CSE 431/531: Algorithm Analysis and Design (Spring 2020)

NP-Completeness

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University at Buffalo

NP-Completeness Theory

- The topics we discussed so far are positive results: how to design efficient algorithms for solving a given problem.
- NP-Completeness provides negative results: some problems can not be solved efficiently.

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Q: Why do we study negative results?

- A given problem X cannot be solved in polynomial time.
- Without knowing it, you will have to keep trying to find polynomial time algorithm for solving X. All our efforts are doomed!

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- A good cut separating problems: for most natural problems, either we have a polynomial time algorithm, or the best algorithm runs in time $\Omega(2^{n^c})$ for some c
- Do not need to worry about the computational model

Outline

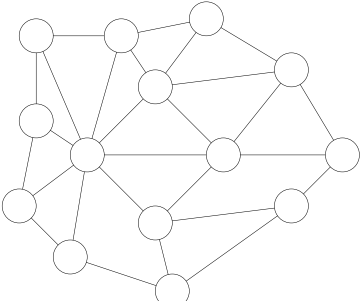
Example: Hamiltonian Cycle Problem

Def. Let G be an undirected graph. A Hamiltonian Cycle (HC) of G is a cycle C in G that passes each vertex of G exactly once.

Hamiltonian Cycle (HC) Problem

Input: graph $G = (V, E)$

Output: whether G contains a Hamiltonian cycle



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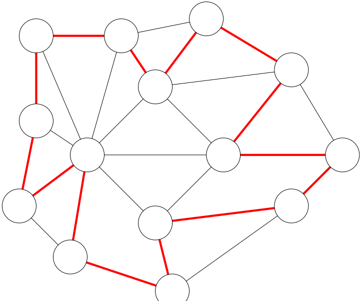

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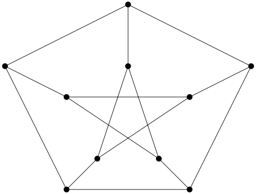

- The graph is called the Petersen Graph. It has no HC.

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- Far away from polynomial time
- HC is NP-hard: it is unlikely that it can be solved in polynomial time.

Maximum Independent Set Problem

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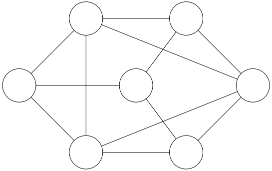

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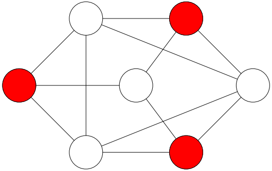

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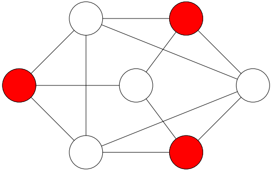

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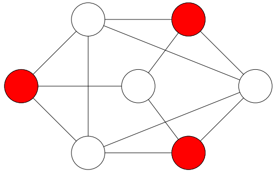

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- Maximum Independent Set is NP-hard

Formula Satisfiability

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Input: boolean formula with n variables, with $\vee, \wedge, \neg$ operators.

Output: whether the boolean formula is satisfiable

- Example: $\neg((\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3))$ is not satisfiable
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Fact For each optimization problem X, there is a decision version X' of the problem. If we have a polynomial time algorithm for the decision version X', we can solve the original problem X in polynomial time.

Optimization to Decision

Shortest Path

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Output: whether there is a path from s to t of length at most L

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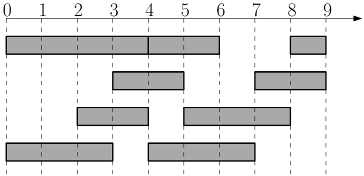

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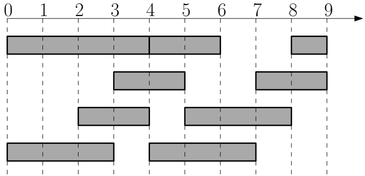

- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$

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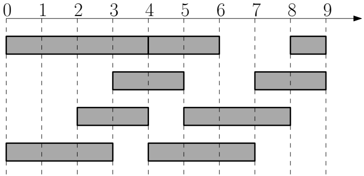

- $(0, 3, 0, 4, 2, 4, 3, 5, 4, 6, 4, 7, 5, 8, 7, 9, 8, 9)$
- Encode the sequence into a binary string as before

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A: No! As long as we are using a "natural" encoding. We only care whether the running time is polynomial or not

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Def. A has a polynomial running time if there is a polynomial function $p(\cdot)$ so that for every string s, the algorithm A terminates on s in at most $p(|s|)$ steps.

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- The decision versions of interval scheduling, shortest path and minimum spanning tree all in P.

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Def. The message Alice sends to Bob is called a certificate, and the algorithm Bob runs is called a certifier.

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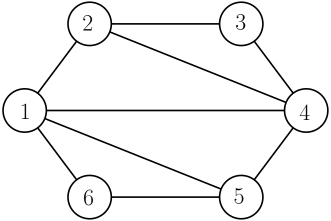

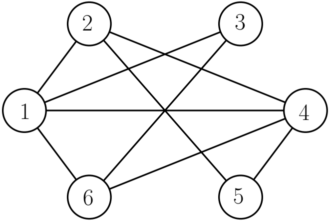

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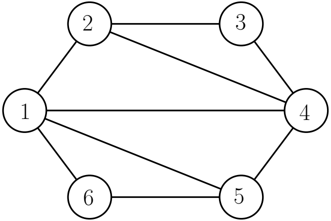

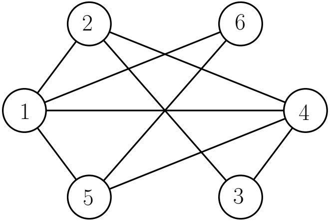

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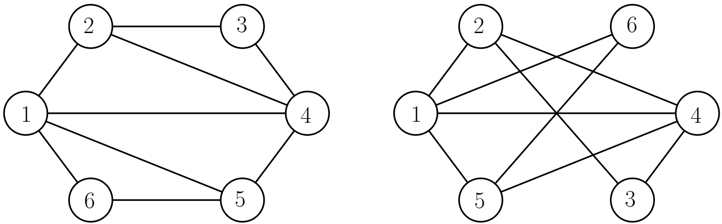

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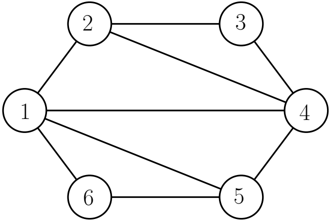

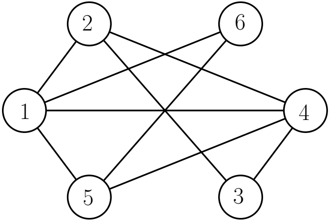

- What is the certificate?
- What is the certifier?

The Complexity Class NP

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t
- there is a polynomial function p such that, $s \in X$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s, t) = 1$.

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Def. The complexity class NP is the set of all problems for which there exists an efficient certifier.

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- $G \in$ HC $\quad\Longleftrightarrow\quad \exists S,\ B(G, S) = 1$

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- Input: two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ on V
- Certificate: a 1-1 function $f : V \to V$
- $|\text{encoding}(f)| \leq p(|\text{encoding}(G_1, G_2)|)$ for some polynomial function p
- Certifier B: $B((G_1, G_2), f) = 1$ if and only if for every $u, v \in V$, we have $(u, v) \in E_1 \Leftrightarrow (f(u), f(v)) \in E_2$.
- Clearly, B runs in polynomial time
- $(G_1, G_2) \in \text{GI} \quad \Longleftrightarrow \quad \exists f,\ B((G_1, G_2), f) = 1$

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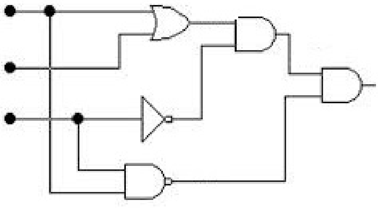
Circuit Satisfiablity (Circuit-Sat) Problem

Input: a circuit with and/or/not gates

Output: whether there is an assignment such that the output is 1?



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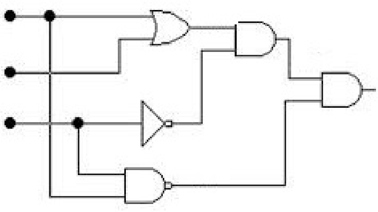

- Is Circuit-Sat $\in$ NP?

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The Complexity Class Co-NP

Def. For a problem X, the problem $\overline{X}$ is the problem such that $s \in \overline{X}$ if and only if $s \notin X$.

Def. Co-NP is the set of decision problems X such that $\overline{X} \in$ NP.

Def. A tautology is a boolean formula that always evaluates to 1.

Tautology Problem

Input: a boolean formula

Output: whether the formula is a tautology

- e.g. $(\neg x_1 \wedge x_2) \vee (\neg x_1 \wedge \neg x_3) \vee x_1 \vee (\neg x_2 \wedge x_3)$ is a tautology

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- Similarly, P $\subseteq$ Co-NP, thus P $\subseteq$ NP $\cap$ Co-NP

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  - if P $\neq$ NP, then HC $\notin$ P
  - HC $\notin$ P, unless P $=$ NP

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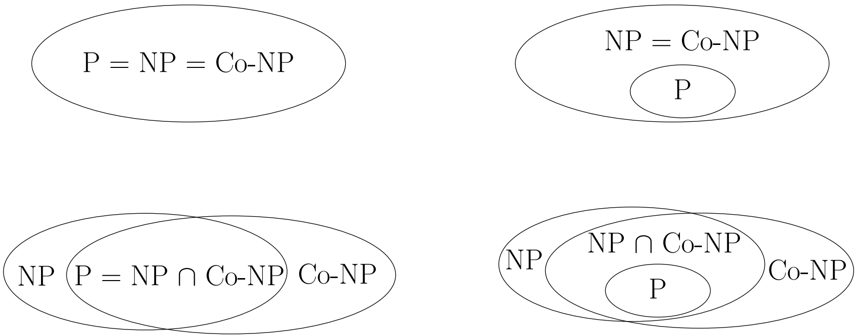
- General belief: NP $\neq$ Co-NP.

4 Possibilities of Relationships

Notice that $X \in \text{NP} \iff \overline{X} \in \text{Co-NP}$ and $\text{P} \subseteq \text{NP} \cap \text{Co-NP}$



- General belief: we are in the 4th scenario

Outline

Polynomial-Time Reducations

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \le_P X$.

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To prove negative results:

Suppose $Y \leq_P X$. If Y cannot be solved in polynomial time, then X cannot be solved in polynomial time.

Polynomial-Time Reduction: Example

Hamiltonian-Path (HP) problem

Input: $G = (V, E)$ and $s, t \in V$

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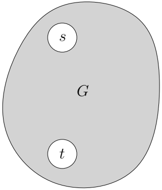

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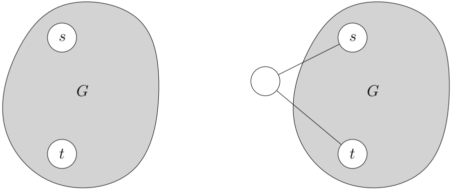

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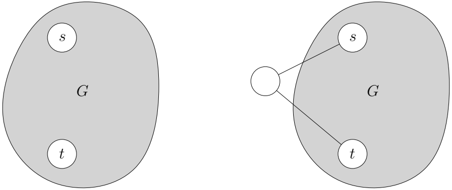

Obs. G has a HP from s to t if and only if graph on right side has a HC.

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- If you believe $\text{P} \neq \text{NP}$, and proved that a problem X is NP-complete (or NP-hard), stop trying to design efficient algorithms for X

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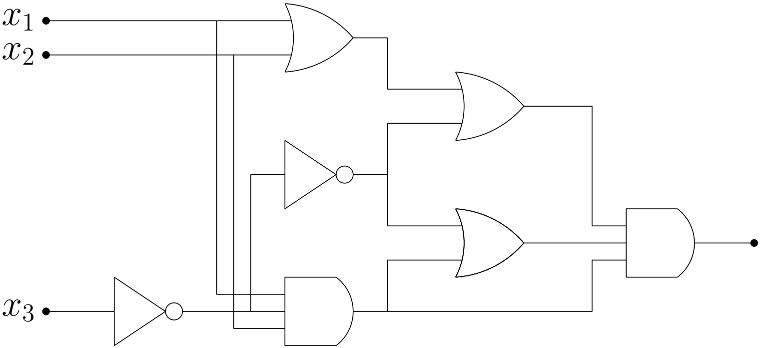
- No! There is indeed a large family of natural NP-complete problems

The First NP-Complete Problem: Circuit-Sat

Circuit Satisfiability (Circuit-Sat)

Input: a circuit

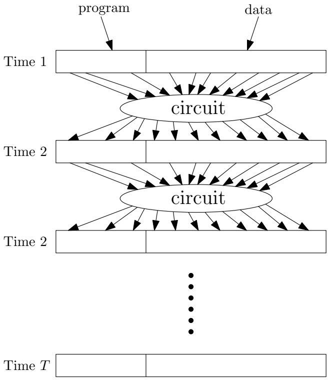
Output: whether the circuit is satisfiable



Circuit-Sat is NP-Complete

- key fact: algorithms can be converted to circuits

Fact Any algorithm that takes n bits as input and outputs $0/1$ with running time $T(n)$ can be converted into a circuit of size $p(T(n))$ for some polynomial function $p(\cdot)$.



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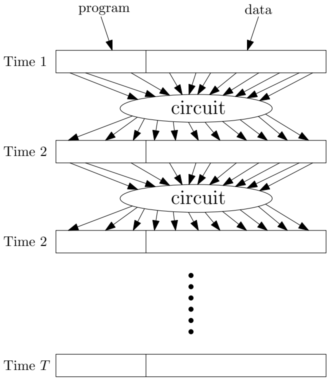

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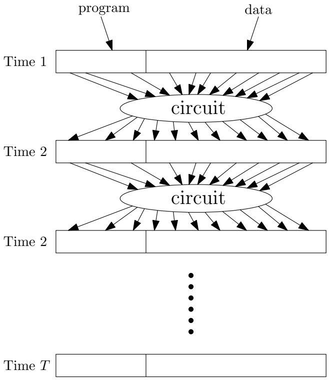

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- We prove HC $\leq_P$ Circuit-Sat as an example.

HC $\leq_P$ Circuit-Sat

check-HC(G, S)

- Let check-HC(G, S) be the certifier for the Hamiltonian cycle problem: check-HC(G, S) returns 1 if S is a Hamiltonian cycle is G and 0 otherwise.

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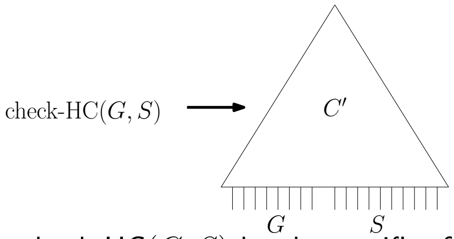

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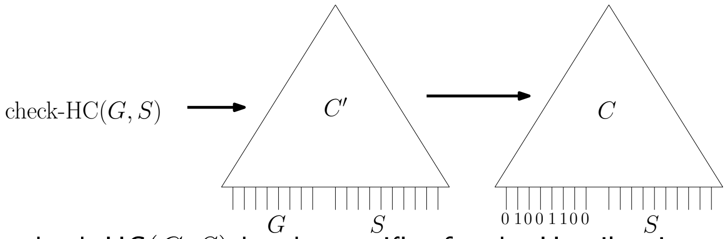

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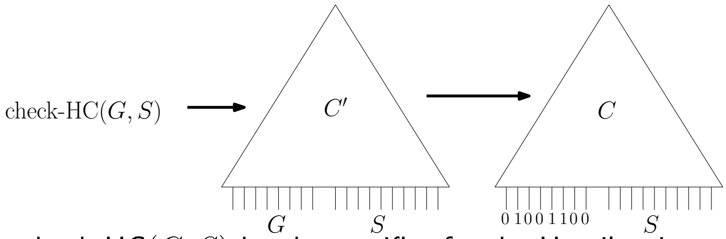

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- G is a yes-instance if and only if C is satisfiable

$Y \leq_P$ Circuit-Sat, For Every $Y \in$NP

- Let check-Y(s, t) be the certifier for problem Y: check-Y(s, t) returns 1 if t is a valid certificate for s.
- s is a yes-instance if and only if there is a t such that check-Y(s, t) returns 1

- Construct a circuit C' for the algorithm check-Y
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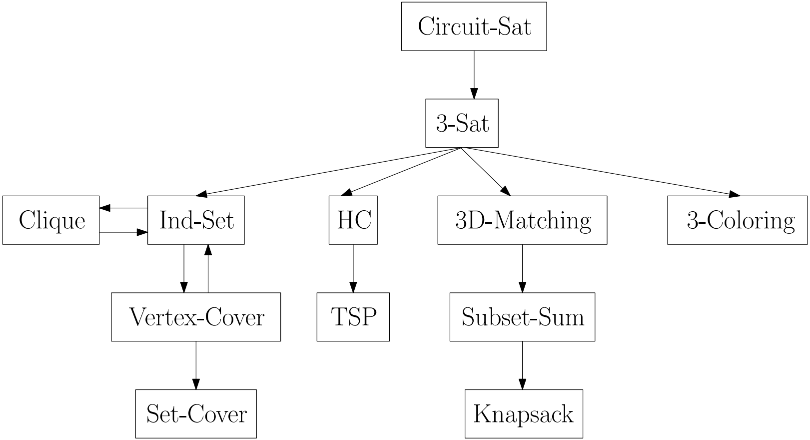
Theorem Circuit-Sat is NP-complete.

Reductions of NP-Complete Problems



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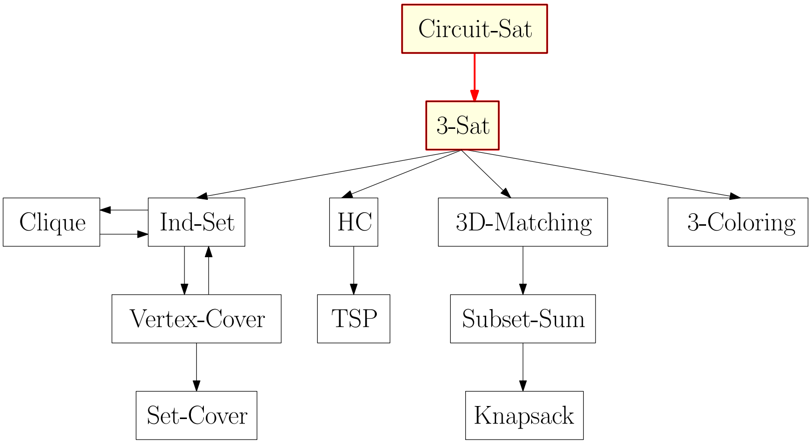

3-Sat

3-CNF (conjunctive normal form) is a special case of formula:

3-Sat

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- Boolean variables: $x_1, x_2, \cdots, x_n$
- Literals: x_i or $\neg x_i$
- Clause: disjunction ("or") of at most 3 literals: $x_3 \vee \neg x_4$, $x_1 \vee x_8 \vee \neg x_9, \quad \neg x_2 \vee \neg x_5 \vee x_7$

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- Boolean variables: $x_1, x_2, \cdots, x_n$
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- Clause: disjunction ("or") of at most 3 literals: $x_3 \vee \neg x_4$, $x_1 \vee x_8 \vee \neg x_9, \quad \neg x_2 \vee \neg x_5 \vee x_7$
- 3-CNF formula: conjunction ("and") of clauses: $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

3-Sat

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Input: a 3-CNF formula

Output: whether the 3-CNF is satisfiable

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- To satisfy a 3-CNF, we need to satisfy all clauses
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3-Sat

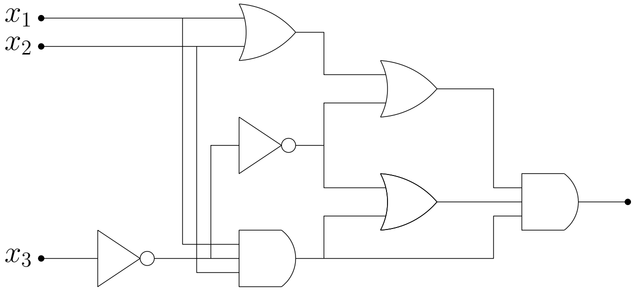
3-Sat

Input: a 3-CNF formula

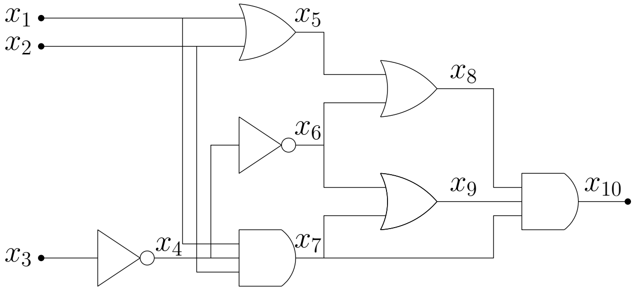
Output: whether the 3-CNF is satisfiable

- To satisfy a 3-CNF, we need to satisfy all clauses
- To satisfy a clause, we need to satisfy at least 1 literal
- Assignment $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$ satisfies $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee \neg x_4)$

Circuit-Sat $\leq_P$ 3-Sat

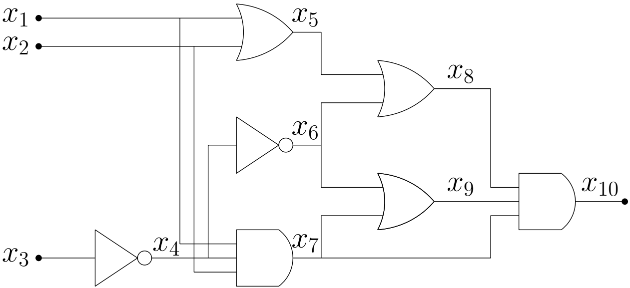


Circuit-Sat $\leq_P$ 3-Sat



- Associate every wire with a new variable

Circuit-Sat $\leq_P$ 3-Sat



- Associate every wire with a new variable
- The circuit is equivalent to the following formula:

$$\begin{aligned}&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)\\ &\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)\\ &\wedge (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}\end{aligned}$$

Circuit-Sat $\leq_P$ 3-Sat

$$\begin{aligned}
&(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4) \\
&\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6) \\
&\wedge (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}
\end{aligned}$$

Convert each clause to a 3-CNF

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

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$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$

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1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
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x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
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0	1	0	0
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1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
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Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$

$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
$$\wedge\, (x_9 = x_6 \vee x_9) \wedge (x_{10} = x_8 \wedge x_9 \wedge x_7) \wedge x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
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Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee x_2 \vee x_5) \quad \wedge$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \land (x_5 = x_1 \lor x_2) \land (x_6 = \neg x_4)$$
$$\land\, (x_7 = x_1 \land x_2 \land x_4) \land (x_8 = x_5 \lor x_6)$$
$$\land\, (x_9 = x_6 \lor x_9) \land (x_{10} = x_8 \land x_9 \land x_7) \land x_{10}$$

Convert each clause to a 3-CNF

$$x_5 = x_1 \lor x_2 \quad \Leftrightarrow$$

$$(x_1 \lor x_2 \lor \neg x_5) \quad \land$$
$$(x_1 \lor \neg x_2 \lor x_5) \quad \land$$
$$(\neg x_1 \lor x_2 \lor x_5) \quad \land$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \lor x_2$
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

Circuit-Sat $\leq_P$ 3-Sat

$$(x_4 = \neg x_3) \wedge (x_5 = x_1 \vee x_2) \wedge (x_6 = \neg x_4)$$
$$\wedge\, (x_7 = x_1 \wedge x_2 \wedge x_4) \wedge (x_8 = x_5 \vee x_6)$$
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Convert each clause to a 3-CNF

$$x_5 = x_1 \vee x_2 \quad \Leftrightarrow$$

$$(x_1 \vee x_2 \vee \neg x_5) \quad \wedge$$
$$(x_1 \vee \neg x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee x_2 \vee x_5) \quad \wedge$$
$$(\neg x_1 \vee \neg x_2 \vee x_5)$$

x_1	x_2	x_5	$x_5 \leftrightarrow x_1 \vee x_2$
0	0	0	1
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Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF

Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\Longleftrightarrow$ Formula $\Longleftrightarrow$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable

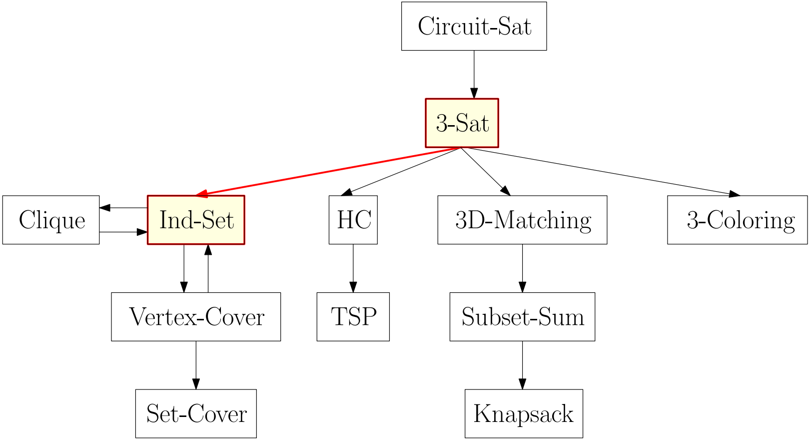
Circuit-Sat $\leq_P$ 3-Sat

- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit

Circuit-Sat $\leq_P$ 3-Sat

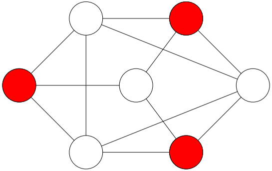
- Circuit $\iff$ Formula $\iff$ 3-CNF
- The circuit is satisfiable if and only if the 3-CNF is satisfiable
- The size of the 3-CNF formula is polynomial (indeed, linear) in the size of the circuit
- Thus, Circuit-Sat $\leq_P$ 3-Sat

Reductions of NP-Complete Problems



Recall: Independent Set Problem

Def. An independent set of $G = (V, E)$ is a subset $I \subseteq V$ such that no two vertices in I are adjacent in G.



Independent Set (Ind-Set) Problem

Input: $G = (V, E), k$

Output: whether there is an independent set of size k in G

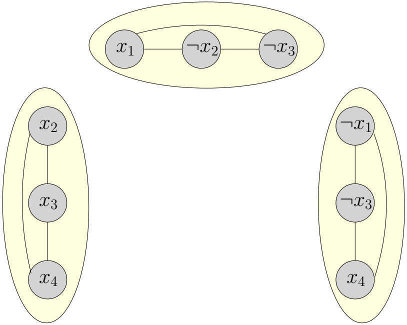
3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

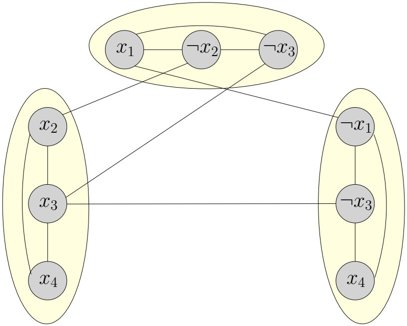
- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group



3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

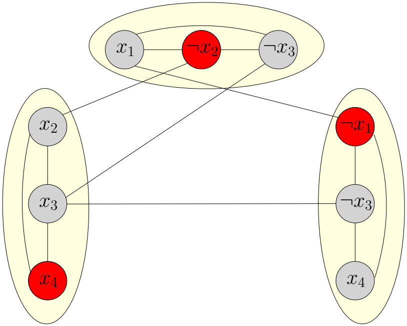
- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals



3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals
- Problem: whether there is an IS of size $k = \#$clauses



3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

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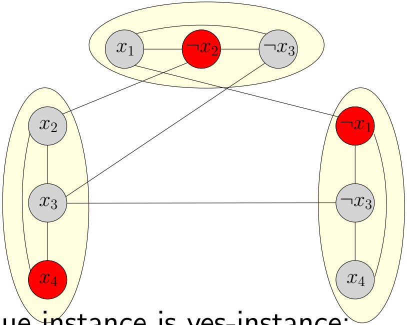
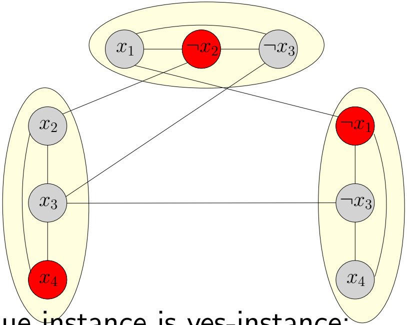


3-Sat instance is yes-instance $\Leftrightarrow$ clique instance is yes-instance:

3-Sat $\leq_P$ Ind-Set

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- A clause $\Rightarrow$ a group of 3 vertices, one for each literal
- An edge between every pair of vertices in same group
- An edge between every pair of contradicting literals
- Problem: whether there is an IS of size $k = \#$clauses

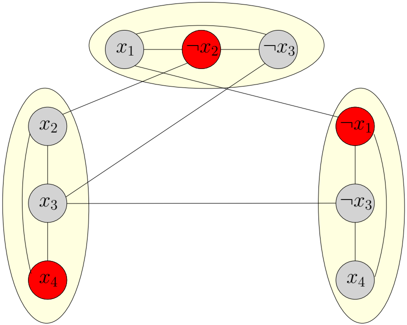


3-Sat instance is yes-instance $\Leftrightarrow$ clique instance is yes-instance:

- satisfying assignment $\Rightarrow$ independent set of size k
- independent set of size k $\Rightarrow$ satisfying assignment

Satisfying Assignment $\Rightarrow$ IS of Size k

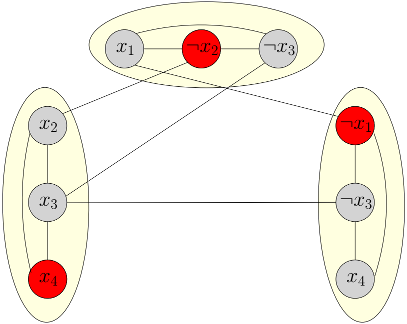
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

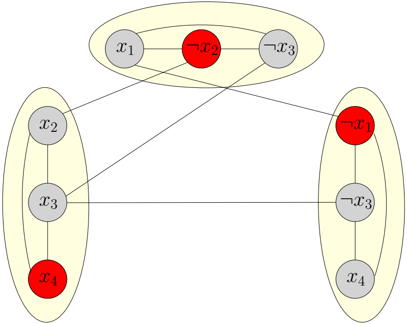
- For every clause, at least 1 literal is satisfied



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

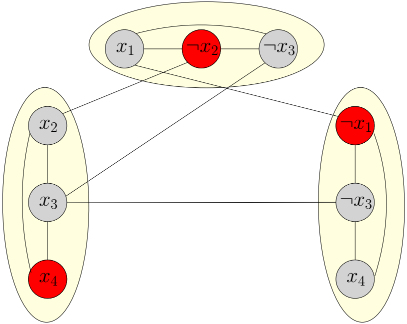
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

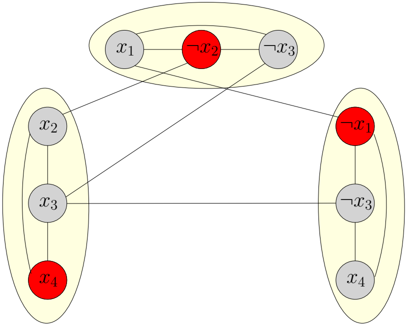
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group



Satisfying Assignment $\Rightarrow$ IS of Size k

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

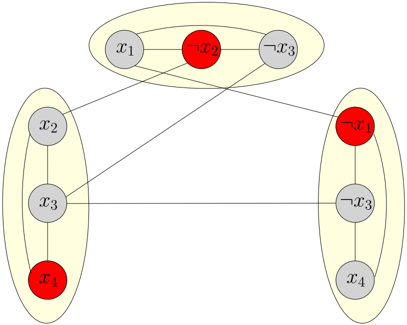
- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group
- No contradictions among the selected literals



Satisfying Assignment $\Rightarrow$ IS of Size k

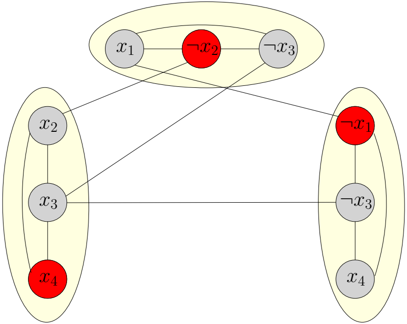
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every clause, at least 1 literal is satisfied
- Pick the vertex correspondent the literal
- So, 1 literal from each group
- No contradictions among the selected literals
- An IS of size k



IS of Size $k \Rightarrow$ Satisfying Assignment

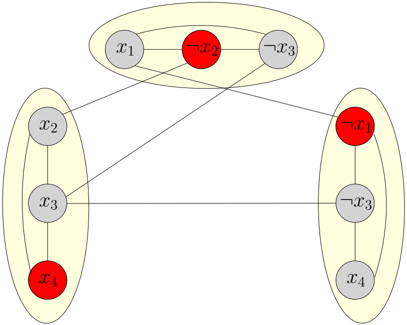
- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

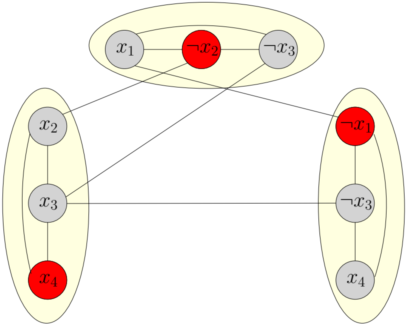
- For every group, exactly one literal is selected in IS



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

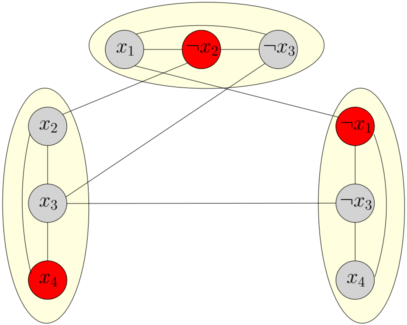
- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
- If x_i is selected in IS, set $x_i = 1$



IS of Size $k \Rightarrow$ Satisfying Assignment

- $(x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_2 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$

- For every group, exactly one literal is selected in IS
- No contradictions among the selected literals
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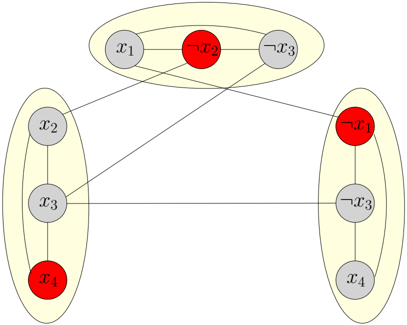

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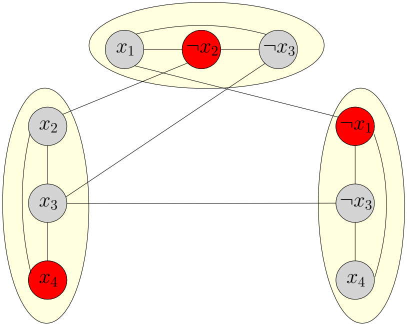
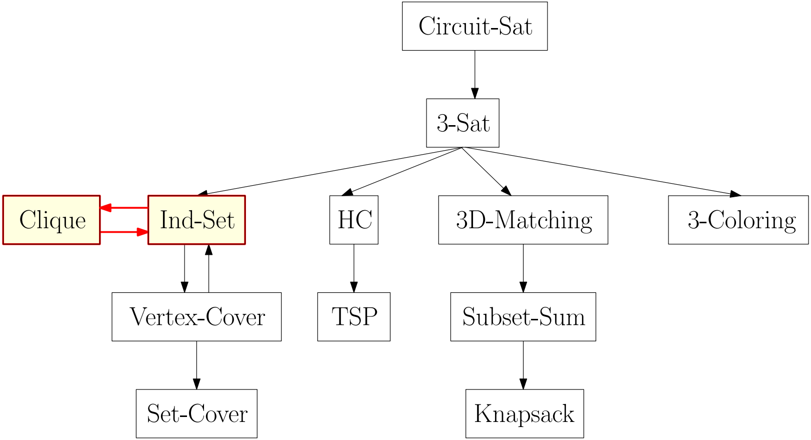
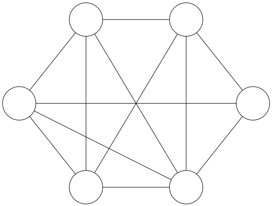


Reductions of NP-Complete Problems



Def. A clique in an undirected graph $G = (V, E)$ is a subset $S \subseteq V$ such that $\forall u, v \in S$ we have $(u, v) \in E$



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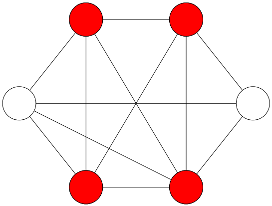

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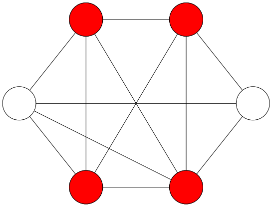

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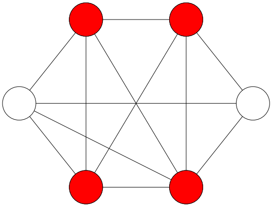

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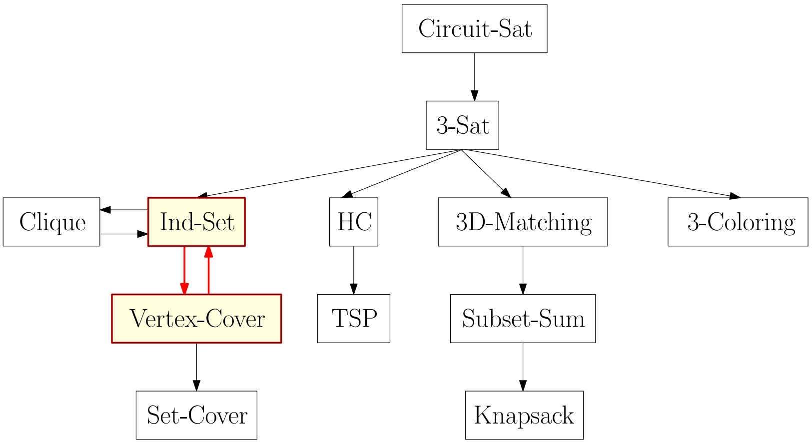
- What is the relationship between Clique and Ind-Set?

Clique $=_P$ Ind-Set

Def. Given a graph $G = (V, E)$, define $\overline{G} = (V, \overline{E})$ be the graph such that $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$.

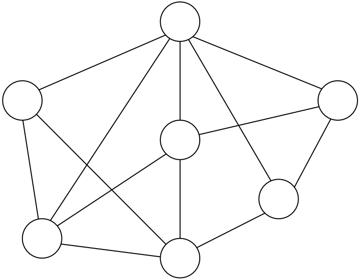
Obs. S is an independent set in G if and only if S is a clique in $\overline{G}$.

Reductions of NP-Complete Problems



Vertex-Cover

Def. Given a graph $G = (V, E)$, a vertex cover of G is a subset $S \subseteq V$ such that for every $(u, v) \in E$ then $u \in S$ or $v \in S$.



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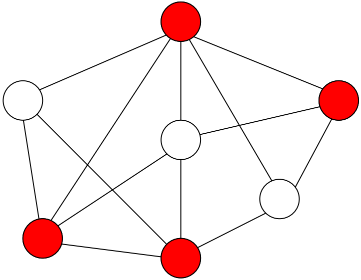

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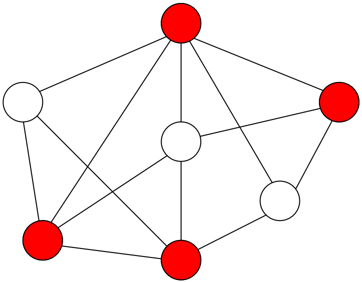

Vertex-Cover Problem

Input: $G = (V, E)$ and integer k

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Vertex-Cover $=_P$ Ind-Set

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Q: What is the relationship between Vertex-Cover and Ind-Set?

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A: S is a vertex-cover of $G = (V, E)$ if and only if $V \setminus S$ is an independent set of G.

A Strategy of Polynomial Reduction

Recall the definition of polynomial time reductions:

Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

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- However, for most reductions, we call algorithm for X only once
- That is, for a given instance s_Y for Y, we only construct one instance s_X for X

A Strategy of Polynomial Reduction

- Given an instance s_Y of problem Y, show how to construct in polynomial time an instance s_X of problem such that:
  - s_Y is a yes-instance of $Y \Rightarrow s_X$ is a yes-instance of X
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- Essentially we have no techniques for proving lower bound for running time

Dealing with NP-Hard Problems

- Faster exponential time algorithms
- Solving the problem for special cases
- Fixed parameter tractability
- Approximation algorithms

Faster Exponential Time Algorithms

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- Brute-force: $O(n! \cdot \text{poly}(n))$
- Better algorithm: $O(2^n \cdot \text{poly}(n))$
- In practice: TSP Solver can solve Euclidean TSP instances with more than 100,000 vertices

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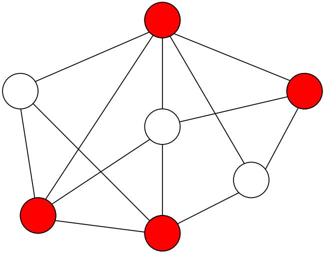

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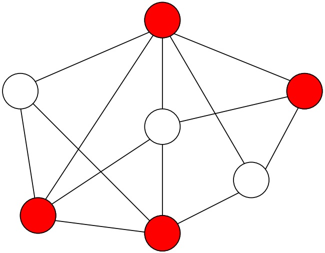

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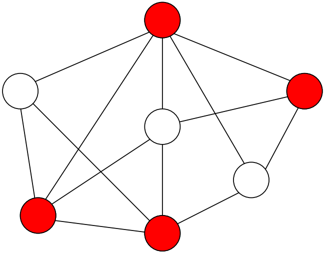

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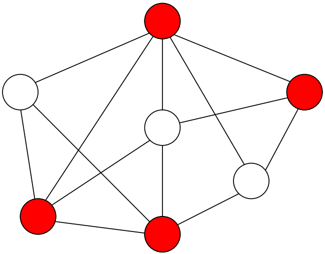

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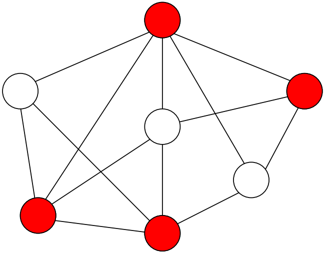
- Vertex-Cover is fixed-parameter tractable.



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- 2-approximation for vertex-cover
- $O(\lg n)$-approximation for set-cover

Outline

Summary

- We consider decision problems
- Inputs are encoded as $\{0, 1\}$-strings

Def. The complexity class P is the set of decision problems X that can be solved in polynomial time.

- Alice has a supercomputer, fast enough to run an exponential time algorithm
- Bob has a slow computer, which can only run a polynomial-time algorithm

Def. (Informal) The complexity class NP is the set of problems for which Alice can convince Bob a yes instance is a yes instance

Summary

Def. B is an efficient certifier for a problem X if

- B is a polynomial-time algorithm that takes two input strings s and t
- there is a polynomial function p such that, $s \in X$ if and only if there is string t such that $|t| \leq p(|s|)$ and $B(s,t) = 1$.

The string t such that $B(s,t) = 1$ is called a certificate.

Def. The complexity class NP is the set of all problems for which there exists an efficient certifier.

Summary

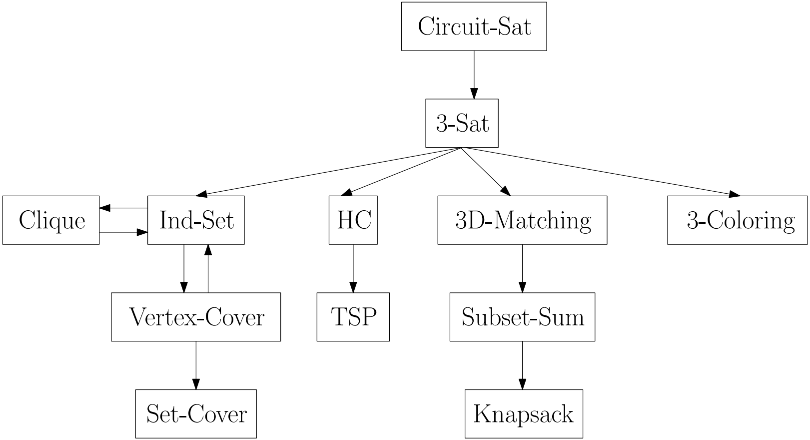
Def. Given a black box algorithm A that solves a problem X, if any instance of a problem Y can be solved using a polynomial number of standard computational steps, plus a polynomial number of calls to A, then we say Y is polynomial-time reducible to X, denoted as $Y \leq_P X$.

Def. A problem X is called NP-complete if

1. $X \in$ NP, and
2. $Y \leq_{\text{P}} X$ for every $Y \in$ NP.

- If any NP-complete problem can be solved in polynomial time, then $P = NP$
- Unless $P = NP$, a NP-complete problem can not be solved in polynomial time

Summary



Circuit-Sat
3-Sat
Clique
Ind-Set
HC
3D-Matching
3-Coloring
Vertex-Cover
TSP
Subset-Sum
Set-Cover
Knapsack

Summary

Proof of NP-Completeness for Circuit-Sat

- Fact 1: a polynomial-time algorithm can be converted to a polynomial-size circuit
- Fact 2: for a problem in NP, there is a efficient certifier.
- Given a problem $X \in$ NP, let $B(s, t)$ be the certifier
- Convert $B(s, t)$ to a circuit and hard-wire s to the input gates
- s is a yes-instance if and only if the resulting circuit is satisfiable

- Proof of NP-Completeness for other problems by reductions