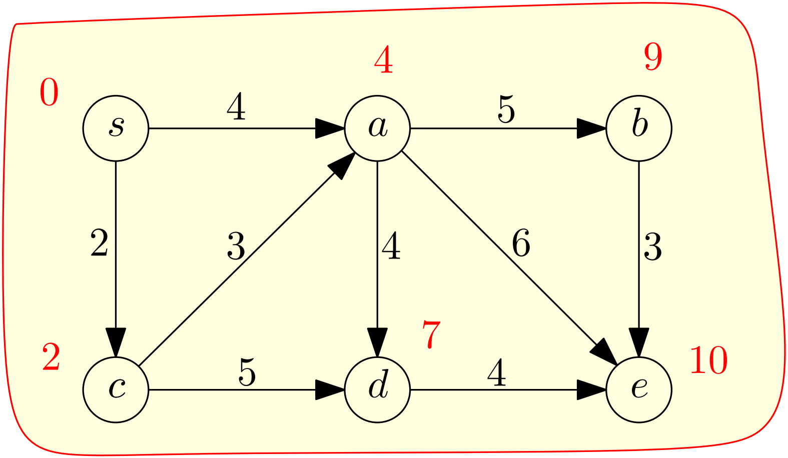
1. replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2. run BFS virtually
3. $\pi(v)$ = vertex from which v is visited
4. $d(v)$ = index of the level containing v

- Problem: $w(u,v)$ may be too large!

Shortest Path Algorithm by Running BFS Virtually

1. $S \leftarrow \{s\}, d(s) \leftarrow 0$
2. while $|S| \leq n$
3. find a $v \notin S$ that minimizes $\displaystyle\min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$
4. $S \leftarrow S \cup \{v\}$
5. $d(v) \leftarrow \min_{u \in S:(u,v) \in E} \{d(u) + w(u,v)\}$

Virtual BFS: Example



Time 10

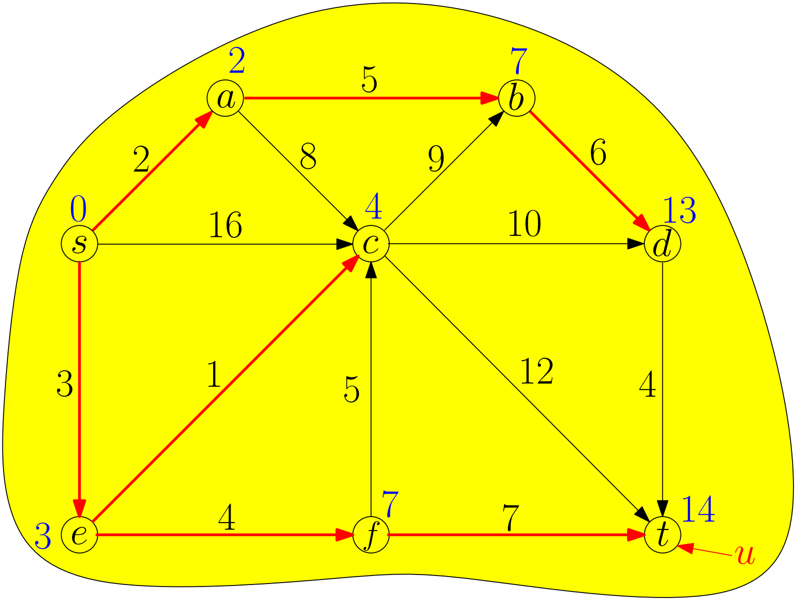
Outline

Dijkstra's Algorithm

Dijkstra(G, w, s)

1. $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
2. while $S \neq V$ do
3. $\quad u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
4. $\quad$ add u to S
5. $\quad$ for each $v \in V \setminus S$ such that $(u, v) \in E$
6. $\quad\quad$ if $d(u) + w(u, v) < d(v)$ then
7. $\quad\quad\quad d(v) \leftarrow d(u) + w(u, v)$
8. $\quad\quad\quad \pi(v) \leftarrow u$
9. return (d, π)

- Running time $= O(n^2)$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Improved Running Time using Priority Queue

Dijkstra(G, w, s)

```
S ← ∅, d(s) ← 0 and d(v) ← ∞ for every v ∈ V \ {s}
Q ← empty queue, for each v ∈ V: Q.insert(v, d(v))
while S ≠ V, do
    u ← Q.extract_min()
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E
        if d(u) + w(u, v) < d(v) then
            d(v) ← d(u) + w(u, v), Q.decrease_key(v, d(v))
            π(v) ← u
return (π, d)
```

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d(v) ← ∞ for every v ∈ V \ {s}
Q ← empty queue, for each v ∈ V: Q.insert(v, d(v))
while S ≠ V, do
    u ← Q.extract_min()
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E
        if w(u, v) < d(v) then
            d(v) ← w(u, v), Q.decrease_key(v, d(v))
            π(v) ← u
return {(u, π(u))|u ∈ V \ {s}}
```

Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
   - Bellman-Ford Algorithm
4. All-Pair Shortest Paths and Floyd-Warshall

Recall: Single Source Shortest Path Problem

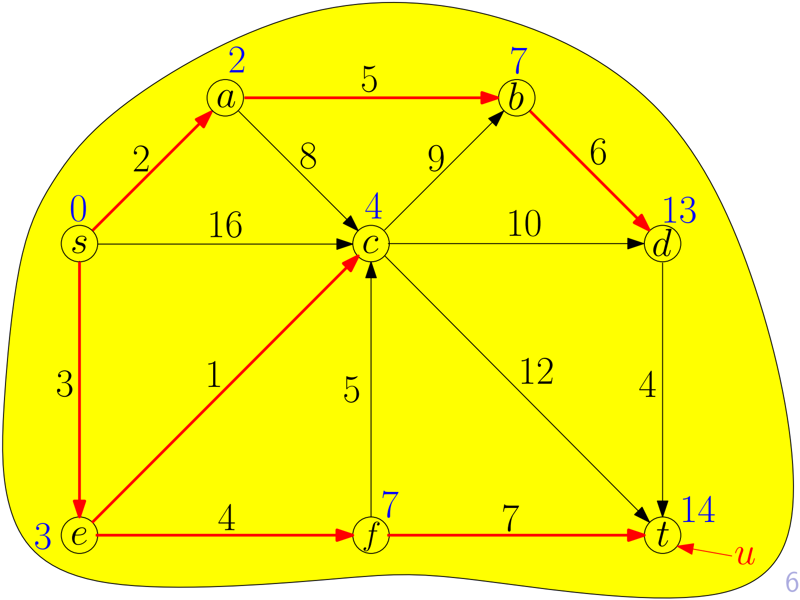
Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

- Algorithm for the problem: Dijkstra's algorithm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Dijkstra's Algorithm Using Priorty Queue

Dijkstra(G, w, s)

$S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d(v))$
while $S \neq V$, do
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$
        if $d(u) + w(u, v) < d(v)$ then
            $d(v) \leftarrow d(u) + w(u, v)$, Q.decrease_key$(v, d(v))$
            $\pi(v) \leftarrow u$
return (π, d)

- Running time $= O(m + n \lg n)$.

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

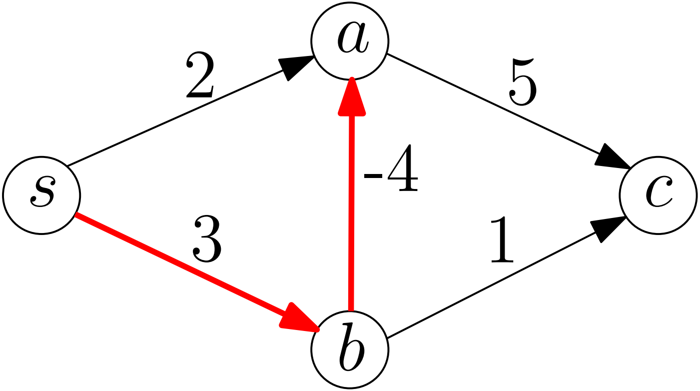
assume all vertices are reachable from s

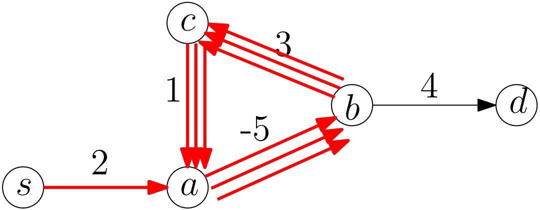
$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

Dijkstra's Algorithm Fails if We Have Negative Weights





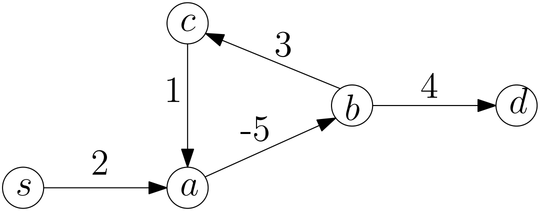
Q: What is the length of the shortest path from s to d?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"



Q: What is the length of the shortest simple path from s to d?

A: 1

- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

Outline

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

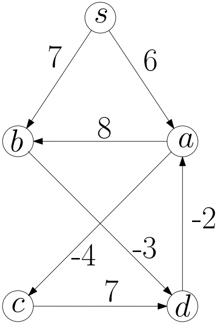
Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

- first try: $f[v]$: length of shortest path from s to v
- issue: do not know in which order we compute $f[v]$'s
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges



- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges
- $f^2[a] = 6$
- $f^3[a] = 2$

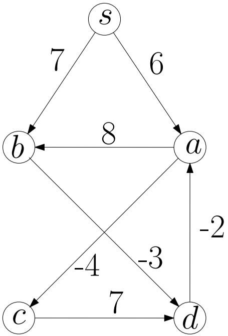
$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \begin{cases} f^{\ell-1}[v] \\ \min_{u:(u,v)\in E} \left(f^{\ell-1}[u] + w(u,v)\right) \end{cases} & \ell > 0 \end{cases}$$

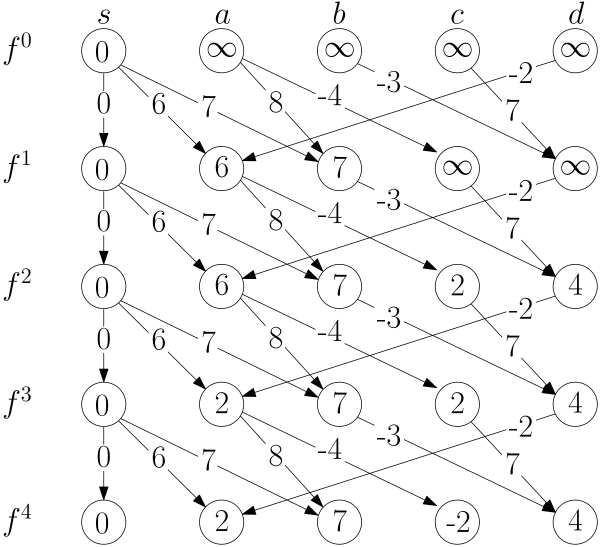
dynamic-programming(G, w, s)

```
f^0[s] ← 0 and f^0[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^{ℓ−1} → f^ℓ
    for each (u, v) ∈ E
        if f^{ℓ−1}[u] + w(u, v) < f^ℓ[v]
            f^ℓ[v] ← f^{ℓ−1}[u] + w(u, v)
return (f^{n−1}[v])_{v∈V}
```

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Dynamic Programming: Example





dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ do
    copy $f^{\ell-1} \rightarrow f^\ell$
    for each $(u,v) \in E$
        if $f^{\ell-1}[u] + w(u,v) < f^\ell[v]$
            $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u,v)$
return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Q: What if there are negative cycles?

Dynamic Programming With Negative Cycle Detection

dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$

for $\ell \leftarrow 1$ to $n-1$ do

    copy $f^{\ell-1} \to f^{\ell}$

    for each $(u, v) \in E$

        if $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$

            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$

for each $(u, v) \in E$

    if $f^{n-1}[u] + w(u, v) < f^{n-1}[v]$

        report "negative cycle exists" and exit

return $(f^{n-1}[v])_{v \in V}$

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

1. $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n-1$ do
3.   for each $(u, v) \in E$
4.   if $f[u] + w(u, v) < f[v]$
5. $f[v] \leftarrow f[u] + w(u, v)$
6. return f

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!
- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

```
f[s] ← 0 and f[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    for each (u, v) ∈ E
        if f[u] + w(u, v) < f[v]
            f[v] ← f[u] + w(u, v)
return f
```

- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v
- Assuming there are no negative cycles, after iteration $n-1$, $f[v] =$ length of shortest path from s to v

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

```
f[s] ← 0 and f[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n do
    updated ← false
    for each (u, v) ∈ E
        if f[u] + w(u, v) < f[v]
            f[v] ← f[u] + w(u, v), π[v] ← u
            updated ← true
    if not updated, then return f
output "negative cycle exists"
```

- $\pi[v]$: the parent of v in the shortest path tree
- Running time $= O(nm)$

Outline

Summary of Shortest Path Algorithms we learned

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

All-Pair Shortest Paths

All Pair Shortest Paths

Input: directed graph $G = (V, E)$,
$w : E \to \mathbb{R}$ (can be negative)

Output: shortest path from u to v for every $u, v \in V$

1. for every starting point $s \in V$ do
2.   run Bellman-Ford(G, w, s)

- Running time $= O(n^2 m)$

Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$
- For simplicity, extend the w values to non-edges:

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

- For now assume there are no negative cycles

Cells for Floyd-Warshall Algorithm

- First try: $f[i,j]$ is length of shortest path from i to j
- Issue: do not know in which order we compute $f[i,j]$'s
- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1,2,3,\cdots,k\}$ as intermediate vertices

$$f^k[i,j] = \begin{cases} w(i,j) & k = 0 \\ \min \begin{cases} f^{k-1}[i,j] \\ f^{k-1}[i,k] + f^{k-1}[k,j] \end{cases} & k = 1,2,\cdots,n \end{cases}$$

Floyd-Warshall(G, w)

$f^0 \leftarrow w$
for $k \leftarrow 1$ to n do
    copy $f^{k-1} \rightarrow f^k$
    for $i \leftarrow 1$ to n do
        for $j \leftarrow 1$ to n do
            if $f^{k-1}[i, k] + f^{k-1}[k, j] < f^k[i, j]$ then
                $f^k[i, j] \leftarrow f^{k-1}[i, k] + f^{k-1}[k, j]$

Floyd-Warshall(G, w)

$f^{\text{old}} \leftarrow w$
for $k \leftarrow 1$ to n do
  copy $f^{\text{old}} \rightarrow f^{\text{new}}$
  for $i \leftarrow 1$ to n do
    for $j \leftarrow 1$ to n do
      if $f^{\text{old}}[i, k] + f^{\text{old}}[k, j] < f^{\text{new}}[i, j]$ then
        $f^{\text{new}}[i, j] \leftarrow f^{\text{old}}[i, k] + f^{\text{old}}[k, j]$

Lemma Assume there are no negative cycles in G. After iteration k, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from i to j that only uses vertices in $\{1, 2, 3, \cdots, k\}$ as intermediate vertices.

- Running time $= O(n^3)$.

Recovering Shortest Paths

Floyd-Warshall(G, w)

$f \leftarrow w$, $\pi[i, j] \leftarrow \perp$ for every $i, j \in V$
for $k \leftarrow 1$ to n do
    for $i \leftarrow 1$ to n do
        for $j \leftarrow 1$ to n do
            if $f[i, k] + f[k, j] < f[i, j]$ then
                $f[i, j] \leftarrow f[i, k] + f[k, j]$, $\pi[i, j] \leftarrow k$

print-path(i, j)

if $\pi[i, j] = \perp$ then
    if $i \neq j$ then print$(i, \text{“,”})$
else
    print-path$(i, \pi[i, j])$, print-path$(\pi[i, j], j)$

Detecting Negative Cycles

Floyd-Warshall(G, w)

```
f ← w, π[i, j] ← ⊥ for every i, j ∈ V
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            if f[i, k] + f[k, j] < f[i, j] then
                f[i, j] ← f[i, k] + f[k, j], π[i, j] ← k
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            if f[i, k] + f[k, j] < f[i, j] then
                report "negative cycle exists" and exit
```

Summary of Shortest Path Algorithms

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
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Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs