CSE 431/531: Algorithm Analysis and Design (Spring 2021)
Dynamic Programming

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo

Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many independent sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Recall: Computing the n-th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

Fib(n)

```
1: $F[0] \leftarrow 0$
2: $F[1] \leftarrow 1$
3: for $i \leftarrow 2$ to $n$ do
4:     $F[i] \leftarrow F[i-1] + F[i-2]$
5: return $F[n]$
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- Store each $F[i]$ for future use.

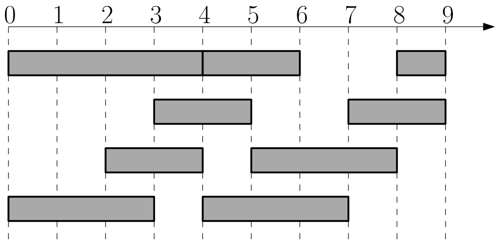
Outline

Recall: Interval Schduling

Input: n jobs, job i with start time s_i and finish time f_i

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs



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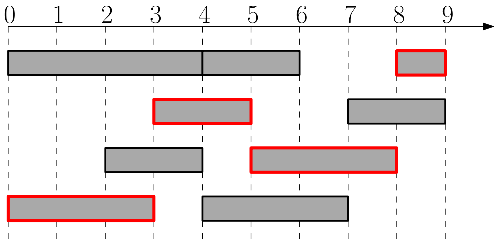

Weighted Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i
each job has a weight (or value) $v_i > 0$
i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-weight subset of mutually compatible jobs

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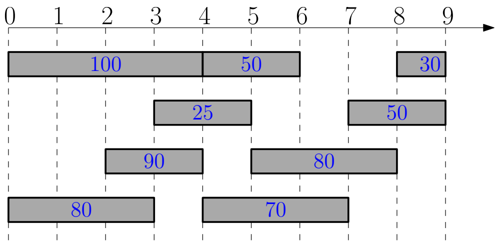

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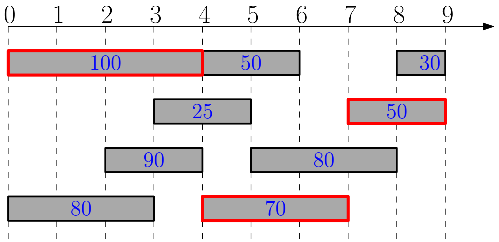

Optimum value $= 220$

Hard to Design a Greedy Algorithm

Q: Which job is safe to schedule?

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- Job with the earliest finish time?

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- Job with the largest weight?

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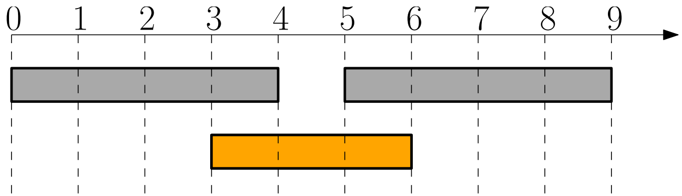

Designing a Dynamic Programming Algorithm

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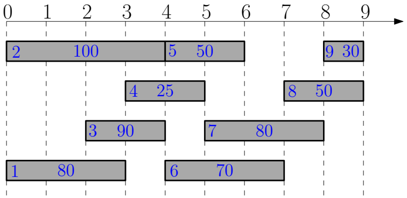
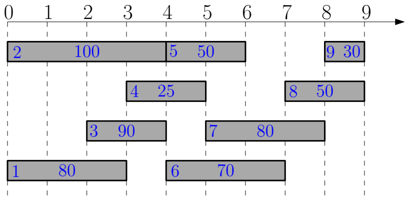


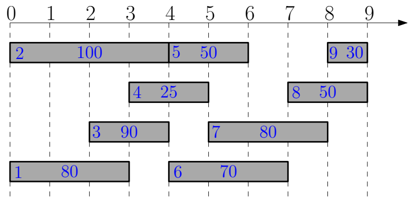
- Sort jobs according to non-decreasing order of finish times

Designing a Dynamic Programming Algorithm



- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

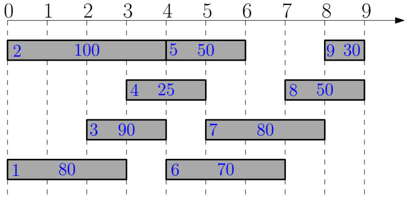
Designing a Dynamic Programming Algorithm



- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

i	$opt[i]$
0	
1	
2	
3	
4	
5	
6	
7	
8	
9	

Designing a Dynamic Programming Algorithm



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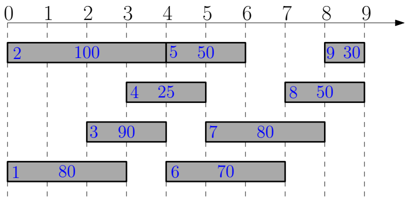
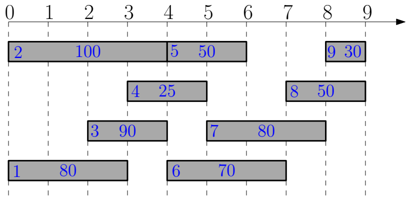


- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

i	$opt[i]$
0	0
1	80
2	
3	
4	
5	
6	
7	
8	
9	

Designing a Dynamic Programming Algorithm



i	$opt[i]$
0	0
1	80
2	100
3	
4	
5	
6	
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9	

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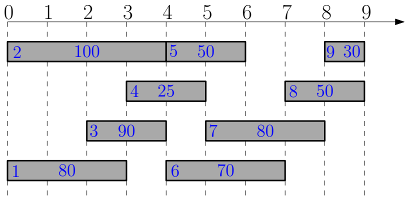

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0	0
1	80
2	100
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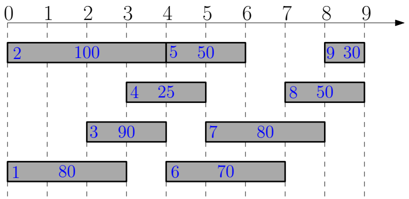

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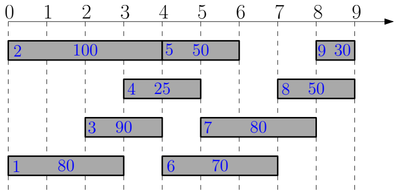
i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
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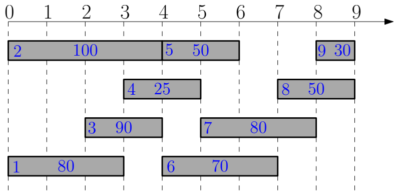
Designing a Dynamic Programming Algorithm



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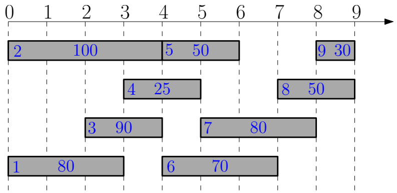
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Designing a Dynamic Programming Algorithm



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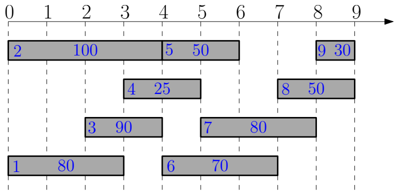

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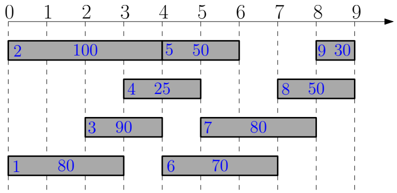

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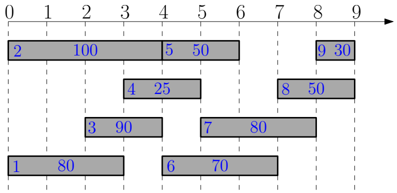

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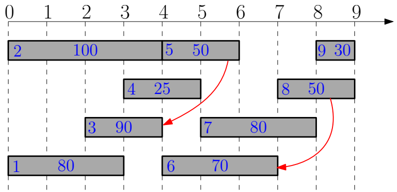
A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \le s_i$

Designing a Dynamic Programming Algorithm



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Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

Designing a Dynamic Programming Algorithm

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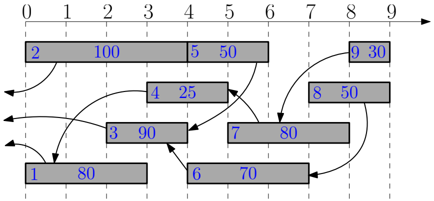

- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] =$
- $opt[3] =$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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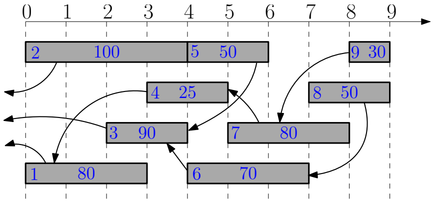

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Designing a Dynamic Programming Algorithm

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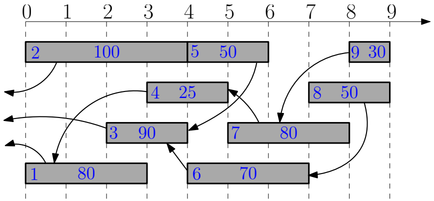

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Designing a Dynamic Programming Algorithm

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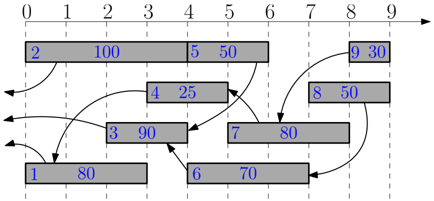

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Designing a Dynamic Programming Algorithm

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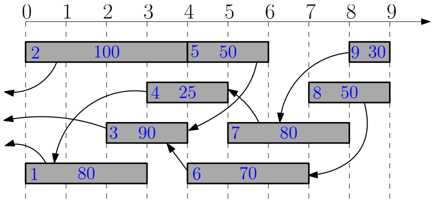

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- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\}$
- $opt[4] =$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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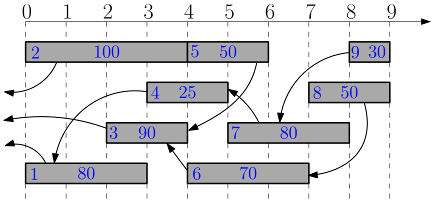

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Designing a Dynamic Programming Algorithm

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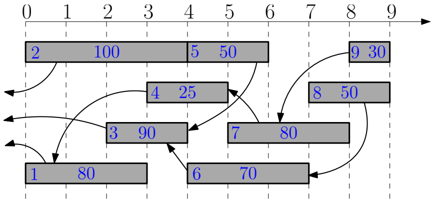

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- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\}$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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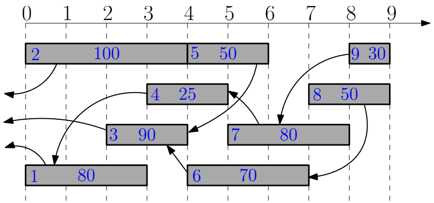

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- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] =$

Designing a Dynamic Programming Algorithm

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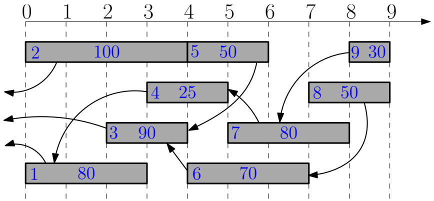

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- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\}$

Designing a Dynamic Programming Algorithm

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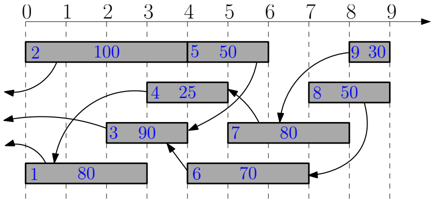

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- $opt[5] = \max\{opt[4], 50 + opt[3]\} = 150$

Designing a Dynamic Programming Algorithm

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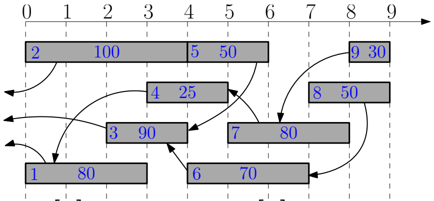
$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



- $opt[0] = 0$, $opt[1] = 80$, $opt[2] = 100$
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Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

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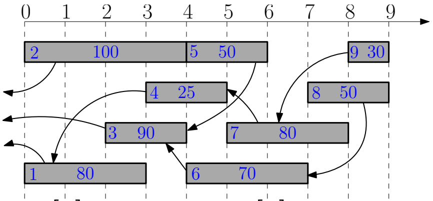

- $opt[0] = 0$, $opt[1] = 80$, $opt[2] = 100$
- $opt[3] = 100$, $opt[4] = 105$, $opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Dynamic Programming

```
1: sort jobs by non-decreasing order of finishing times
2: compute $p_1, p_2, \cdots, p_n$
3: $opt[0] \leftarrow 0$
4: for $i \leftarrow 1$ to $n$ do
5:     $opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$
```

Dynamic Programming

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5:     $opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$
```

- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

How Can We Recover the Optimum Schedule?

```
1: sort jobs by non-decreasing order of
   finishing times
2: compute $p_1, p_2, \cdots, p_n$
3: $opt[0] \leftarrow 0$
4: for $i \leftarrow 1$ to $n$ do
5:     if $opt[i-1] \geq v_i + opt[p_i]$ then
6:         $opt[i] \leftarrow opt[i-1]$
7:
8:     else
9:         $opt[i] \leftarrow v_i + opt[p_i]$
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8:     else
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10:        $b[i] \leftarrow$ Y
```
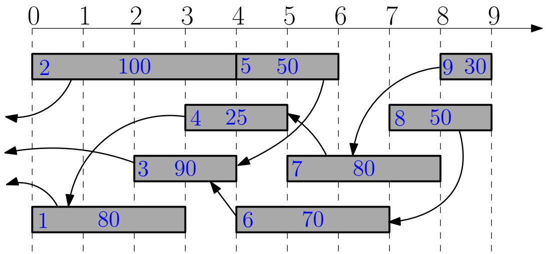

```
1: $i \leftarrow n, S \leftarrow \emptyset$
2: while $i \neq 0$ do
3:     if $b[i] =$ N then
4:         $i \leftarrow i - 1$
5:     else
6:         $S \leftarrow S \cup \{i\}$
7:         $i \leftarrow p_i$
8: return $S$
```

Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	
2	100	
3	100	
4	105	
5	150	
6	170	
7	185	
8	220	
9	220	



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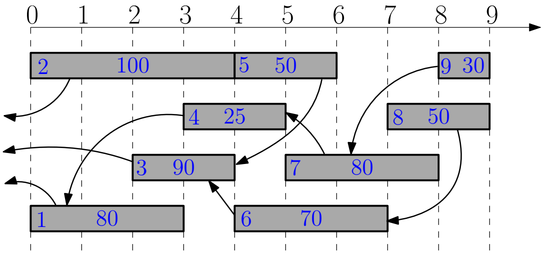

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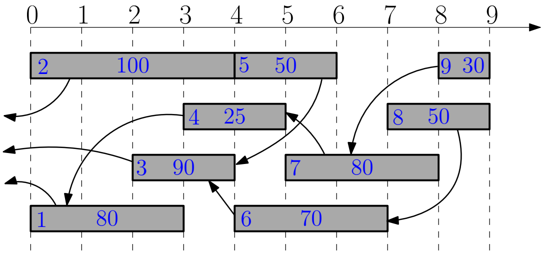

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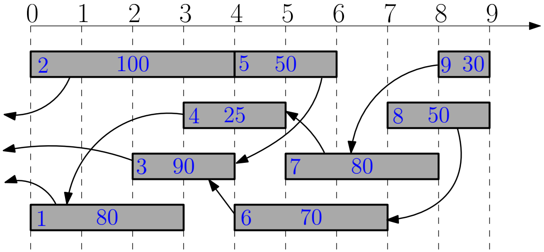

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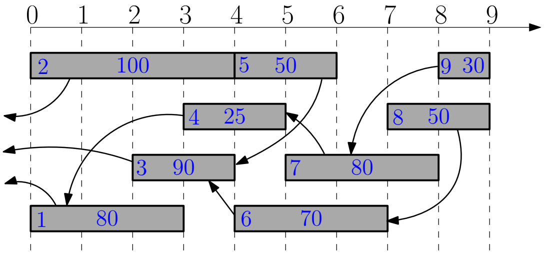

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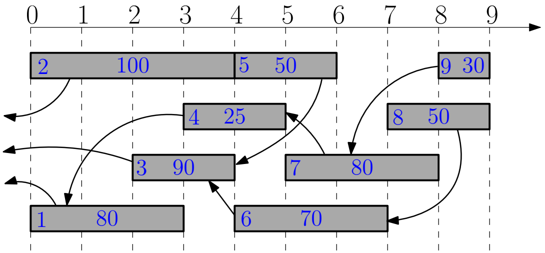

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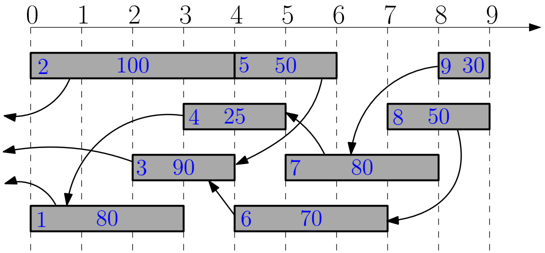

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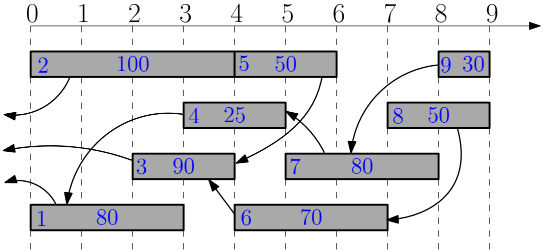
7	185	Y
8	220	
9	220	



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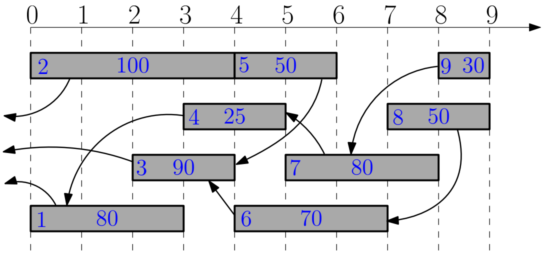
7	185	Y
8	220	Y
9	220	



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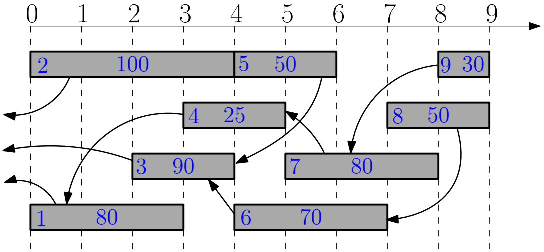

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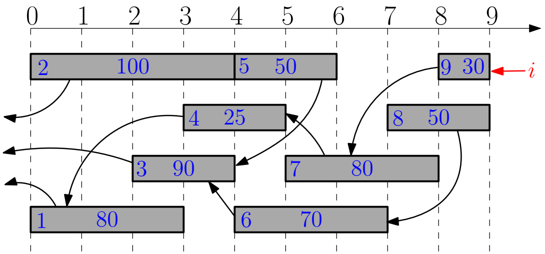

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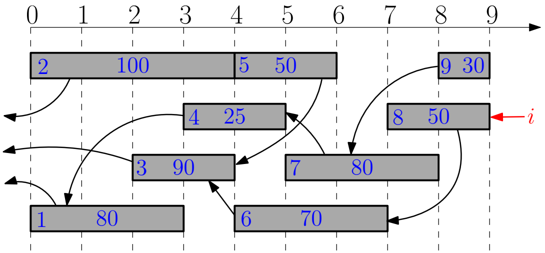

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8	220	Y
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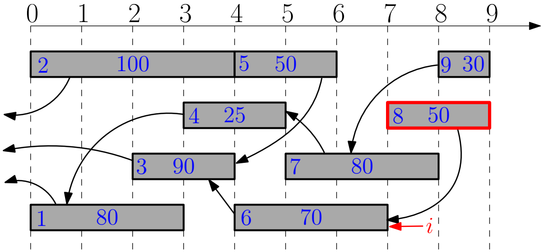

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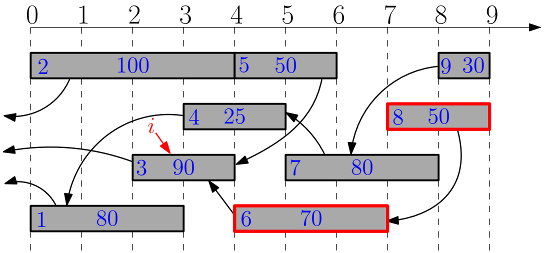

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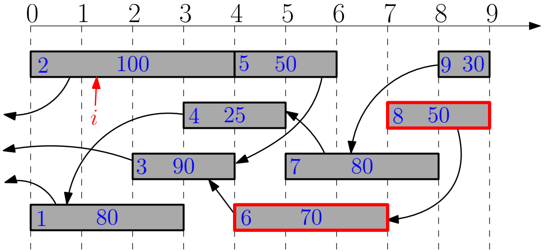

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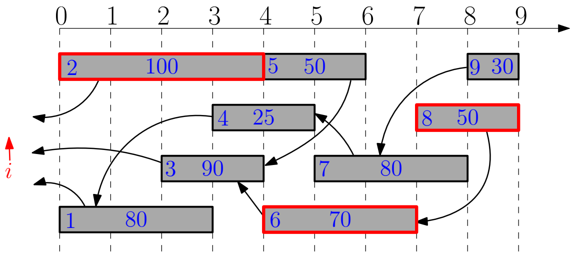

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Outline

Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \leq W.$$

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- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

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Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$

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- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

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A: No. $W = 100, n = 3, w = (51, 50, 50)$.

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A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Q: What if we change "non-increasing" to "non-decreasing"?

A: No. $W = 100, n = 3, w = (1, 50, 50)$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

Design a Dynamic Programming Algorithm

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- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Q: The value of the optimum solution that contains i?

A: $opt[i-1, W'-w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} & i = 0 \\ & i > 0, w_i > W' \\ & i > 0, w_i \leq W' \end{cases}$$

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Dynamic Programming

```
1: for $W' \leftarrow 0$ to $W$ do
2:     $opt[0, W'] \leftarrow 0$
3: for $i \leftarrow 1$ to $n$ do
4:     for $W' \leftarrow 0$ to $W$ do
5:         $opt[i, W'] \leftarrow opt[i-1, W']$
6:         if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ then
7:             $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
8: return $opt[n, W]$
```

Recover the Optimum Set

```
1: for $W' \leftarrow 0$ to $W$ do
2:     $opt[0, W'] \leftarrow 0$
3: for $i \leftarrow 1$ to $n$ do
4:     for $W' \leftarrow 0$ to $W$ do
5:         $opt[i, W'] \leftarrow opt[i-1, W']$
6:         $b[i, W'] \leftarrow$ N
7:         if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ then
8:             $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
9:             $b[i, W'] \leftarrow$ Y
10: return $opt[n, W]$
```

Recover the Optimum Set

```
1: $i \leftarrow n, W' \leftarrow W, S \leftarrow \emptyset$
2: **while** $i > 0$ **do**
3:     **if** $b[i, W'] = \mathsf{Y}$ **then**
4:         $W' \leftarrow W' - w_i$
5:         $S \leftarrow S \cup \{i\}$
6:     $i \leftarrow i - 1$
7: **return** $S$
```

Running Time of Algorithm

```
1: for $W' \leftarrow 0$ to $W$ do
2:     $opt[0, W'] \leftarrow 0$
3: for $i \leftarrow 1$ to $n$ do
4:     for $W' \leftarrow 0$ to $W$ do
5:         $opt[i, W'] \leftarrow opt[i-1, W']$
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- Running time is $O(nW)$

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8: return $opt[n, W]$
```

- Running time is $O(nW)$
- Running time is pseudo-polynomial because it depends on value of the input integers.

Avoiding Unncessary Computation and Memory Using Memoized Algorithm and Hash Map

compute-opt(i, W')

```
1: if $opt[i, W'] \neq \bot$ then return $opt[i, W']$
2: if $i = 0$ then $r \leftarrow 0$
3: else
4:     $r \leftarrow$ compute-opt$(i-1, W')$
5:     if $w_i \leq W'$ then
6:         $r' \leftarrow$ compute-opt$(i-1, W'-w_i) + w_i$
7:         if $r' > r$ then $r \leftarrow r'$
8: $opt[i, W'] \leftarrow r$
9: return $r$
```

- Use hash map for opt

Outline

Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \le W.$$

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$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \le W.$$

- Motivation: you have budget W, and want to buy a subset of items of maximum total value

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.
- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \cdots, W$.

$$opt[i, W'] = \begin{cases} & i = 0 \\ & i > 0, w_i > W' \\ & i > 0, w_i \leq W' \end{cases}$$

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Exercise: Items with 3 Parameters

Input: integer bounds $W > 0, Z > 0$,
a set of n items, each with an integer weight $w_i > 0$
a size $z_i > 0$ for each item i
a value $v_i > 0$ for each item i

Output: a subset S of items that

maximizes $\sum_{i \in S} v_i$ s.t.

$$\sum_{i \in S} w_i \le W \text{ and } \sum_{i \in S} z_i \le Z$$

Outline

Subsequence

- $A = bacdca$
- $C = adca$

Subsequence

- $A = bacdca$
- $C = adca$
- C is a subsequence of A

Subsequence

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Def. Given two sequences $A[1 .. n]$ and $C[1 .. t]$ of letters, C is called a subsequence of A if there exists integers $1 \leq i_1 < i_2 < i_3 < \ldots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \cdots, t$.

Subsequence

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- Exercise: how to check if sequence C is a subsequence of A?

Longest Common Subsequence

Input: $A[1..n]$ and $B[1..m]$

Output: the longest common subsequence of A and B

Example:

- $A = `bacdca'$
- $B = `adbcda'$

Longest Common Subsequence

Input: $A[1 .. n]$ and $B[1 .. m]$

Output: the longest common subsequence of A and B

Example:

- $A = `bacdca'$
- $B = `adbcda'$
- LCS$(A, B) = `adca'$

Longest Common Subsequence

Input: $A[1 \,..\, n]$ and $B[1 \,..\, m]$

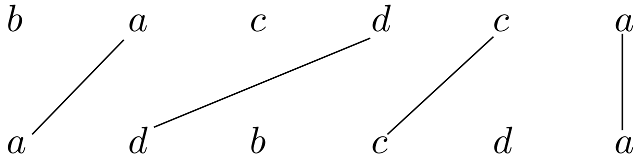
Output: the longest common subsequence of A and B

Example:

- $A = `bacdca'$
- $B = `adbcda'$
- $\text{LCS}(A, B) = `adca'$

- Applications: edit distance (diff), similarity of DNAs

Matching View of LCS



- Goal of LCS: find a maximum-size non-crossing matching between letters in A and letters in B.

Reduce to Subproblems

- $A = `bacdca'$
- $B = `adbcda'$

Reduce to Subproblems

- $A = \text{`}bacdca'$
- $B = \text{`}adbcda'$

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$

- either the last letter of A is not matched:

- or the last letter of B is not matched:

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$
- either the last letter of A is not matched:
- need to compute LCS($`bacd', `adbcd'$)
- or the last letter of B is not matched:

Reduce to Subproblems

- $A = `bacdc'$
- $B = `adbcd'$
- either the last letter of A is not matched:
- need to compute LCS($`bacd', `adbcd'$)
- or the last letter of B is not matched:
- need to compute LCS($`bacdc', `adbc'$)

Dynamic Programming for LCS

- $opt[i, j], 0 \le i \le n, 0 \le j \le m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.

Dynamic Programming for LCS

- $opt[i, j], 0 \le i \le n, 0 \le j \le m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 \mathinner{..} i]$ and $B[1 \mathinner{..} j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} & \text{if } A[i] = B[j] \\ & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
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$$opt[i, j] = \begin{cases} opt[i-1, j-1] + 1 & \text{if } A[i] = B[j] \\ & \text{if } A[i] \neq B[j] \\ & \end{cases}$$

Dynamic Programming for LCS

- $opt[i,j], 0 \le i \le n, 0 \le j \le m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
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- if $i > 0, j > 0$, then

$$opt[i,j] = \begin{cases} opt[i-1,j-1]+1 & \text{if } A[i] = B[j] \\ \max\begin{cases} opt[i-1,j] \\ opt[i,j-1] \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

```
for $j \leftarrow 0$ to $m$ do
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to $n$ do
    $opt[i, 0] \leftarrow 0$
    for $j \leftarrow 1$ to $m$ do
        if $A[i] = B[j]$ then
            $opt[i, j] \leftarrow opt[i-1, j-1] + 1$
        else if $opt[i, j-1] \geq opt[i-1, j]$ then
            $opt[i, j] \leftarrow opt[i, j-1]$
        else
            $opt[i, j] \leftarrow opt[i-1, j]$
```

Dynamic Programming for LCS

```
for $j \leftarrow 0$ to $m$ do
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to $n$ do
    $opt[i, 0] \leftarrow 0$
    for $j \leftarrow 1$ to $m$ do
        if $A[i] = B[j]$ then
            $opt[i, j] \leftarrow opt[i-1, j-1] + 1$, $\pi[i, j] \leftarrow$ "$\nwarrow$"
        else if $opt[i, j-1] \geq opt[i-1, j]$ then
            $opt[i, j] \leftarrow opt[i, j-1]$, $\pi[i, j] \leftarrow$ "$\leftarrow$"
        else
            $opt[i, j] \leftarrow opt[i-1, j]$, $\pi[i, j] \leftarrow$ "$\uparrow$"
```

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$						
2	0 $\perp$						
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1	0 $\perp$	0 $\leftarrow$					
2	0 $\perp$						
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1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$				
2	0 $\perp$						
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1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$			
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1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$						
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1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$			
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1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$						
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1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
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2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
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2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$			
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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
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3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
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0	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$
1	$0\ \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0\ \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0\ \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0\ \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$		
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0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
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3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	
5	0 $\perp$						
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A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$
1	$0\ \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0\ \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0\ \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0\ \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0\ \perp$						
6	$0\ \perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$						
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$					
6	$0 \perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$				
6	$0 \perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$			
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$		
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$						

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$					

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$
1	$0\ \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0\ \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0\ \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0\ \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0\ \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0\ \perp$	$1 \nwarrow$	$2 \uparrow$				

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$			

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$	$0\ \perp$
1	$0\ \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0\ \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0\ \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0\ \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0\ \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0\ \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$		

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	4 $\nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
4	$0 \perp$	$1 \uparrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$
5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$	$0 \perp$
1	$0 \perp$	$0 \leftarrow$	$0 \leftarrow$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$
2	$0 \perp$	$1 \nwarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$
3	$0 \perp$	$1 \uparrow$	$1 \leftarrow$	$1 \leftarrow$	$2 \nwarrow$	$2 \leftarrow$	$2 \leftarrow$
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5	$0 \perp$	$1 \uparrow$	$2 \uparrow$	$2 \leftarrow$	$3 \nwarrow$	$3 \leftarrow$	$3 \leftarrow$
6	$0 \perp$	$1 \nwarrow$	$2 \uparrow$	$2 \leftarrow$	$3 \uparrow$	$3 \leftarrow$	$4 \nwarrow$

Find Common Subsequence

```
1: $i \leftarrow n, j \leftarrow m, S \leftarrow$ “”
2: while $i > 0$ and $j > 0$ do
3:     if $\pi[i, j] =$ “$\nwarrow$” then
4:         $S \leftarrow A[i] \bowtie S, i \leftarrow i - 1, j \leftarrow j - 1$
5:     else if $\pi[i, j] =$ “$\uparrow$” then
6:         $i \leftarrow i - 1$
7:     else
8:         $j \leftarrow j - 1$
9: return $S$
```

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B?

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Obs. $\#$OPs $=$ length(A) $+$ length(B) - 2 $\cdot$ length(LCS(A, B))

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Example:

- $A =$ ocurrance, $B =$ occurrence.
- 2 operations: insert 'c', change 'a' to 'e'

- Not related to LCS any more

Edit Distance (with Replacing)

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: edit distance between $A[1 .. i]$ and $B[1 .. j]$.

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$$opt[i,j] = \begin{cases} opt[i-1, j-1] & \text{if } A[i] = B[j] \\ \min \begin{cases} opt[i-1, j] + 1 \\ opt[i, j-1] + 1 \\ opt[i-1, j-1] + 1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Exercise: Longest Palindrome

Def. A palindrome is a string which reads the same backward or forward.

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Example:

- Input: acbcedeacab
- Output: acedeca

Outline

Computing the Length of LCS

```
1: for $j \leftarrow 0$ to $m$ do
2:     $opt[0, j] \leftarrow 0$
3: for $i \leftarrow 1$ to $n$ do
4:     $opt[i, 0] \leftarrow 0$
5:     for $j \leftarrow 1$ to $m$ do
6:         if $A[i] = B[j]$ then
7:             $opt[i, j] \leftarrow opt[i-1, j-1] + 1$
8:         else if $opt[i, j-1] \geq opt[i-1, j]$ then
9:             $opt[i, j] \leftarrow opt[i, j-1]$
10:        else
11:            $opt[i, j] \leftarrow opt[i-1, j]$
```

Obs. The i-th row of table only depends on $(i-1)$-th row.

Reducing Space to $O(n+m)$

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Q: How to use this observation to reduce space?

Reducing Space to $O(n+m)$

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Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i-1)$-th row and the i-th row.

Linear Space Algorithm to Compute Length of LCS

```
for $j \leftarrow 0$ to $m$ do
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to $n$ do
    $opt[i \bmod 2, 0] \leftarrow 0$
    for $j \leftarrow 1$ to $m$ do
        if $A[i] = B[j]$ then
            $opt[i \bmod 2, j] \leftarrow opt[i-1 \bmod 2, j-1] + 1$
        else if $opt[i \bmod 2, j-1] \geq opt[i-1 \bmod 2, j]$ then
            $opt[i \bmod 2, j] \leftarrow opt[i \bmod 2, j-1]$
        else
            $opt[i \bmod 2, j] \leftarrow opt[i-1 \bmod 2, j]$
return $opt[n \bmod 2, m]$
```

How to Recover LCS Using Linear Space?

- Only keep the last two rows: only know how to match $A[n]$

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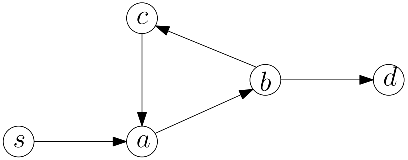
  - Space: $O(m+n)$
  - Time: $O(nm)$

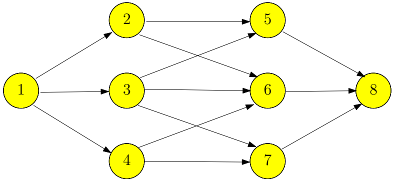
Outline

Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



not a DAG



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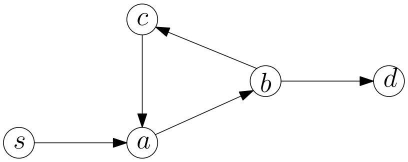

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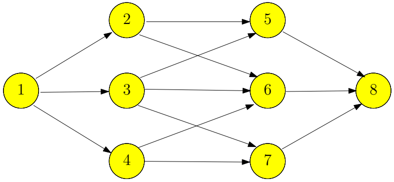


a DAG

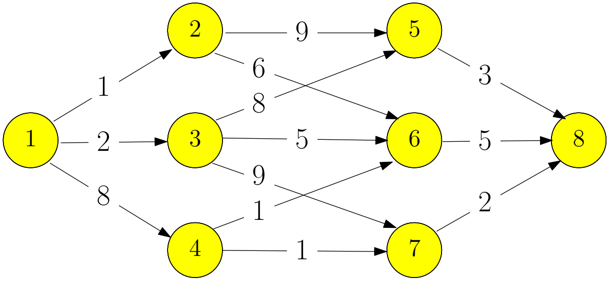
Lemma A directed graph is a DAG if and only its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i, for every $i \in V$



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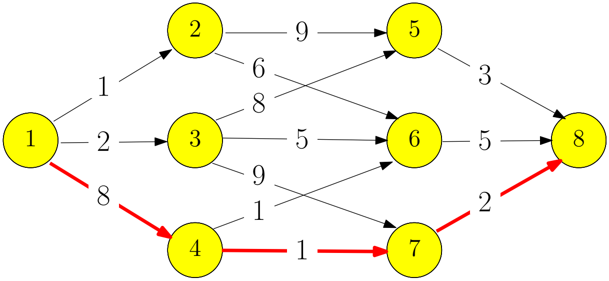

Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} & i = 1 \\ & i = 2, 3, \cdots, n \end{cases}$$

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- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i)\in E} \{f(j) + w(j,i)\} & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

Shortest Paths in DAG

```
1: $f[1] \leftarrow 0$
2: **for** $i \leftarrow 2$ to $n$ **do**
3:     $f[i] \leftarrow \infty$
4:     **for** each incoming edge $(j, i)$ of $i$ **do**
5:         **if** $f[j] + w(j, i) < f[i]$ **then**
6:             $f[i] \leftarrow f[j] + w(j, i)$
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7:             $\pi(i) \leftarrow j$
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6:             $f[i] \leftarrow f[j] + w(j, i)$
7:             $\pi(i) \leftarrow j$
```

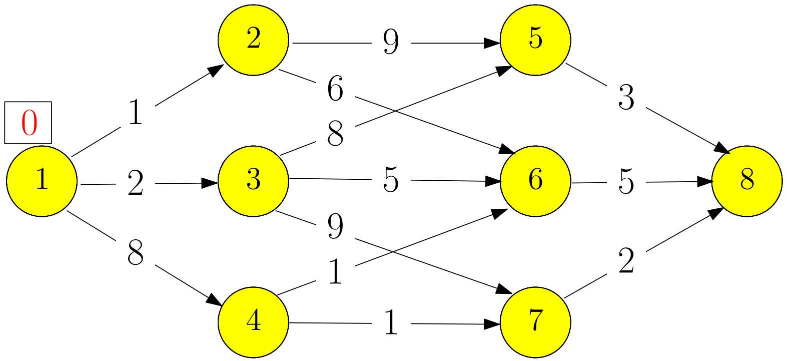
print-path(t)

```
1: if $t = 1$ then
2:     print(1)
3:     return
4: print-path($\pi(t)$)
5: print(",", $t$)
```

Example

$$\begin{tikzcd}
 & 2 \arrow[r, "9"] \arrow[dr, "6"] & 5 \arrow[dr, "3"] & \\
1 \arrow[ur, "1"] \arrow[r, "2"] \arrow[dr, "8"] & 3 \arrow[ur, "8"] \arrow[r, "5"] \arrow[dr, "9"] & 6 \arrow[r, "5"] & 8 \\
 & 4 \arrow[ur, "1"] \arrow[r, "1"] & 7 \arrow[ur, "2"] &
\end{tikzcd}$$

Example



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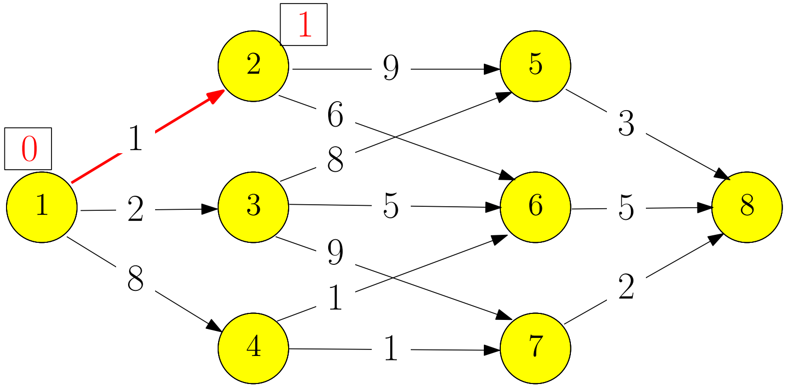

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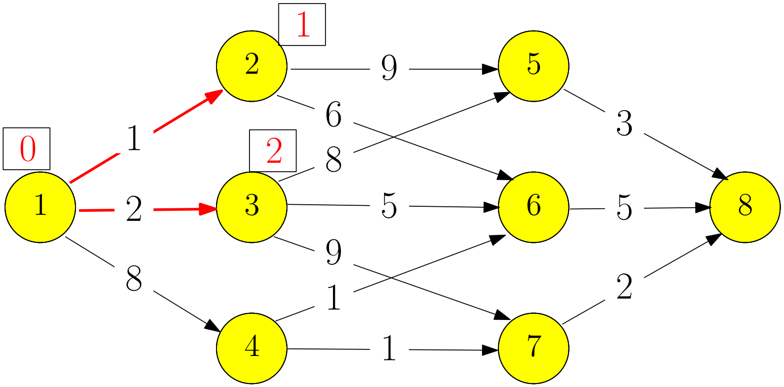

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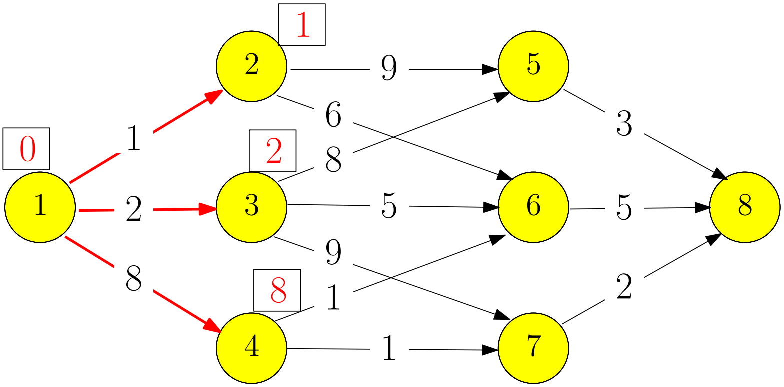

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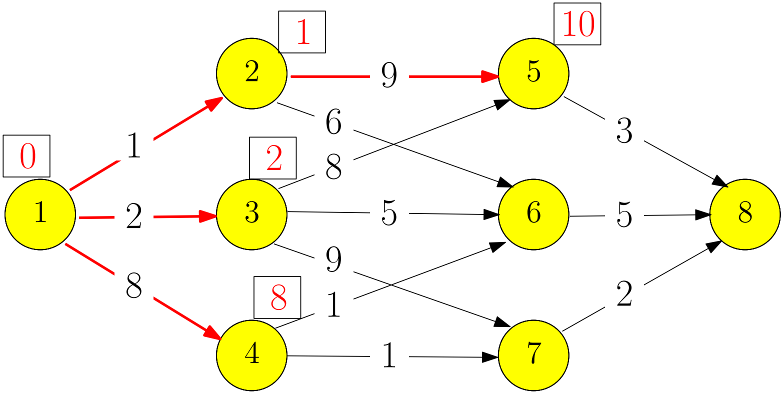

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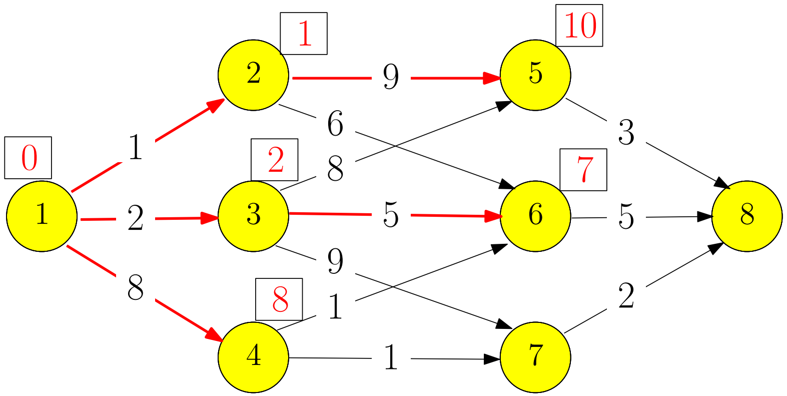

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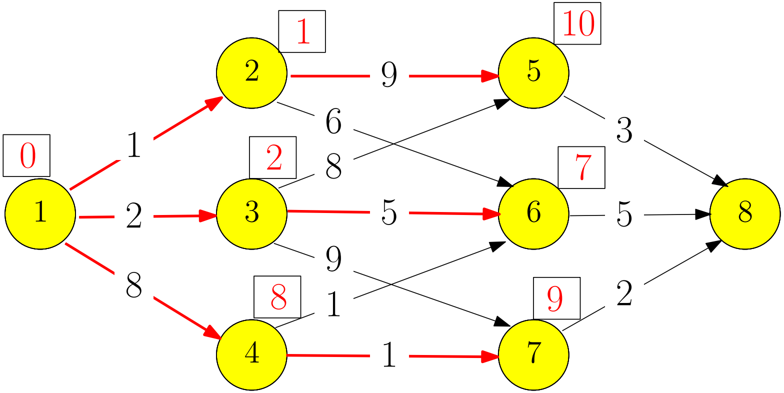

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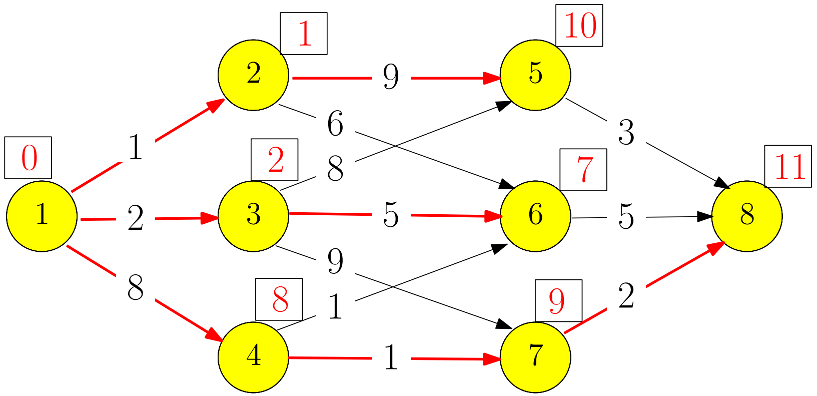

Variant: Heaviest Path in a Directed Acyclic Graph

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Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the path with the largest weight (the heaviest path) from 1 to n.

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$$f[i] = \begin{cases} & i = 1 \\ & i = 2, 3, \cdots, n \end{cases}$$

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Outline

Matrix Chain Multiplication

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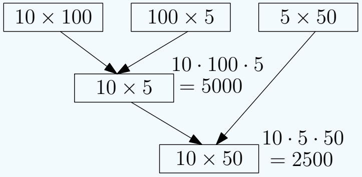
Input: n matrices $A_1, A_2, \cdots, A_n$ of sizes $r_1 \times c_1, r_2 \times c_2, \cdots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \cdots, n-1$.

Output: the order of computing $A_1 A_2 \cdots A_n$ with the minimum number of multiplications

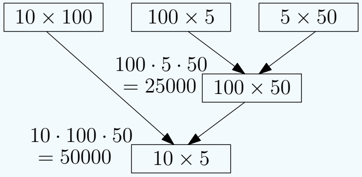
Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$



cost = 5000 + 2500 = 7500



cost = 25000 + 50000 = 75000

- $(A_1A_2)A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1(A_2A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

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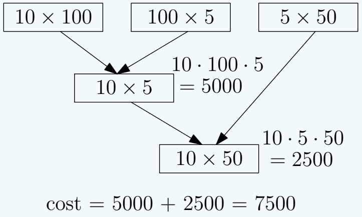

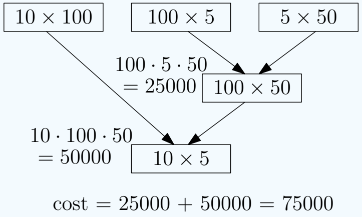

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$$opt[i, j] = \begin{cases} 0 & i = j \\ \min_{k: i \le k < j} \left(opt[i, k] + opt[k+1, j] + r_i c_k c_j \right) & i < j \end{cases}$$

Matrix Chain Multiplication: Design DP

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

```
1: let $opt[i,i] \leftarrow 0$ for every $i = 1, 2, \cdots, n$
2: for $\ell \leftarrow 2$ to $n$ do
3:     for $i \leftarrow 1$ to $n - \ell + 1$ do
4:         $j \leftarrow i + \ell - 1$
5:         $opt[i,j] \leftarrow \infty$
6:         for $k \leftarrow i$ to $j - 1$ do
7:             if $opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j]$ then
8:                 $opt[i,j] \leftarrow opt[i,k] + opt[k+1,j] + r_i c_k c_j$
9: return $opt[1,n]$
```

Recover the Optimum Way of Multiplication

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

```
1: let $opt[i,i] \leftarrow 0$ for every $i = 1, 2, \cdots, n$
2: for $\ell \leftarrow 2$ to $n$ do
3:     for $i \leftarrow 1$ to $n - \ell + 1$ do
4:         $j \leftarrow i + \ell - 1$
5:         $opt[i,j] \leftarrow \infty$
6:         for $k \leftarrow i$ to $j - 1$ do
7:             if $opt[i,k] + opt[k+1,j] + r_i c_k c_j < opt[i,j]$ then
8:                 $opt[i,j] \leftarrow opt[i,k] + opt[k+1,j] + r_i c_k c_j$
9:                 $\pi[i,j] \leftarrow k$
10: return $opt[1,n]$
```

Constructing Optimal Solution

Print-Optimal-Order(i, j)

```
1: if $i = j$ then
2:     print("A"$_i$)
3: else
4:     print("(")
5:     Print-Optimal-Order$(i, \pi[i, j])$
6:     Print-Optimal-Order$(\pi[i, j] + 1, j)$
7:     print(")")
```

Outline

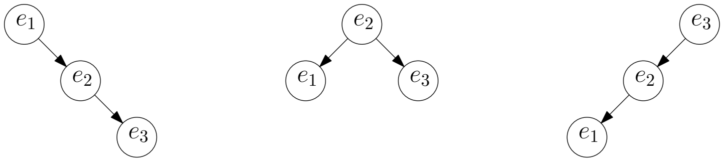
Optimum Binary Search Tree

- n elements $e_1 < e_2 < e_3 < \cdots < e_n$
- e_i has frequency f_i
- goal: build a binary search tree for $\{e_1, e_2, \cdots, e_n\}$ with the minimum accessing cost:

$$\sum_{i=1}^{n} f_i \times (\text{depth of } e_i \text{ in the tree})$$

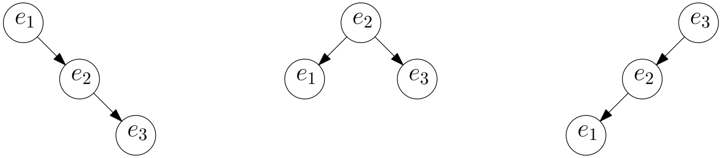
Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$



Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$



- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

Optimum Binary Search Tree

- Example: $f_1 = 10, f_2 = 5, f_3 = 3$

$$\begin{tikzcd}
e_1 \arrow[dr] & & \\
& e_2 \arrow[dr] & \\
& & e_3
\end{tikzcd}
\qquad
\begin{tikzcd}
& e_2 \arrow[dl] \arrow[dr] & \\
e_1 & & e_3
\end{tikzcd}
\qquad
\begin{tikzcd}
& & e_3 \arrow[dl] \\
& e_2 \arrow[dl] & \\
e_1 & &
\end{tikzcd}$$

- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

- suppose we decided to let e_i be the root
- $e_1, e_2, \cdots, e_{i-1}$ are on left sub-tree
- $e_{i+1}, e_{i+2}, \cdots, e_n$ are on right sub-tree

- d_j: depth of e_j in our tree
- C, C_L, C_R: cost of tree, left sub-tree and right sub-tree respectively

$$\begin{aligned} C &= \sum_{j=1}^{n} f_j d_j = \sum_{j=1}^{n} f_j + \sum_{j=1}^{n} f_j(d_j - 1) \\ &= \sum_{j=1}^{n} f_j + \sum_{j=1}^{i-1} f_j(d_j - 1) + \sum_{j=i+1}^{n} f_j(d_j - 1) \\ &= \sum_{j=1}^{n} f_j + C_L + C_R \end{aligned}$$

$$C = \sum_{j=1}^{n} f_j + C_L + C_R$$

- In order to minimize C, need to minimize C_L and C_R respectively
- $opt_{i,j}$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$
- for every $i \in \{1, 2, \cdots, n, n+1\}$: $opt[i, i-1] = 0$
- for every i, j such that $1 \leq i \leq j \leq n$,

$$opt[i,j] = \sum_{k=i}^{j} f_k + \min_{k: i \leq k \leq j} \big(opt[i, k-1] + opt[k+1, j] \big)$$

Outline

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many independent sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j]$ = value of instance defined by $A[1..i]$ and $B[1..j]$
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Matrix chain multiplication, optimum binary search tree: $opt[i, j]$ = value of instances defined by matrices i to j