CSE 431/531: Algorithm Analysis and Design (Spring 2021)

Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo

Outline

CSE 431/531: Algorithm Analysis and Design

- Course Webpage (contains schedule, policies, homeworks and slides):
  `http://www.cse.buffalo.edu/~shil/courses/CSE531/`
- Please sign up course on Piazza via link on course webpage
  - announcements, polls, asking/answering questions

CSE 431/531: Algorithm Analysis and Design

- Time
  - MoWeFr, 9:10am-10:00am
- All lectures are virtual
- Instructor:
  - Shi Li, shil@buffalo.edu
- TAs:
  - Xiangyu Guo
  - Alesandro Baccarini

You should already have/know:

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- Mathematical Background
    - basic reasoning skills, inductive proofs

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- Basic data Structures
  - linked lists, arrays
  - stacks, queues
- Some Programming Experience
  - C, C++, Java or Python

You Will Learn

- Classic algorithms for classic problems
  - Sorting, shortest paths, minimum spanning tree, $\cdots$

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  - Dynamic programming
  - $\cdots$
- NP-completeness

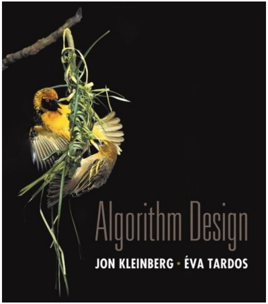
Tentative Schedule (42 Lectures)

See the course webpage.

Textbook

Textbook (Highly Recommended):

- Algorithm Design, 1st Edition, by *Jon Kleinberg* and *Eva Tardos*



Other Reference Books

- Introduction to Algorithms, Third Edition, *Thomas Cormen, Charles Leiserson, Rondald Rivest, Clifford Stein*

Reading Before Classes

- Highly recommended: read the correspondent sections from the textbook (or reference book) before classes
  - Sections for each lecture can be found on the course webpage.
- Slides and example problems for recitations will be posted on the course webpage before class

Grading

- 40% for theory homeworks
  - 8 points × 5 theory homeworks
- 20% for programming problems
  - 10 points × 2 programming problems
- 40% for final exam

For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussions
  - Must write down solutions on your own, in your own words
  - Write down names of students you collaborated with

For Homeworks, You Are Not Allowed to

- Use external resources
  - Can't Google or ask questions online for solutions
  - Can't read posted solutions from other algorithm course webpages
- Copy solutions from other students

For Programming Problems

- Need to implement the algorithms by yourself
- Can not copy codes from others or the Internet
- We use Moss (`https://theory.stanford.edu/~aiken/moss/`) to detect similarity of programs

Late Policy

- You have 1 “late credit”, using it allows you to submit an assignment solution for three days
- With no special reasons, no other late submissions will be accepted

- Final Exam will be closed-book
- Per Departmental Policy on Academia Integrity Violations, penalty for AI violation is:
    - “F” for the course
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Questions?

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1. Syllabus
2. Introduction
    - What is an Algorithm?
    - Example: Insertion Sort
    - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.

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- Computational problem: specifies the input/output relationship.
- An algorithm solves a computational problem if it produces the correct output for any given input.

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

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- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$

Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of a and b

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
- $\gcd(270, 210) = \gcd(210, 270 \bmod 210) = \gcd(210, 60)$
- $(270, 210) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)$

Examples

Sorting

Input: sequence of n numbers $(a_1, a_2, \cdots, a_n)$

Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

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- Algorithms: insertion sort, merge sort, quicksort, ...

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Output: a shortest path from s to t in G

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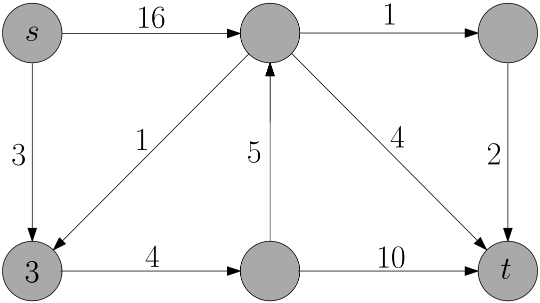

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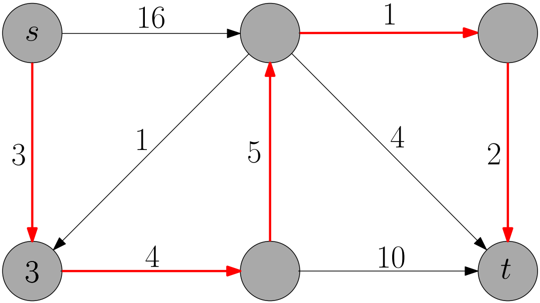

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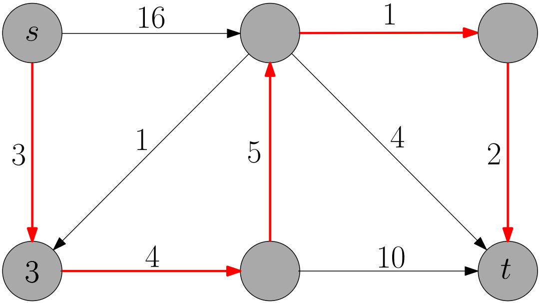

- Algorithm: Dijkstra's algorithm

Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, using a particular programming language

Pseudo-Code

Pseudo-Code:

```
Euclidean($a, b$)
while $b > 0$ do
    $(a, b) \leftarrow (b, a \bmod b)$
return $a$
```

C++ program:

```
int Euclidean(int a, int b){
    int c;
     while (b > 0){
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```

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  4. it is fun!

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Sorting Problem

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Output: a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

Example:

- Input: $53, 12, 35, 21, 59, 15$
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Insertion-Sort

- At the end of j-th iteration, the first j numbers are sorted.

iteration 1: $53, 12, 35, 21, 59, 15$

iteration 2: $12, 53, 35, 21, 59, 15$

iteration 3: $12, 35, 53, 21, 59, 15$

iteration 4: $12, 21, 35, 53, 59, 15$

iteration 5: $12, 21, 35, 53, 59, 15$

iteration 6: $12, 15, 21, 35, 53, 59$

Example:

- Input: $53, 12, 35, 21, 59, 15$
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insertion-sort(A, n)

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for $j \leftarrow 2$ to $n$ do
    $key \leftarrow A[j]$
    $i \leftarrow j - 1$
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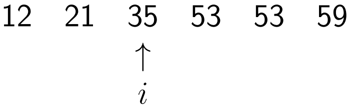

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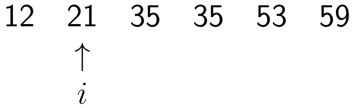

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Outline

Analysis of Insertion Sort

- Correctness
- Running time

Correctness of Insertion Sort

- Invariant: after iteration j of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

$$\text{after } j = 1 : 53, 12, 35, 21, 59, 15$$

$$\text{after } j = 2 : 12, 53, 35, 21, 59, 15$$

$$\text{after } j = 3 : 12, 35, 53, 21, 59, 15$$

$$\text{after } j = 4 : 12, 21, 35, 53, 59, 15$$

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$$\text{after } j = 6 : 12, 15, 21, 35, 53, 59$$

Analyzing Running Time of Insertion Sort

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  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.

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- Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
- A2: Worst-case analysis:
  - Running time for size $n =$ worst running time over all possible arrays of length n

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?

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- A: They do not matter!

Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
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Important idea: asymptotic analysis

- Focus on growth of running-time as a function, not any particular value.

Asymptotic Analysis: O-notation

Informal way to define O-notation:

- Ignoring lower order terms
- Ignoring leading constant

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- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$

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- how we measure the running time: seconds or # instructions?

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  - program 1 requires 10 instructions, or 10^{-8} seconds
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- to execute $a \leftarrow b + c$:
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  - program 2 requires 2 instructions, or 10^{-9} seconds
  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation

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- For Algorithm 2: if we increase n by a factor of 2, running time increases by a factor of 2

Asymptotic Analysis of Insertion Sort

insertion-sort(A, n)

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for $j \leftarrow 2$ to $n$ do
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- Can we do better than insertion sort asymptotically?
- Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time

- Remember to sign up for Piazza.

Questions?

Outline

1. Syllabus
2. Introduction
    - What is an Algorithm?
    - Example: Insertion Sort
    - Analysis of Insertion Sort
3. Asymptotic Notations
4. Common Running times

Asymptotically Positive Functions

Def. $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

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- We only consider asymptotically positive functions.

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O-Notation For a function $g(n)$,

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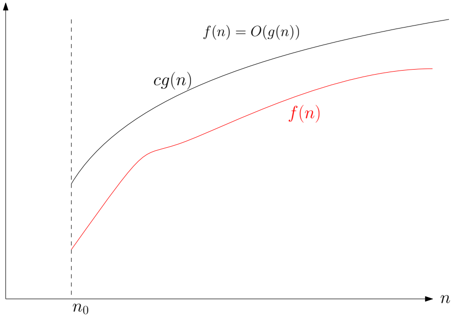

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Proof.

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)$$
$$= -n^2 + 40n \leq 0.$$
$$3n^2 + 2n \leq c(n^2 - 10n)$$

□

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- Analogy: Mike is a student. ~~A student is Mike.~~

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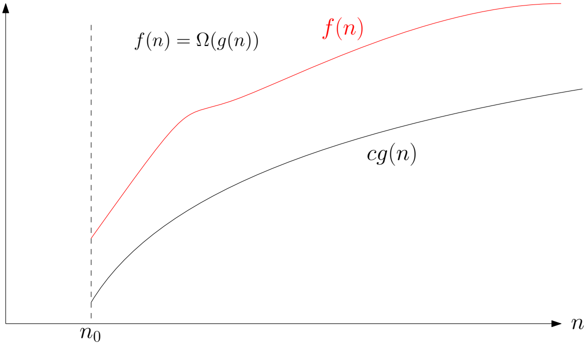

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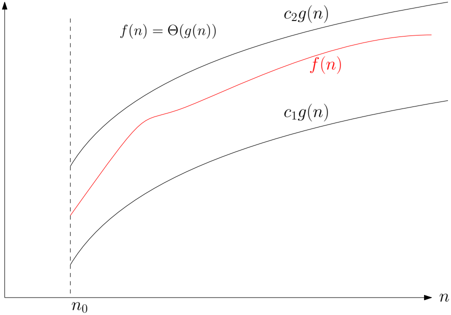

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$$f(n) = n^2$$

$$g(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ n^3 & \text{if } n \text{ is even} \end{cases}$$

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- In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.
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- $3n^2 - 10n - 5 = O(n^2)$ is the most natural since n^2 is the simplest term we can have inside $O(\cdot)$.

Notice that O denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.
- We say: the running time of the insertion sort algorithm is $O(n^2)$ and the bound is tight.

Notice that O denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.
- We say: the running time of the insertion sort algorithm is $O(n^2)$ and the bound is tight.
- We do not use Ω and Θ very often when we upper bound running times.

Exercise

For each pair of functions f, g in the following table, indicate whether f is O, Ω or Θ of g.

f	g	O	Ω	Θ
$n^3 - 100n$	$5n^2 + 3n$			
$3n - 50$	$n^2 - 7n$			
$n^2 - 100n$	$5n^2 + 30n$			
$\log_2 n$	$\log_{10} n$			
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Asymptotic Notations	O	Ω	Θ	o	ω
Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

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Comparison Relations	$\leq$	$\geq$	$=$	$<$	$>$

Questions?

Outline

$O(n)$ (Linear) Running Time

Computing the sum of n numbers

sum(A, n)

```
$S \leftarrow 0$
for $i \leftarrow 1$ to $n$
    $S \leftarrow S + A[i]$
return $S$
```

$O(n)$ (Linear) Running Time

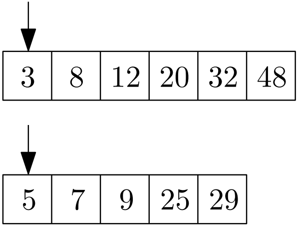
- Merge two sorted arrays

3	8	12	20	32	48

5	7	9	25	29

$O(n)$ (Linear) Running Time

- Merge two sorted arrays



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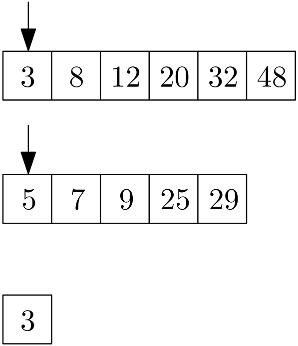

3	8	12	20	32	48

5	7	9	25	29

3

$O(n)$ (Linear) Running Time

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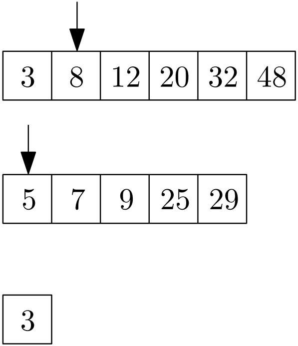

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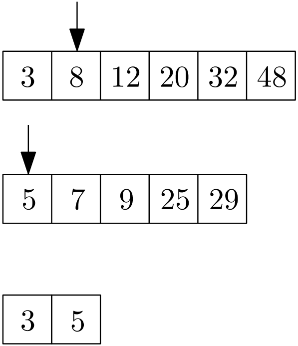

3	8	12	20	32	48

5	7	9	25	29

3	5

$O(n)$ (Linear) Running Time

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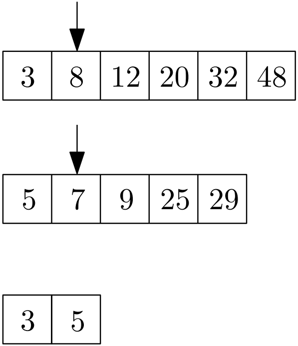

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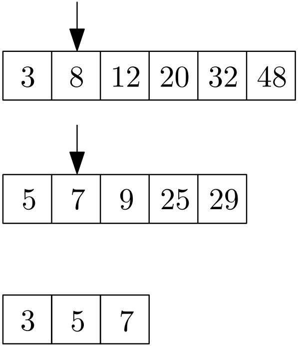

3	8	12	20	32	48

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3	5	7

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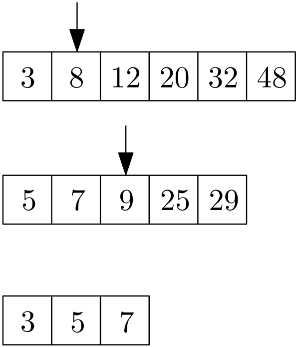

3	8	12	20	32	48

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3	5	7

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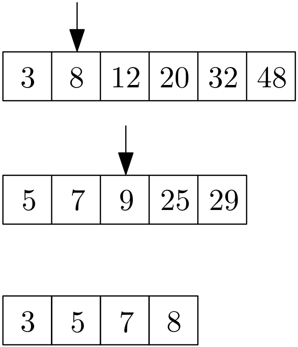

3	8	12	20	32	48

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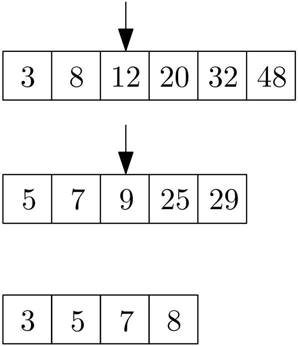

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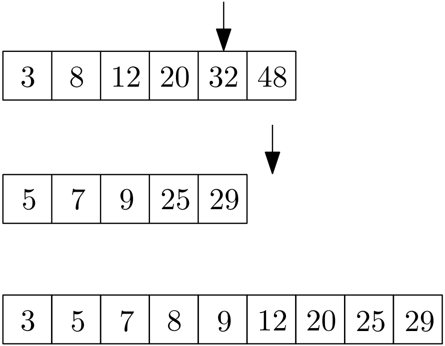

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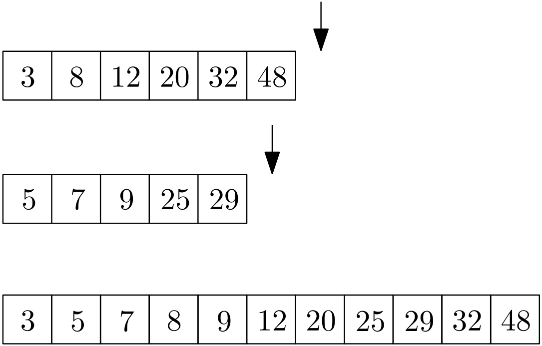

3	8	12	20	32	48

5	7	9	25	29

3	5	7	8	9	12	20	25	29	32	48

$O(n)$ (Linear) Running Time

```
merge$(B, C, n_1, n_2)$        \\ $B$ and $C$ are sorted, with length $n_1$ and $n_2$
$A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ and $j \leq n_2$ do
    if $B[i] \leq C[j]$ then
        append $B[i]$ to $A$; $i \leftarrow i + 1$
    else
        append $C[j]$ to $A$; $j \leftarrow j + 1$
if $i \leq n_1$ then append $B[i..n_1]$ to $A$
if $j \leq n_2$ then append $C[j..n_2]$ to $A$
return $A$
```

$O(n)$ (Linear) Running Time

merge(B, C, n_1, n_2) \\ B and C are sorted, with length n_1 and n_2

```
$A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
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if $i \leq n_1$ then append $B[i..n_1]$ to $A$
if $j \leq n_2$ then append $C[j..n_2]$ to $A$
return $A$
```

Running time $= O(n)$ where $n = n_1 + n_2$.

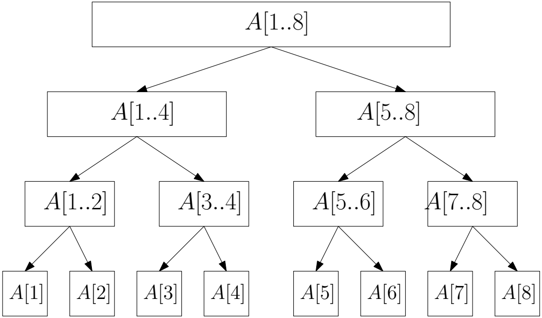
$O(n \log n)$ Running Time

merge-sort(A, n)

```
if $n = 1$ then
    return $A$
else
    $B \leftarrow$ merge-sort$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
    $C \leftarrow$ merge-sort$\Big(A\big[\lfloor n/2 \rfloor + 1..n\big], n - \lfloor n/2 \rfloor\Big)$
return merge$(B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor)$
```

$O(n \log n)$ Running Time

- Merge-Sort



$O(n \log n)$ Running Time

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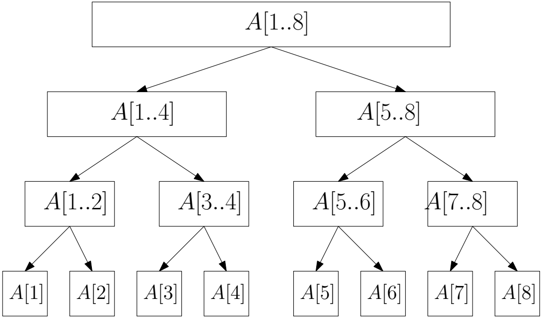
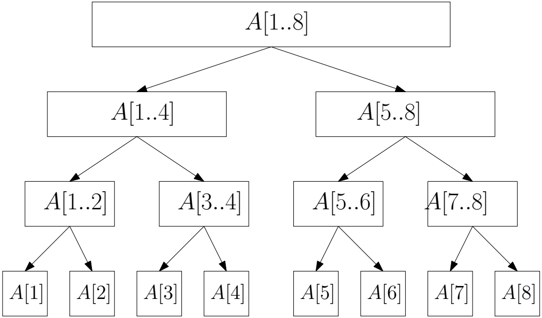


- Each level takes running time $O(n)$

$O(n \log n)$ Running Time

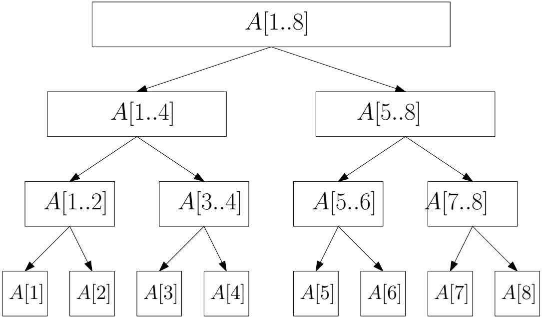
- Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\log n)$ levels

$O(n \log n)$ Running Time

- Merge-Sort



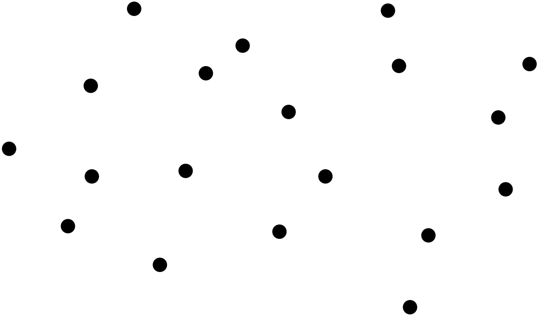
- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest



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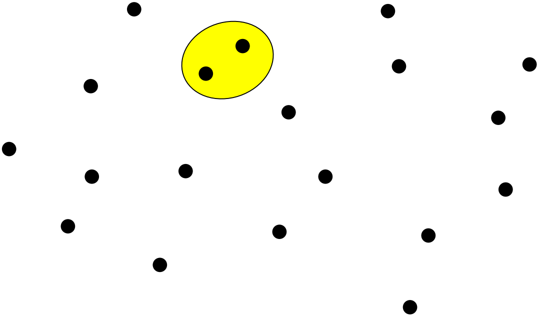

$O(n^2)$ (Quardatic) Running Time

Closest Pair

Input: n points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

Output: the pair of points that are closest

closest-pair(x, y, n)

```
$bestd \leftarrow \infty$
for $i \leftarrow 1$ to $n - 1$ do
    for $j \leftarrow i + 1$ to $n$ do
        $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
        if $d < bestd$ then
            $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
return $(besti, bestj)$
```

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for i ← 1 to n − 1 do
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        d ← $\sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
        if d < bestd then
            besti ← i, bestj ← j, bestd ← d
return (besti, bestj)
```

Closest pair can be solved in $O(n \log n)$ time!

$O(n^3)$ (Cubic) Running Time

Multiply two matrices of size $n \times n$

matrix-multiplication(A, B, n)

```
$C \leftarrow$ matrix of size $n \times n$, with all entries being $0$
for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        for $k \leftarrow 1$ to $n$ do
            $C[i,k] \leftarrow C[i,k] + A[i,j] \times B[j,k]$
return $C$
```

$O(n^k)$ Running Time for Integer $k \geq 4$

Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

$O(n^k)$ Running Time for Integer $k \geq 4$

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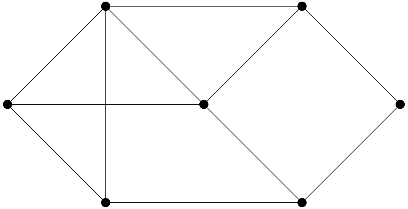

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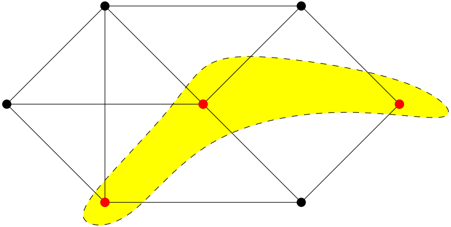

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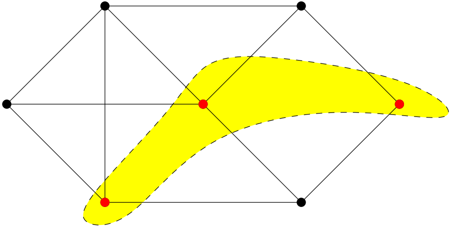


Independent set of size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

$O(n^k)$ Running Time for Integer $k \geq 4$

Independent Set of Size k

Input: graph $G = (V, E)$

Output: whether there is an independent set of size k

independent-set($G = (V, E)$)

```
for every set $S \subseteq V$ of size $k$ do
    $b \leftarrow$ true
    for every $u, v \in S$ do
        if $(u, v) \in E$ then $b \leftarrow$ false
    if $b$ return true
return false
```

Running time $= O(\frac{n^k}{k!} \times k^2) = O(n^k)$ (assume k is a constant)

Beyond Polynomial Time: 2^n

Maximum Independent Set Problem

Input: graph $G = (V, E)$

Output: the maximum independent set of G

max-independent-set($G = (V, E)$)

```
$R \leftarrow \emptyset$
for every set $S \subseteq V$ do
    $b \leftarrow$ true
    for every $u, v \in S$ do
        if $(u, v) \in E$ then $b \leftarrow$ false
    if $b$ and $|S| > |R|$ then $R \leftarrow S$
return $R$
```

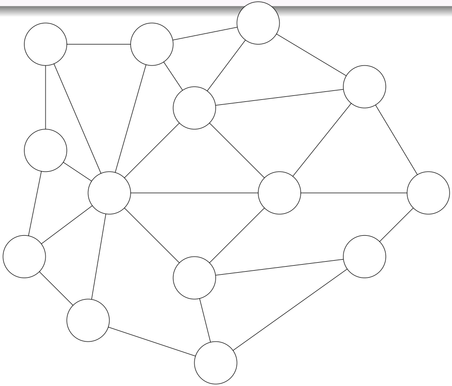
Running time $= O(2^n n^2)$.

Beyond Polynomial Time: $n!$

Hamiltonian Cycle Problem

Input: a graph with n vertices

Output: a cycle that visits each node exactly once,
or say no such cycle exists



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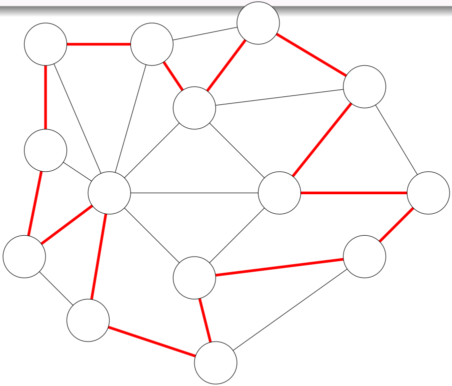

Beyond Polynomial Time: $n!$

Hamiltonian$(G = (V, E))$

```
for every permutation $(p_1, p_2, \cdots, p_n)$ of $V$ do
    $b \leftarrow$ true
    for $i \leftarrow 1$ to $n - 1$ do
        if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
    if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
    if $b$ then return $(p_1, p_2, \cdots, p_n)$
return "No Hamiltonian Cycle"
```

Running time $= O(n! \times n)$

$O(\log n)$ (Logarithmic) Running Time

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- Binary search
  - Input: sorted array A of size n, an integer t;
  - Output: whether t appears in A.

$O(\log n)$ (Logarithmic) Running Time

- Binary search
  - Input: sorted array A of size n, an integer t;
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- E.g, search 35 in the following array:

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

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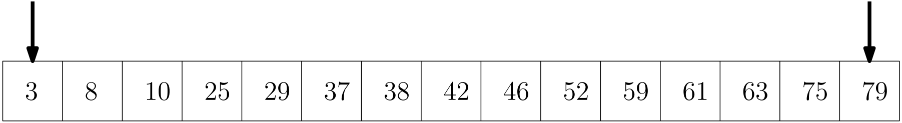

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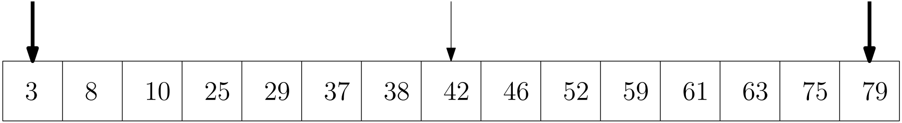

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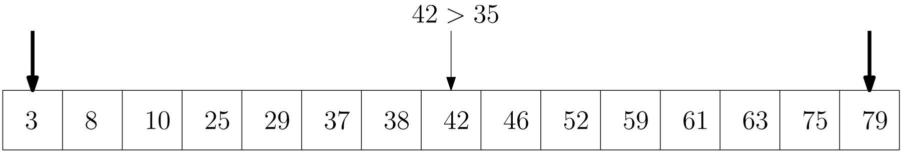

$42 > 35$

3	8	10	25	29	37	38	42	46	52	59	61	63	75	79

$O(\log n)$ (Logarithmic) Running Time

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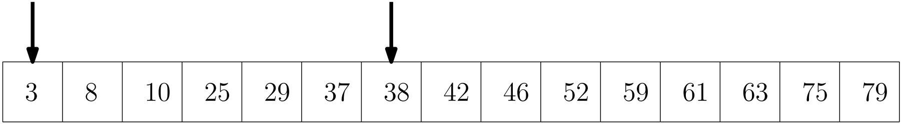

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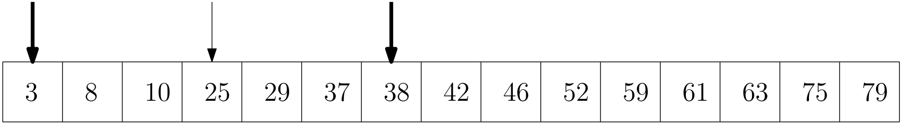

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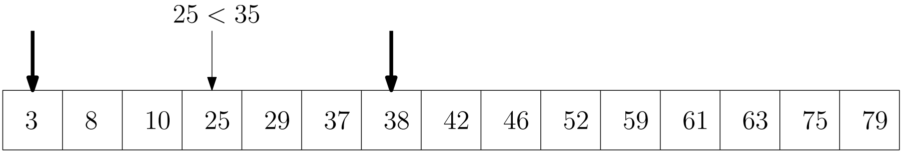

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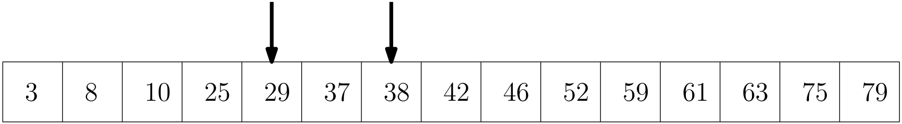

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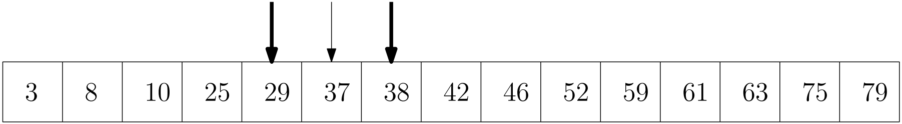

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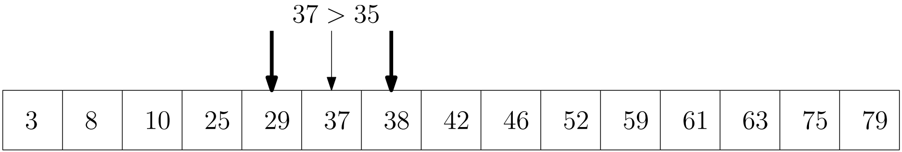

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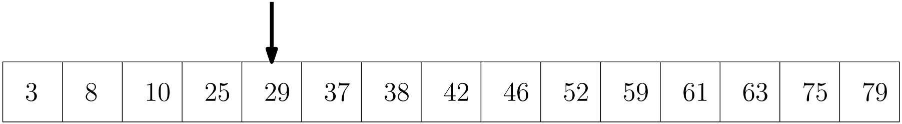

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Running time $= O(\log n)$

Comparing the Orders

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Terminologies

When we talk about upper bound on running time:

- Logarithmic time: $O(\log n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant k
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$

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- Using asymptotic analysis allows us to ignore the leading constants and lower order terms
- Makes our life much easier! (E.g., the leading constant depends on the implementation, complier and computer architecture of computer.)

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- e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

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- For "natural" algorithms, constants are not so big!
- So, for reasonably large n, algorithm with lower order running time beats algorithm with higher order running time.