CSE 431/531: Algorithm Analysis and Design (Spring 2022)

Dynamic Programming

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Paradigms for Designing Algorithms

Greedy algorithm

- Make a greedy choice
- Prove that the greedy choice is safe
- Reduce the problem to a sub-problem and solve it iteratively
- Usually for optimization problems

Divide-and-conquer

- Break a problem into many independent sub-problems
- Solve each sub-problem separately
- Combine solutions for sub-problems to form a solution for the original one
- Usually used to design more efficient algorithms

Paradigms for Designing Algorithms

Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Recall: Computing the n-th Fibonacci Number

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

Fib(n)

```
1: F[0] ← 0
2: F[1] ← 1
3: for i ← 2 to n do
4:     F[i] ← F[i − 1] + F[i − 2]
5: return F[n]
```

- Store each $F[i]$ for future use.

Outline

1. Weighted Interval Scheduling
2. Subset Sum Problem
3. Knapsack Problem
4. Longest Common Subsequence
   - Longest Common Subsequence in Linear Space
5. Shortest Paths in Directed Acyclic Graphs
6. Matrix Chain Multiplication
7. Optimum Binary Search Tree
8. Summary

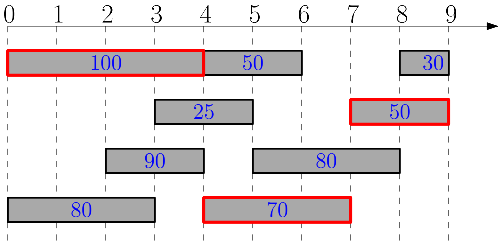
Recall: Interval Schduling

Input: n jobs, job i with start time s_i and finish time f_i

each job has a weight (or value) $v_i > 0$

i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: a maximum-size subset of mutually compatible jobs



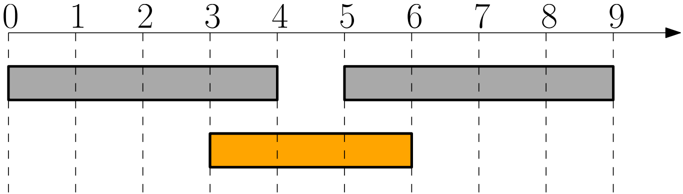
Optimum value = 220

Hard to Design a Greedy Algorithm

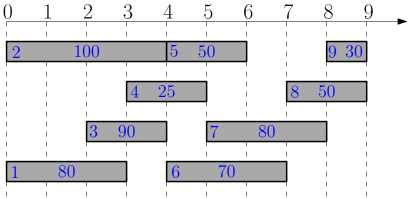
Q: Which job is safe to schedule?

- Job with the earliest finish time? No, we are ignoring weights
- Job with the largest weight? No, we are ignoring times
- Job with the largest $\frac{\text{weight}}{\text{length}}$?

  No, when weights are equal, this is the shortest job



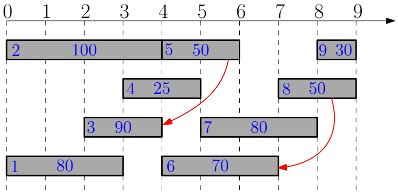
Designing a Dynamic Programming Algorithm



i	$opt[i]$
0	0
1	80
2	100
3	100
4	105
5	150
6	170
7	185
8	220
9	220

- Sort jobs according to non-decreasing order of finish times
- $opt[i]$: optimal value for instance only containing jobs $\{1, 2, \cdots, i\}$

Designing a Dynamic Programming Algorithm



- Focus on instance $\{1, 2, 3, \cdots, i\}$,
- $opt[i]$: optimal value for the instance
- assume we have computed $opt[0], opt[1], \cdots, opt[i-1]$

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $p_i =$ the largest j such that $f_j \leq s_i$

Designing a Dynamic Programming Algorithm

Q: The value of optimal solution that does not contain i?

A: $opt[i-1]$

Q: The value of optimal solution that contains job i?

A: $v_i + opt[p_i]$, $\quad p_i =$ the largest j such that $f_j \leq s_i$

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

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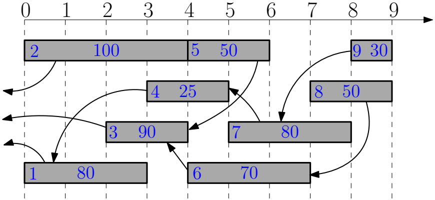
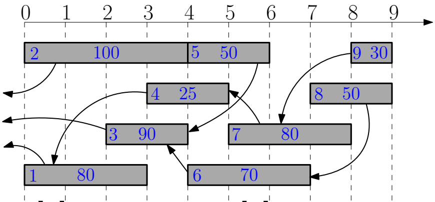


- $opt[0] = 0$
- $opt[1] = \max\{opt[0], 80 + opt[0]\} = 80$
- $opt[2] = \max\{opt[1], 100 + opt[0]\} = 100$
- $opt[3] = \max\{opt[2], 90 + opt[0]\} = 100$
- $opt[4] = \max\{opt[3], 25 + opt[1]\} = 105$
- $opt[5] = \max\{opt[4], 50 + opt[3]\} = 150$

Designing a Dynamic Programming Algorithm

Recursion for $opt[i]$:

$$opt[i] = \max\{opt[i-1], v_i + opt[p_i]\}$$



- $opt[0] = 0, \quad opt[1] = 80, \quad opt[2] = 100$
- $opt[3] = 100, \quad opt[4] = 105, \quad opt[5] = 150$
- $opt[6] = \max\{opt[5], 70 + opt[3]\} = 170$
- $opt[7] = \max\{opt[6], 80 + opt[4]\} = 185$
- $opt[8] = \max\{opt[7], 50 + opt[6]\} = 220$
- $opt[9] = \max\{opt[8], 30 + opt[7]\} = 220$

Dynamic Programming

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i] \leftarrow \max\{opt[i-1], v_i + opt[p_i]\}$

- Running time sorting: $O(n \lg n)$
- Running time for computing p: $O(n \lg n)$ via binary search
- Running time for computing $opt[n]$: $O(n)$

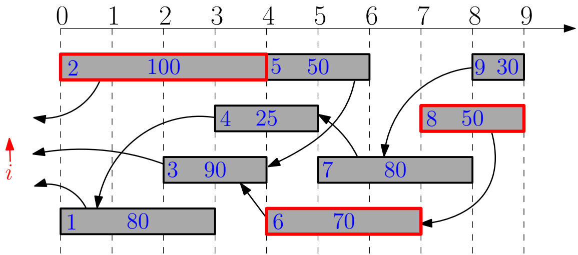
How Can We Recover the Optimum Schedule?

sort jobs by non-decreasing order of finishing times
compute $p_1, p_2, \cdots, p_n$
$opt[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
  if $opt[i-1] \geq v_i + opt[p_i]$ **then**
    $opt[i] \leftarrow opt[i-1]$
    $b[i] \leftarrow$ N
  else
    $opt[i] \leftarrow v_i + opt[p_i]$
    $b[i] \leftarrow$ Y

$i \leftarrow n, S \leftarrow \emptyset$
while $i \neq 0$ **do**
  if $b[i] =$ N **then**
    $i \leftarrow i - 1$
  else
    $S \leftarrow S \cup \{i\}$
    $i \leftarrow p_i$
return S

Recovering Optimum Schedule: Example

i	$opt[i]$	$b[i]$
0	0	$\perp$
1	80	Y
2	100	Y
3	100	N
4	105	Y
5	150	Y
6	170	Y
7	185	Y
8	220	Y
9	220	N



Dynamic Programming

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- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

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Subset Sum Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} w_i \qquad \text{s.t.} \sum_{i \in S} w_i \le W.$$

- Motivation: you have budget W, and want to buy a subset of items, so as to spend as much money as possible.

Example:

- $W = 35, n = 5, w = (14, 9, 17, 10, 13)$
- Optimum: $S = \{1, 2, 4\}$ and $14 + 9 + 10 = 33$

Greedy Algorithms for Subset Sum

Candidate Algorithm:

- Sort according to non-increasing order of weights
- Select items in the order as long as the total weight remains below W

Q: Does candidate algorithm always produce optimal solutions?

A: No. $W = 100, n = 3, w = (51, 50, 50)$.

Q: What if we change "non-increasing" to "non-decreasing"?

A: No. $W = 100, n = 3, w = (1, 50, 50)$

Design a Dynamic Programming Algorithm

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

Q: The value of the optimum solution that does not contain i?

A: $opt[i-1, W']$

Q: The value of the optimum solution that contains i?

A: $opt[i-1, W'-w_i] + w_i$

Dynamic Programming

- Consider the instance: $i, W', (w_1, w_2, \cdots, w_i)$;
- $opt[i, W']$: the optimum value of the instance

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max\left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W'-w_i] + w_i \end{array} \right\} & i > 0, w_i \le W' \end{cases}$$

Dynamic Programming

for $W' \leftarrow 0$ to W **do**
    $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    for $W' \leftarrow 0$ to W **do**
        $opt[i, W'] \leftarrow opt[i-1, W']$
        if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ **then**
            $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
return $opt[n, W]$

Recover the Optimum Set

for $W' \leftarrow 0$ to W **do**
    $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    for $W' \leftarrow 0$ to W **do**
        $opt[i, W'] \leftarrow opt[i-1, W']$
        $b[i, W'] \leftarrow$ N
        if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ **then**
            $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
            $b[i, W'] \leftarrow$ Y
return $opt[n, W]$

Recover the Optimum Set

$i \leftarrow n, W' \leftarrow W, S \leftarrow \emptyset$
while $i > 0$ **do**
    if $b[i, W'] = \mathsf{Y}$ **then**
        $W' \leftarrow W' - w_i$
        $S \leftarrow S \cup \{i\}$
    $i \leftarrow i - 1$
return S

Running Time of Algorithm

for $W' \leftarrow 0$ to W **do**
    $opt[0, W'] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    for $W' \leftarrow 0$ to W **do**
        $opt[i, W'] \leftarrow opt[i-1, W']$
        if $w_i \leq W'$ and $opt[i-1, W'-w_i] + w_i \geq opt[i, W']$ **then**
            $opt[i, W'] \leftarrow opt[i-1, W'-w_i] + w_i$
return $opt[n, W]$

- Running time is $O(nW)$
- Running time is pseudo-polynomial because it depends on value of the input integers.

Avoiding Unncessary Computation and Memory Using Memoized Algorithm and Hash Map

compute-opt(i, W')

if $opt[i, W'] \neq \bot$ **then return** $opt[i, W']$
if $i = 0$ **then** $r \leftarrow 0$
else
    $r \leftarrow$ compute-opt$(i-1, W')$
    if $w_i \leq W'$ **then**
        $r' \leftarrow$ compute-opt$(i-1, W'-w_i) + w_i$
        if $r' > r$ **then** $r \leftarrow r'$
$opt[i, W'] \leftarrow r$
return r

- Use hash map for opt

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Knapsack Problem

Input: an integer bound $W > 0$

a set of n items, each with an integer weight $w_i > 0$

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes} \sum_{i \in S} v_i \qquad \text{s.t.} \sum_{i \in S} w_i \le W.$$

- Motivation: you have budget W, and want to buy a subset of items of maximum total value

DP for Knapsack Problem

- $opt[i, W']$: the optimum value when budget is W' and items are $\{1, 2, 3, \cdots, i\}$.
- If $i = 0$, $opt[i, W'] = 0$ for every $W' = 0, 1, 2, \cdots, W$.

$$opt[i, W'] = \begin{cases} 0 & i = 0 \\ opt[i-1, W'] & i > 0, w_i > W' \\ \max \left\{ \begin{array}{c} opt[i-1, W'] \\ opt[i-1, W'-w_i] + v_i \end{array} \right\} & i > 0, w_i \leq W' \end{cases}$$

Exercise: Items with 3 Parameters

Input: integer bounds $W > 0$, $Z > 0$,

a set of n items, each with an integer weight $w_i > 0$

a size $z_i > 0$ for each item i

a value $v_i > 0$ for each item i

Output: a subset S of items that

$$\text{maximizes } \sum_{i \in S} v_i \qquad \text{s.t.}$$

$$\sum_{i \in S} w_i \leq W \text{ and } \sum_{i \in S} z_i \leq Z$$

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Subsequence

- $A = bacdca$
- $C = adca$
- C is a subsequence of A

Def. Given two sequences $A[1 .. n]$ and $C[1 .. t]$ of letters, C is called a subsequence of A if there exists integers $1 \leq i_1 < i_2 < i_3 < \ldots < i_t \leq n$ such that $A[i_j] = C[j]$ for every $j = 1, 2, 3, \cdots, t$.

- Exercise: how to check if sequence C is a subsequence of A?

Longest Common Subsequence

Input: $A[1 .. n]$ and $B[1 .. m]$

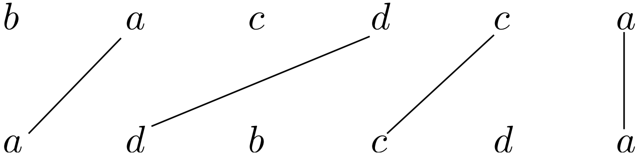
Output: the longest common subsequence of A and B

Example:

- $A = `bacdca'$
- $B = `adbcda'$
- $\text{LCS}(A, B) = `adca'$

- Applications: edit distance (diff), similarity of DNAs

Matching View of LCS



- Goal of LCS: find a maximum-size non-crossing matching between letters in A and letters in B.

Reduce to Subproblems

- $A = `bacdca'$
- $B = `adbcda'$
- either the last letter of A is not matched:
- need to compute LCS($`bacd'$, $`adbcd'$)
- or the last letter of B is not matched:
- need to compute LCS($`bacdc'$, $`adbc'$)

Dynamic Programming for LCS

- $opt[i, j], 0 \leq i \leq n, 0 \leq j \leq m$: length of longest common sub-sequence of $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ or $j = 0$, then $opt[i, j] = 0$.
- if $i > 0, j > 0$, then

$$opt[i, j] = \begin{cases} opt[i-1, j-1] + 1 & \text{if } A[i] = B[j] \\ \max\begin{cases} opt[i-1, j] \\ opt[i, j-1] \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Dynamic Programming for LCS

for $j \leftarrow 0$ to m **do**
    $opt[0, j] \leftarrow 0$
for $i \leftarrow 1$ to n **do**
    $opt[i, 0] \leftarrow 0$
    for $j \leftarrow 1$ to m **do**
        if $A[i] = B[j]$ **then**
            $opt[i, j] \leftarrow opt[i-1, j-1] + 1$, $\pi[i, j] \leftarrow$ "$\nwarrow$"
        else if $opt[i, j-1] \geq opt[i-1, j]$ **then**
            $opt[i, j] \leftarrow opt[i, j-1]$, $\pi[i, j] \leftarrow$ "$\leftarrow$"
        else
            $opt[i, j] \leftarrow opt[i-1, j]$, $\pi[i, j] \leftarrow$ "$\uparrow$"

Example

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	4 $\nwarrow$

Example: Find Common Subsequence

	1	2	3	4	5	6
A	b	a	c	d	c	a
B	a	d	b	c	d	a

	0	1	2	3	4	5	6
0	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$	0 $\perp$
1	0 $\perp$	0 $\leftarrow$	0 $\leftarrow$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$
2	0 $\perp$	1 $\nwarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$
3	0 $\perp$	1 $\uparrow$	1 $\leftarrow$	1 $\leftarrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$
4	0 $\perp$	1 $\uparrow$	2 $\nwarrow$	2 $\leftarrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$
5	0 $\perp$	1 $\uparrow$	2 $\uparrow$	2 $\leftarrow$	3 $\nwarrow$	3 $\leftarrow$	3 $\leftarrow$
6	0 $\perp$	1 $\nwarrow$	2 $\uparrow$	2 $\leftarrow$	3 $\uparrow$	3 $\leftarrow$	4 $\nwarrow$

Find Common Subsequence

$i \leftarrow n, j \leftarrow m, S \leftarrow ()$

while $i > 0$ and $j > 0$ **do**

    if $\pi[i, j] =$ "$\nwarrow$" **then**

        add $A[i]$ to beginning of S, $i \leftarrow i - 1, j \leftarrow j - 1$

    else if $\pi[i, j] =$ "$\uparrow$" **then**

        $i \leftarrow i - 1$

    else

        $j \leftarrow j - 1$

return S

Variants of Problem

Edit Distance with Insertions and Deletions

Input: a string A

each time we can delete a letter from A or insert a letter to A

Output: minimum number of operations (insertions or deletions) we need to change A to B?

Example:

- A = ocurrance, B = occurrence
- 3 operations: insert 'c', remove 'a' and insert 'e'

Obs. #OPs = length(A) + length(B) - 2 · length(LCS(A, B))

Variants of Problem

Edit Distance with Insertions, Deletions and Replacing

Input: a string A,

each time we can delete a letter from A, insert a letter to A or change a letter

Output: how many operations do we need to change A to B?

Example:

- $A =$ ocurrance, $B =$ occurrence.
- 2 operations: insert 'c', change 'a' to 'e'

- Not related to LCS any more

Edit Distance (with Replacing)

- $opt[i,j], 0 \le i \le n, 0 \le j \le m$: edit distance between $A[1 .. i]$ and $B[1 .. j]$.
- if $i = 0$ then $opt[i,j] = j$; if $j = 0$ then $opt[i,j] = i$.
- if $i > 0, j > 0$, then

$$opt[i,j] = \begin{cases} opt[i-1,j-1] & \text{if } A[i] = B[j] \\ \min \begin{cases} opt[i-1,j]+1 \\ opt[i,j-1]+1 \\ opt[i-1,j-1]+1 \end{cases} & \text{if } A[i] \neq B[j] \end{cases}$$

Exercise: Longest Palindrome

Def. A palindrome is a string which reads the same backward or forward.

- example: “racecar”, “wasitacaroracatisaw”, ”putitup”

Longest Palindrome Subsequence

Input: a sequence A

Output: the longest subsequence C of A that is a palindrome.

Example:

- Input: acbcedeacab
- Output: acedeca

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Computing the Length of LCS

```
for j ← 0 to m do
    opt[0, j] ← 0
for i ← 1 to n do
    opt[i, 0] ← 0
    for j ← 1 to m do
        if A[i] = B[j] then
            opt[i, j] ← opt[i − 1, j − 1] + 1
        else if opt[i, j − 1] ≥ opt[i − 1, j] then
            opt[i, j] ← opt[i, j − 1]
        else
            opt[i, j] ← opt[i − 1, j]
```

Obs. The i-th row of table only depends on $(i-1)$-th row.

Reducing Space to $O(n+m)$

Obs. The i-th row of table only depends on $(i-1)$-th row.

Q: How to use this observation to reduce space?

A: We only keep two rows: the $(i-1)$-th row and the i-th row.

Linear Space Algorithm to Compute Length of LCS

```
for j ← 0 to m do
    opt[0, j] ← 0
for i ← 1 to n do
    opt[i mod 2, 0] ← 0
    for j ← 1 to m do
        if A[i] = B[j] then
            opt[i mod 2, j] ← opt[i − 1 mod 2, j − 1] + 1
        else if opt[i mod 2, j − 1] ≥ opt[i − 1 mod 2, j] then
            opt[i mod 2, j] ← opt[i mod 2, j − 1]
        else
            opt[i mod 2, j] ← opt[i − 1 mod 2, j]
return opt[n mod 2, m]
```

How to Recover LCS Using Linear Space?

- Only keep the last two rows: only know how to match $A[n]$
- Can recover the LCS using n rounds: time $= O(n^2m)$
- Using Divide and Conquer + Dynamic Programming:
  - Space: $O(m+n)$
  - Time: $O(nm)$

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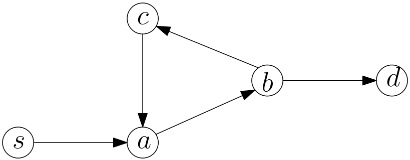
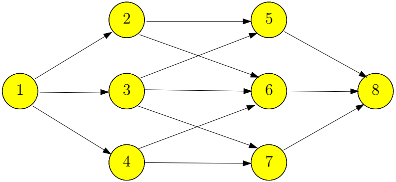
Directed Acyclic Graphs

Def. A directed acyclic graph (DAG) is a directed graph without (directed) cycles.



not a DAG



a DAG

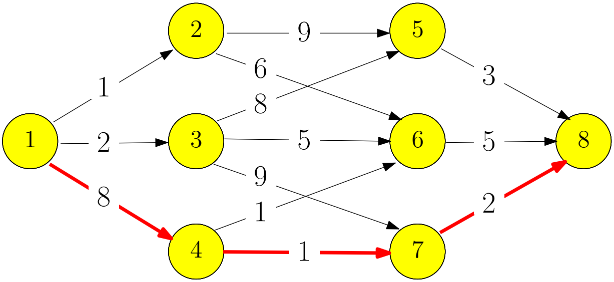
Lemma A directed graph is a DAG if and only its vertices can be topologically sorted.

Shortest Paths in DAG

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the shortest path from 1 to i, for every $i \in V$



Shortest Paths in DAG

- $f[i]$: length of the shortest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \min_{j:(j,i)\in E} \{f(j) + w(j,i)\} & i = 2, 3, \cdots, n \end{cases}$$

Shortest Paths in DAG

- Use an adjacency list for incoming edges of each vertex i

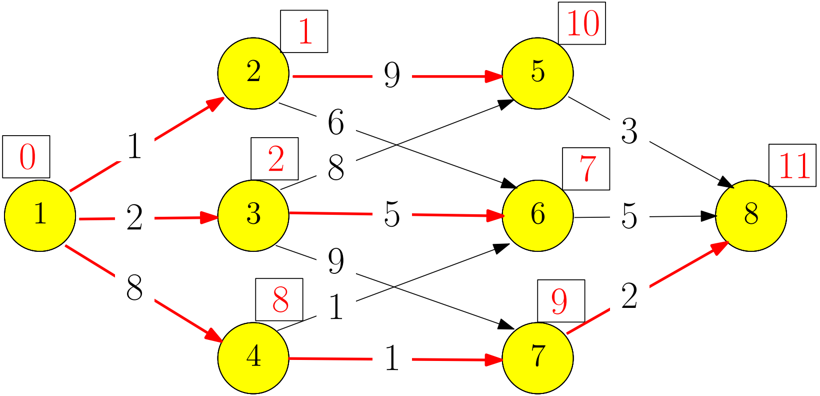
Shortest Paths in DAG

$f[1] \leftarrow 0$
for $i \leftarrow 2$ to n **do**
    $f[i] \leftarrow \infty$
    for each incoming edge (j, i) of i **do**
        if $f[j] + w(j, i) < f[i]$ **then**
            $f[i] \leftarrow f[j] + w(j, i)$
            $\pi(i) \leftarrow j$

print-path(t)

if $t = 1$ **then**
    print(1)
    return
print-path$(\pi(t))$
print(",", t)

Example



Variant: Heavest Path in a Directed Acyclic Graph

Heaviest Path in a Directed Acyclic Graph

Input: directed acyclic graph $G = (V, E)$ and $w : E \to \mathbb{R}$.

Assume $V = \{1, 2, 3 \cdots, n\}$ is topologically sorted: if $(i, j) \in E$, then $i < j$

Output: the path with the largest weight (the heaviest path) from 1 to n.

- $f[i]$: weight of the heaviest path from 1 to i

$$f[i] = \begin{cases} 0 & i = 1 \\ \max_{j:(j,i)\in E} \{f(j) + w(j,i)\} & i = 2, 3, \cdots, n \end{cases}$$

Outline

1. Weighted Interval Scheduling
2. Subset Sum Problem
3. Knapsack Problem
4. Longest Common Subsequence
   - Longest Common Subsequence in Linear Space
5. Shortest Paths in Directed Acyclic Graphs
6. Matrix Chain Multiplication
7. Optimum Binary Search Tree
8. Summary

Matrix Chain Multiplication

Matrix Chain Multiplication

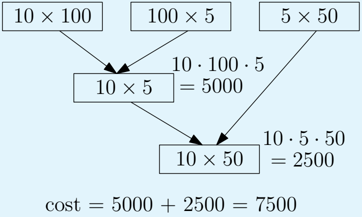
Input: n matrices $A_1, A_2, \cdots, A_n$ of sizes $r_1 \times c_1, r_2 \times c_2, \cdots, r_n \times c_n$, such that $c_i = r_{i+1}$ for every $i = 1, 2, \cdots, n-1$.

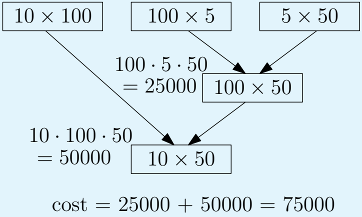
Output: the order of computing $A_1 A_2 \cdots A_n$ with the minimum number of multiplications

Fact Multiplying two matrices of size $r \times k$ and $k \times c$ takes $r \times k \times c$ multiplications.

Example:

- $A_1 : 10 \times 100, \quad A_2 : 100 \times 5, \quad A_3 : 5 \times 50$





- $(A_1A_2)A_3$: $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
- $A_1(A_2A_3)$: $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Matrix Chain Multiplication: Design DP

- Assume the last step is $(A_1 A_2 \cdots A_i)(A_{i+1} A_{i+2} \cdots A_n)$
- Cost of last step: $r_1 \times c_i \times c_n$
- Optimality for sub-instances: we need to compute $A_1 A_2 \cdots A_i$ and $A_{i+1} A_{i+2} \cdots A_n$ optimally
- $opt[i,j]$: the minimum cost of computing $A_i A_{i+1} \cdots A_j$

$$opt[i,j] = \begin{cases} 0 & i = j \\ \min_{k: i \le k < j} \left(opt[i,k] + opt[k+1,j] + r_i c_k c_j \right) & i < j \end{cases}$$

Matrix Chain Multiplication: Design DP

matrix-chain-multiplication$(n, r[1..n], c[1..n])$

let $opt[i, i] \leftarrow 0$ for every $i = 1, 2, \cdots, n$
for $\ell \leftarrow 2$ to n **do**
    for $i \leftarrow 1$ to $n - \ell + 1$ **do**
        $j \leftarrow i + \ell - 1$
        $opt[i, j] \leftarrow \infty$
        for $k \leftarrow i$ to $j - 1$ **do**
            if $opt[i, k] + opt[k + 1, j] + r_i c_k c_j < opt[i, j]$ **then**
                $opt[i, j] \leftarrow opt[i, k] + opt[k + 1, j] + r_i c_k c_j$
                $\pi[i, j] \leftarrow k$
return $opt[1, n]$

Constructing Optimal Solution

Print-Optimal-Order(i, j)

if $i = j$ **then**
    print("A"$_i$)
else
    print("(")
    Print-Optimal-Order($i, \pi[i, j]$)
    Print-Optimal-Order($\pi[i, j] + 1, j$)
    print(")")

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[1,2] &= opt[1,1] + opt[2,2] + 3 \times 5 \times 2 = 30, && \pi[1,2] = 1 \\
opt[2,3] &= opt[2,2] + opt[3,3] + 5 \times 2 \times 6 = 60, && \pi[2,3] = 2 \\
opt[3,4] &= opt[3,3] + opt[4,4] + 2 \times 6 \times 9 = 108, && \pi[3,4] = 3 \\
opt[4,5] &= opt[4,4] + opt[5,5] + 6 \times 9 \times 4 = 216, && \pi[4,5] = 4 \\
opt[1,3] &= \min\{opt[1,1] + opt[2,3] + 3 \times 5 \times 6, \\
&\qquad opt[1,2] + opt[3,3] + 3 \times 2 \times 6\} \\
&= \min\{0 + 60 + 90, 30 + 0 + 36\} = 66, && \pi[1,3] = 2 \\
opt[2,4] &= \min\{opt[2,2] + opt[3,4] + 5 \times 2 \times 9, \\
&\qquad opt[2,3] + opt[4,4] + 5 \times 6 \times 9\} \\
&= \min\{0 + 108 + 90, 60 + 0 + 270\} = 198, && \pi[2,4] = 2,
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$
\begin{aligned}
opt[3,5] &= \min\{opt[3,3] + opt[4,5] + 2 \times 6 \times 4, \\
&\qquad opt[3,4] + opt[5,5] + 2 \times 9 \times 4\} \\
&= \min\{0 + 216 + 48, 108 + 0 + 72\} = 180, \\
\pi[3,5] &= 4, \\
opt[1,4] &= \min\{opt[1,1] + opt[2,4] + 3 \times 5 \times 9, \\
&\qquad opt[1,2] + opt[3,4] + 3 \times 2 \times 9, \\
&\qquad opt[1,3] + opt[4,4] + 3 \times 6 \times 9\} \\
&= \min\{0 + 198 + 135, 30 + 108 + 54, 66 + 0 + 162\} = 192, \\
\pi[1,4] &= 2,
\end{aligned}
$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

$$\begin{aligned}
opt[2,5] &= \min\{opt[2,2] + opt[3,5] + 5 \times 2 \times 4,\\
&\qquad opt[2,3] + opt[4,5] + 5 \times 6 \times 4,\\
&\qquad opt[2,4] + opt[5,5] + 5 \times 9 \times 4\}\\
&= \min\{0 + 180 + 40, 60 + 216 + 120, 198 + 0 + 180\} = 220,\\
opt[1,5] &= \min\{opt[1,1] + opt[2,5] + 3 \times 5 \times 4,\\
&\qquad opt[1,2] + opt[3,5] + 3 \times 2 \times 4,\\
&\qquad opt[1,3] + opt[4,5] + 3 \times 6 \times 4,\\
&\qquad opt[1,4] + opt[5,5] + 3 \times 9 \times 4\}\\
&= \min\{0 + 220 + 60, 30 + 180 + 24,\\
&\qquad 66 + 216 + 72, 192 + 0 + 108\}\\
&= 234,\\
\pi[1,5] &= 2.
\end{aligned}$$

matrix	A_1	A_2	A_3	A_4	A_5
size	3×5	5×2	2×6	6×9	9×4

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, /	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, /	60, 2	198, 2	220, 2
$i = 3$			0, /	108, 3	180, 4
$i = 4$				0, /	216, 4
$i = 5$					0, /

opt, π	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0, /	30, 1	66, 2	192, 2	234, 2
$i = 2$		0, /	60, 2	198, 2	220, 2
$i = 3$			0, /	108, 3	180, 4
$i = 4$				0, /	216, 4
$i = 5$					0, /

Print-Optimal-Order(1,5)
  Print-Optimal-Order(1, 2)
    Print-Optimal-Order(1, 1)
    Print-Optimal-Order(2, 2)
  Print-Optimal-Order(3, 5)
    Print-Optimal-Order(3, 4)
      Print-Optimal-Order(3, 3)
      Print-Optimal-Order(4, 4)
    Print-Optimal-Order(5, 5)

Optimum way for multiplication: $((A_1 A_2)((A_3 A_4) A_5))$

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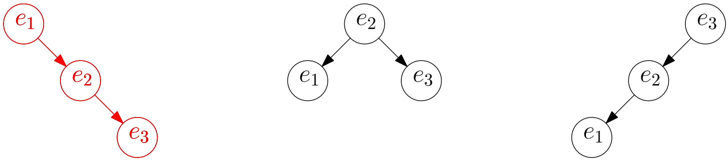
Optimum Binary Search Tree

- n elements $e_1 < e_2 < e_3 < \cdots < e_n$
- e_i has frequency f_i
- goal: build a binary search tree for $\{e_1, e_2, \cdots, e_n\}$ with the minimum accessing cost:

$$\sum_{i=1}^{n} f_i \times (\text{depth of } e_i \text{ in the tree})$$

Optimum Binary Search Tree

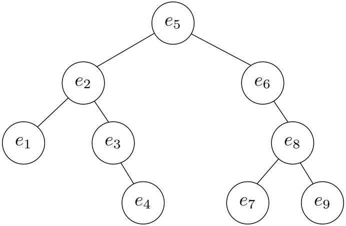
- Example: $f_1 = 10, f_2 = 5, f_3 = 3$



- $10 \times 1 + 5 \times 2 + 3 \times 3 = 29$
- $10 \times 2 + 5 \times 1 + 3 \times 2 = 31$
- $10 \times 3 + 5 \times 2 + 3 \times 1 = 43$

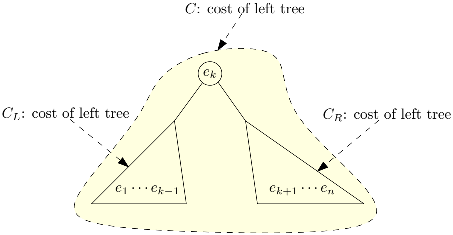
- suppose we decided to let e_k be the root
- $e_1, e_2, \cdots, e_{k-1}$ are on left sub-tree
- $e_{k+1}, e_{k+2}, \cdots, e_n$ are on right sub-tree

- d_j: depth of e_j in our tree
- C, C_L, C_R: cost of tree, left sub-tree and right sub-tree



- $d_1 = 3, d_2 = 2, d_3 = 3, d_4 = 4, d_5 = 1$,
- $d_6 = 2, d_7 = 4, d_8 = 3, d_9 = 4$,
- $C = 3f_1 + 2f_2 + 3f_3 + 4f_4 + f_5 + 2f_6 + 4f_7 + 3f_8 + 4f_9$
- $C_L = 2f_1 + f_2 + 2f_3 + 3f_4$
- $C_R = f_6 + 3f_7 + 2f_8 + 3f_9$

- $C = C_L + C_R + \sum_{j=1}^{9} f_j Z$



$$
\begin{aligned}
C = \sum_{\ell=1}^{n} f_\ell d_\ell &= \sum_{\ell=1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= \sum_{\ell=1}^{k-1} f_\ell (d_\ell - 1) + \sum_{\ell=k+1}^{n} f_\ell (d_\ell - 1) + \sum_{\ell=1}^{n} f_\ell \\
&= C_L + C_R + \sum_{\ell=1}^{n} f_\ell
\end{aligned}
$$

$$C = C_L + C_R + \sum_{\ell=1}^{n} f_\ell$$

- In order to minimize C, need to minimize C_L and C_R respectively
- $opt[i, j]$: the optimum cost for the instance $(f_i, f_{i+1}, \cdots, f_j)$

$$opt[1, n] = \min_{k:1 \le k \le n} (opt[1, k-1] + opt[k+1, n]) + \sum_{\ell=1}^{n} f_\ell$$

- In general, $opt[i, j] =$

$$\begin{cases} 0 & \text{if } i = j+1 \\ \min_{k: i \le k \le j} \big(opt[i, k-1] + opt[k+1, j]\big) + \sum_{\ell=i}^{j} f_\ell & \text{if } i \le j \end{cases}$$

Optimum Binary Search Tree

$fsum[0] \leftarrow 0$
for $i \leftarrow 1$ to n **do** $fsum[i] \leftarrow fsum[i-1] + f_i$ $\qquad \triangleright\ fsum[i] = \sum_{j=1}^{i} f_j$
for $i \leftarrow 0$ to n **do** $opt[i+1, i] \leftarrow 0$
for $\ell \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to $n - \ell + 1$ **do**
        $j \leftarrow i + \ell - 1$, $opt[i, j] \leftarrow \infty$
        for $k \leftarrow i$ to j **do**
            if $opt[i, k-1] + opt[k+1, j] < opt[i, j]$ **then**
                $opt[i, j] \leftarrow opt[i, k-1] + opt[k+1, j]$
                $\pi[i, j] \leftarrow k$
        $opt[i, j] \leftarrow opt[i, j] + fsum[j] - fsum[i-1]$

Printing the Tree

Print-Tree(i, j)

```
if i > j then
    return
else
    print('(')
    Print-Tree(i, π[i, j] − 1)
    print(π[i, j])
    Print-Tree(π[i, j] + 1, j)
    print(')')
```

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Dynamic Programming

- Break up a problem into many overlapping sub-problems
- Build solutions for larger and larger sub-problems
- Use a table to store solutions for sub-problems for reuse

Comparison with greedy algorithms

- Greedy algorithm: each step is making a small progress towards constructing the solution
- Dynamic programming: the whole solution is constructed in the last step

Comparison with divide and conquer

- Divide and conquer: an instance is broken into many independent sub-instances, which are solved separately.
- Dynamic programming: the sub-instances we constructed are overlapping.

Definition of Cells for Problems We Learnt

- Weighted interval scheduling: $opt[i]$ = value of instance defined by jobs $\{1, 2, \cdots, i\}$
- Subset sum, knapsack: $opt[i, W']$ = value of instance with items $\{1, 2, \cdots, i\}$ and budget W'
- Longest common subsequence: $opt[i, j]$ = value of instance defined by $A[1..i]$ and $B[1..j]$
- Shortest paths in DAG: $f[v]$ = length of shortest path from s to v
- Matrix chain multiplication, optimum binary search tree: $opt[i, j]$ = value of instances defined by matrices i to j