CSE 431/531: Algorithm Analysis and Design (Spring 2022)

Graph Algorithms

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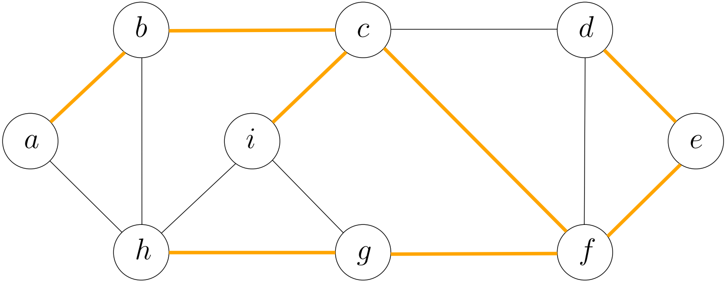
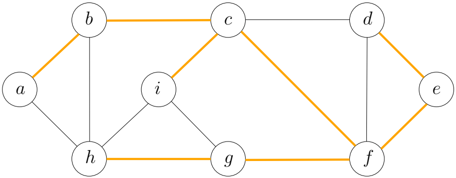
Outline

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight

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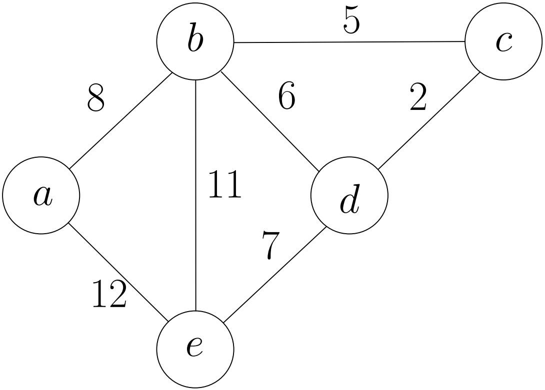

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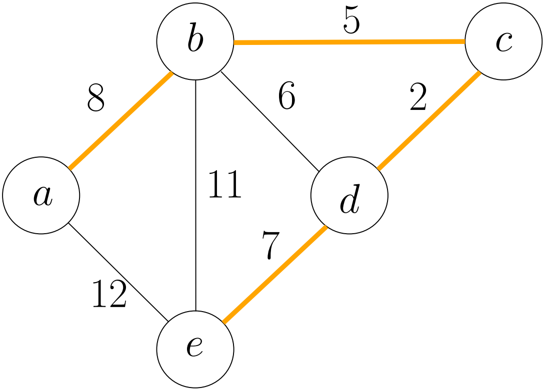

Recall: Steps of Designing A Greedy Algorithm

- Design a "reasonable" strategy
- Prove that the reasonable strategy is "safe" (key, usually done by "exchanging argument")
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

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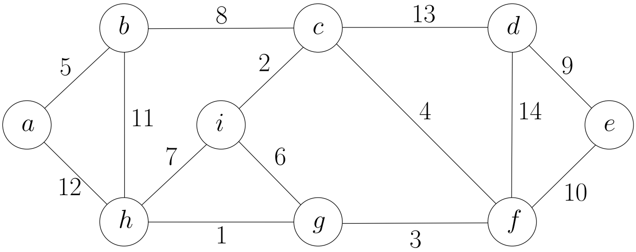
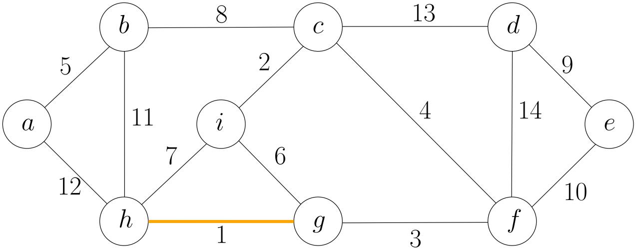
Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

Outline



Q: Which edge can be safely included in the MST?



Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

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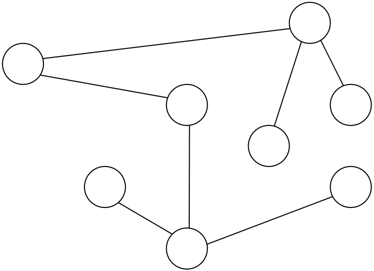
Proof.

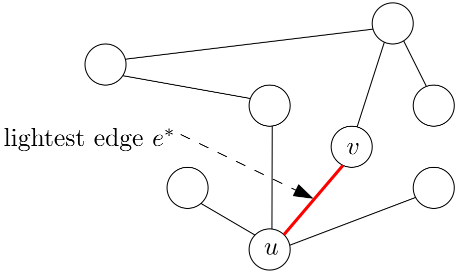
- Take a minimum spanning tree T



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

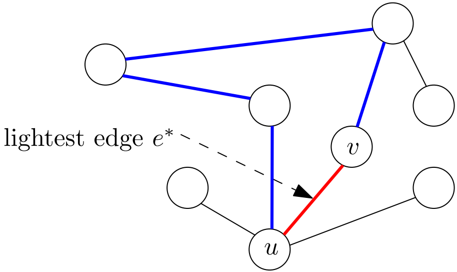
- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
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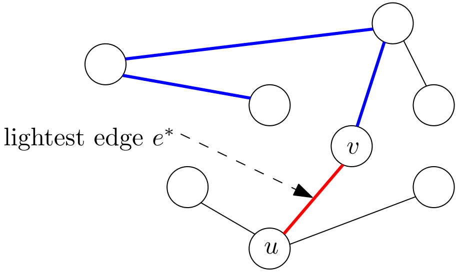
- Remove any edge e in the path to obtain tree T'



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Proof.

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- Assume the lightest edge e^* is not in T
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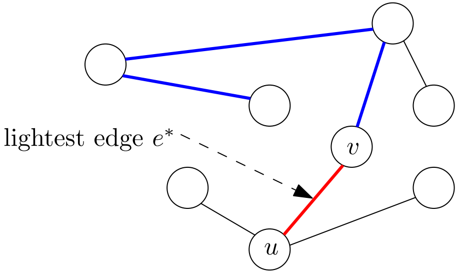
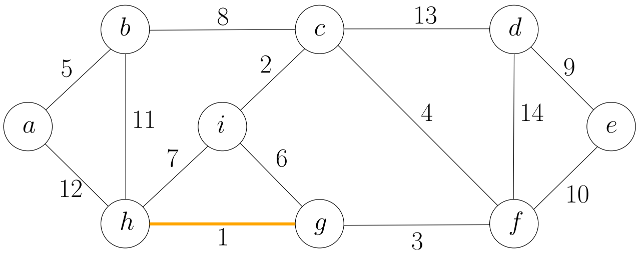
- $w(e^*) \le w(e) \implies w(T') \le w(T)$: T' is also a MST $\square$



Is the Residual Problem Still a MST Problem?



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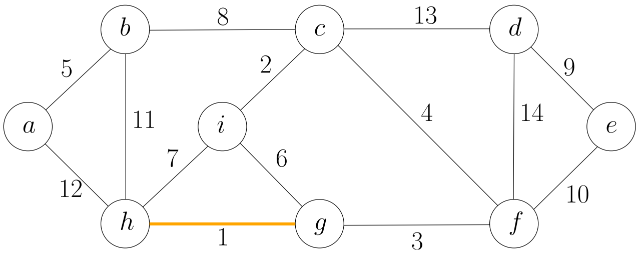

- Residual problem: find the minimum spanning tree that contains edge (g, h)

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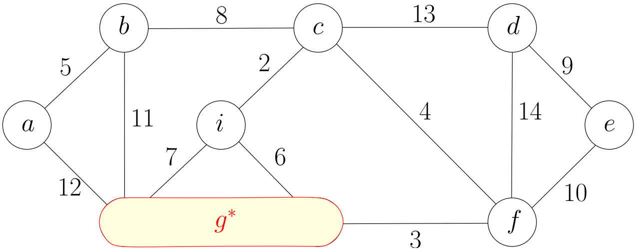
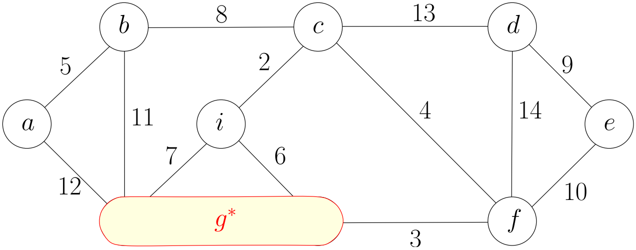


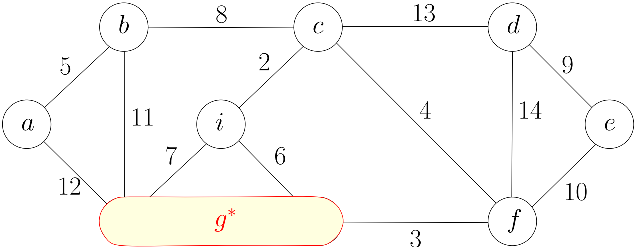
- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)

Is the Residual Problem Still a MST Problem?



- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



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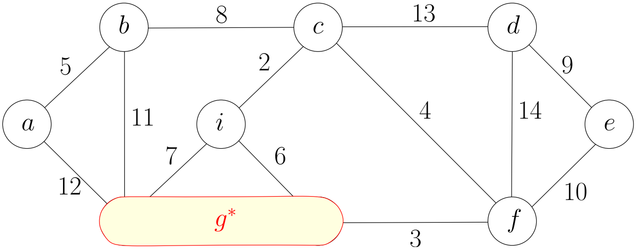

- Remove u and v from the graph, and add a new vertex u^*

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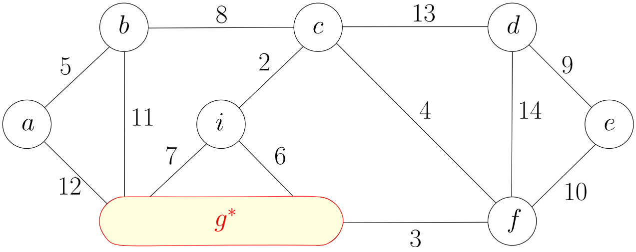

- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E

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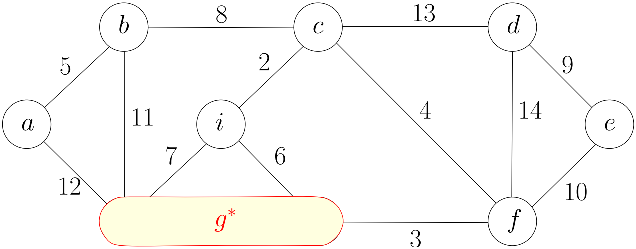

- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)

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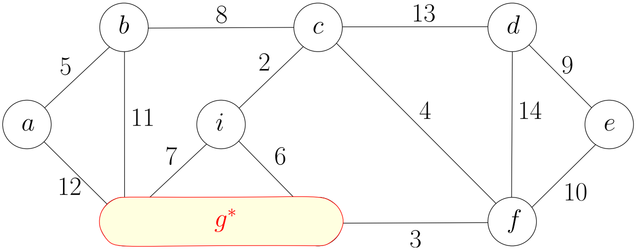
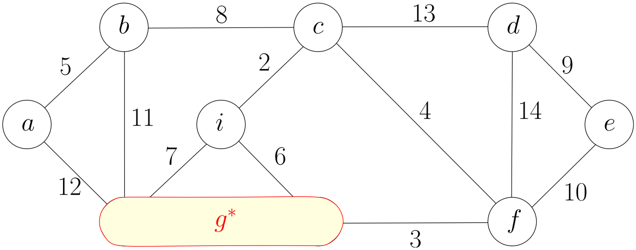


- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

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2. Contract e^* and update G be the contracted graph

Q: What edges are removed due to contractions?

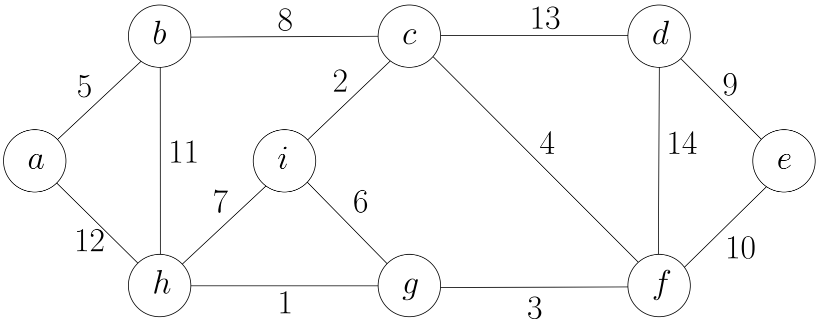
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

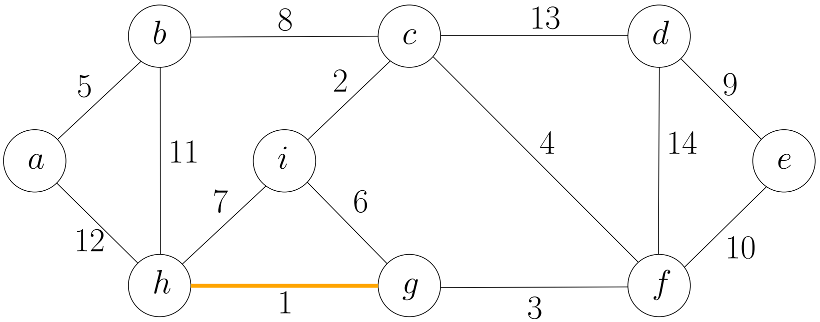
$F \leftarrow \emptyset$
sort edges in E in non-decreasing order of weights w
for each edge (u, v) in the order **do**
    if u and v are not connected by a path of edges in F **then**
        $F \leftarrow F \cup \{(u, v)\}$
return (V, F)

Kruskal's Algorithm: Example



Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

Kruskal's Algorithm: Example



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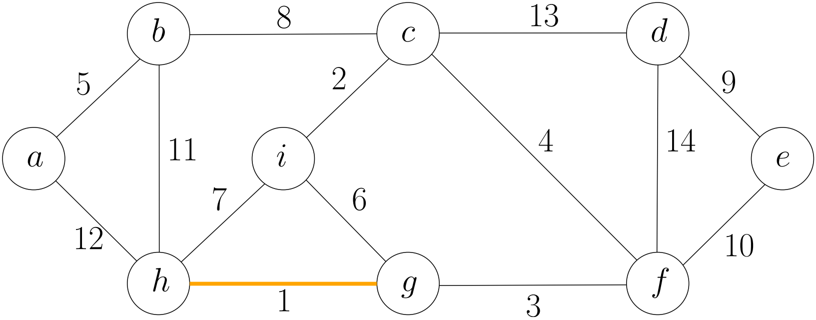

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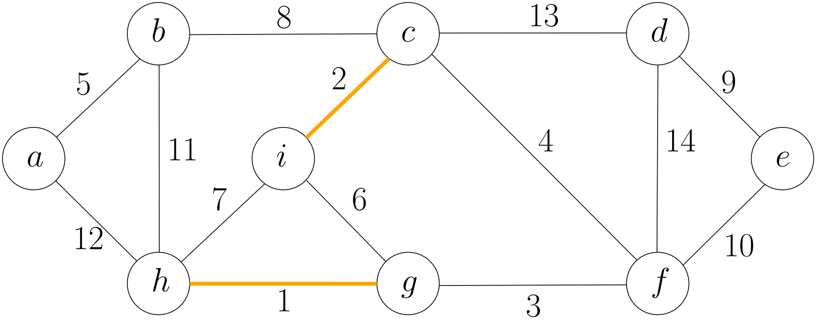

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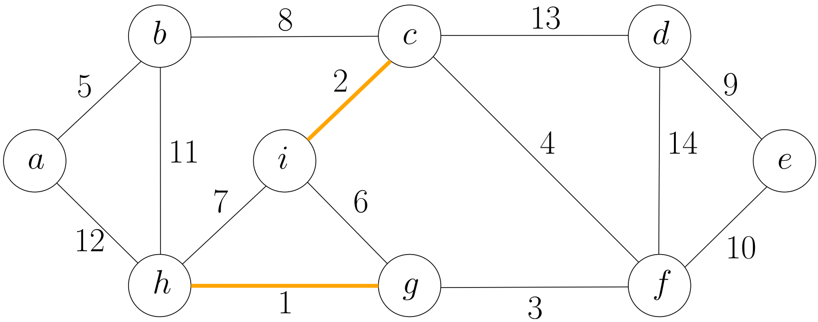

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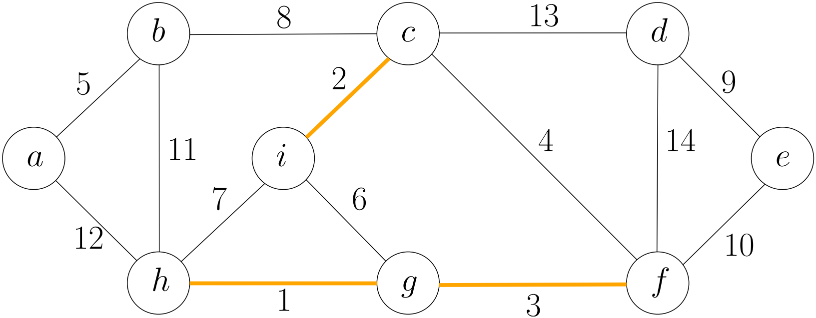

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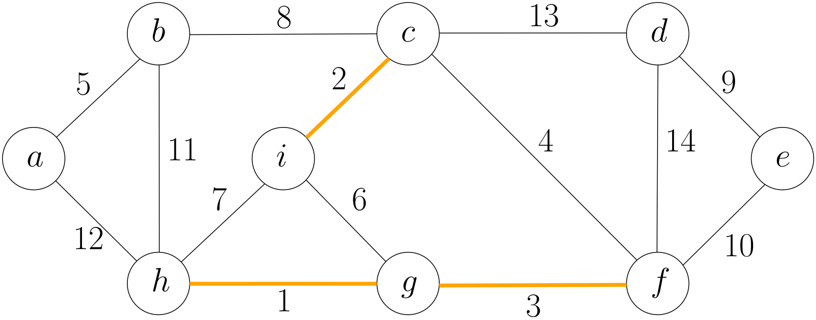

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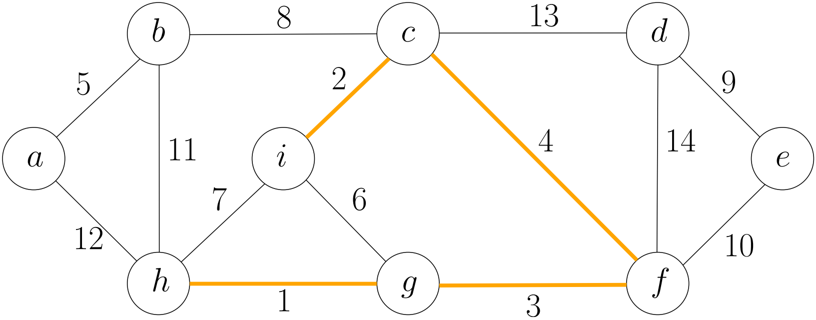

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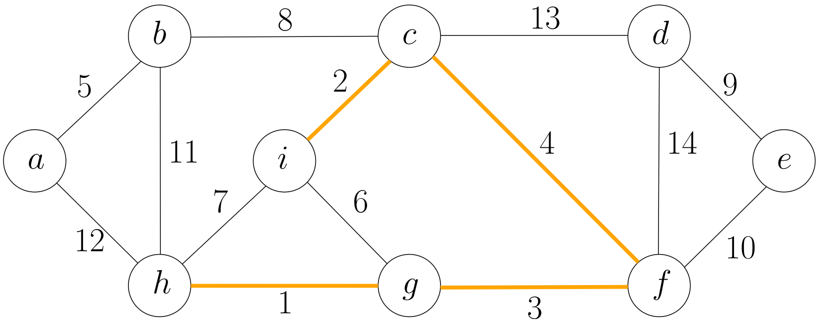

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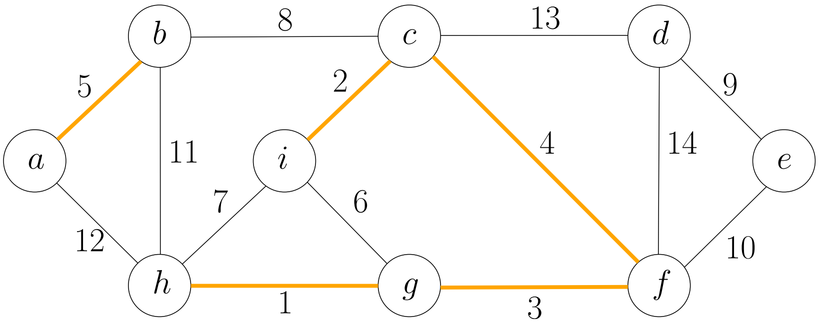

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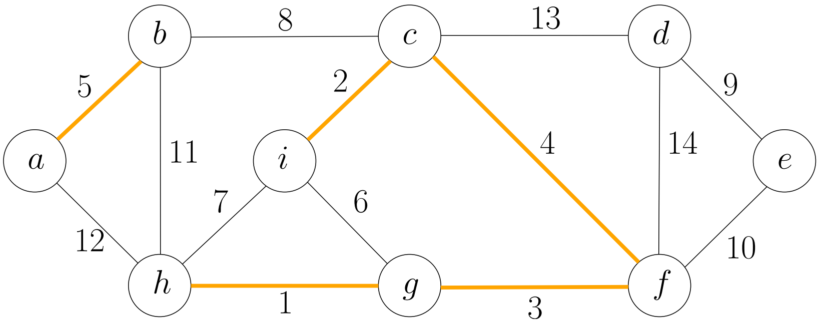

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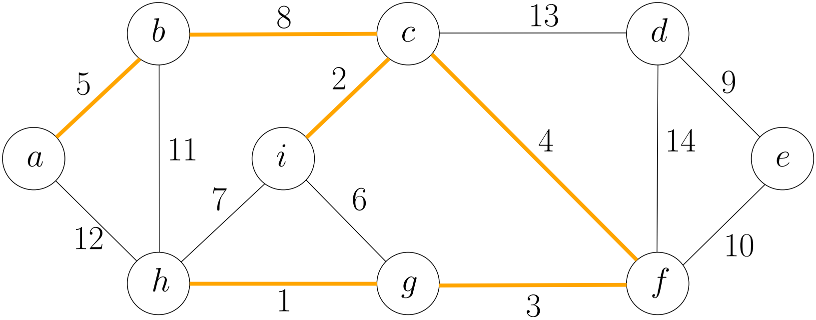

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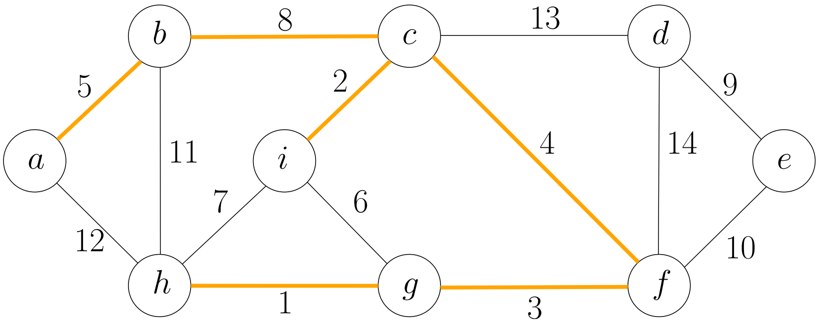

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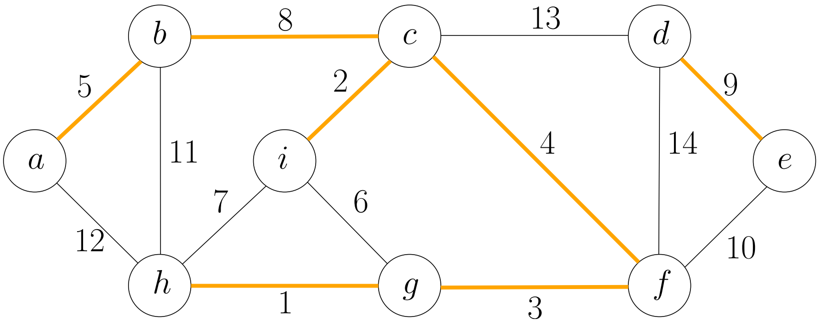

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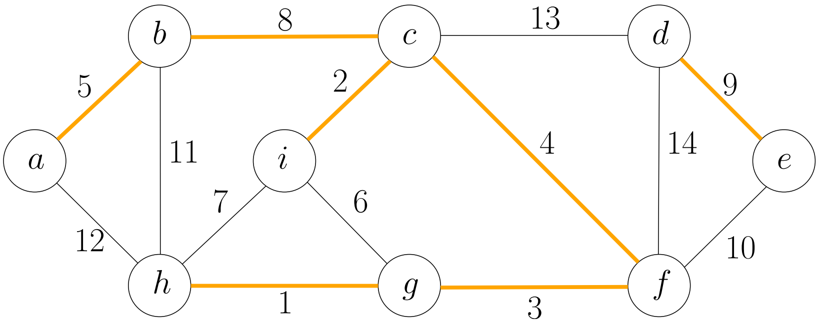

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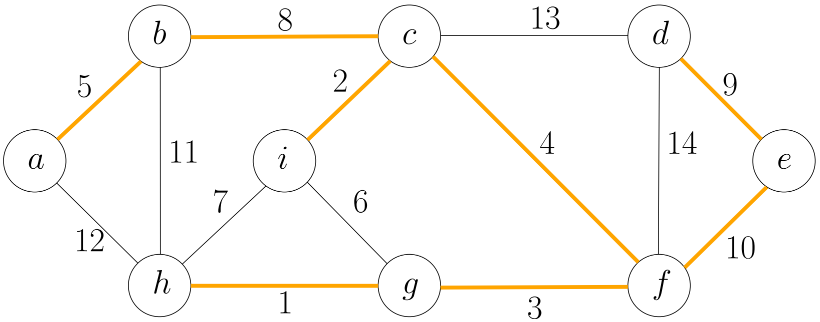

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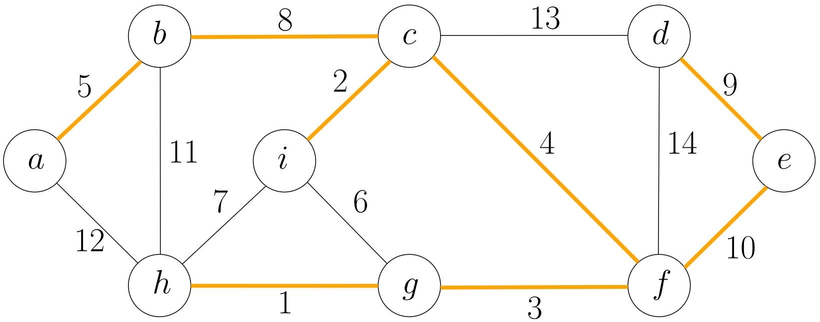


Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

```
F ← ∅
𝒮 ← {{v} : v ∈ V}
sort the edges of E in non-decreasing order of weights w
for each edge (u, v) ∈ E in the order do
    S_u ← the set in 𝒮 containing u
    S_v ← the set in 𝒮 containing v
    if S_u ≠ S_v then
        F ← F ∪ {(u, v)}
        𝒮 ← 𝒮 \ {S_u} \ {S_v} ∪ {S_u ∪ S_v}
return (V, F)
```

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

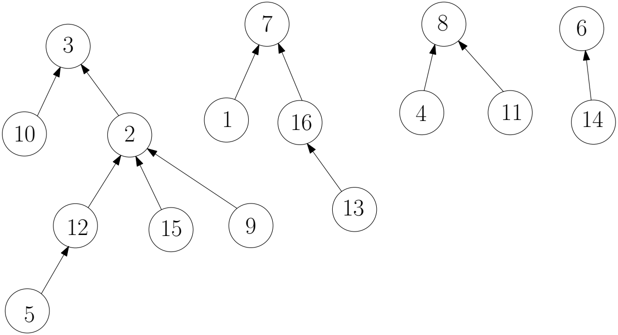
1: $F \leftarrow \emptyset$
2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of E in non-decreasing order of weights w
4: **for** each edge $(u, v) \in E$ in the order **do**
5: $\quad S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6: $\quad S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7: $\quad$ **if** $S_u \neq S_v$ **then**
8: $\quad\quad F \leftarrow F \cup \{(u, v)\}$
9: $\quad\quad \mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: **return** (V, F)

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

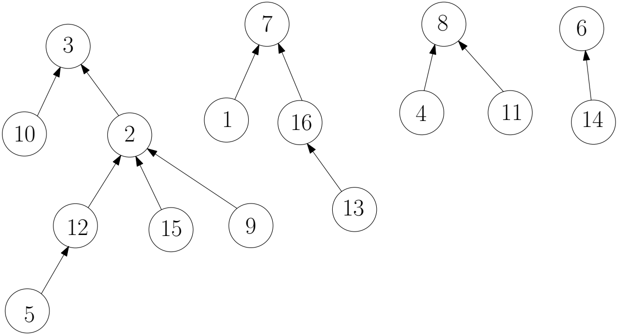
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$

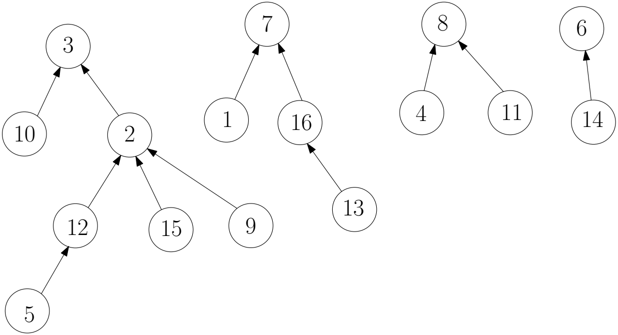


- $par[i]$: parent of i, ($par[i] = \perp$ if i is a root).

Union-Find Data Structure

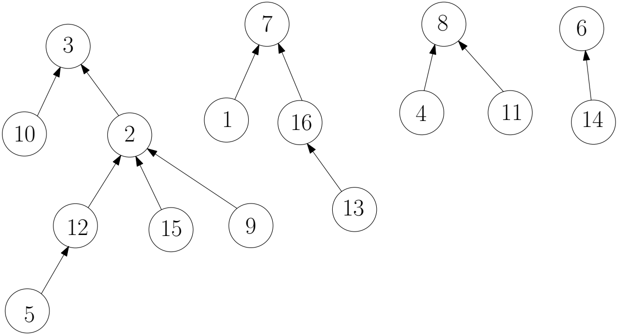


Union-Find Data Structure



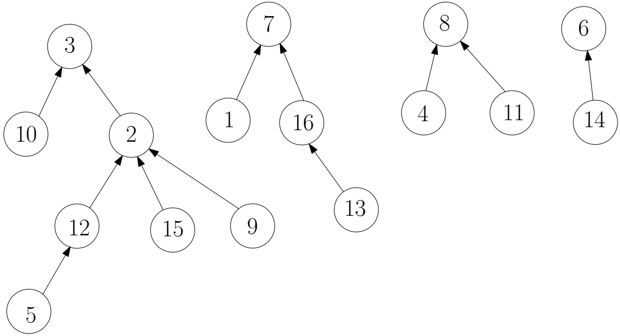
- Q: how can we check if u and v are in the same set?

Union-Find Data Structure



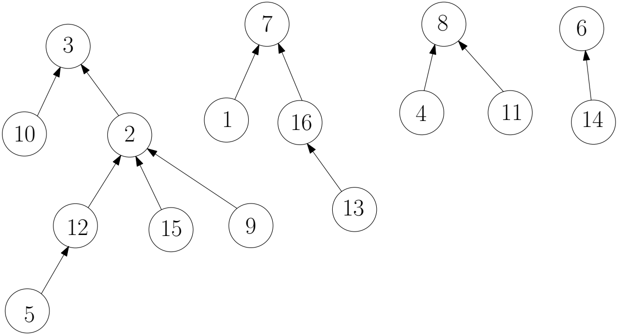
- Q: how can we check if u and v are in the same set?
- A: Check if $\text{root}(u) = \text{root}(v)$.

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if $\text{root}(u) = \text{root}(v)$.
- $\text{root}(u)$: the root of the tree containing u

Union-Find Data Structure



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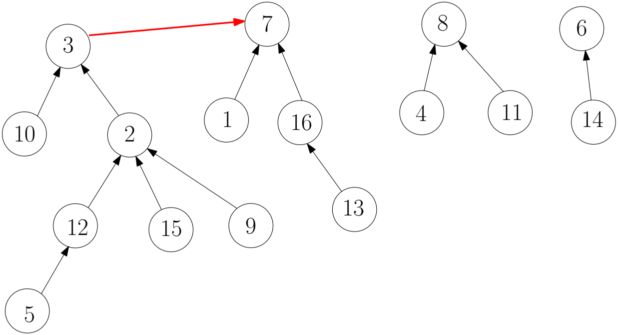
- root(u): the root of the tree containing u
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).
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- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure

root(v)

```
if par[v] = ⊥ then
    return v
else
    return root(par[v])
```

root(v)

if $par[v] = \perp$ **then**
    return v
else
    return root($par[v]$)

- Problem: the tree might too deep; running time might be large

Union-Find Data Structure

root(v)

```
if par[v] = ⊥ then
    return v
else
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- Problem: the tree might too deep; running time might be large
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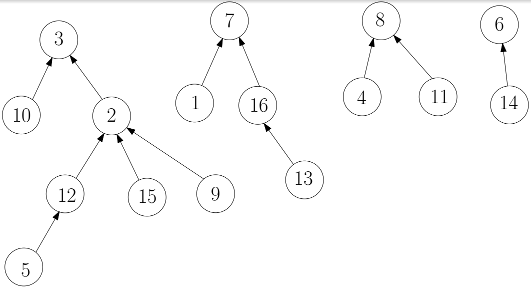

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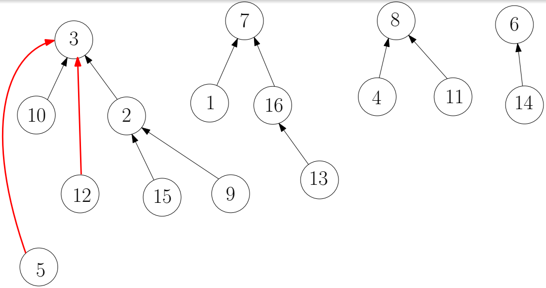

MST-Kruskal(G, w)

```
F ← ∅
𝒮 ← {{v} : v ∈ V}
sort the edges of E in non-decreasing order of weights w
for each edge (u, v) ∈ E in the order do
    S_u ← the set in 𝒮 containing u
    S_v ← the set in 𝒮 containing v
    if S_u ≠ S_v then
        F ← F ∪ {(u, v)}
        𝒮 ← 𝒮 \ {S_u} \ {S_v} ∪ {S_u ∪ S_v}
return (V, F)
```

MST-Kruskal(G, w)

$F \leftarrow \emptyset$

for every $v \in V$ **do**: $par[v] \leftarrow \perp$

sort the edges of E in non-decreasing order of weights w

for each edge $(u, v) \in E$ in the order **do**

    $u' \leftarrow \text{root}(u)$

    $v' \leftarrow \text{root}(v)$

    if $u' \neq v'$ **then**

        $F \leftarrow F \cup \{(u, v)\}$

        $par[u'] \leftarrow v'$

return (V, F)

MST-Kruskal(G, w)

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2: **for** every $v \in V$ **do**: $par[v] \leftarrow \perp$
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4: **for** each edge $(u, v) \in E$ in the order **do**
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7:     **if** $u' \neq v'$ **then**
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9:         $par[u'] \leftarrow v'$
10: **return** (V, F)

- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

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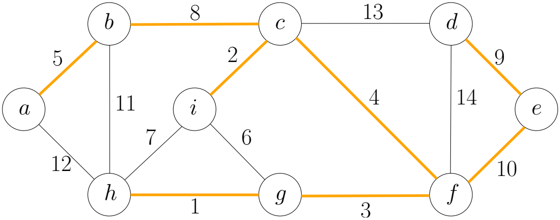
- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time = time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



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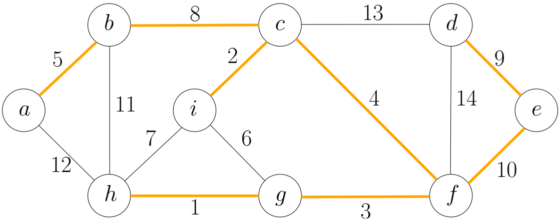

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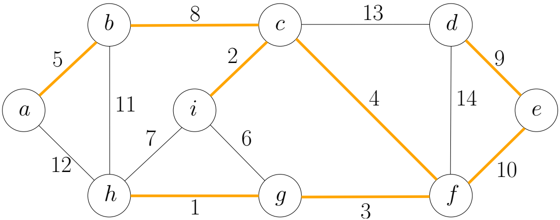

- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree

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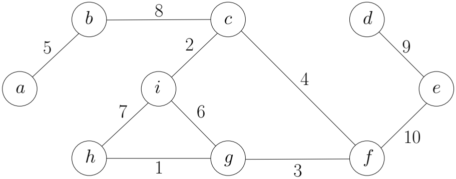

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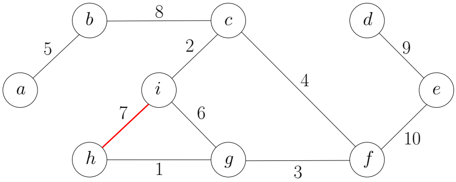

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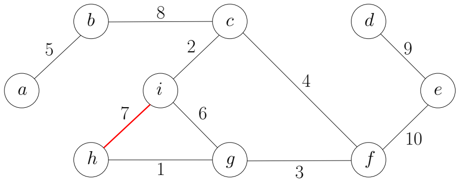

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A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

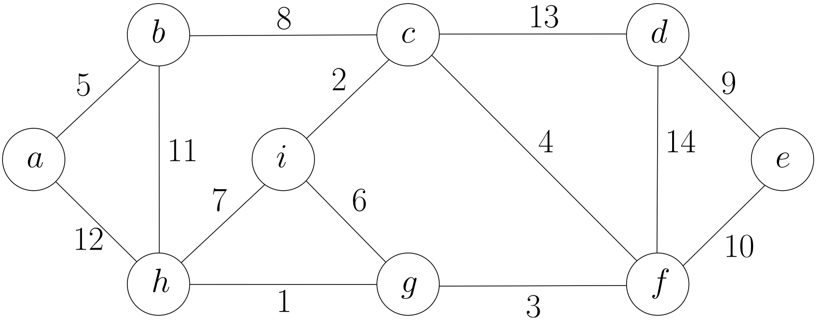
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

$F \leftarrow E$
sort E in non-increasing order of weights
for every e in this order **do**
    if $(V, F \setminus \{e\})$ is connected **then**
        $F \leftarrow F \setminus \{e\}$
return (V, F)

Reverse Kruskal's Algorithm: Example



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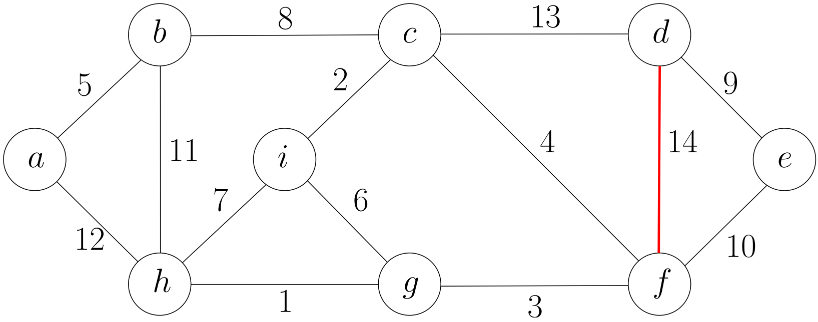

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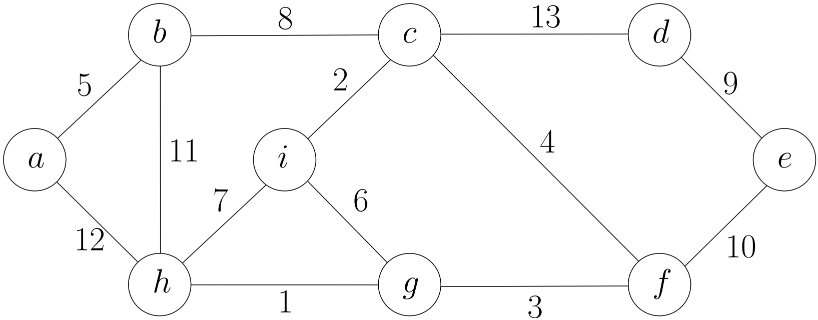

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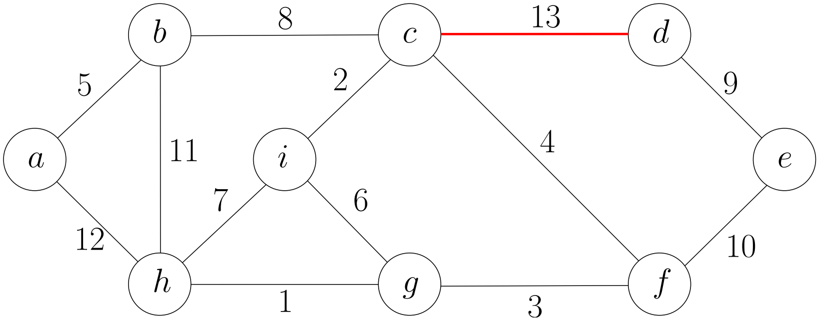

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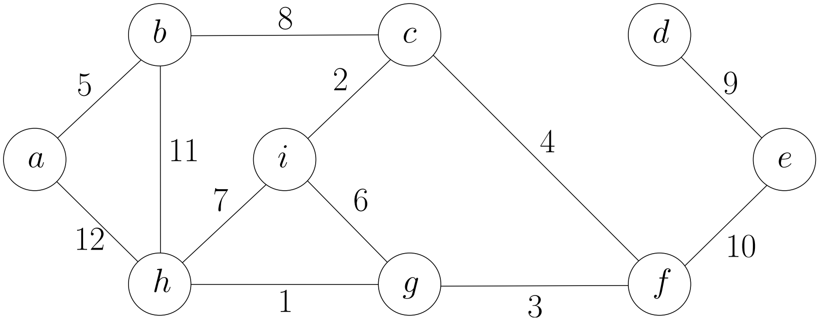

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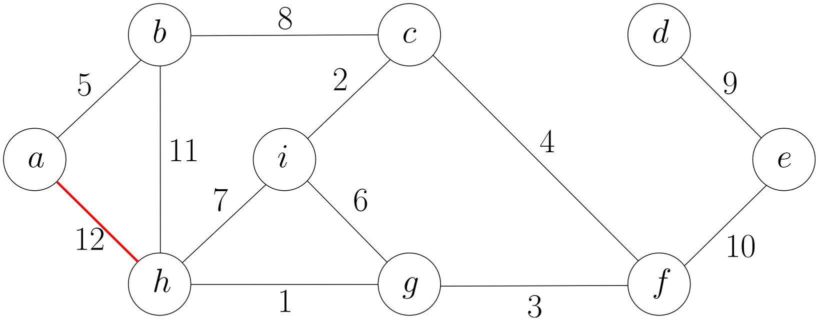

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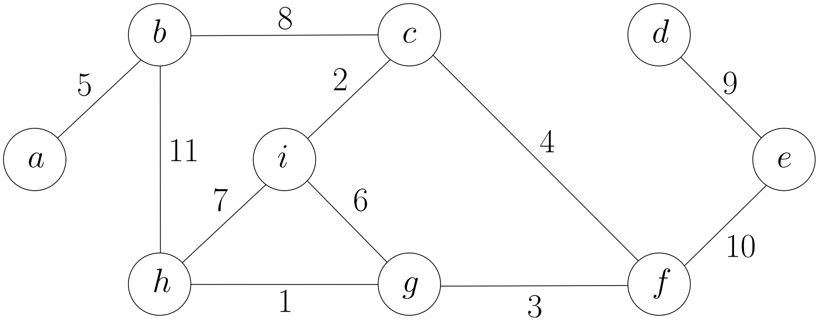

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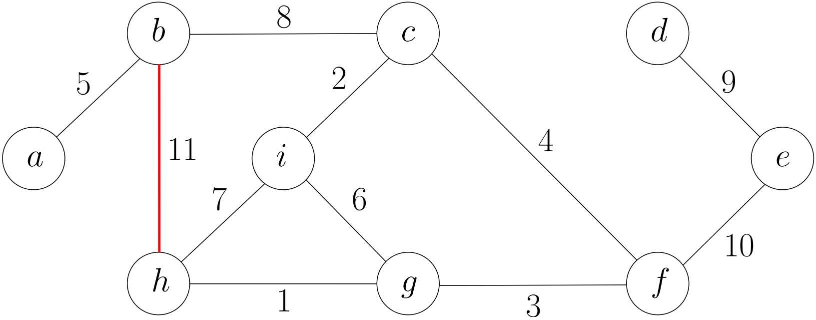

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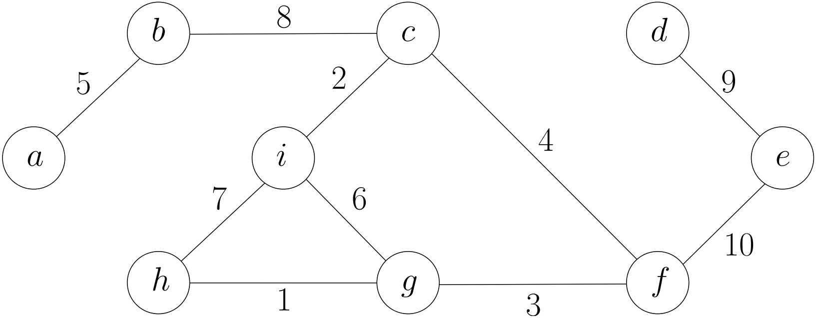

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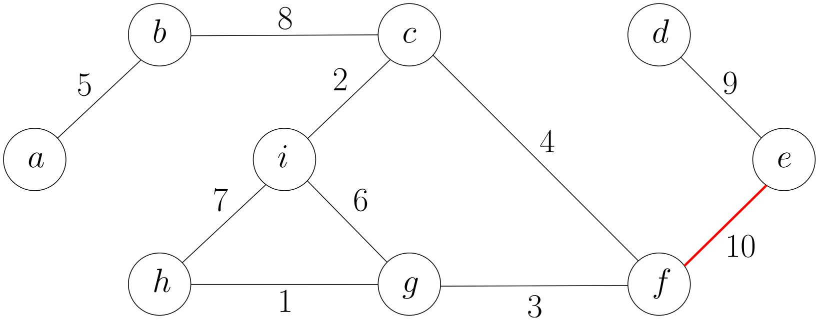

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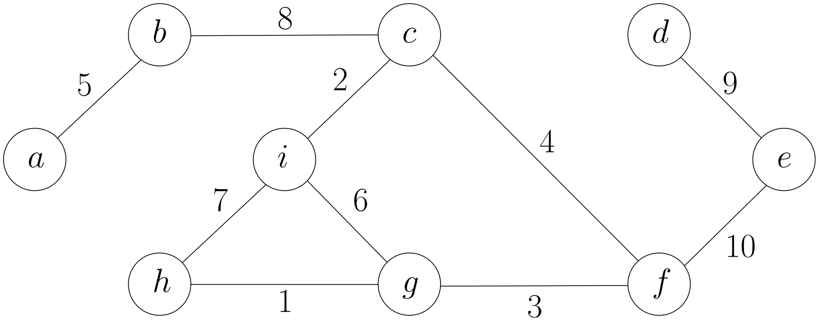

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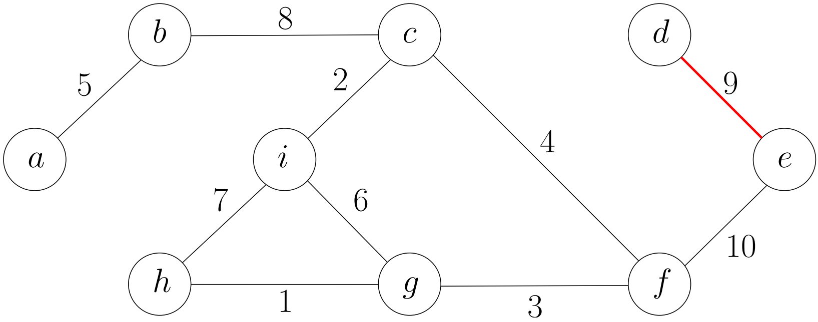

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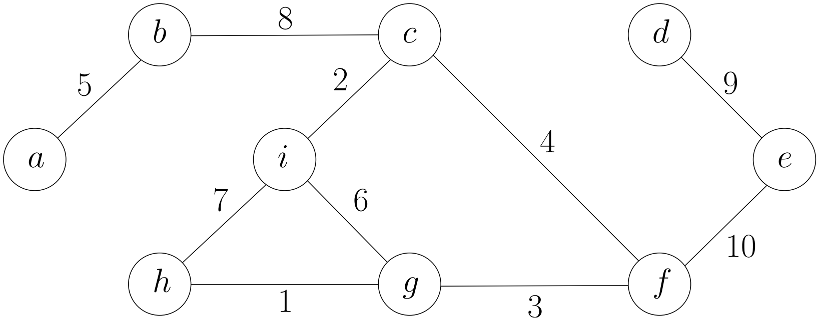

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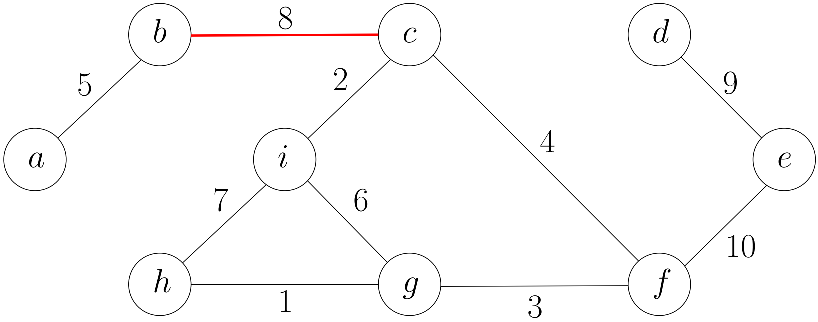

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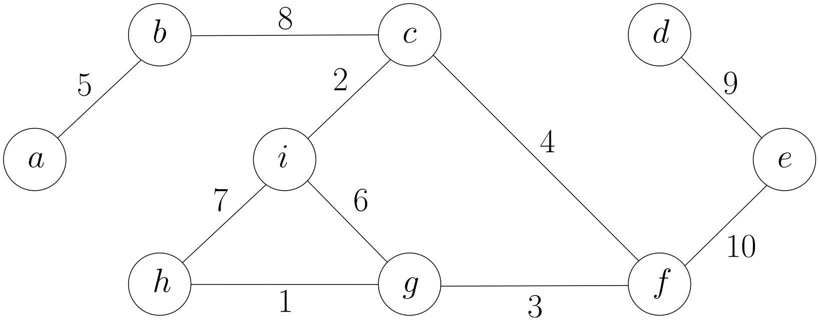

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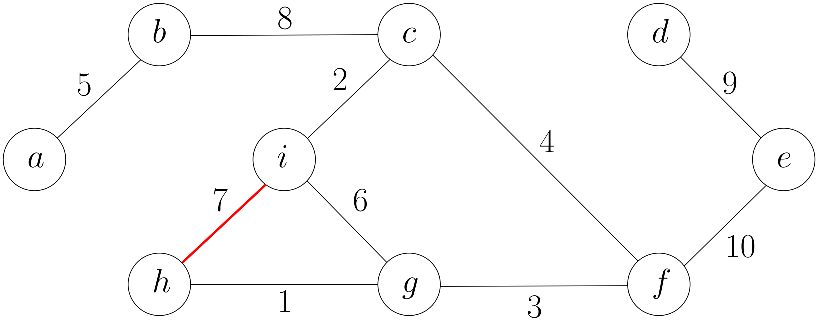

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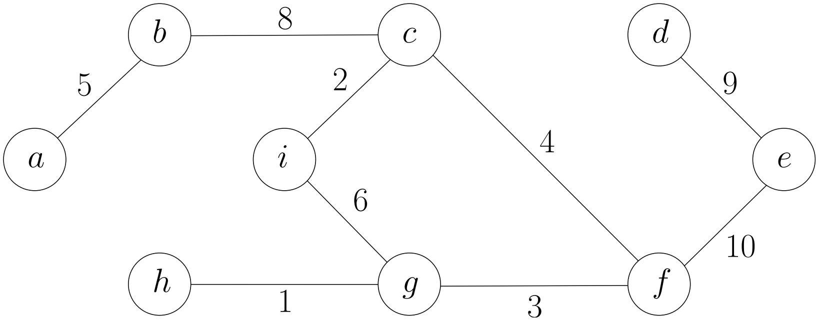

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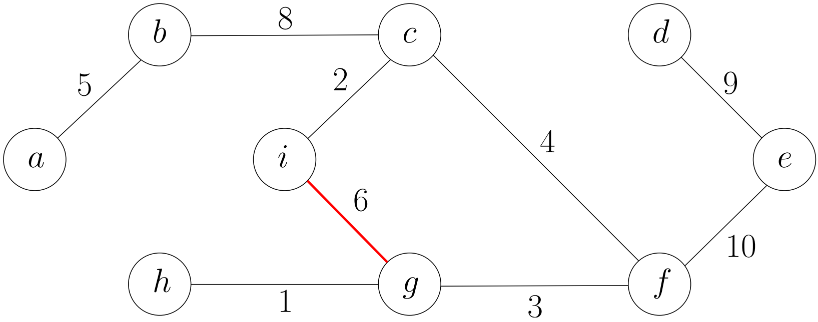

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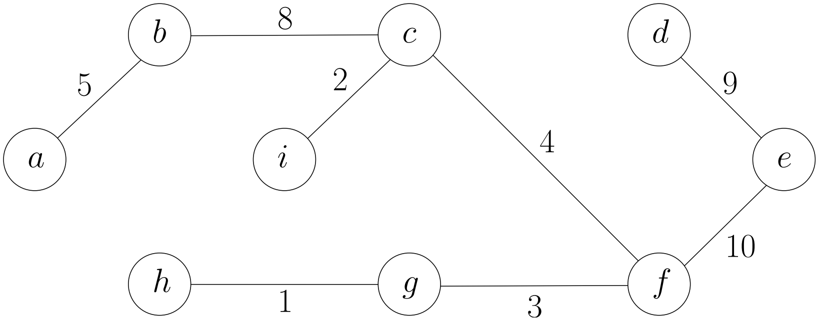


Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm

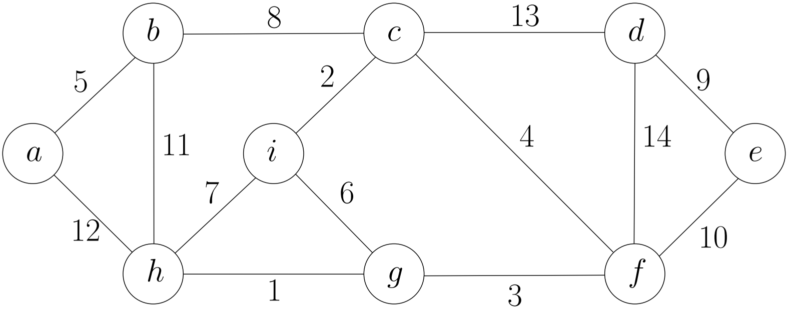
2. Single Source Shortest Paths
   - Dijkstra's Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall

Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



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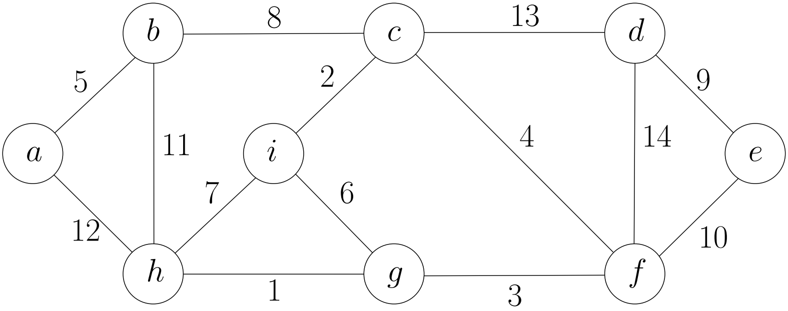

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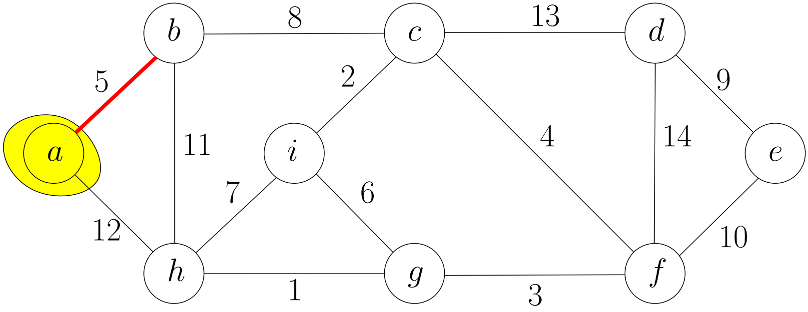

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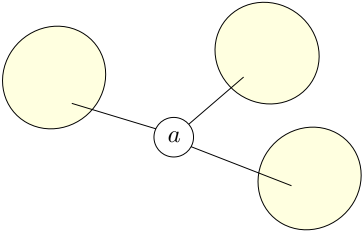

Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

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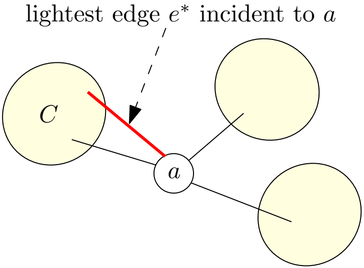

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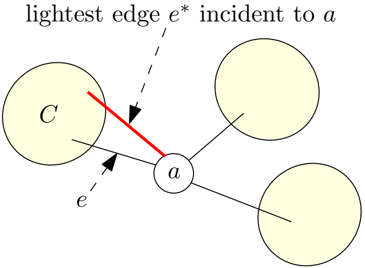

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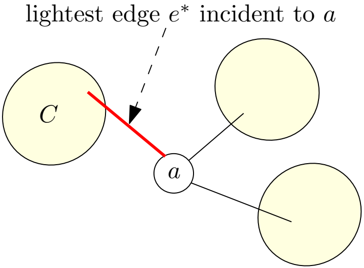

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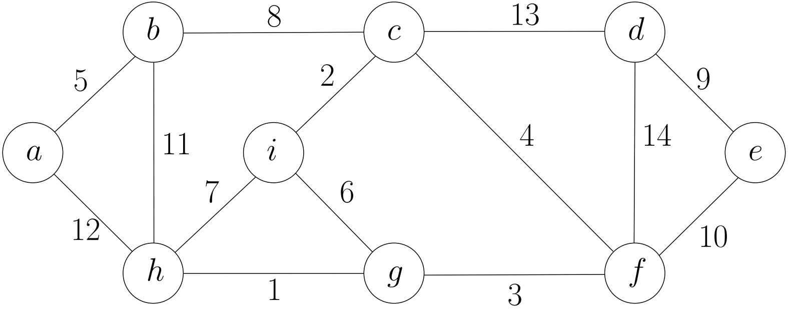
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ $\square$

Prim's Algorithm: Example



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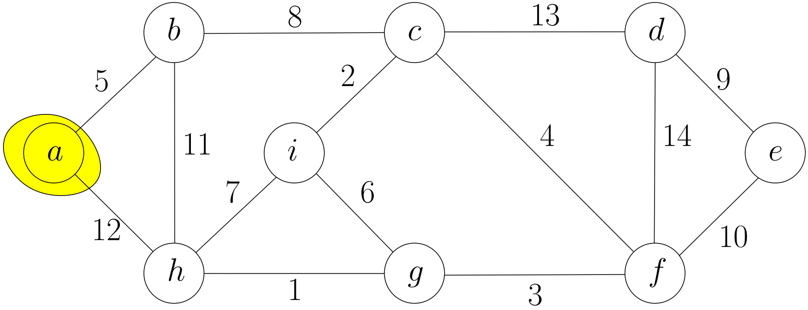

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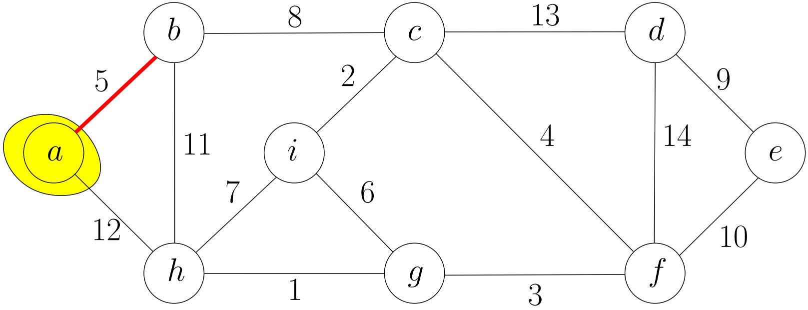

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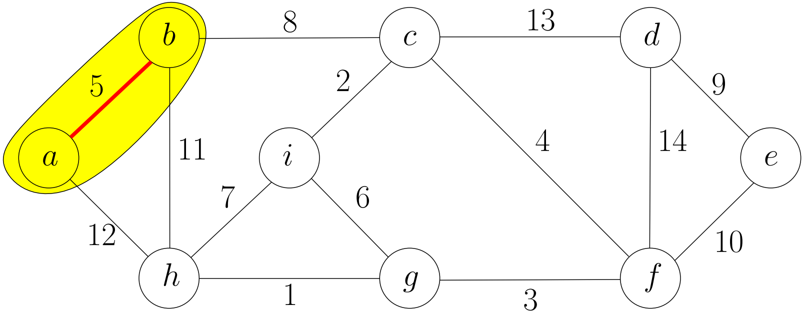

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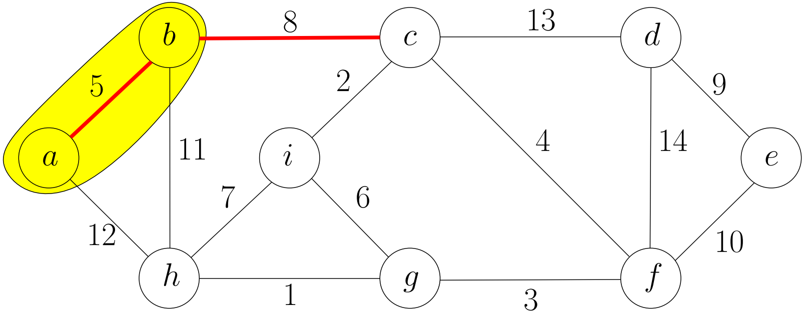

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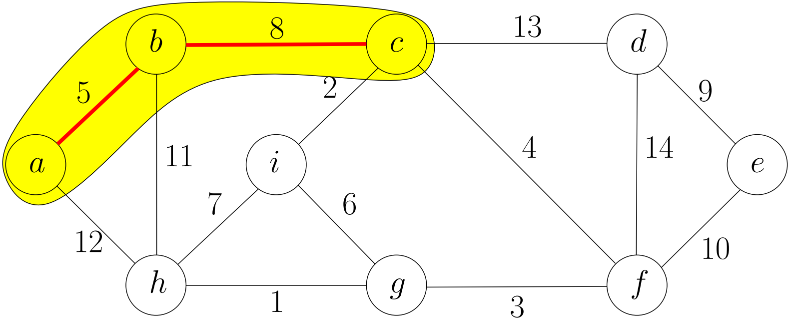

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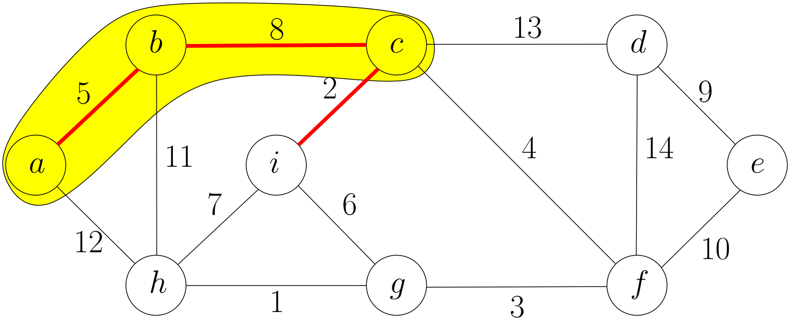

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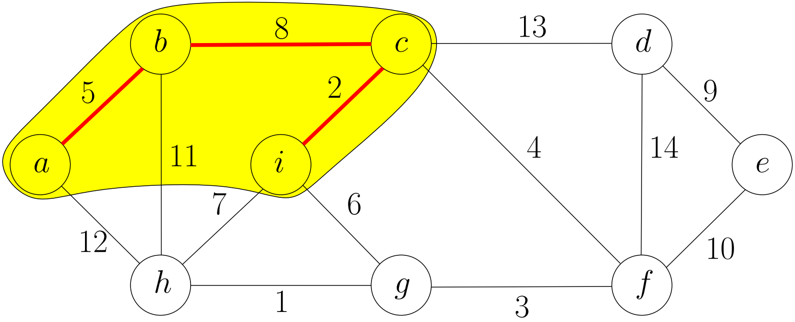

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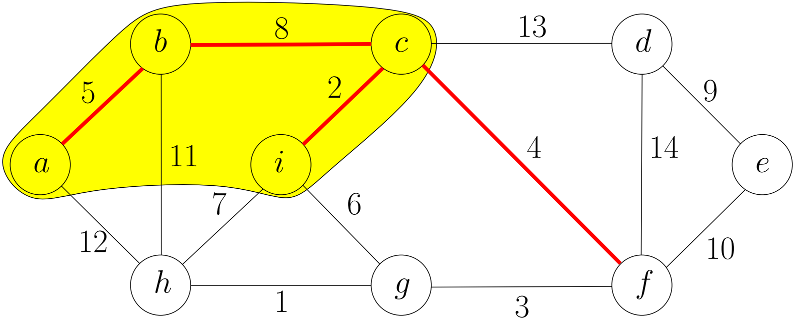

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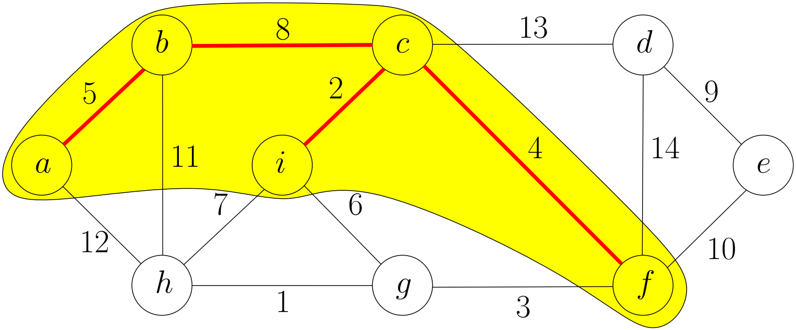

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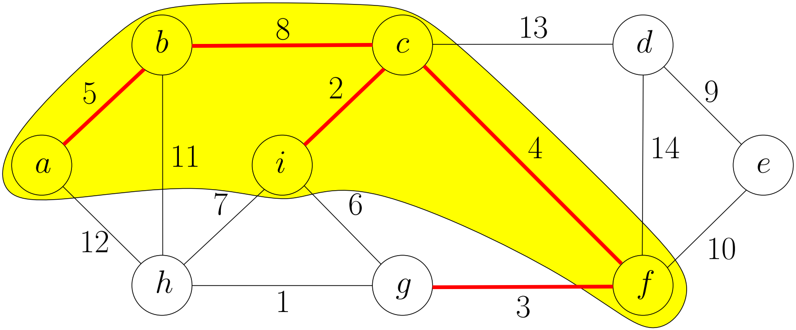

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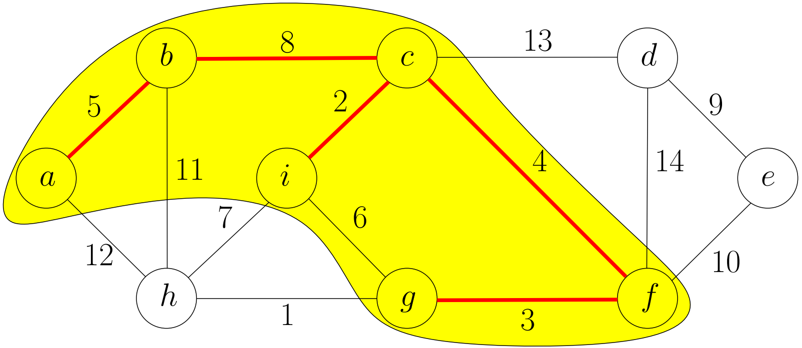

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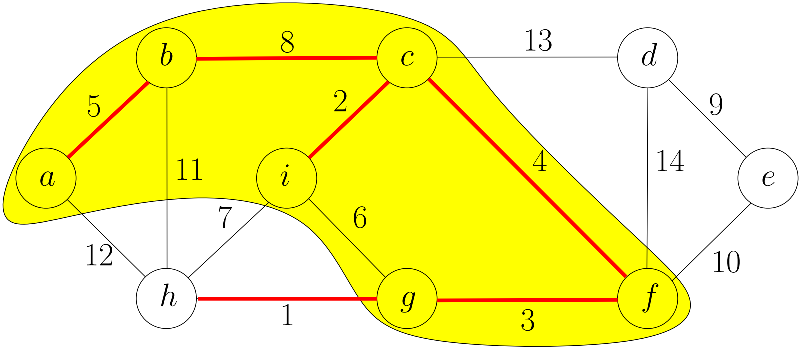

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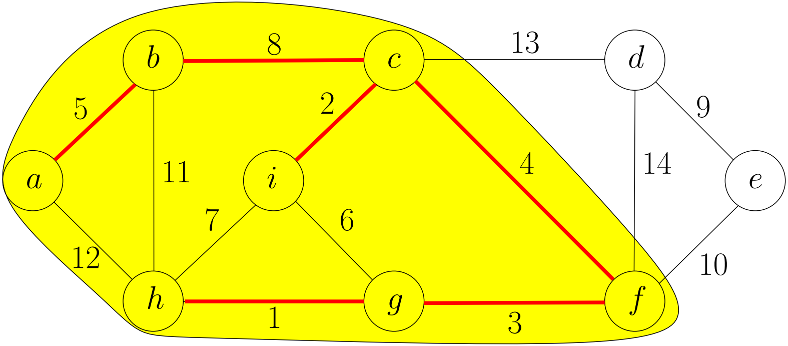

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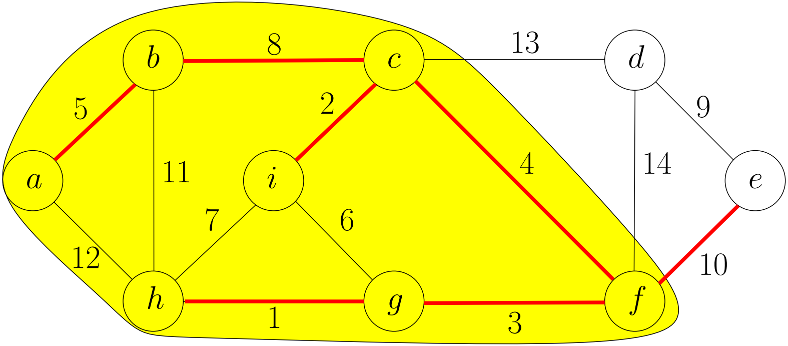

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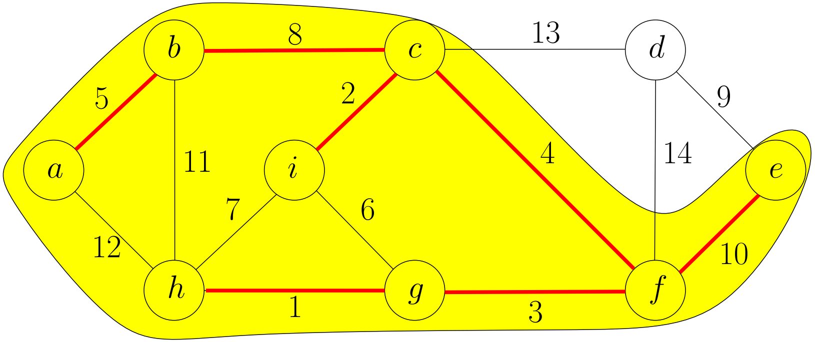

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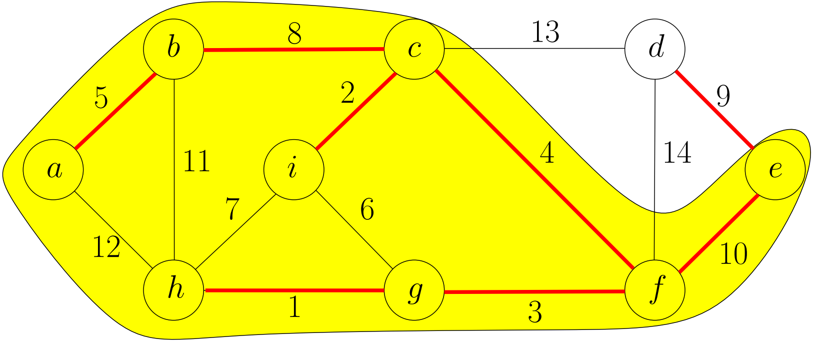

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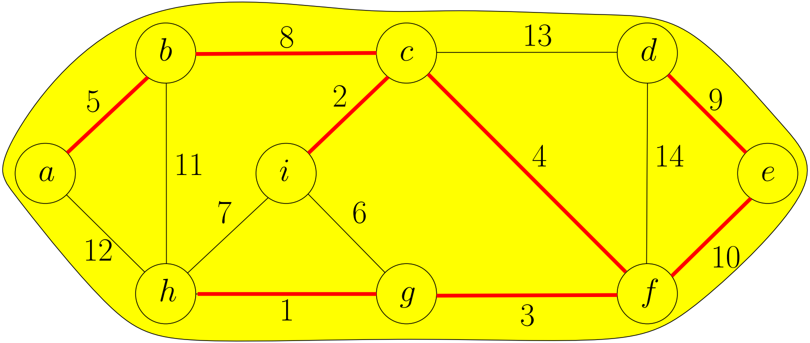


Greedy Algorithm

MST-Greedy1(G, w)

```
1: S ← {s}, where s is arbitrary vertex in V
2: F ← ∅
3: while S ≠ V do
4:     (u, v) ← lightest edge between S and V \ S,
                              where u ∈ S and v ∈ V \ S
5:     S ← S ∪ {v}
6:     F ← F ∪ {(u, v)}
7: return (V, F)
```

Greedy Algorithm

MST-Greedy1(G, w)

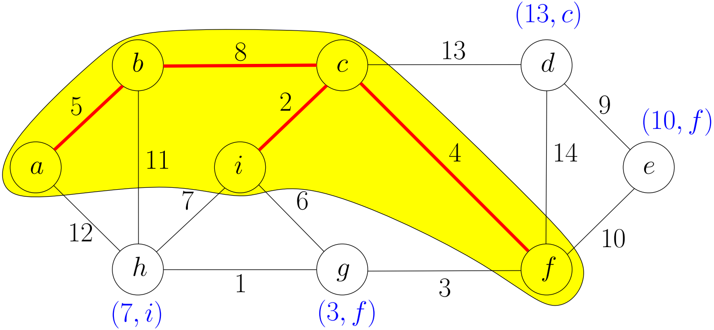
$S \leftarrow \{s\}$, where s is arbitrary vertex in V
$F \leftarrow \emptyset$
while $S \neq V$ **do**
  $(u, v) \leftarrow$ lightest edge between S and $V \setminus S$,
    where $u \in S$ and $v \in V \setminus S$
  $S \leftarrow S \cup \{v\}$
  $F \leftarrow F \cup \{(u, v)\}$
return (V, F)

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi[v], v)$ is the lightest edge between v and S



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In every iteration

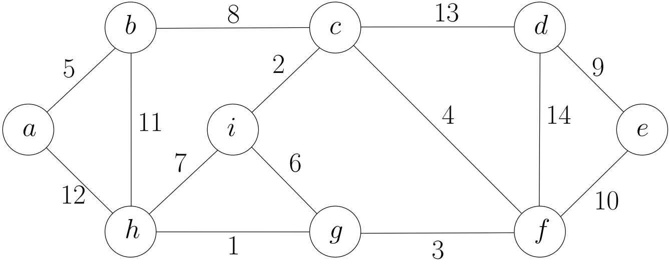
- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$
            $\pi[v] \leftarrow u$
return $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$

Example



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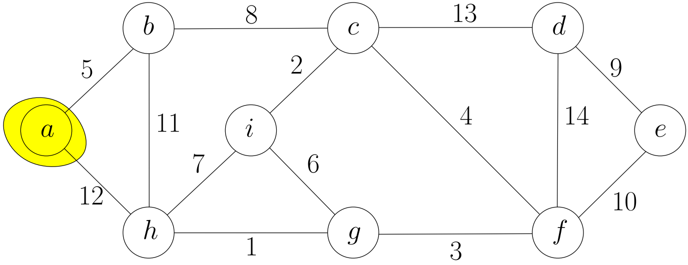

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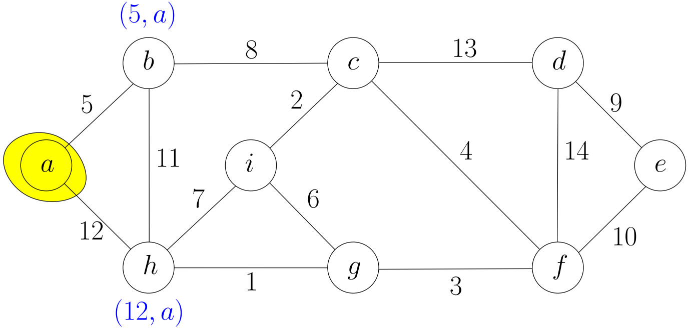

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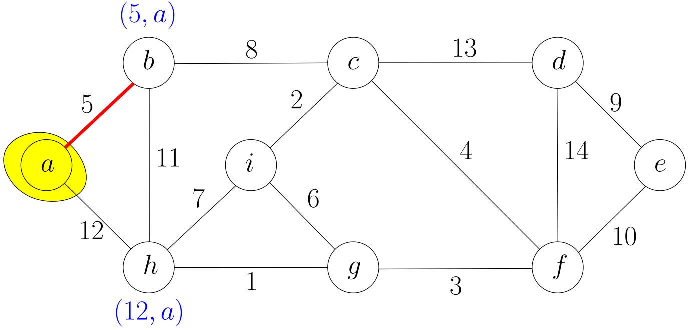

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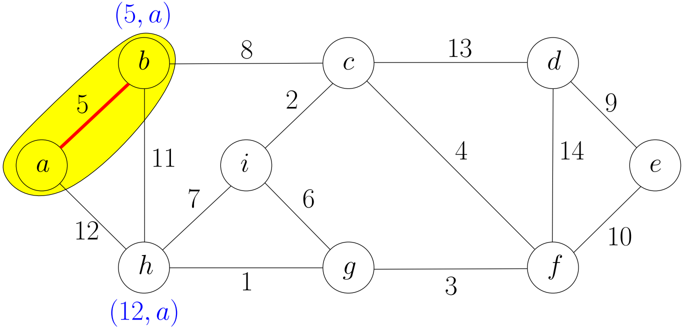

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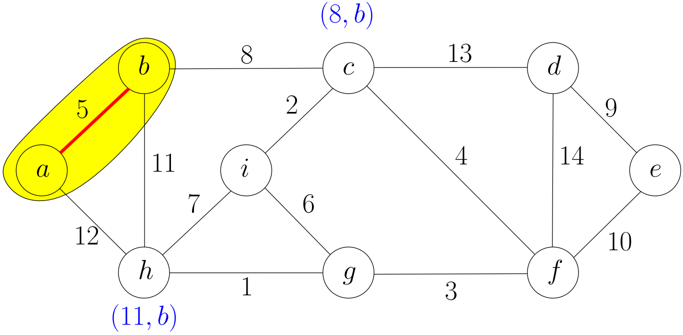

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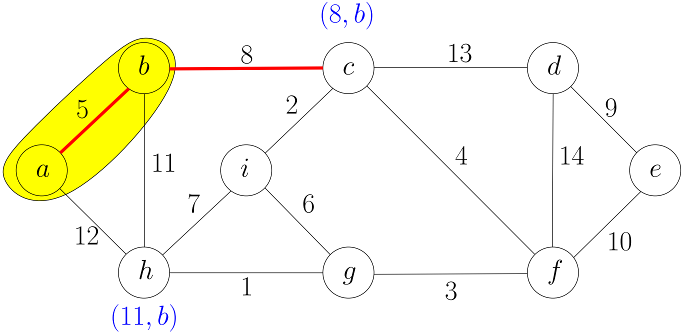

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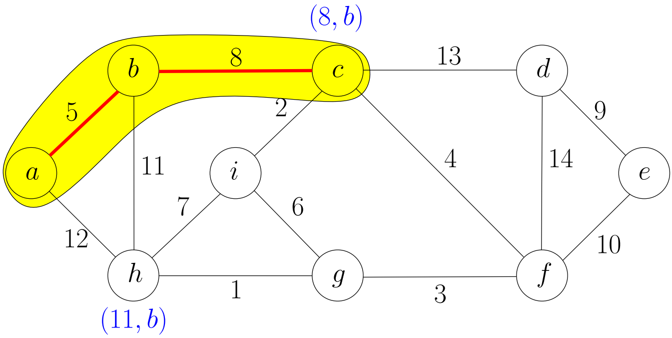

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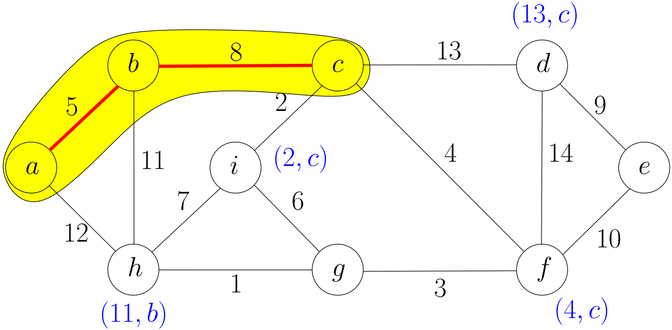

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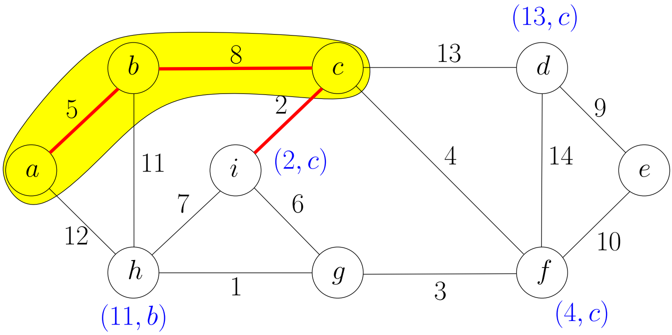

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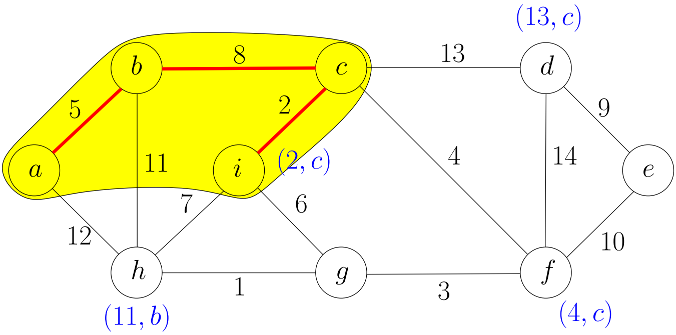

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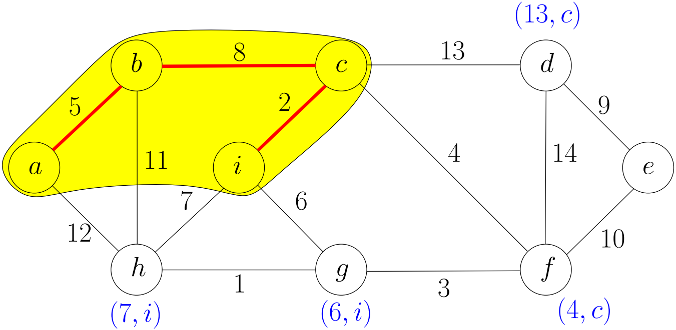

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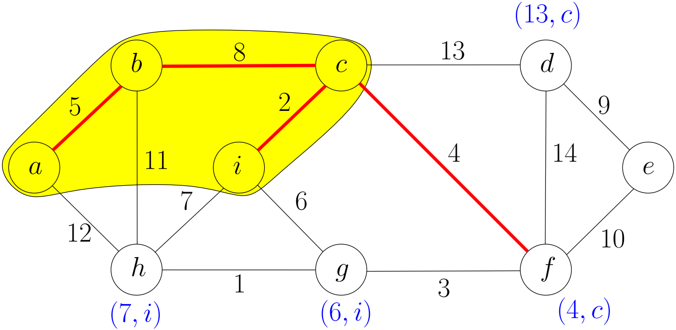

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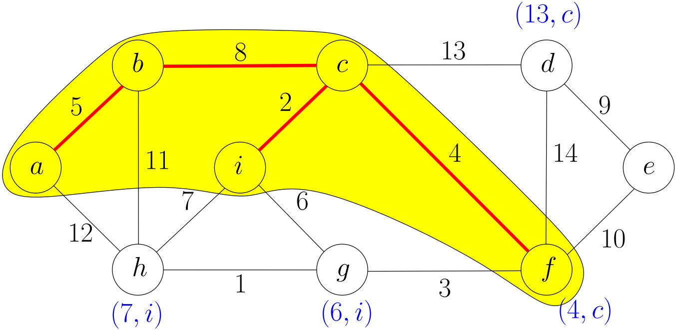

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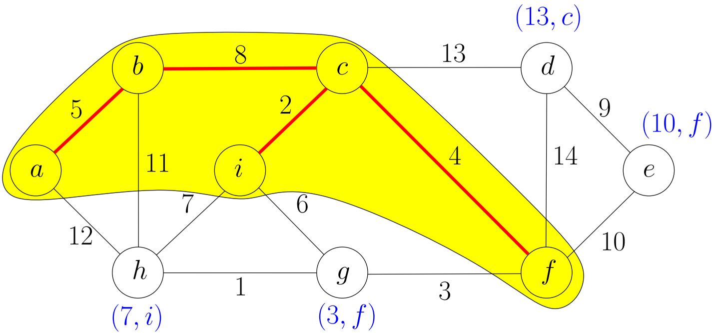

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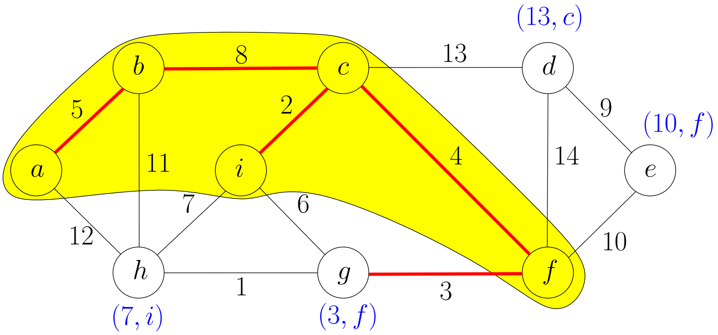

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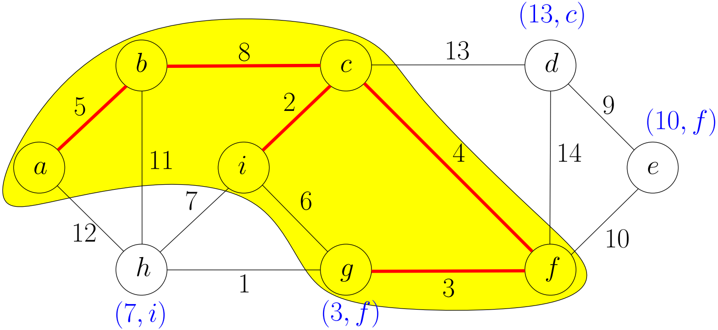

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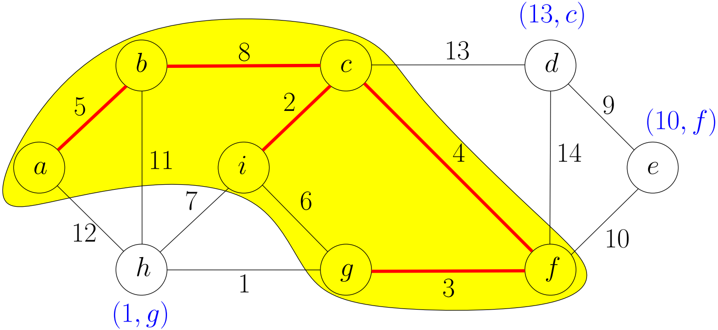

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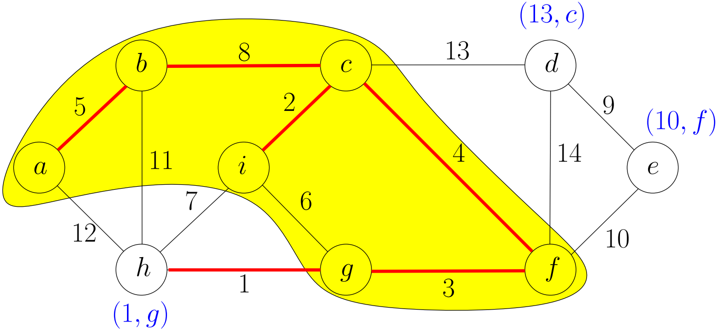

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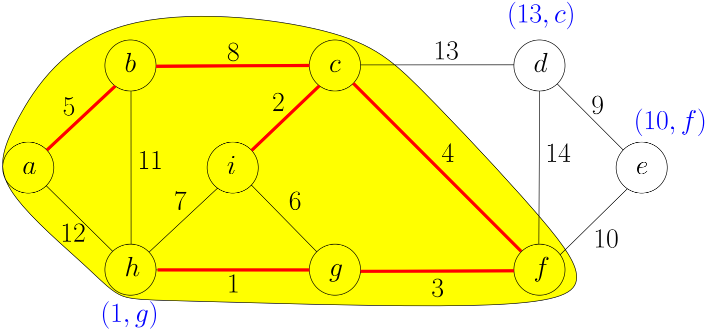

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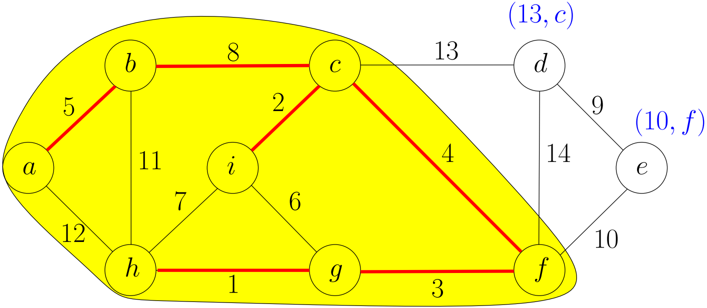

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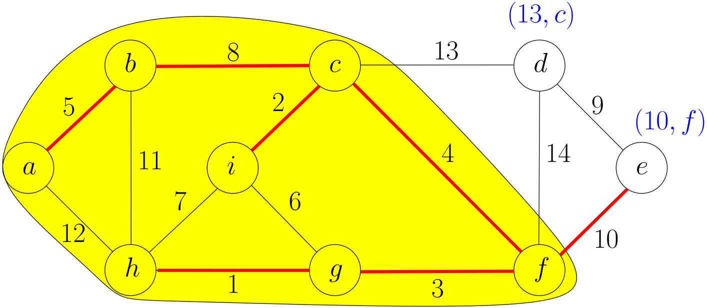

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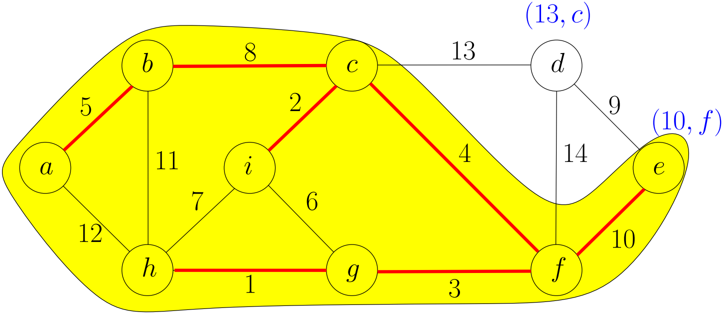

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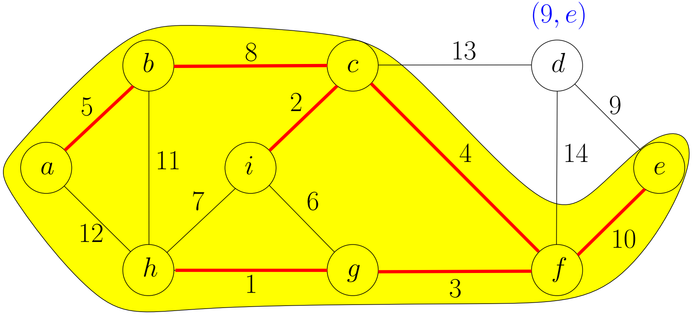

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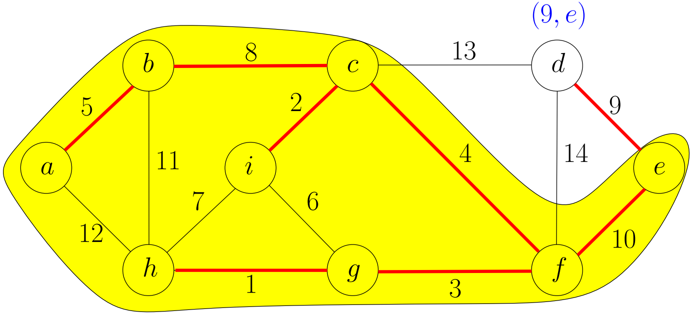

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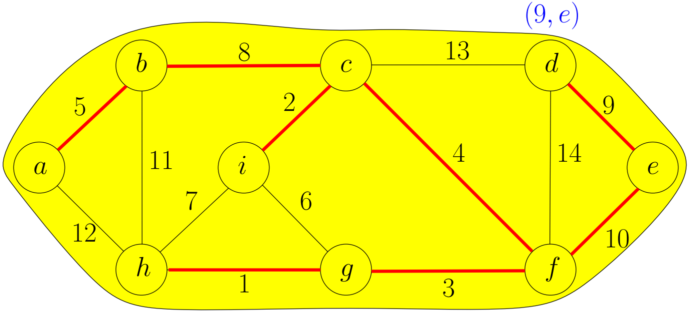

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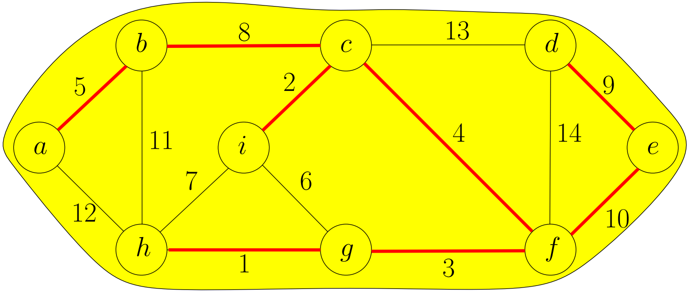

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In every iteration

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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value — extract_min
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values. — decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

Prim's Algorithm

MST-Prim(G, w)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}

while S ≠ V do
    u ← vertex in V \ S with the minimum d[u]
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if w(u, v) < d[v] then
            d[v] ← w(u, v)
            π[v] ← u
return {(u, π[u])|u ∈ V \ {s}}
```

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
Q ← empty queue, for each v ∈ V: Q.insert(v, d[v])
while S ≠ V do
    u ← Q.extract_min()
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if w(u, v) < d[v] then
            d[v] ← w(u, v), Q.decrease_key(v, d[v])
            π[v] ← u
return {(u, π[u])|u ∈ V \ {s}}
```

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+ O(m)\times$ (time for decrease_key)

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concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
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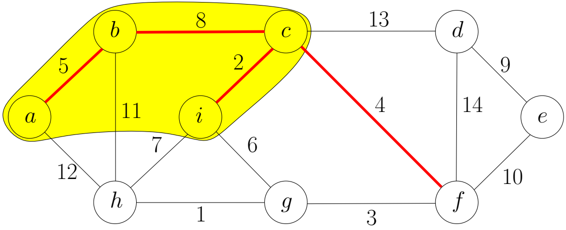

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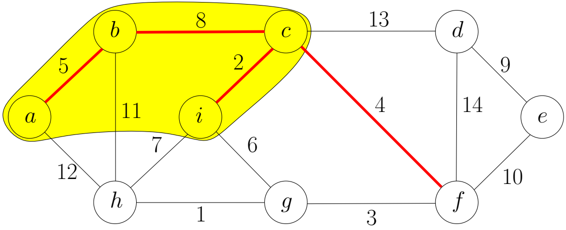

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

"Evidence" for $e \in$ MST or $e \notin$ MST

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- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
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Thus, the minimum spanning tree is unique with assumption.

Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
4. All-Pair Shortest Paths and Floyd-Warshall

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$
$w : E \to \mathbb{R}_{\geq 0}$

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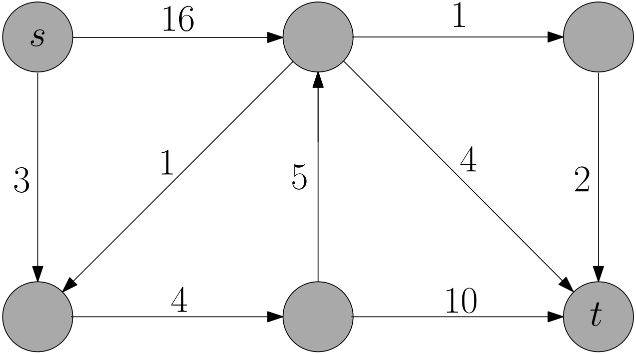

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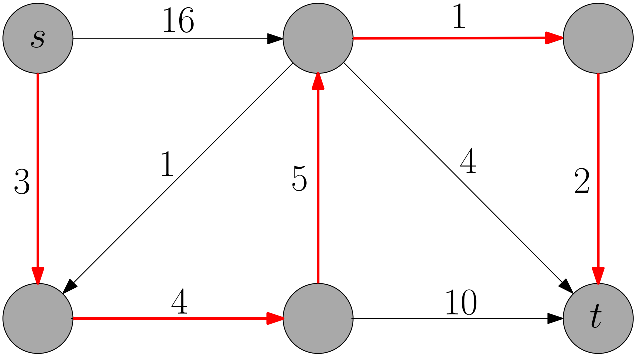

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Input: (directed or undirected) graph $G = (V, E)$, $s \in V$

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Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi[v], v \in V \setminus s$: the parent of v in shortest path tree

$d[v], v \in V \setminus s$: the length of shortest path from s to v

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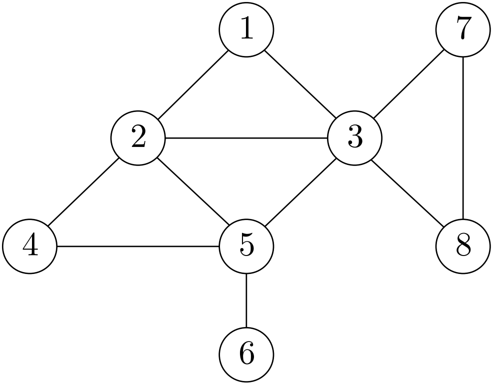

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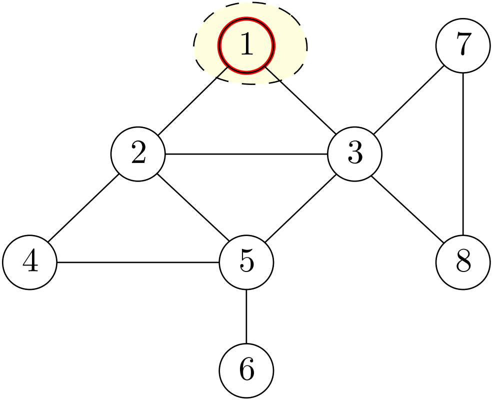

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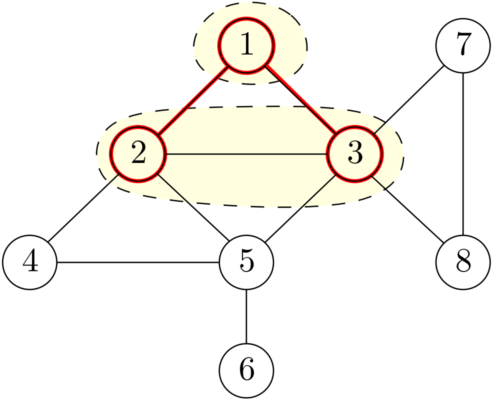

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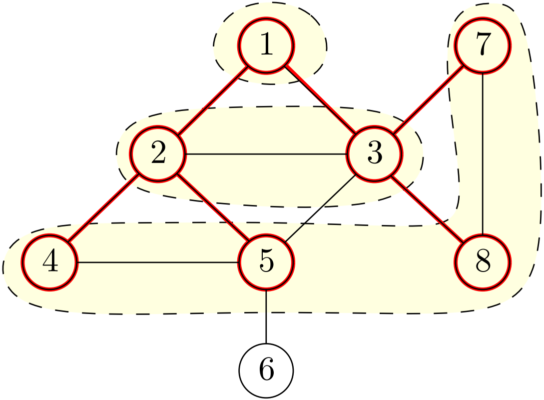

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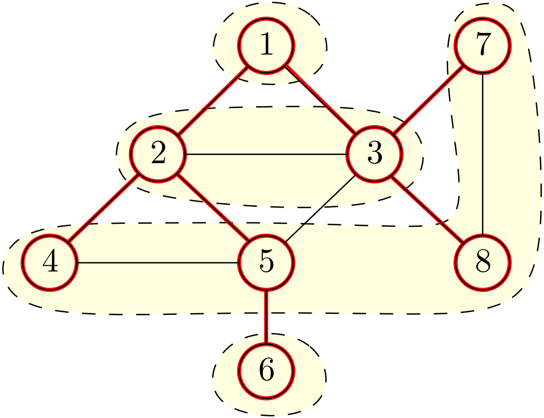

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Shortest Path Algorithm by Running BFS

1: replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2: run BFS
3: $\pi[v] \leftarrow$ vertex from which v is visited
4: $d[v] \leftarrow$ index of the level containing v

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4: $d[v] \leftarrow$ index of the level containing v

- Problem: $w(u,v)$ may be too large!

Assumption Weights $w(u,v)$ are integers (w.l.o.g).

- An edge of weight $w(u,v)$ is equivalent to a pah of $w(u,v)$ unit-weight edges

Shortest Path Algorithm by Running BFS

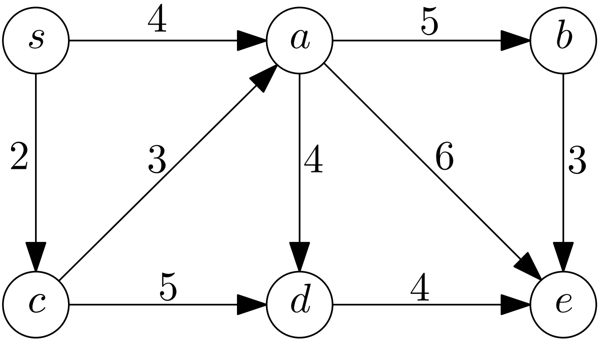
1: replace (u,v) of length $w(u,v)$ with a path of $w(u,v)$ unit-weight edges, for every $(u,v) \in E$
2: run BFS virtually
3: $\pi[v] \leftarrow$ vertex from which v is visited
4: $d[v] \leftarrow$ index of the level containing v

- Problem: $w(u,v)$ may be too large!

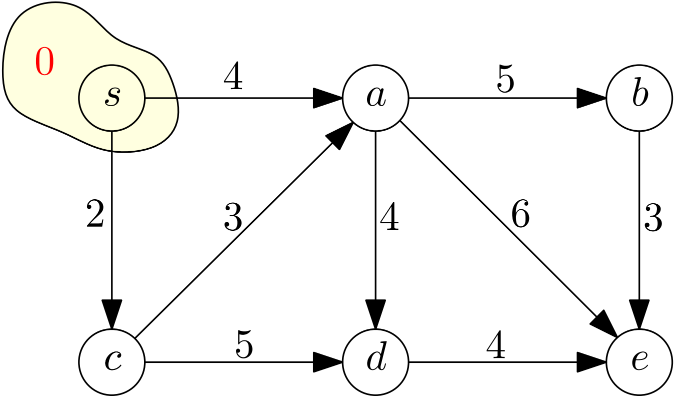
Shortest Path Algorithm by Running BFS Virtually

$S \leftarrow \{s\}, d(s) \leftarrow 0$
while $|S| \leq n$ **do**
    find a $v \notin S$ that minimizes $\min_{u \in S:(u,v) \in E} \{d[u] + w(u, v)\}$
    $S \leftarrow S \cup \{v\}$
    $d[v] \leftarrow \min_{u \in S:(u,v) \in E} \{d[u] + w(u, v)\}$

Virtual BFS: Example

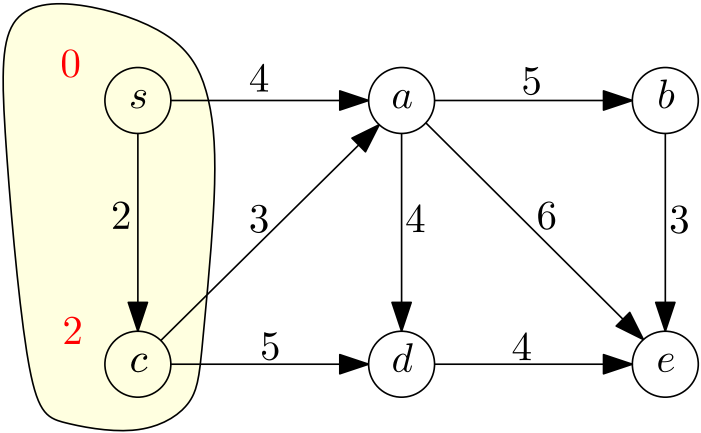


Virtual BFS: Example



Time 0

Virtual BFS: Example



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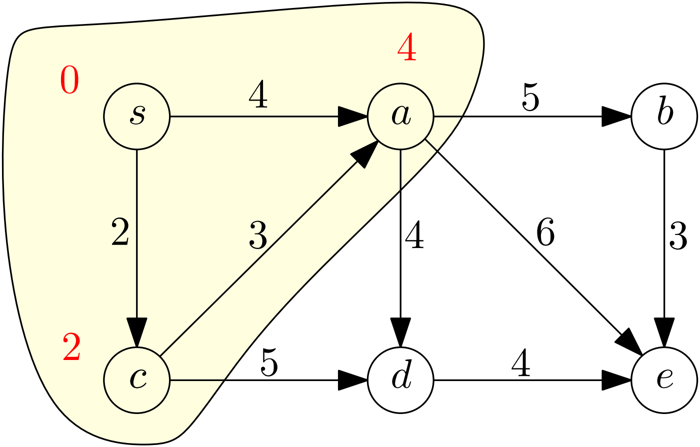
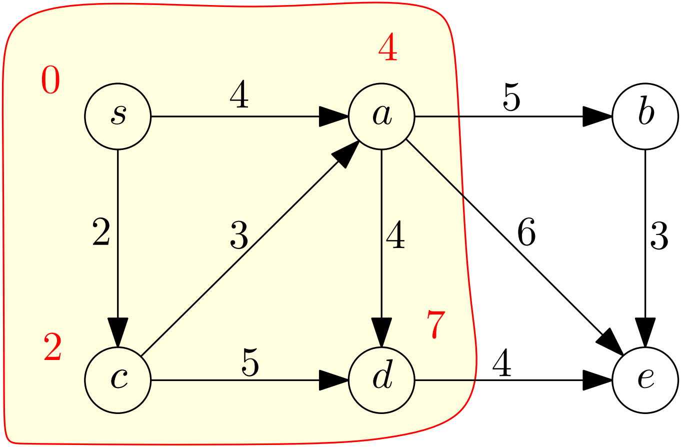
Virtual BFS: Example



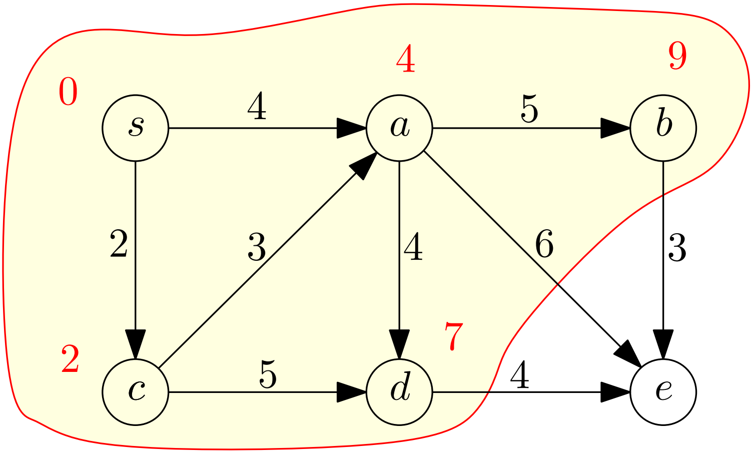
Time 4

Virtual BFS: Example



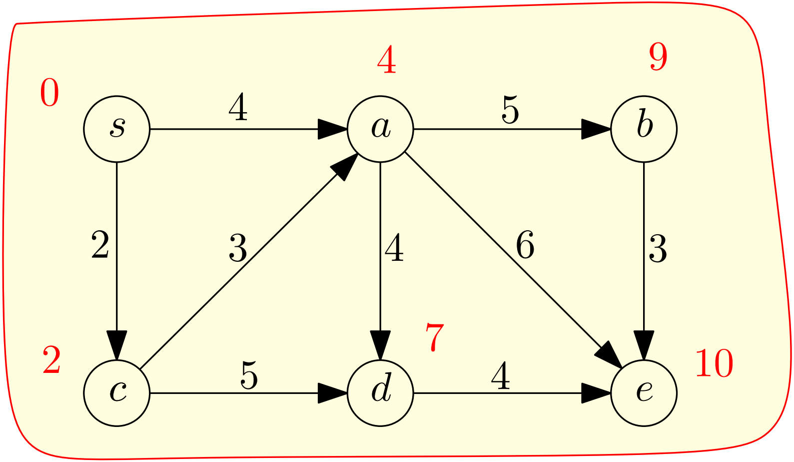
Time 7

Virtual BFS: Example



Time 9

Virtual BFS: Example



Time 10

Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
4. All-Pair Shortest Paths and Floyd-Warshall

Dijkstra's Algorithm

Dijkstra(G, w, s)

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    add u to S
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
return (d, π)

Dijkstra's Algorithm

Dijkstra(G, w, s)

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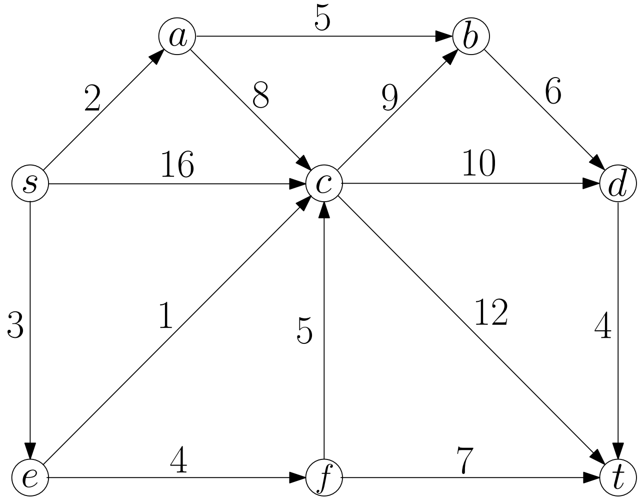
return (d, π)

- Running time $= O(n^2)$



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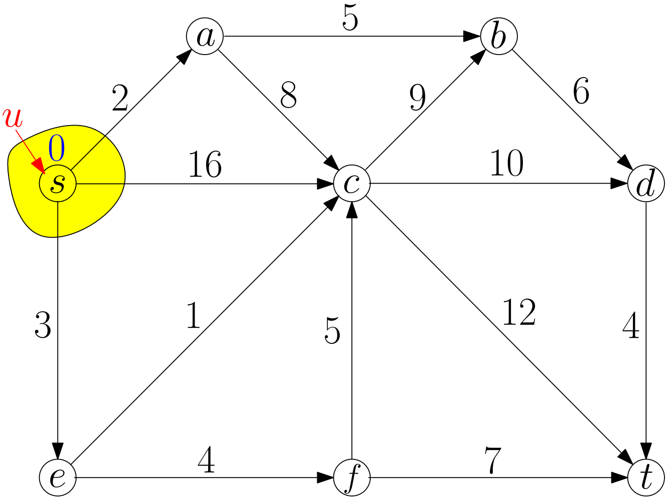

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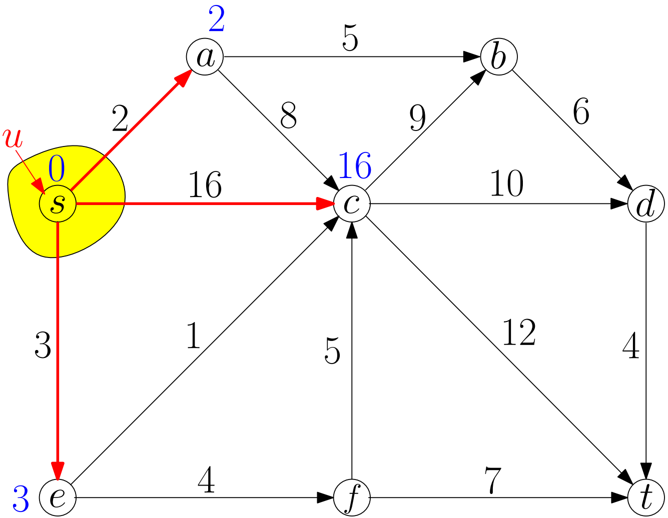
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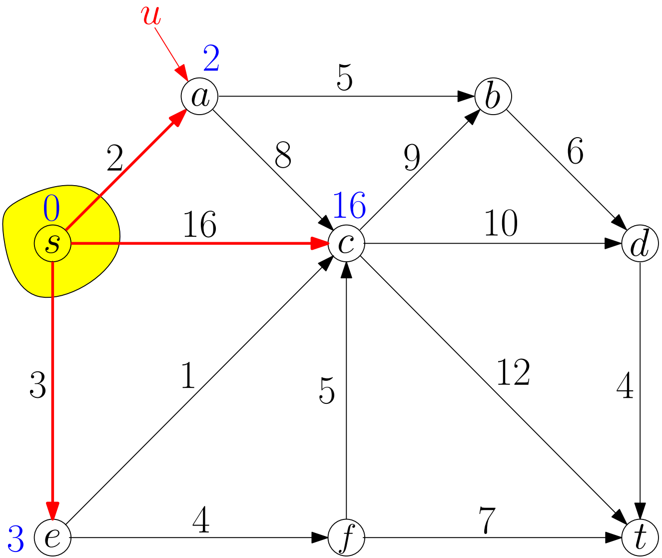
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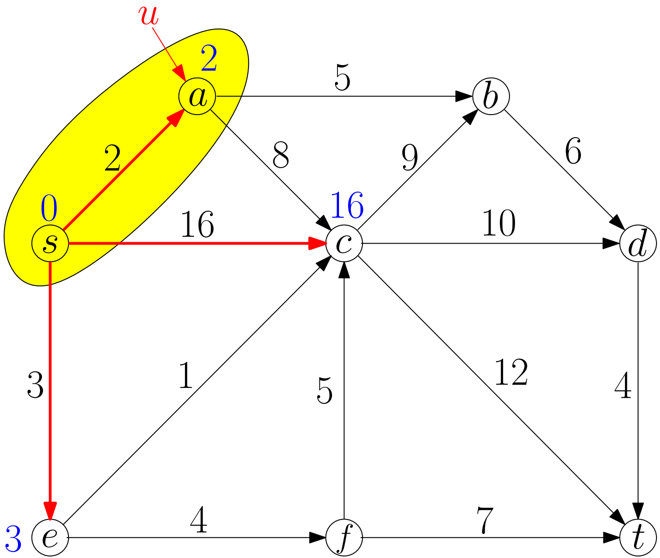
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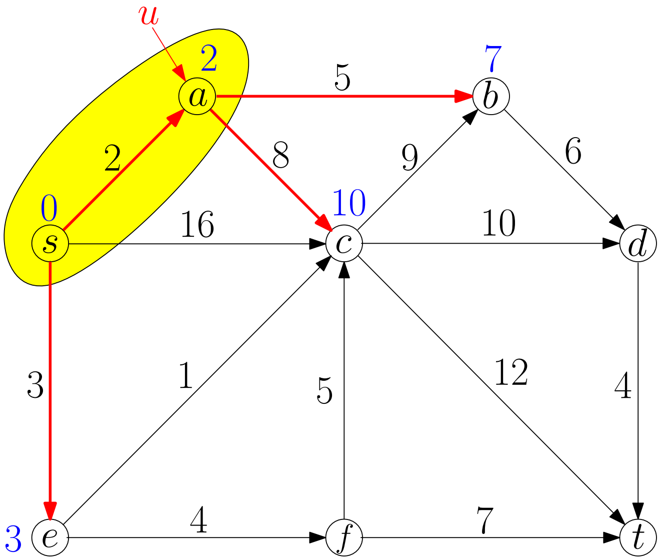
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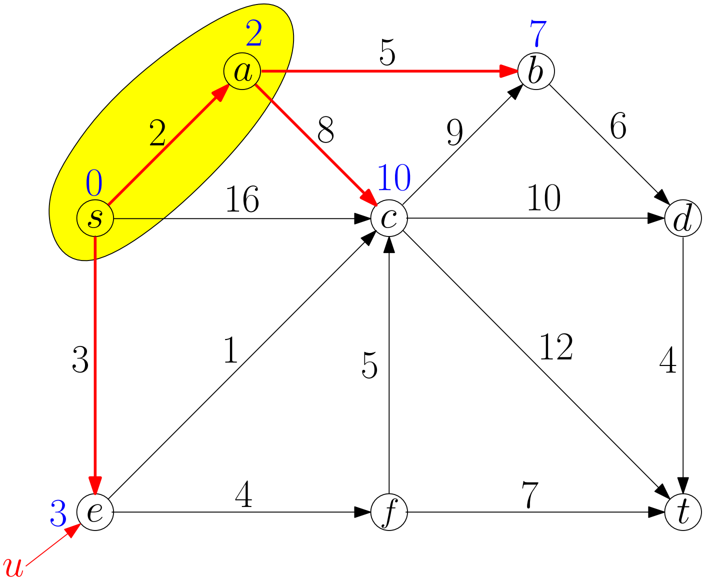

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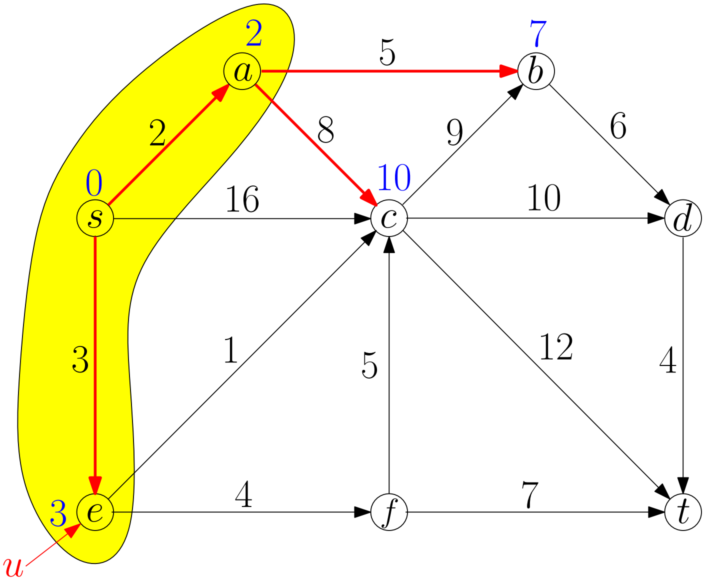

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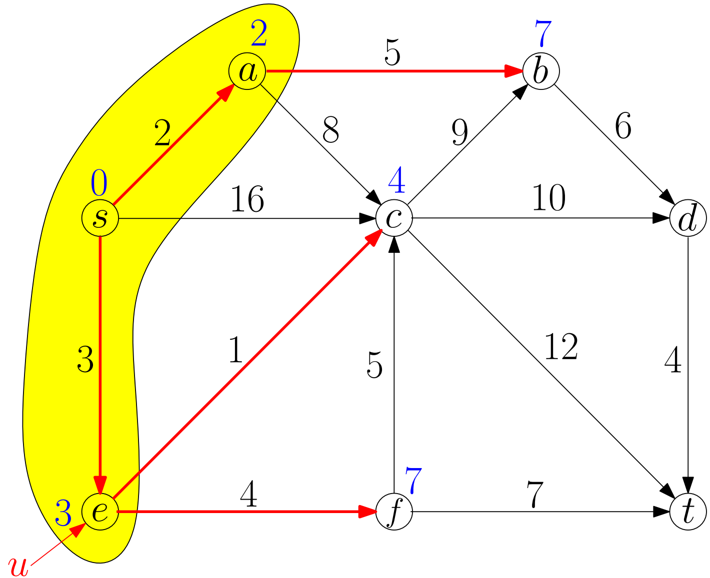
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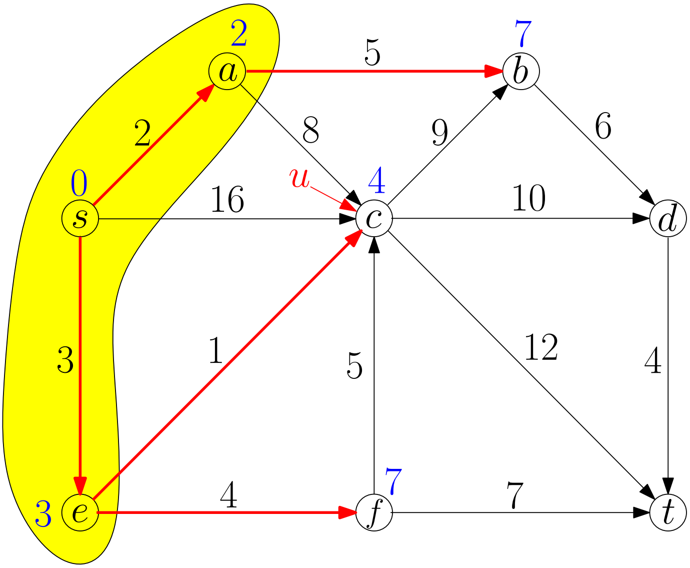

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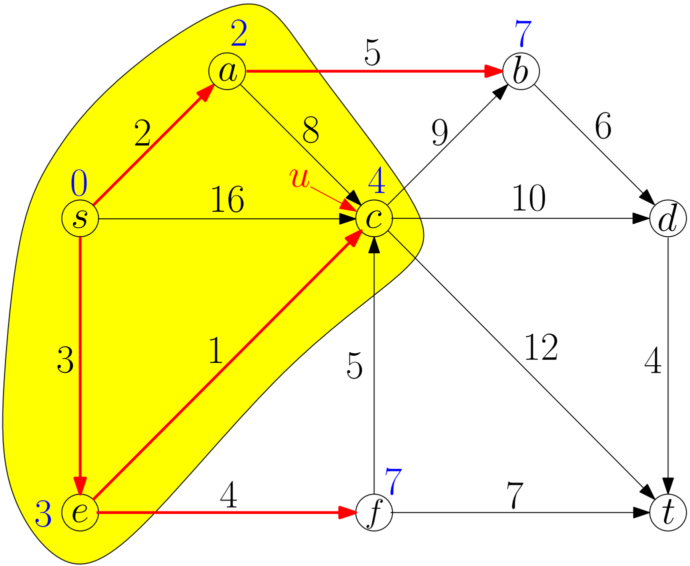

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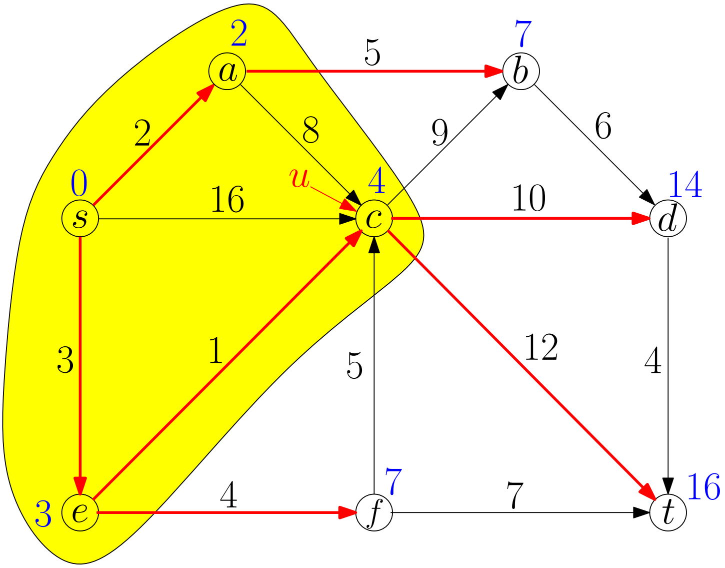

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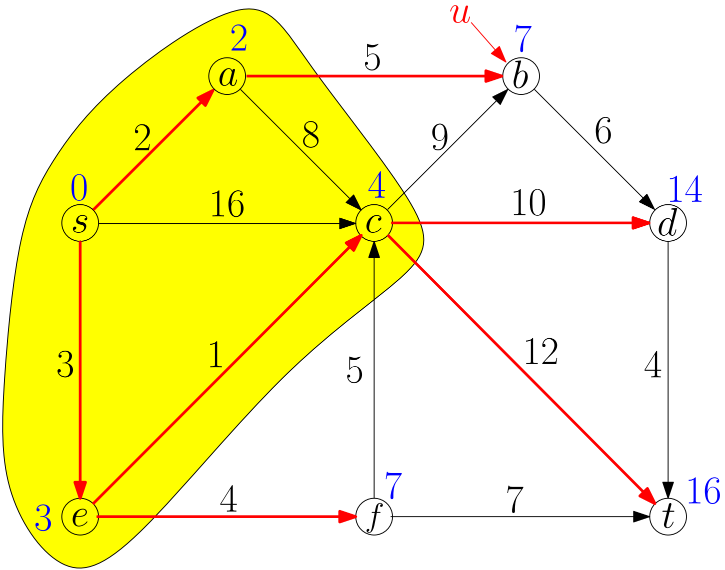

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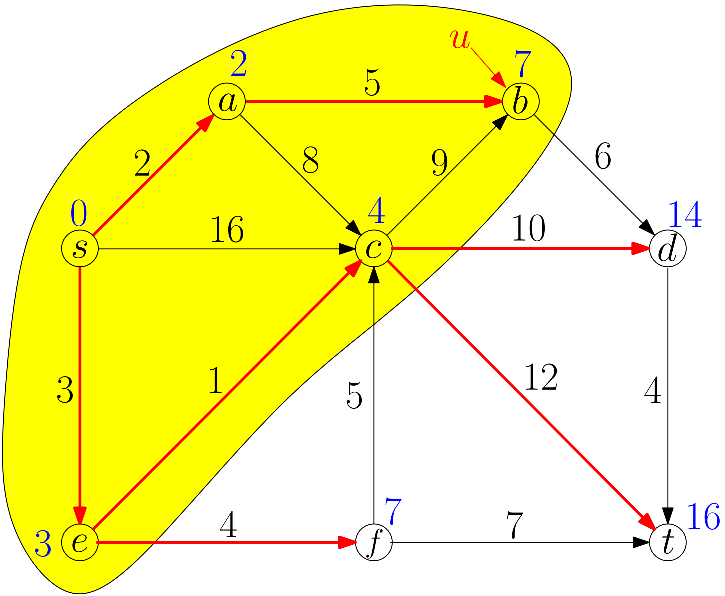

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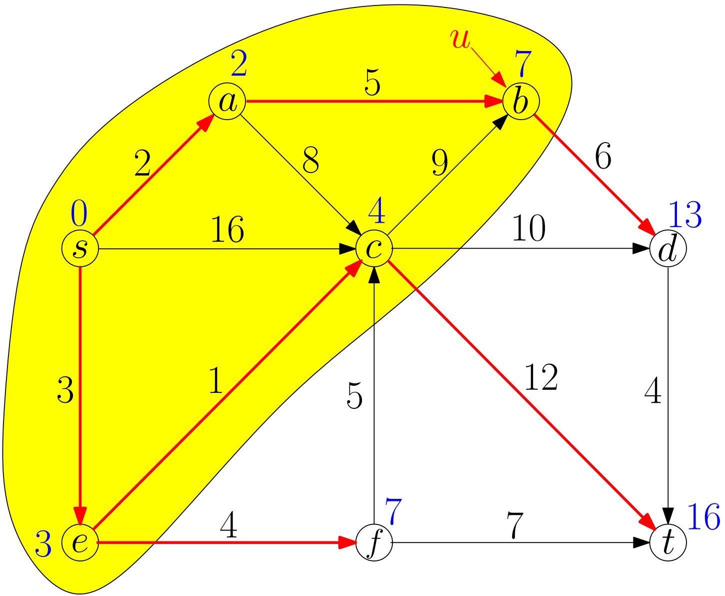

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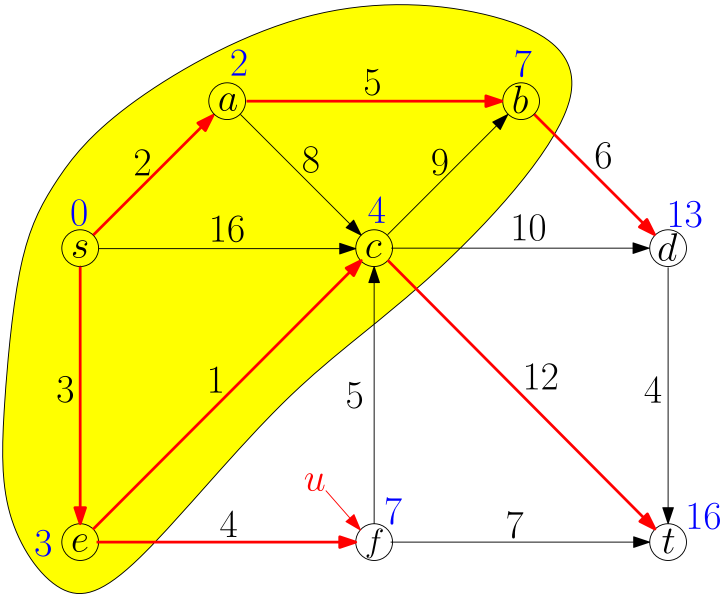

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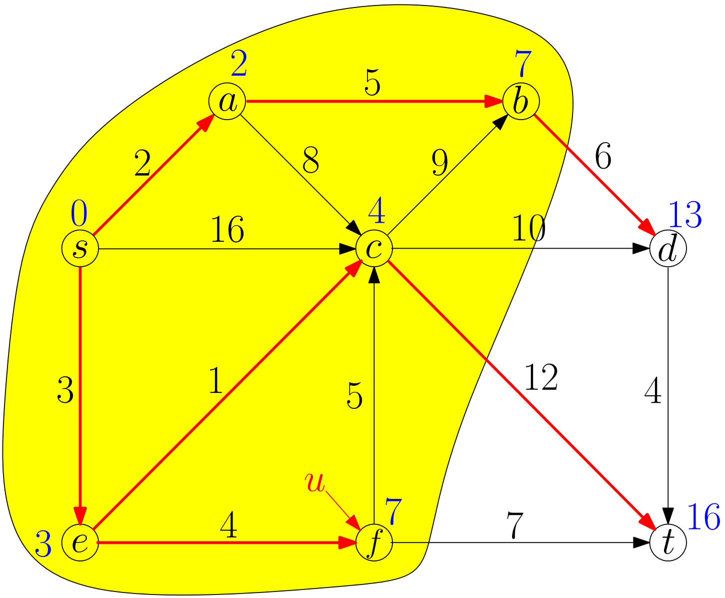

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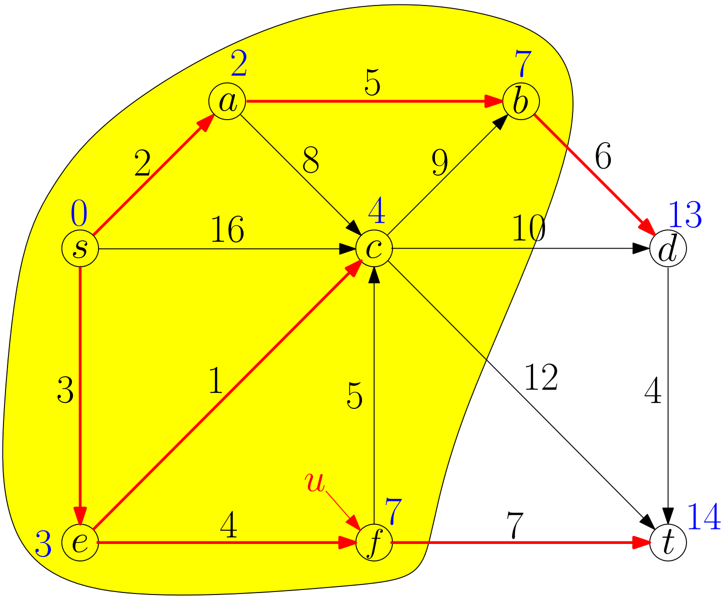

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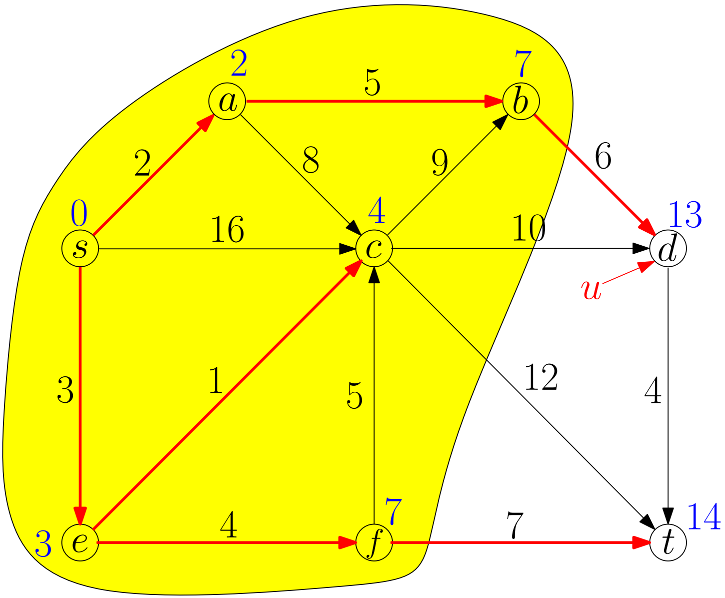

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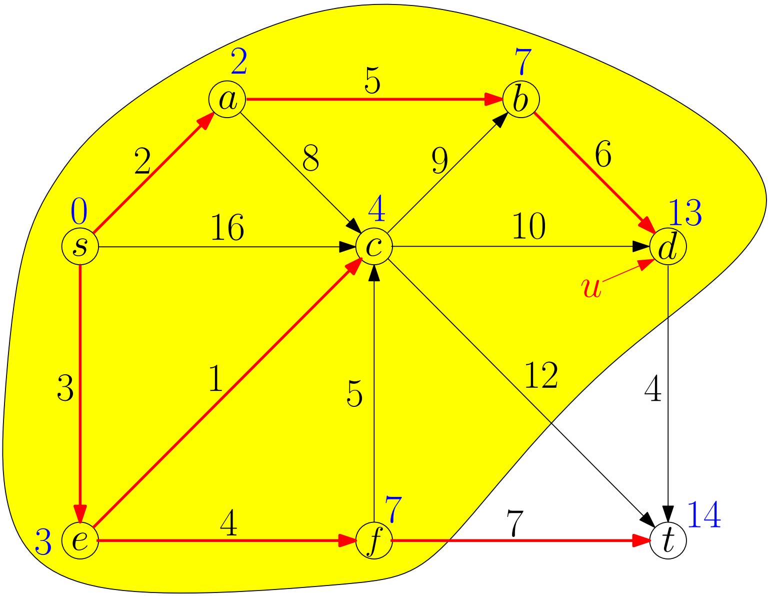

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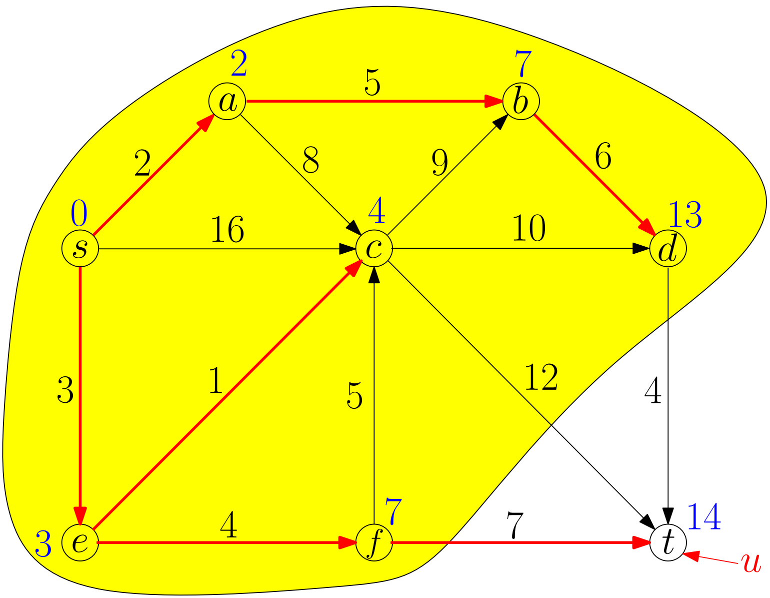

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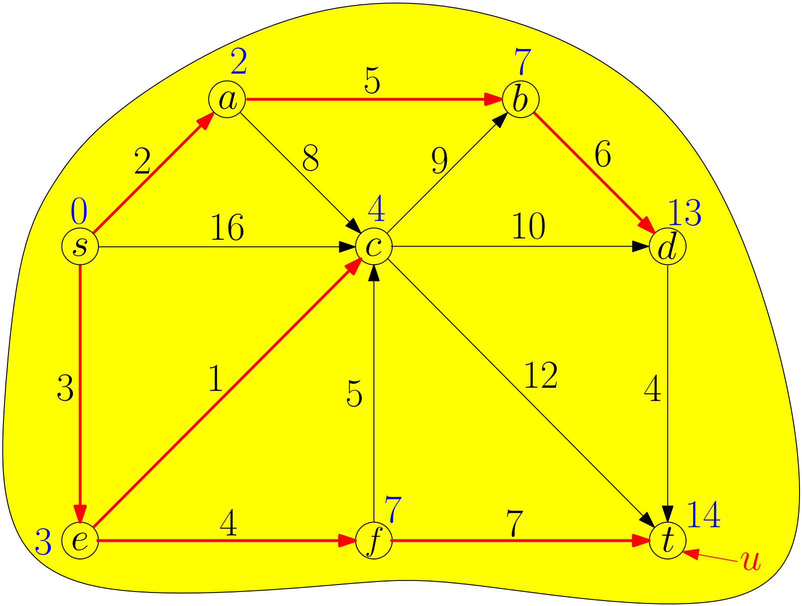
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Improved Running Time using Priority Queue

Dijkstra(G, w, s)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
Q ← empty queue, for each v ∈ V: Q.insert(v, d[v])
while S ≠ V do
    u ← Q.extract_min()
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if d[u] + w(u, v) < d[v] then
            d[v] ← d[u] + w(u, v), Q.decrease_key(v, d[v])
            π[v] ← u
return (π, d)
```

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
  $u \leftarrow Q$.extract_min()
  $S \leftarrow S \cup \{u\}$
  for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
    if $w(u, v) < d[v]$ **then**
      $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$
      $\pi[v] \leftarrow u$
return $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$

Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

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Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

Single Source Shortest Paths, Weights May be Negative

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- In transition graphs, negative weights make sense

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- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)

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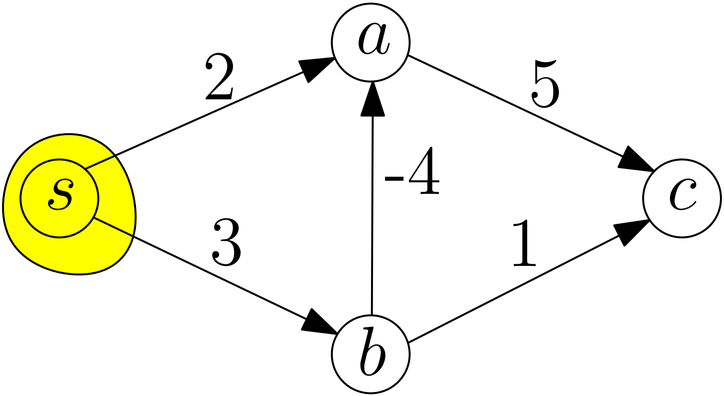
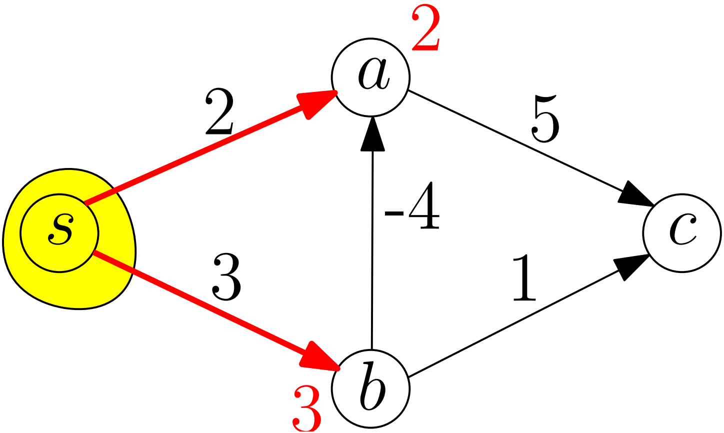
- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

Dijkstra's Algorithm Fails if We Have Negative Weights



Dijkstra's Algorithm Fails if We Have Negative Weights



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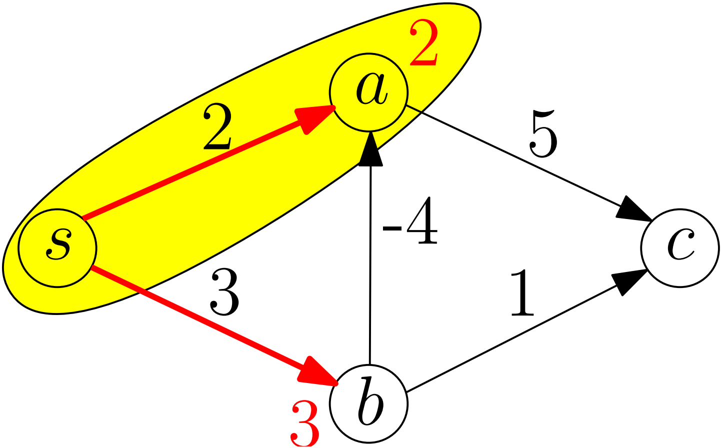
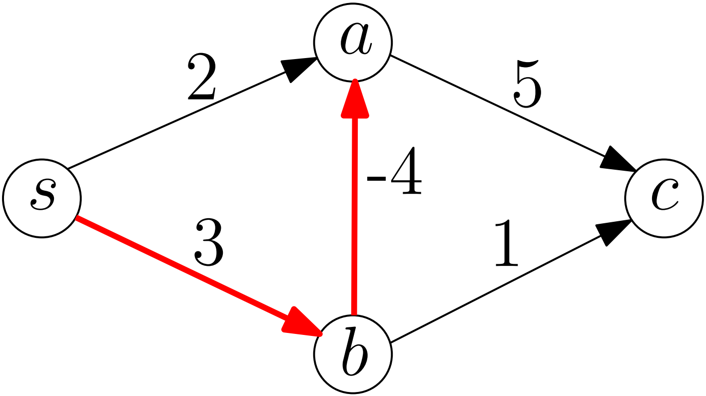
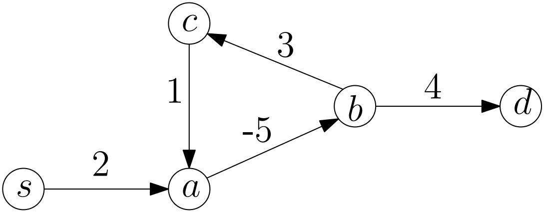


Dijkstra's Algorithm Fails if We Have Negative Weights

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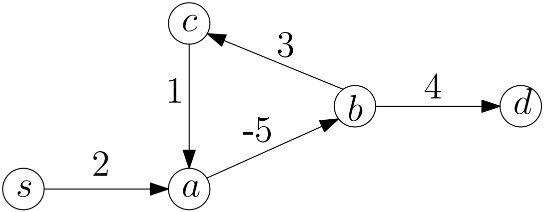
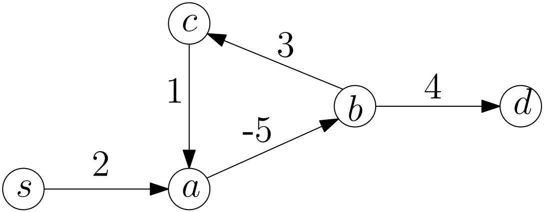


Q: What is the length of the shortest path from s to d?



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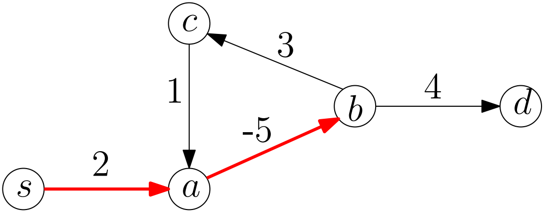
A: $-\infty$



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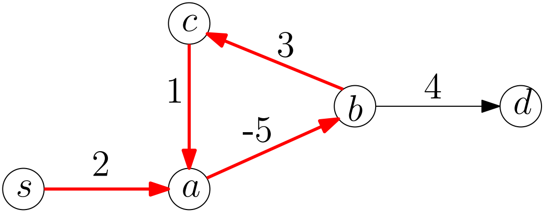

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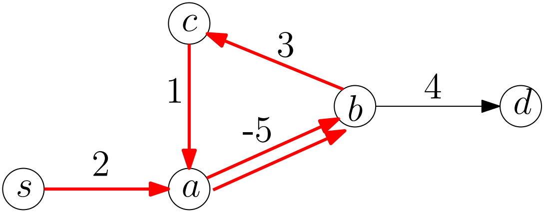

Q: What is the length of the shortest path from s to d?

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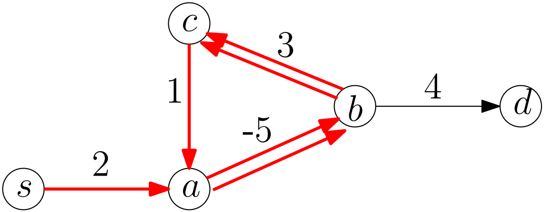

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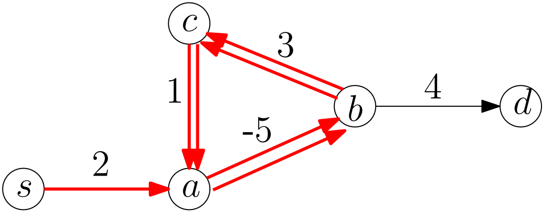

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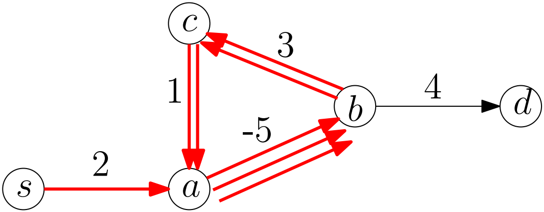

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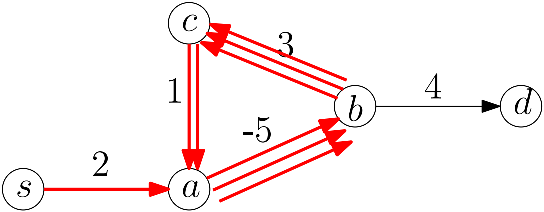

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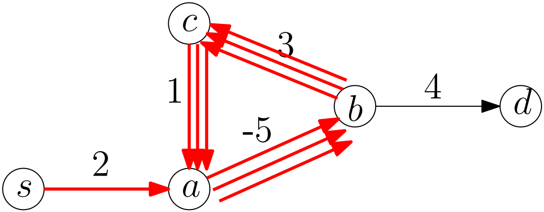

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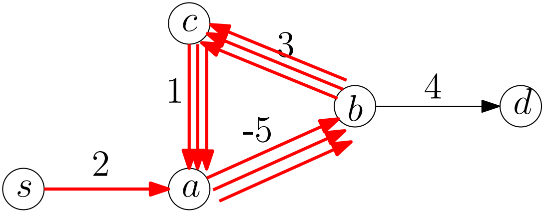

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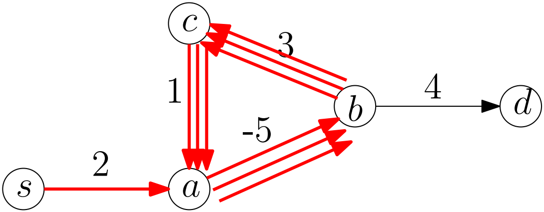
Def. A negative cycle is a cycle in which the total weight of edges is negative.



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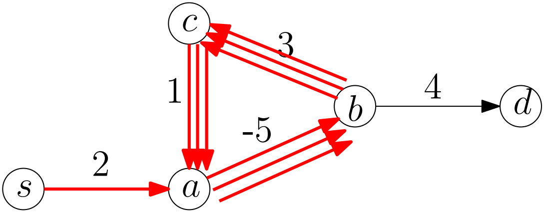
Dealing with Negative Cycles



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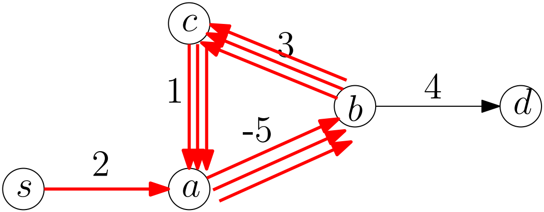
Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or



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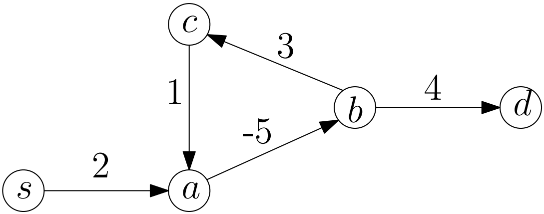
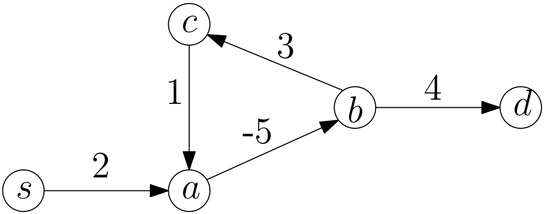
Dealing with Negative Cycles

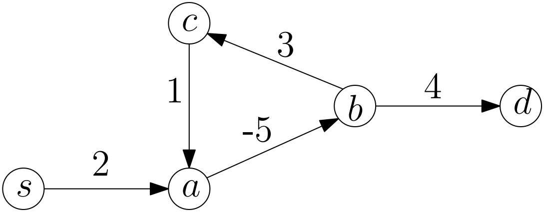
- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"



c
3
1
4
b
d
-5
2
s
a

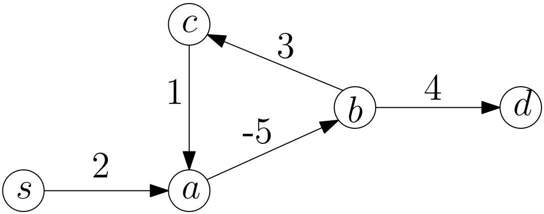


Q: What is the length of the shortest simple path from s to d?



Q: What is the length of the shortest simple path from s to d?

A: 1



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- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

Defining Cells of Table

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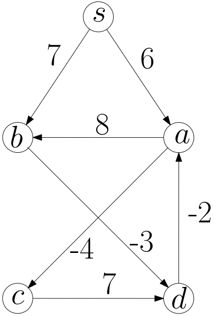

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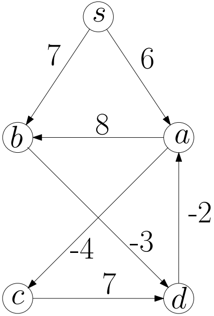

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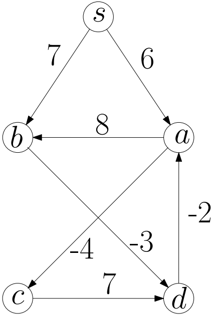

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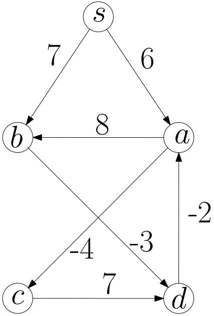

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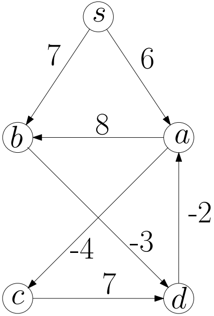

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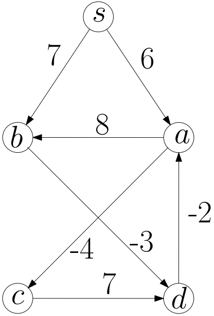

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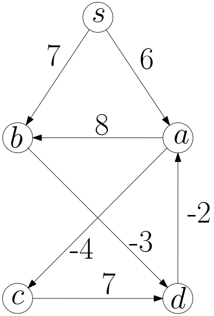
$$f^{\ell}[v] = \begin{cases} & \ell = 0, v = s \\ & \ell = 0, v \neq s \\ & \ell > 0 \end{cases}$$



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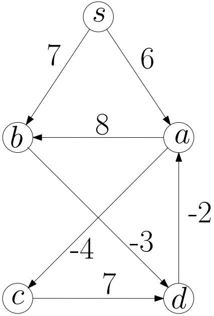

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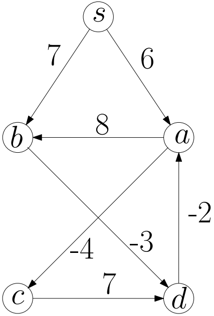

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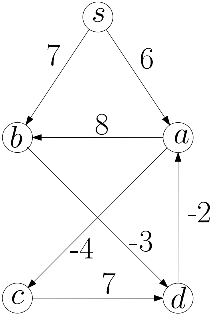

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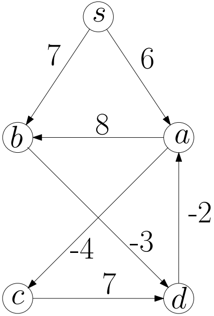

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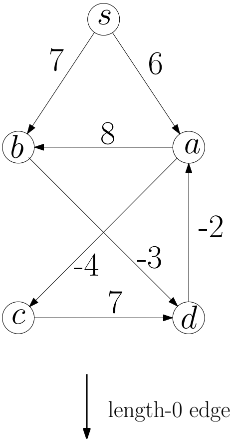
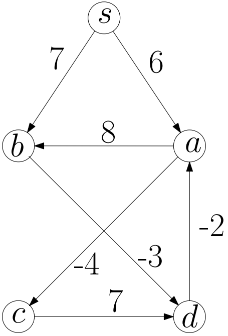
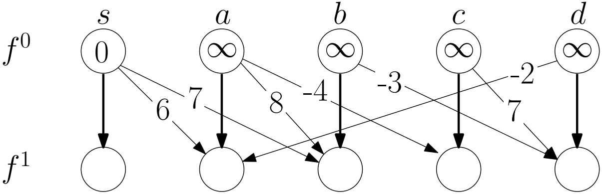
Dynamic Programming: Example

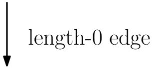


	s	a	b	c	d
f^0	0	∞	∞	∞	∞

Dynamic Programming: Example







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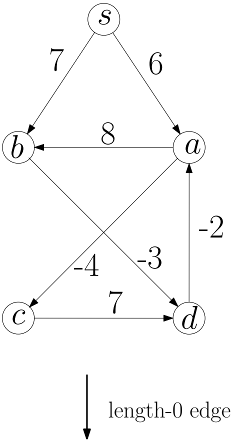

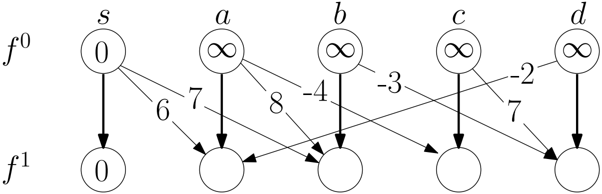

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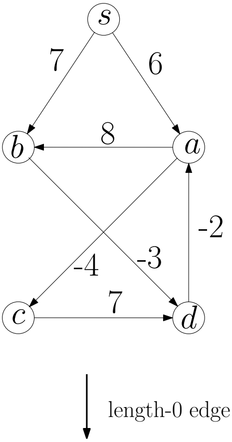

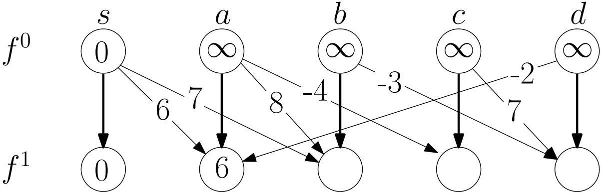

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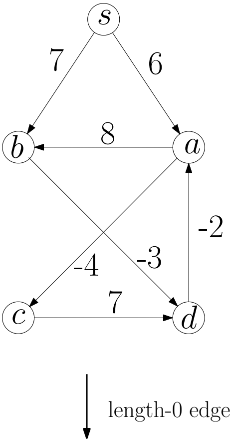

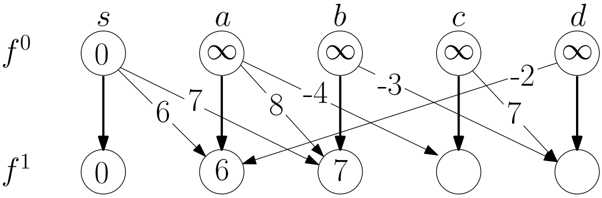

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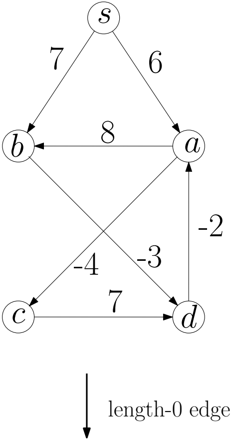

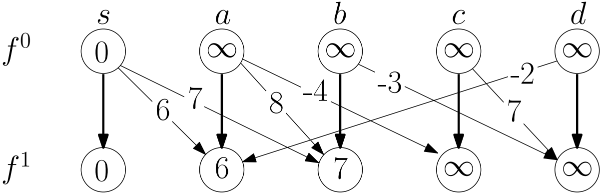

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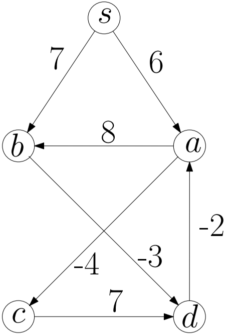
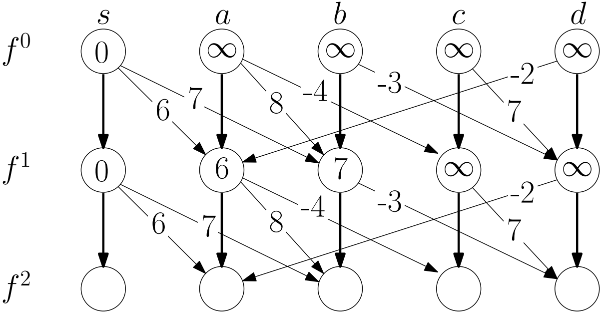
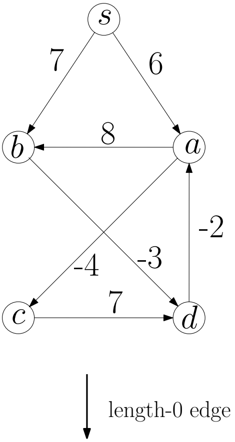
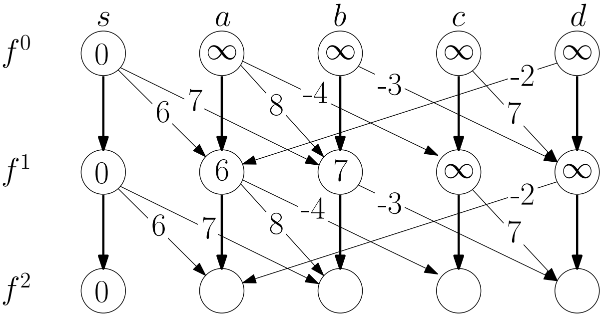




length-0 edge

Dynamic Programming: Example





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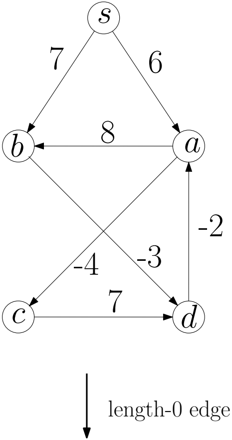

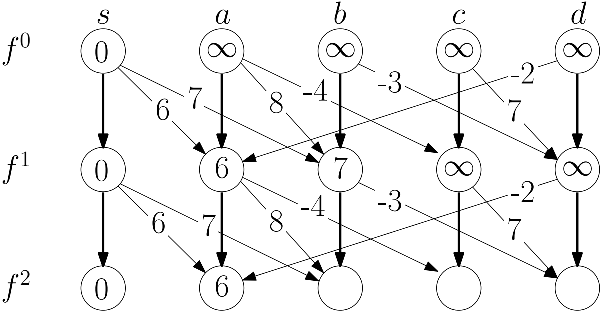

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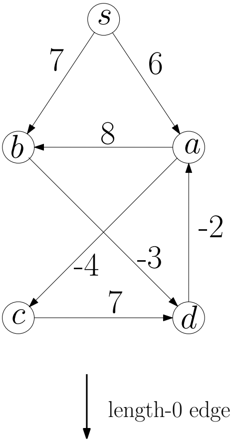

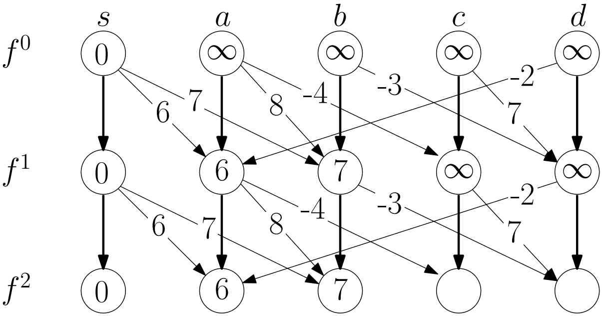

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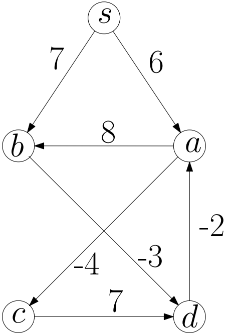

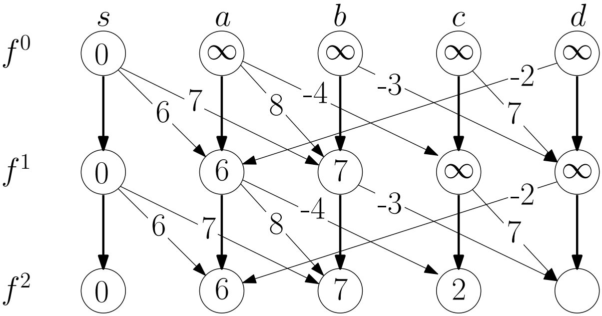

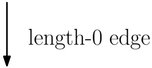

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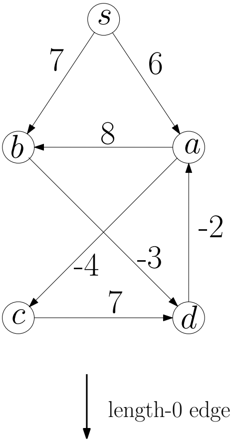

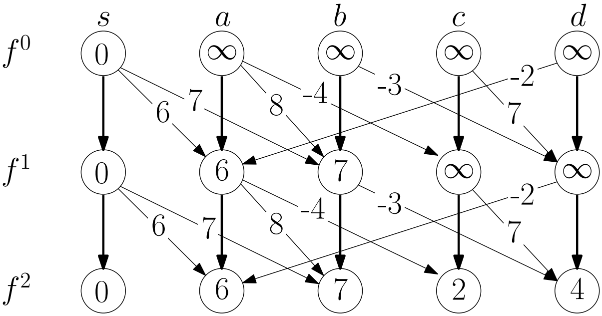

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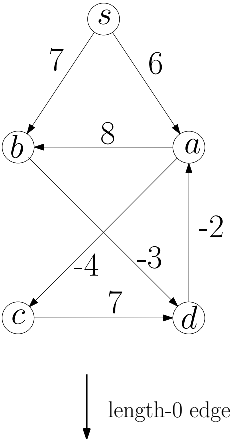

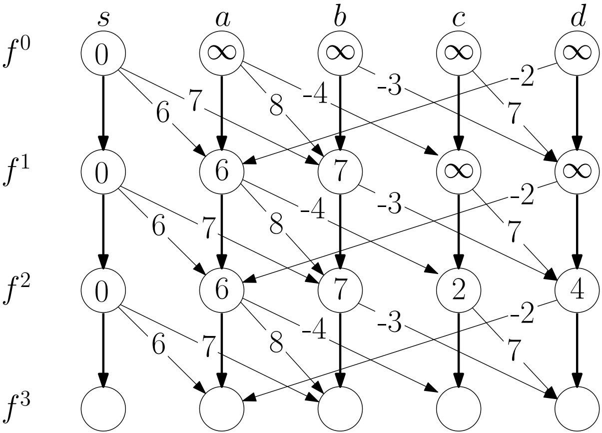

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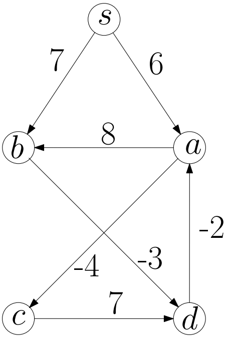

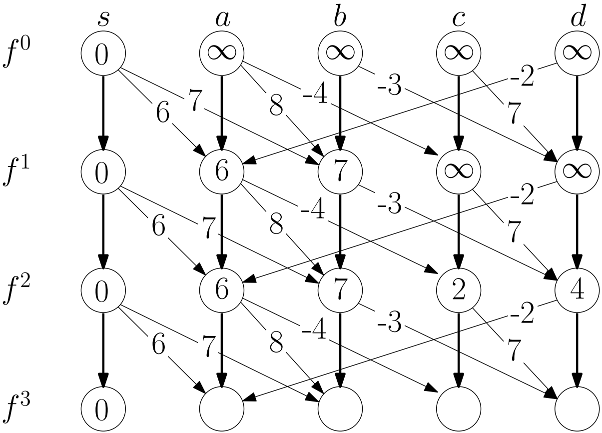

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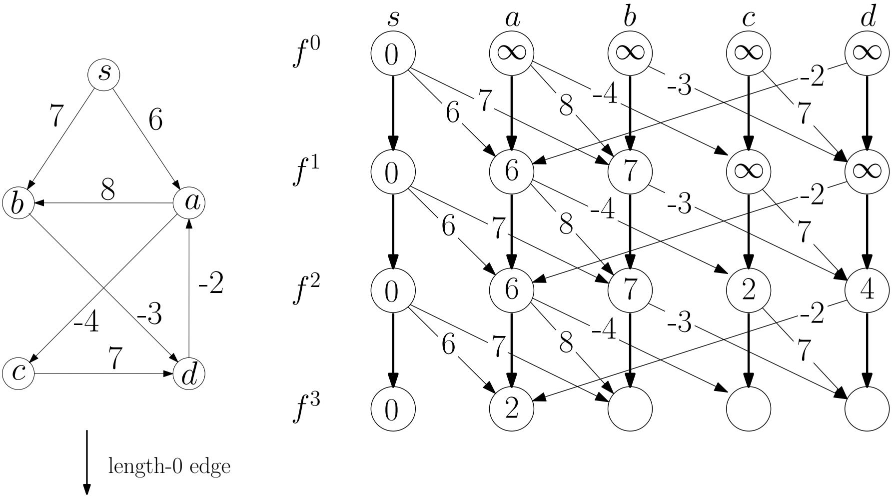
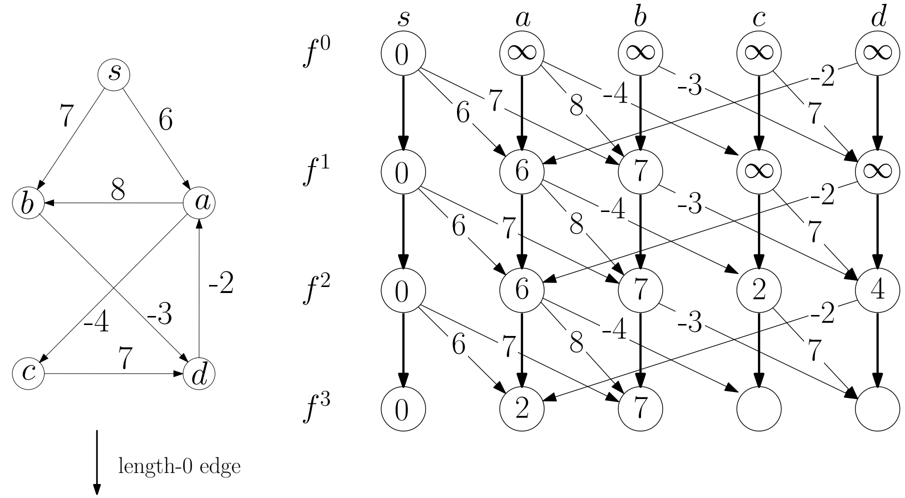


Dynamic Programming: Example



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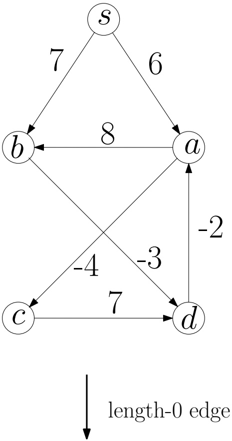
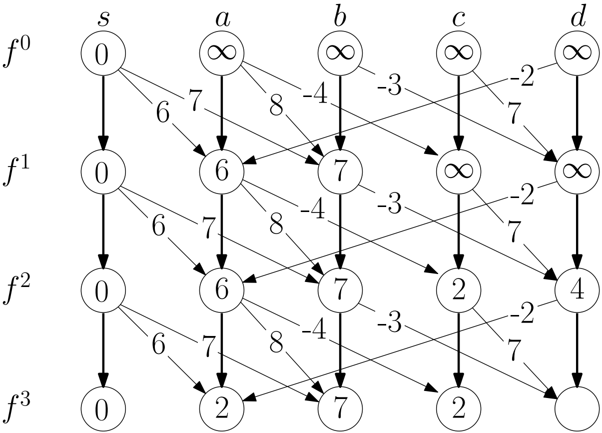
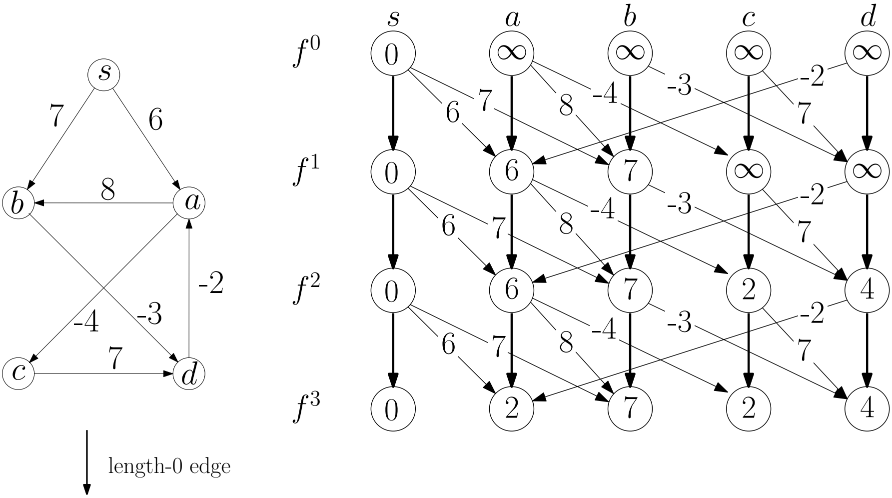
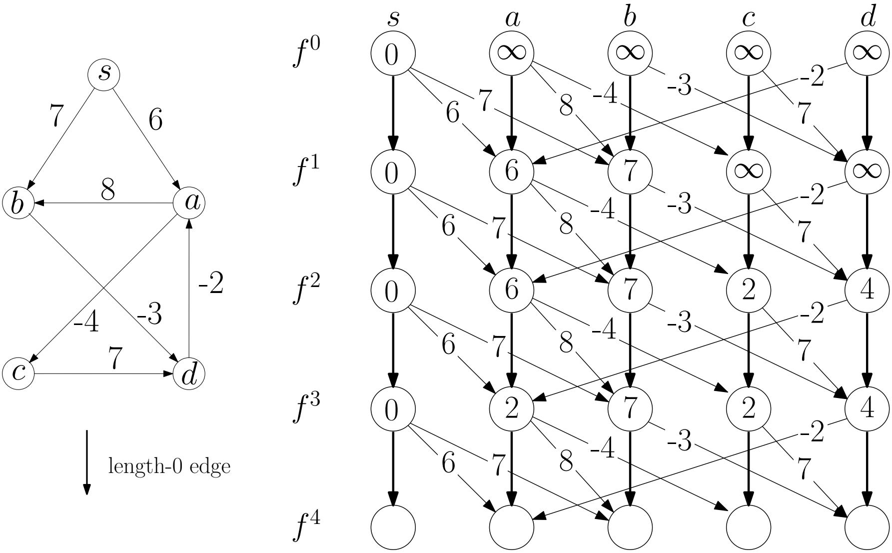




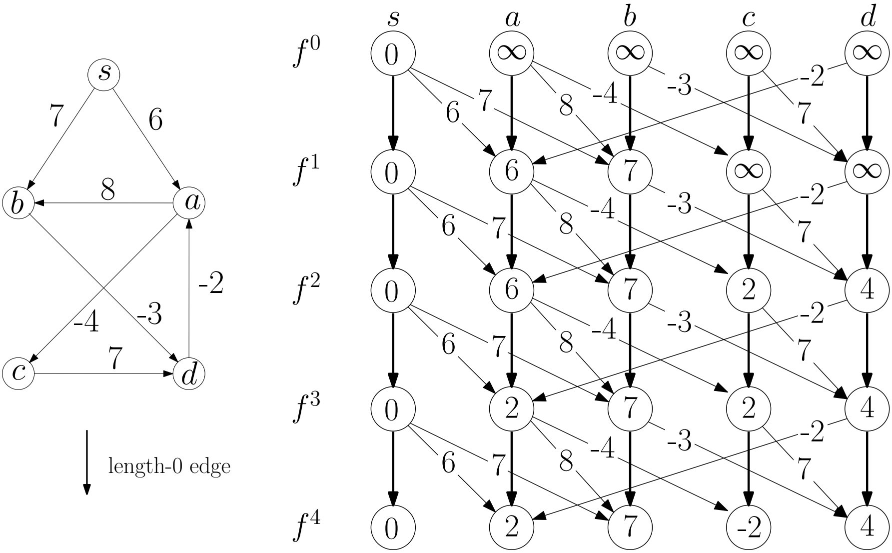
Dynamic Programming: Example



Dynamic Programming: Example



Dynamic Programming: Example



dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \rightarrow f^\ell$
    for each $(u, v) \in E$ **do**
        if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ **then**
            $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
return $(f^{n-1}[v])_{v \in V}$

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Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

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Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Proof.

If there is a path containing at least n edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. □

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

$f^{\text{old}}[s] \leftarrow 0$ and $f^{\text{old}}[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\text{old}} \to f^{\text{new}}$
    for each $(u, v) \in E$ **do**
        if $f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v]$ **then**
            $f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)$
    copy $f^{\text{new}} \to f^{\text{old}}$
return f^{old}

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

```
f^old[s] ← 0 and f^old[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^old → f^new
    for each (u, v) ∈ E do
        if f^old[u] + w(u, v) < f^new[v] then
            f^new[v] ← f^old[u] + w(u, v)
    copy f^new → f^old
return f^old
```

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors
- only need 1 vector!

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f \rightarrow f$
    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$
    copy $f \rightarrow f$
return f

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors
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Dynamic Programming with Better Space Usage

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Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

```
f[s] ← 0 and f[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    for each (u, v) ∈ E do
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            f[v] ← f[u] + w(u, v)
return f
```

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            f[v] ← f[u] + w(u, v)
return f
```

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration

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```

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- This is OK: it can only "accelerate" the process!

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

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for $\ell \leftarrow 1$ to $n-1$ **do**
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- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$
return f

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!
- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v

Bellman-Ford Algorithm

- After iteration ℓ:

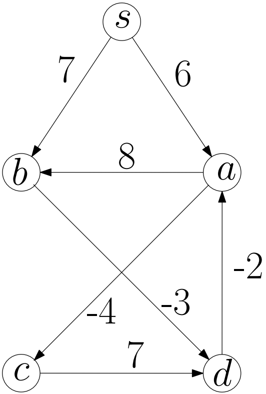
$$\begin{aligned} &\text{length of shortest } s\text{-}v \text{ path} \\ \le\ & f[v] \\ \le\ & \text{length of shortest } s\text{-}v \text{ path using at most } \ell \text{ edges} \end{aligned}$$

- Assuming there are no negative cycles:

$$\begin{aligned} &\text{length of shortest } s\text{-}v \text{ path} \\ =\ & \text{length of shortest } s\text{-}v \text{ path using at most } n-1 \text{ edges} \end{aligned}$$

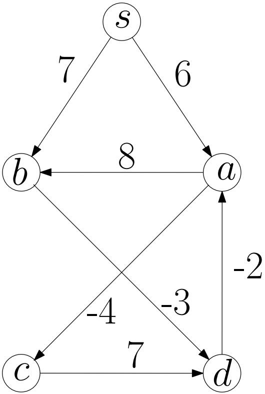
- So, assuming there are no negative cycles, after iteration $n-1$:

$$f[v] = \text{length of shortest } s\text{-}v \text{ path}$$



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



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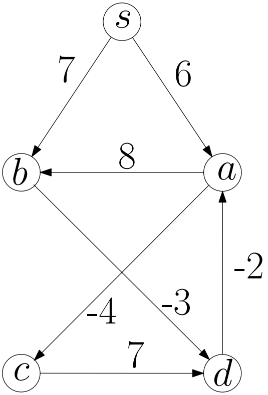
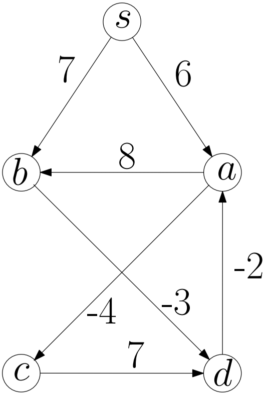
vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



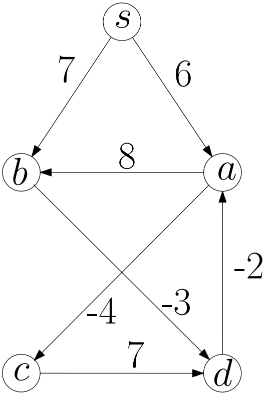
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	∞	∞	∞



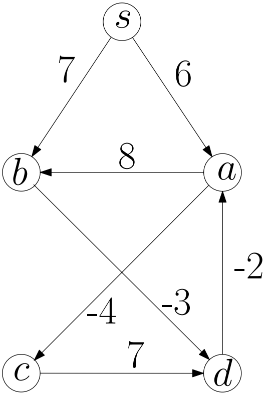
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	∞	∞	∞



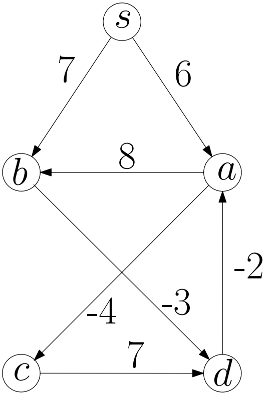
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	∞	∞



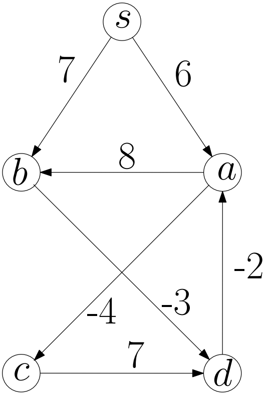
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	7	∞	∞



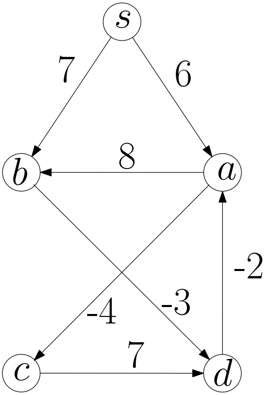
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	7	∞	∞



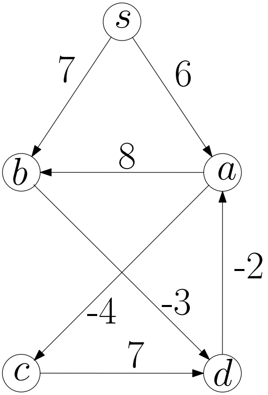
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	2	∞



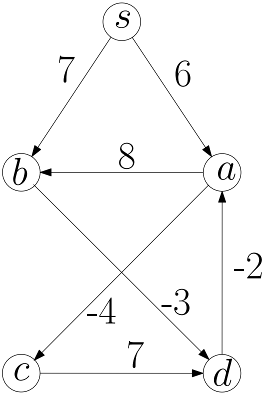
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	2	∞



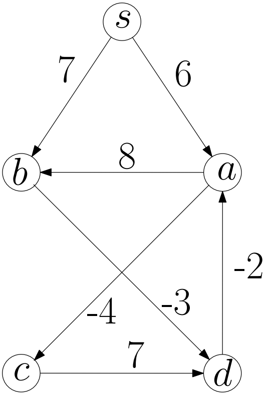
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	7	2	4



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	2	4



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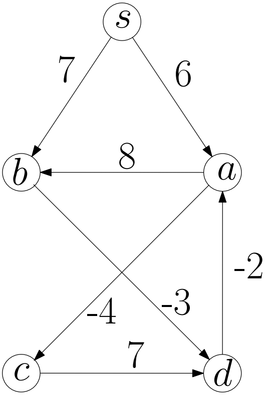
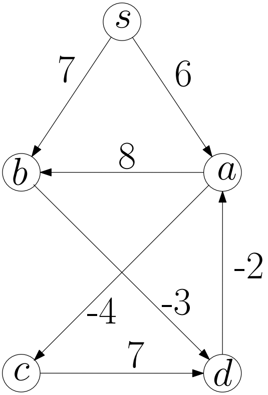


- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

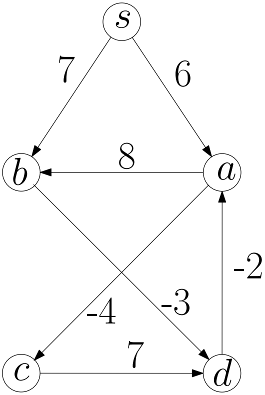
vertices	s	a	b	c	d
f	0	2	7	2	4



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	2	7	2	4

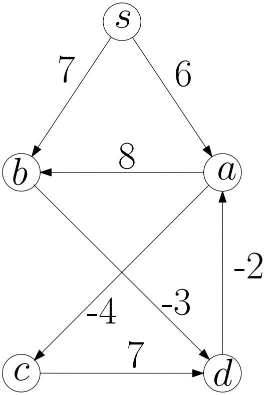
- end of iteration 1: 0, 2, 7, 2, 4



- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	2	7	2	4

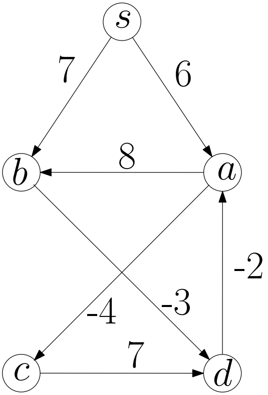
- end of iteration 1: 0, 2, 7, 2, 4



- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	2	7	2	4

- end of iteration 1: 0, 2, 7, 2, 4



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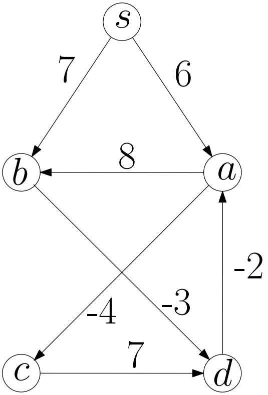
vertices	s	a	b	c	d
f	0	2	7	2	4

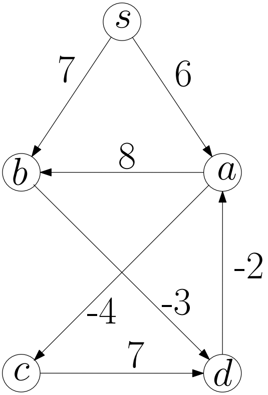
- end of iteration 1: 0, 2, 7, 2, 4



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	2	7	2	4

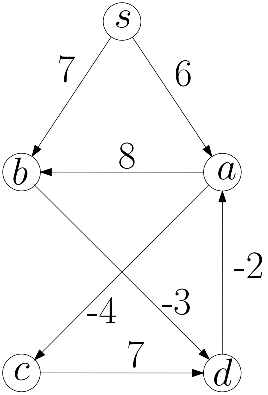
- end of iteration 1: 0, 2, 7, 2, 4



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	2	7	-2	4

- end of iteration 1: 0, 2, 7, 2, 4



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vertices	s	a	b	c	d
f	0	2	7	-2	4

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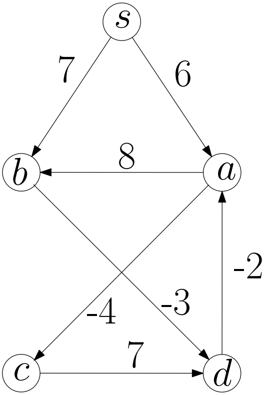

- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	2	7	-2	4

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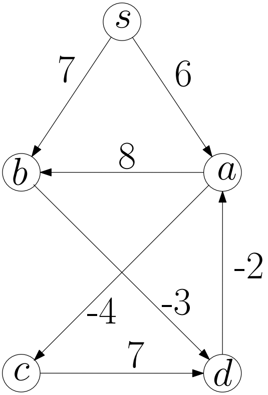

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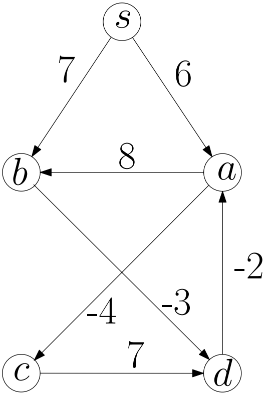
- end of iteration 1: 0, 2, 7, 2, 4



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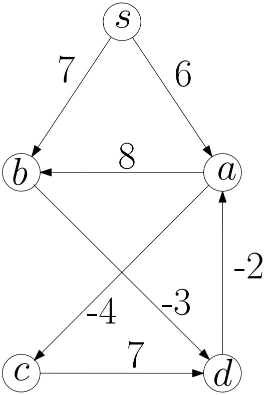
vertices	s	a	b	c	d
f	0	2	7	-2	4

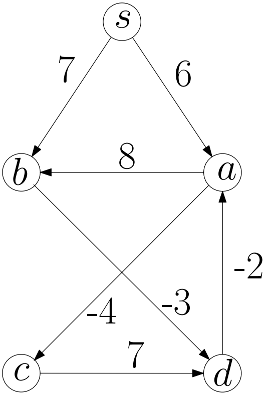
- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4



- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	2	7	-2	4

- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	2	7	-2	4

- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4
- Algorithm terminates in 3 iterations, instead of 4.

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to n **do**
    $updated \leftarrow$ false
    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$
            $updated \leftarrow$ true
    if not $updated$, then return f
output "negative cycle exists"

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

```
f[s] ← 0 and f[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n do
    updated ← false
    for each (u, v) ∈ E do
        if f[u] + w(u, v) < f[v] then
            f[v] ← f[u] + w(u, v), π[v] ← u
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```

- $\pi[v]$: the parent of v in the shortest path tree

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to n **do**
    $updated \leftarrow$ false
    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
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- $\pi[v]$: the parent of v in the shortest path tree
- Running time $= O(nm)$

Outline

All-Pair Shortest Paths

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Input: directed graph $G = (V, E)$,

$w : E \to \mathbb{R}$ (can be negative)

Output: shortest path from u to v for every $u, v \in V$

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- Running time $= O(n^2 m)$

Summary of Shortest Path Algorithms we learned

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n\log n+m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs

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$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

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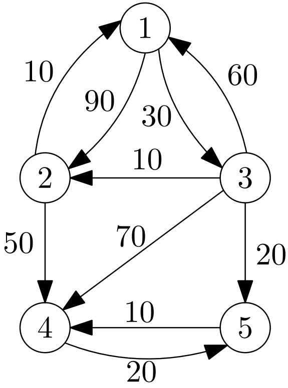
- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

Example for Definition of $f^k[i,j]$'s



$$f^0[1,4] = \infty$$

$$f^1[1,4] = \infty$$

$$f^2[1,4] = 140 \qquad (1 \to 2 \to 4)$$

$$f^3[1,4] = 90 \qquad (1 \to 3 \to 2 \to 4)$$

$$f^4[1,4] = 90 \qquad (1 \to 3 \to 2 \to 4)$$

$$f^5[1,4] = 60 \qquad (1 \to 3 \to 5 \to 4)$$

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

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$$f^k[i,j] = \begin{cases} w(i,j) & k = 0 \\ \min \begin{cases} f^{k-1}[i,j] \\ \end{cases} & k = 1, 2, \cdots, n \end{cases}$$

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$$f^k[i,j] = \begin{cases} w(i,j) & k = 0 \\ \min \begin{cases} f^{k-1}[i,j] \\ f^{k-1}[i,k] + f^{k-1}[k,j] \end{cases} & k = 1, 2, \cdots, n \end{cases}$$

Floyd-Warshall(G, w)

```
f^0 ← w
for k ← 1 to n do
    copy f^{k-1} → f^k
    for i ← 1 to n do
        for j ← 1 to n do
            if f^{k-1}[i,k] + f^{k-1}[k,j] < f^k[i,j] then
                f^k[i,j] ← f^{k-1}[i,k] + f^{k-1}[k,j]
```

Floyd-Warshall(G, w)

```
f^old ← w
for k ← 1 to n do
    copy f^old → f^new
    for i ← 1 to n do
        for j ← 1 to n do
            if f^old[i,k] + f^old[k,j] < f^new[i,j] then
                f^new[i,j] ← f^old[i,k] + f^old[k,j]
```

Floyd-Warshall(G, w)

$f^{\text{old}} \leftarrow w$
for $k \leftarrow 1$ to n **do**
    copy $f^{\text{old}} \rightarrow f^{\text{new}}$
    for $i \leftarrow 1$ to n **do**
        for $j \leftarrow 1$ to n **do**
            if $f^{\text{old}}[i,k] + f^{\text{old}}[k,j] < f^{\text{new}}[i,j]$ **then**
                $f^{\text{new}}[i,j] \leftarrow f^{\text{old}}[i,k] + f^{\text{old}}[k,j]$

Floyd-Warshall(G, w)

$f \leftarrow w$
for $k \leftarrow 1$ to n **do**
  copy $f \rightarrow f$
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Lemma Assume there are no negative cycles in G. After iteration k, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from i to j that only uses vertices in $\{1, 2, 3, \cdots, k\}$ as intermediate vertices.

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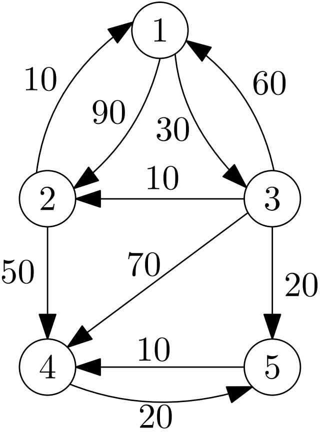
- Running time $= O(n^3)$.



	1	2	3	4	5
1	0	90	30	∞	∞
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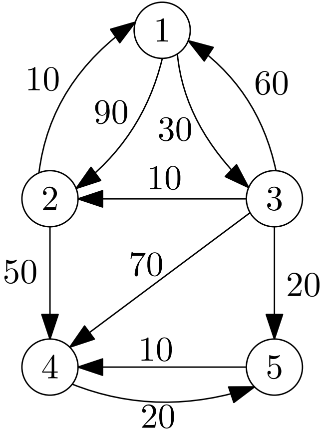
-



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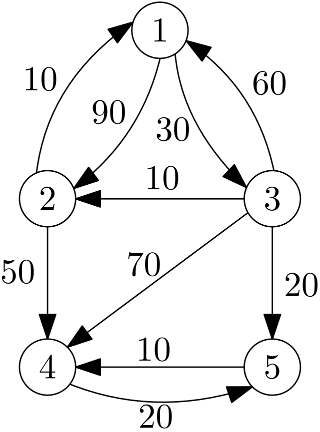
- $i = 2,\ k = 1,\ j = 3$



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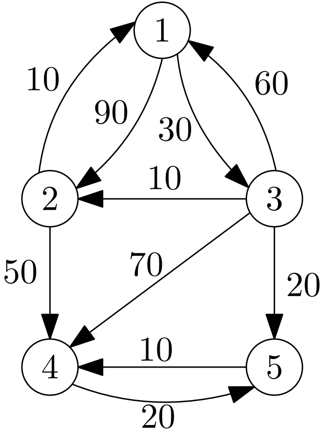

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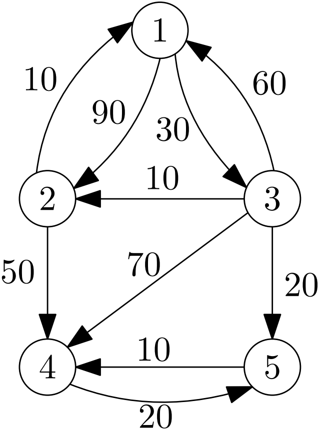

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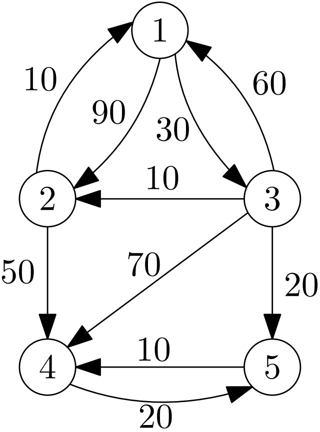

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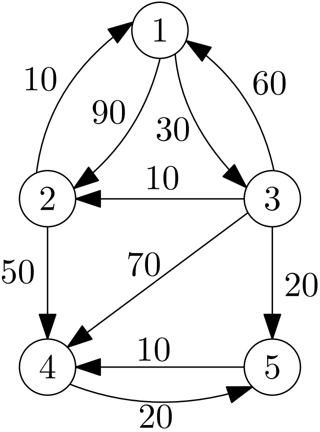

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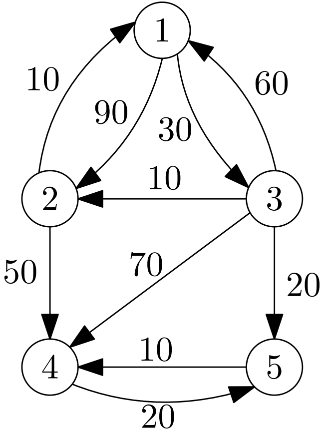

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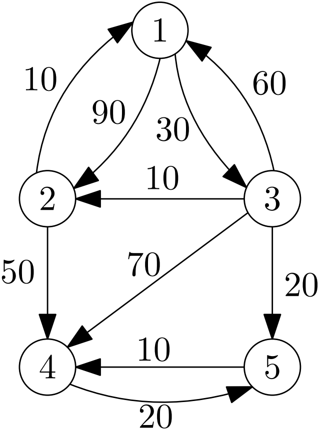

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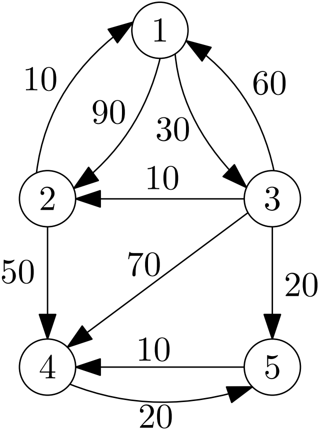

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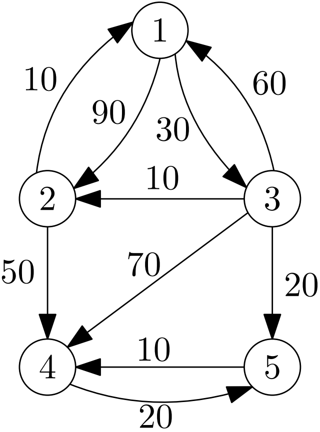

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Recovering Shortest Paths

Floyd-Warshall(G, w)

```
f ← w, π[i, j] ← ⊥ for every i, j ∈ V
for k ← 1 to n do
    for i ← 1 to n do
        for j ← 1 to n do
            if f[i, k] + f[k, j] < f[i, j] then
                f[i, j] ← f[i, k] + f[k, j], π[i, j] ← k
```

Recovering Shortest Paths

Floyd-Warshall(G, w)

$f \leftarrow w$, $\pi[i,j] \leftarrow \perp$ for every $i, j \in V$
for $k \leftarrow 1$ to n **do**
  for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to n **do**
      if $f[i,k] + f[k,j] < f[i,j]$ **then**
        $f[i,j] \leftarrow f[i,k] + f[k,j]$, $\pi[i,j] \leftarrow k$

print-path(i, j)

if $\pi[i,j] = \perp$ **then** then
  if $i \neq j$ **then** print$(i, \text{“,”})$
else
  print-path$(i, \pi[i,j])$, print-path$(\pi[i,j], j)$

Detecting Negative Cycles

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  - **for** $i \leftarrow 1$ to n **do**
    - **for** $j \leftarrow 1$ to n **do**
      - **if** $f[i, k] + f[k, j] < f[i, j]$ **then**
        - report "negative cycle exists" and exit

Summary of Shortest Path Algorithms

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
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- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs