CSE 431/531: Algorithm Analysis and Design (Fall 2022)

# Divide-and-Conquer

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# Outline

## Greedy Algorithm

- mainly for combinatorial optimization problems
- trivial algorithm runs in exponential time
- greedy algorithm gives an efficient algorithm
- main focus of analysis: correctness of algorithm

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## Divide-and-Conquer

- not necessarily for combinatorial optimization problems
- trivial algorithm already runs in polynomial time
- divide-and-conquer gives a more efficient algorithm
- main focus of analysis: running time

# Divide-and-Conquer

- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance

### merge-sort$(A, n)$

```
if $n = 1$ then
    return $A$
else
    $B \leftarrow$ merge-sort$\Big(A\big[1..\lfloor n/2\rfloor\big], \lfloor n/2\rfloor\Big)$
    $C \leftarrow$ merge-sort$\Big(A\big[\lfloor n/2\rfloor + 1..n\big], \lceil n/2\rceil\Big)$
    return merge$(B, C, \lfloor n/2\rfloor, \lceil n/2\rceil)$
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6:     return merge$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
```
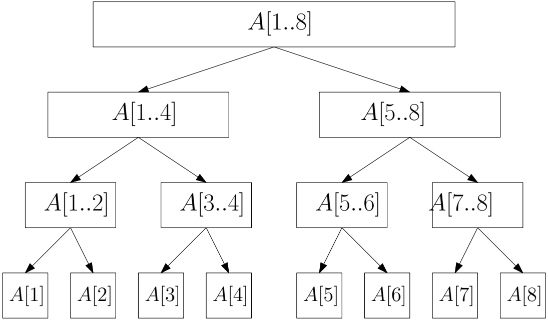

- Divide: trivial
- Conquer: 4, 5
- Combine: 6

# Running Time for Merge-Sort



- Each level takes running time $O(n)$
- There are $O(\lg n)$ levels
- Running time $= O(n \lg n)$
- Better than insertion sort

# Running Time for Merge-Sort Using Recurrence

- $T(n) =$ running time for sorting $n$ numbers,then

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + O(n) & \text{if } n \geq 2 \end{cases}$$

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- Even simpler: $T(n) = 2T(n/2) + O(n)$. (Implicit assumption: $T(n) = O(1)$ if $n$ is at most some constant.)
- Solving this recurrence, we have $T(n) = O(n \lg n)$ (we shall show how later)

# Outline

**Def.** Given an array $A$ of $n$ integers, an inversion in $A$ is a pair $(i, j)$ of indices such that $i < j$ and $A[i] > A[j]$.

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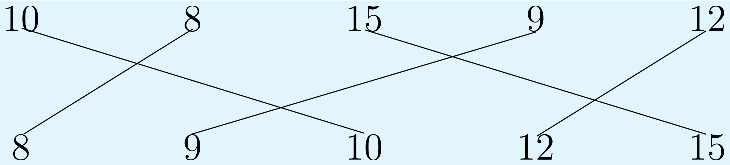

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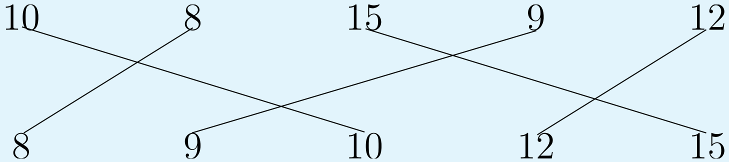


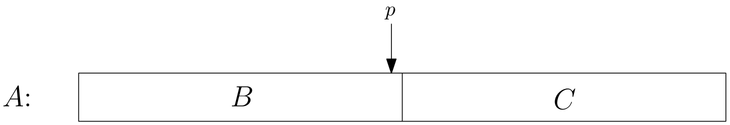
- 4 inversions (for convenience, using numbers, not indices): $(10, 8), (10, 9), (15, 9), (15, 12)$

# Naive Algorithm for Counting Inversions

**count-inversions($A, n$)**

```
$c \leftarrow 0$
for every $i \leftarrow 1$ to $n - 1$ do
    for every $j \leftarrow i + 1$ to $n$ do
        if $A[i] > A[j]$ then $c \leftarrow c + 1$
return $c$
```

# Divide-and-Conquer



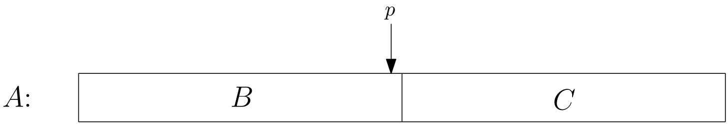
- $p = \lfloor n/2 \rfloor, B = A[1..p], C = A[p+1..n]$
- $$\#\mathsf{invs}(A) = \#\mathsf{invs}(B) + \#\mathsf{invs}(C) + m$$ $$m = \Big|\big\{(i,j) : B[i] > C[j]\big\}\Big|$$

**Q:** How fast can we compute $m$, via trivial algorithm?

**A:** $O(n^2)$

- Can not improve the $O(n^2)$ time for counting inversions.

# Divide-and-Conquer



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- $$\#\text{invs}(A) = \#\text{invs}(B) + \#\text{invs}(C) + m$$ $$m = \big|\{(i,j) : B[i] > C[j]\}\big|$$

**Lemma** If both $B$ and $C$ are sorted, then we can compute $m$ in $O(n)$ time!

# Counting Inversions between $B$ and $C$

Count pairs $i, j$ such that $B[i] > C[j]$:

$B$:

| | | | | | |
|---|---|---|---|---|---|
| 3 | 8 | 12 | 20 | 32 | 48 |

total$=$ 0

$C$:

| | | | | |
|---|---|---|---|---|
| 5 | 7 | 9 | 25 | 29 |

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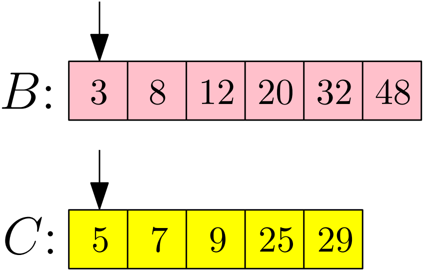
Count pairs $i, j$ such that $B[i] > C[j]$:



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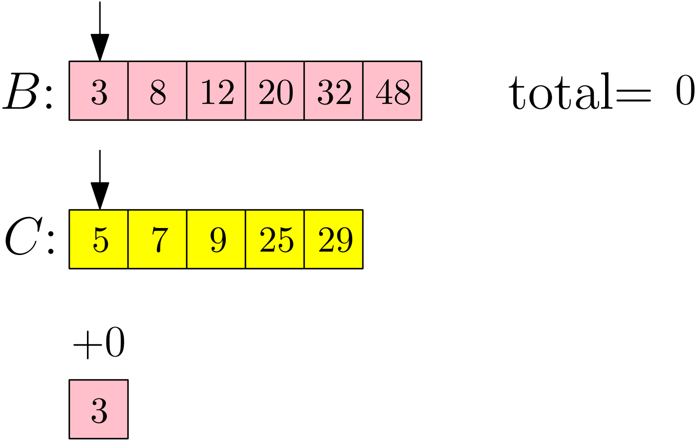

$B$: 3 8 12 20 32 48

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$+0$

3

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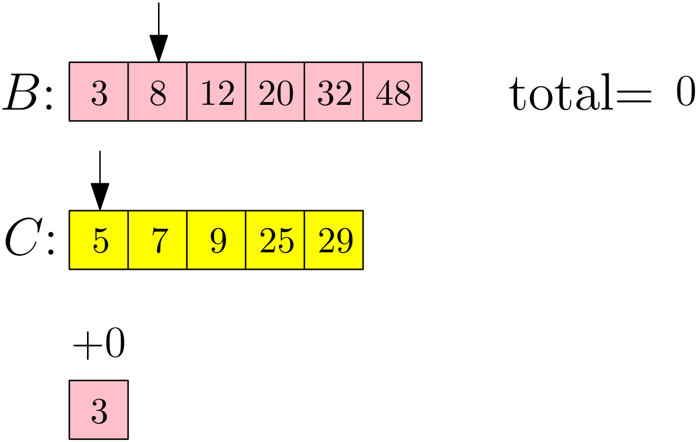

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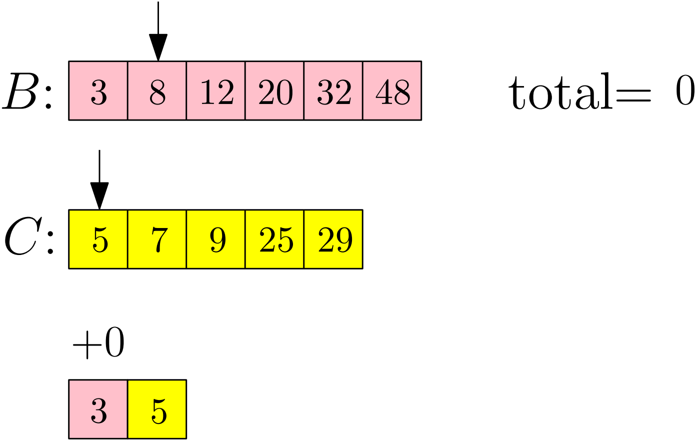

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| | |
|---|---|
| 3 | 5 |

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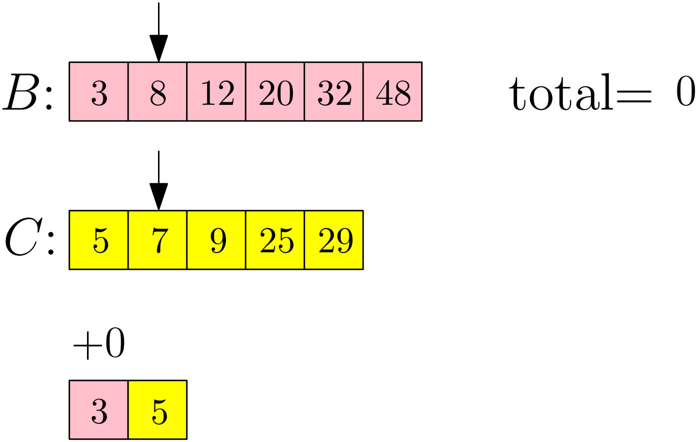

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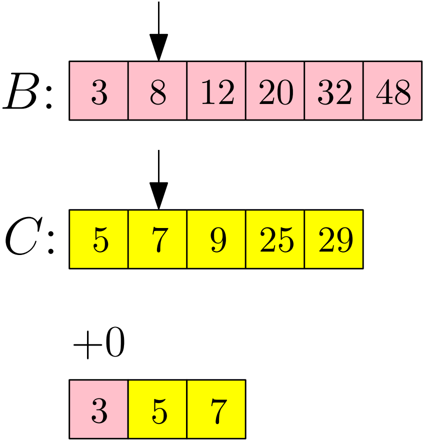

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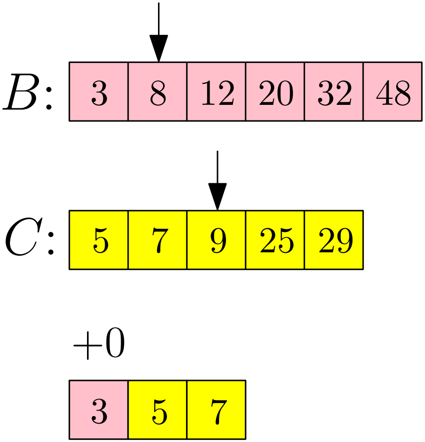

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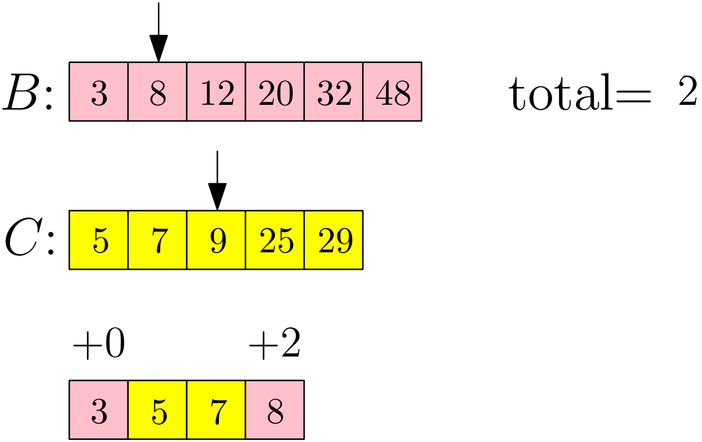

$B$:

| | | | | | |
|---|---|---|---|---|---|
| 3 | 8 | 12 | 20 | 32 | 48 |

total$=$ 2

$C$:

| | | | | |
|---|---|---|---|---|
| 5 | 7 | 9 | 25 | 29 |

$+0$ $+2$

| | | | |
|---|---|---|---|
| 3 | 5 | 7 | 8 |

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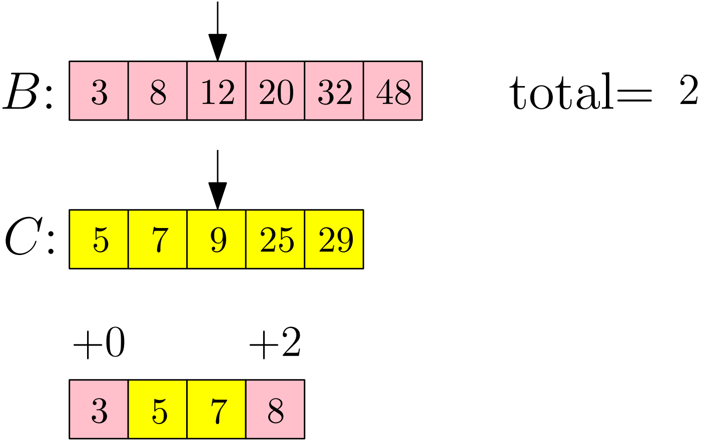

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total$=$ 2

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3 5 7 8

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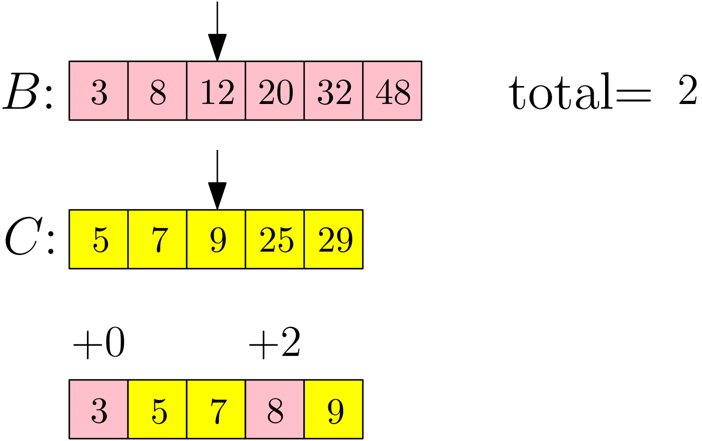

$B$: 3 8 12 20 32 48

total= 2

$C$: 5 7 9 25 29

+0 +2

3 5 7 8 9

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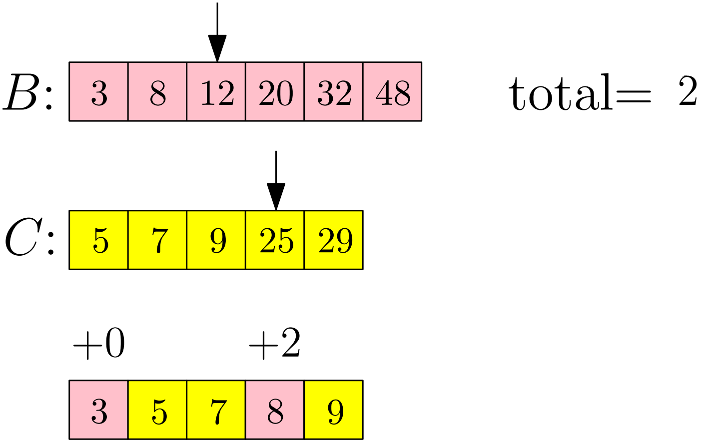

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| | | | | |
|---|---|---|---|---|
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$+0$ $+2$

| | | | | |
|---|---|---|---|---|
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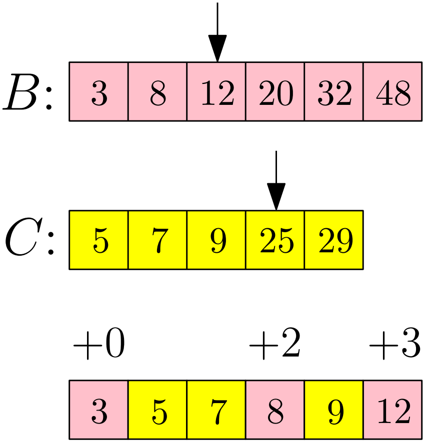

$B$: 3 | 8 | 12 | 20 | 32 | 48

$C$: 5 | 7 | 9 | 25 | 29

$+0$ $+2$ $+3$

3 | 5 | 7 | 8 | 9 | 12

total$=$ 5

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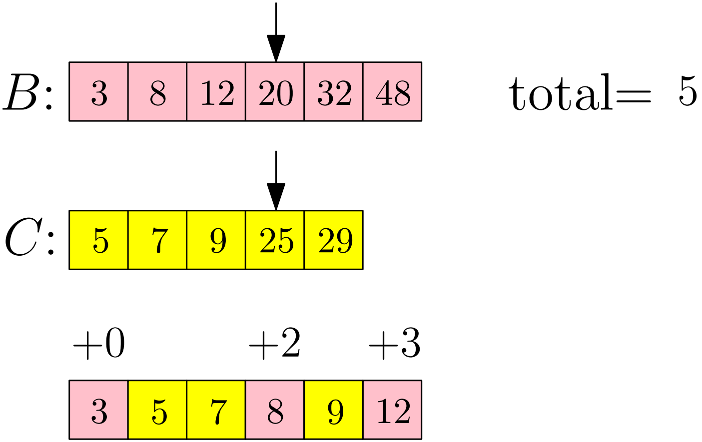
Count pairs $i, j$ such that $B[i] > C[j]$:



$B$:

| | | | | | |
|---|---|---|---|---|---|
| 3 | 8 | 12 | 20 | 32 | 48 |

$\text{total} = 5$

$C$:

| | | | | |
|---|---|---|---|---|
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$+0$ $+2$ $+3$

| | | | | | |
|---|---|---|---|---|---|
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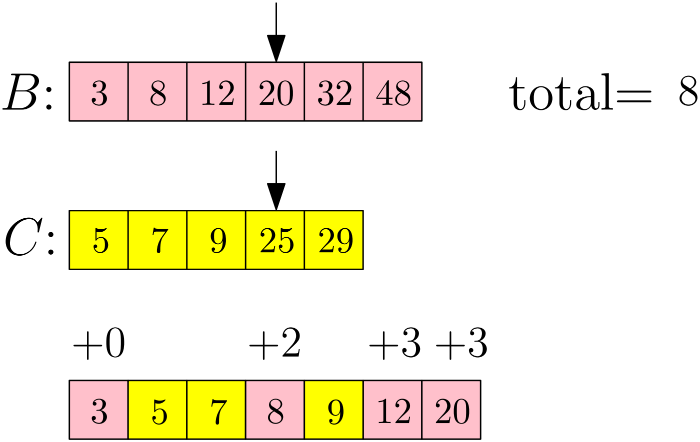

$B$: 3 8 12 20 32 48

total= 8

$C$: 5 7 9 25 29

$+0$ $+2$ $+3$ $+3$

3 5 7 8 9 12 20

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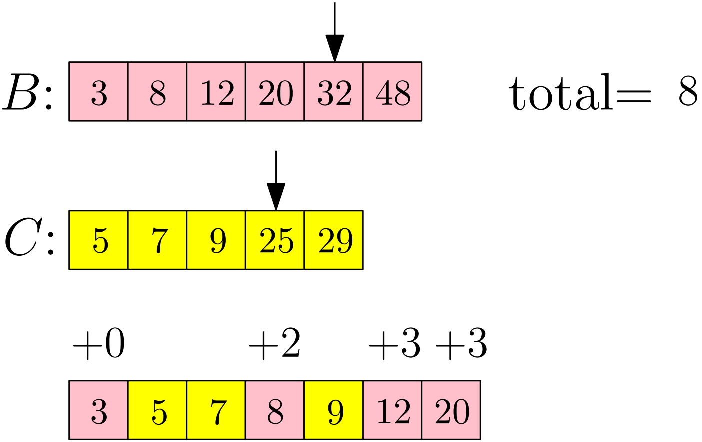

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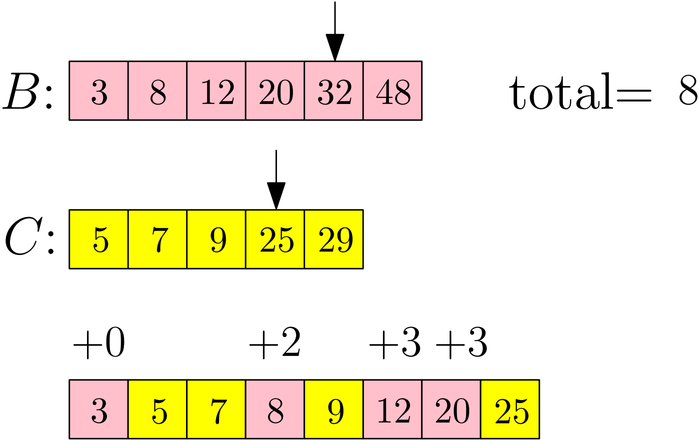

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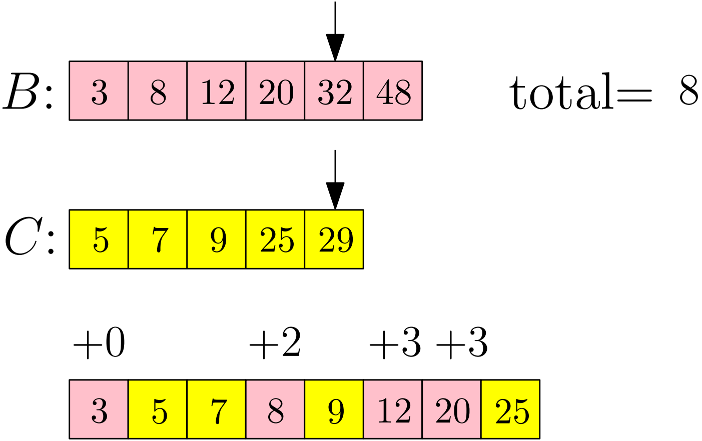

$B$: 3 8 12 20 32 48

total= 8

$C$: 5 7 9 25 29

+0 +2 +3 +3

3 5 7 8 9 12 20 25

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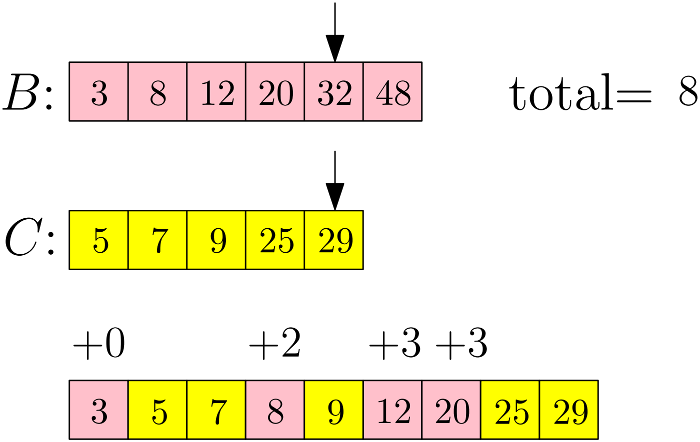

$B$: 3 | 8 | 12 | 20 | 32 | 48

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$+0$ $+2$ $+3$ $+3$

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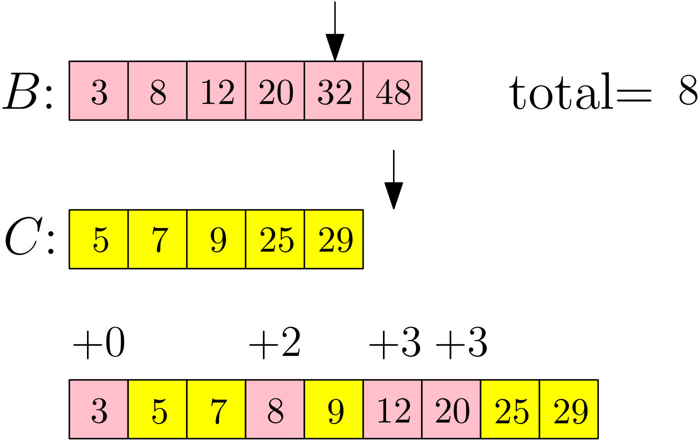

$B$: 3 | 8 | 12 | 20 | 32 | 48

total= 8

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+0 +2 +3 +3

3 | 5 | 7 | 8 | 9 | 12 | 20 | 25 | 29

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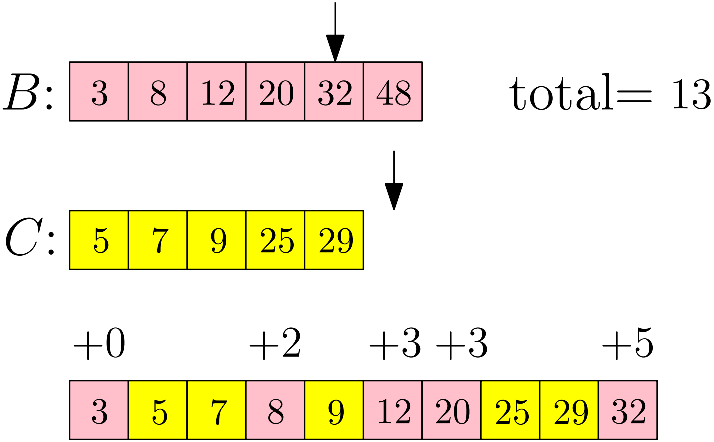

$B$: 3, 8, 12, 20, 32, 48

total $= 13$

$C$: 5, 7, 9, 25, 29

| +0 | | | +2 | | +3 | +3 | | | +5 |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 5 | 7 | 8 | 9 | 12 | 20 | 25 | 29 | 32 |

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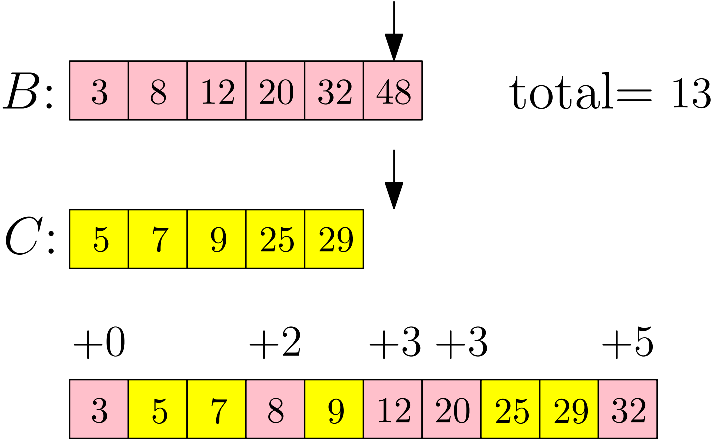
Count pairs $i, j$ such that $B[i] > C[j]$:



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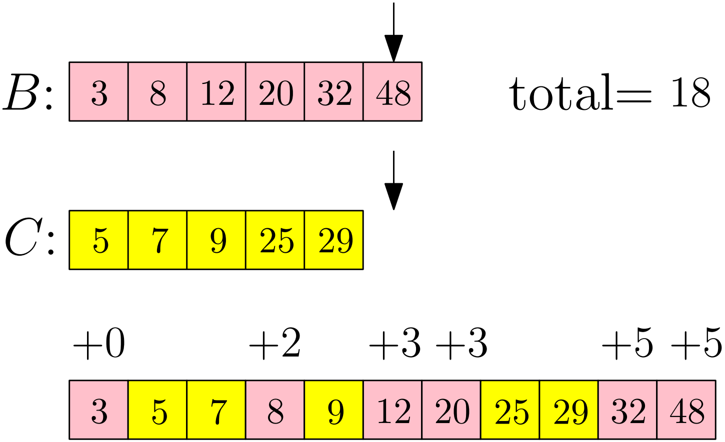


$B$: 3 8 12 20 32 48

total= 18

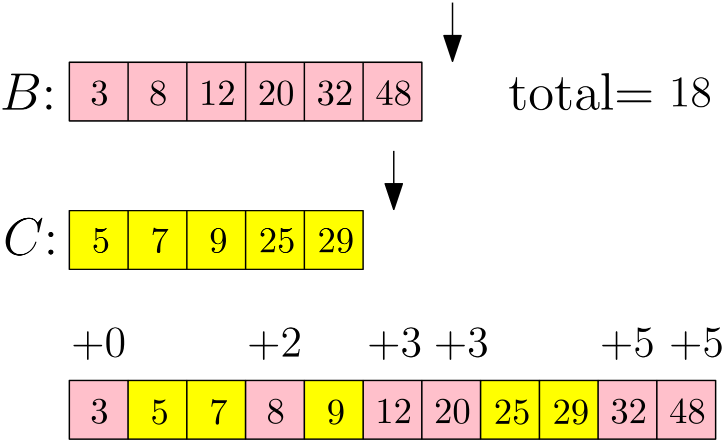
$C$: 5 7 9 25 29

+0 +2 +3 +3 +5 +5

3 5 7 8 9 12 20 25 29 32 48

# Counting Inversions between $B$ and $C$

Count pairs $i, j$ such that $B[i] > C[j]$:



$B$: 3 8 12 20 32 48

total$= 18$

$C$: 5 7 9 25 29

$+0$ $+2$ $+3$ $+3$ $+5$ $+5$

3 5 7 8 9 12 20 25 29 32 48

# Count Inversions between $B$ and $C$

- Procedure that merges $B$ and $C$ and counts inversions between $B$ and $C$ at the same time

```
merge-and-count($B, C, n_1, n_2$)
$count \leftarrow 0$;
$A \leftarrow$ array of size $n_1 + n_2$; $i \leftarrow 1$; $j \leftarrow 1$
while $i \leq n_1$ or $j \leq n_2$ do
    if $j > n_2$ or ($i \leq n_1$ and $B[i] \leq C[j]$) then
        $A[i + j - 1] \leftarrow B[i]$; $i \leftarrow i + 1$
        $count \leftarrow count + (j - 1)$
    else
        $A[i + j - 1] \leftarrow C[j]$; $j \leftarrow j + 1$
return $(A, count)$
```

# Sort and Count Inversions in $A$

- A procedure that returns the sorted array of $A$ and counts the number of inversions in $A$:

**sort-and-count($A, n$)**

```
if $n = 1$ then
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\left(A\left[1..\lfloor n/2 \rfloor\right], \lfloor n/2 \rfloor\right)$
    $(C, m_2) \leftarrow$ sort-and-count$\left(A\left[\lfloor n/2 \rfloor + 1..n\right], \lceil n/2 \rceil\right)$
    $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2 \rfloor, \lceil n/2 \rceil)$
    return $(A, m_1 + m_2 + m_3)$
```

# Sort and Count Inversions in $A$

- A procedure that returns the sorted array of $A$ and counts the number of inversions in $A$:

**sort-and-count$(A, n)$**

```
1: if $n = 1$ then
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4:     $(B, m_1) \leftarrow$ sort-and-count$\left(A\left[1..\lfloor n/2\rfloor\right], \lfloor n/2\rfloor\right)$
5:     $(C, m_2) \leftarrow$ sort-and-count$\left(A\left[\lfloor n/2\rfloor + 1..n\right], \lceil n/2\rceil\right)$
6:     $(A, m_3) \leftarrow$ merge-and-count$(B, C, \lfloor n/2\rfloor, \lceil n/2\rceil)$
7:     return $(A, m_1 + m_2 + m_3)$
```

- Divide: trivial
- Conquer: 4, 5
- Combine: 6, 7

### sort-and-count$(A, n)$

```
if $n = 1$ then
    return $(A, 0)$
else
    $(B, m_1) \leftarrow$ sort-and-count$\Big(A\big[1..\lfloor n/2 \rfloor\big], \lfloor n/2 \rfloor\Big)$
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    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
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# Quicksort vs Merge-Sort

| | **Merge Sort** | **Quicksort** |
|---|---|---|
| Divide | Trivial | Separate small and big numbers |
| Conquer | Recurse | Recurse |
| Combine | Merge 2 sorted arrays | Trivial |

# Quicksort Example

**Assumption** We can choose median of an array of size $n$ in $O(n)$ time.

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quicksort$(A, n)$

```
if $n \le 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ array of elements in $A$ that are less than $x$        \\ Divide
$A_R \leftarrow$ array of elements in $A$ that are greater than $x$     \\ Divide
$B_L \leftarrow$ quicksort$(A_L,$ length of $A_L)$                      \\ Conquer
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$t \leftarrow$ number of times $x$ appear $A$
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2. Choose a pivot randomly and pretend it is the median (it is practical)

# Quicksort Using A Random Pivot

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- In theory: assume they can.

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**Lemma** The expected running time of the algorithm is $O(n \lg n)$.

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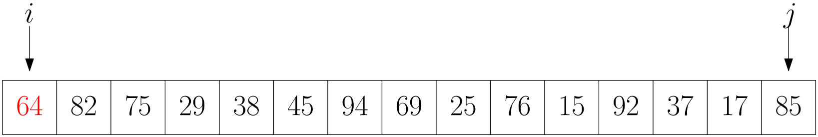

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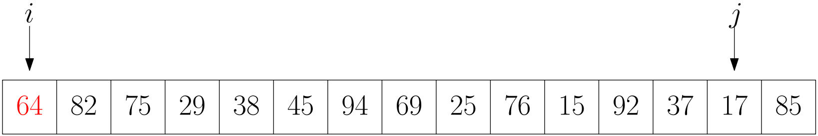

$i$ $j$

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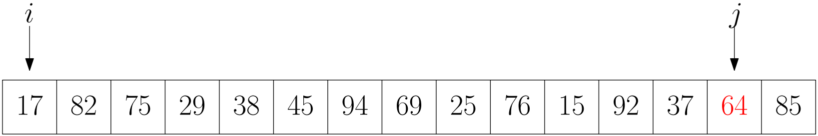

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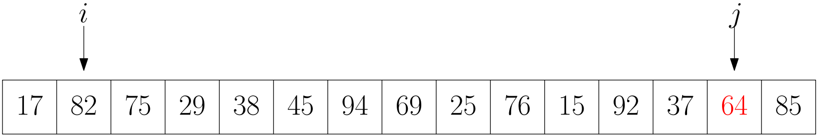

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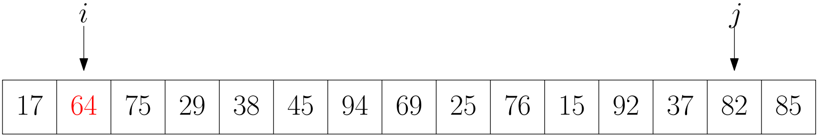

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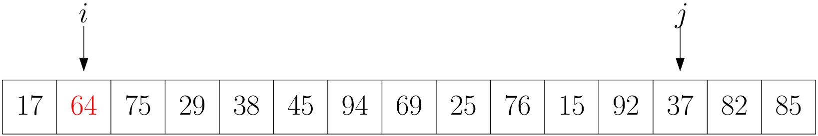

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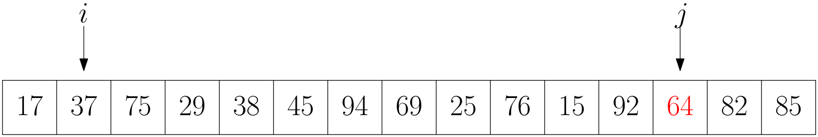

|  | $i$ |  |  |  |  |  |  |  |  |  |  | $j$ |  |  |
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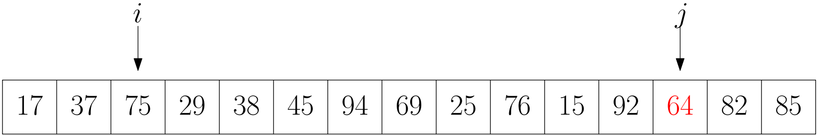

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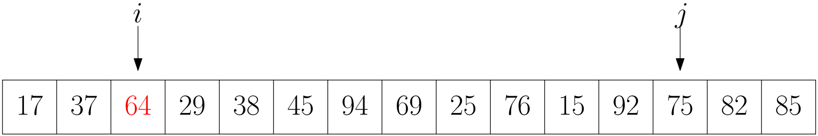

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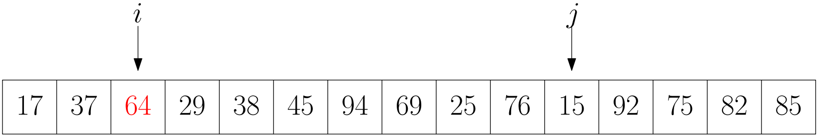

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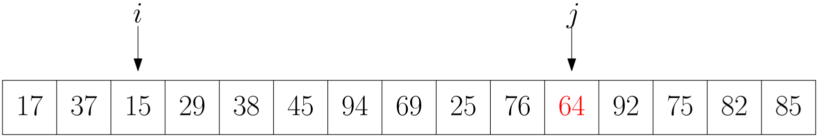

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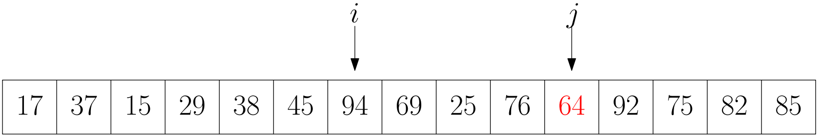

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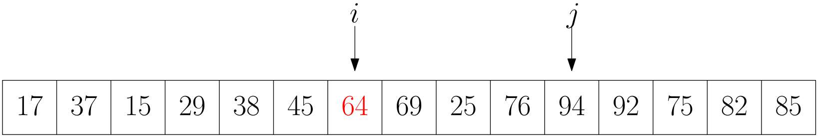

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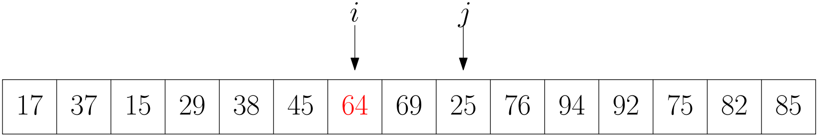

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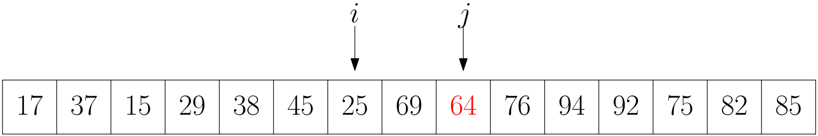

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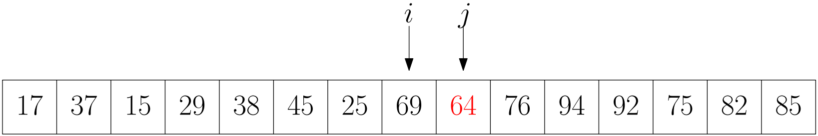

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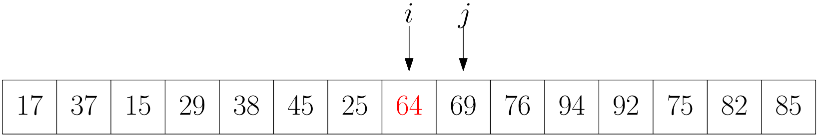

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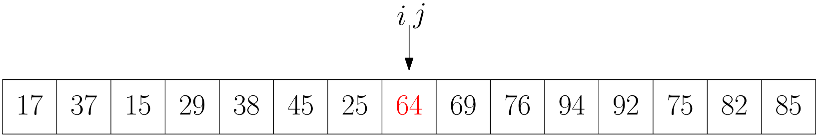

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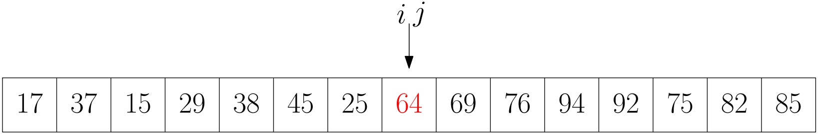

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- To partition the array into two parts, we only need $O(1)$ extra space.

### partition$(A, \ell, r)$

```
$p \leftarrow$ random integer between $\ell$ and $r$, swap $A[p]$ and $A[\ell]$
$i \leftarrow \ell, j \leftarrow r$
while true do
    while $i < j$ and $A[i] < A[j]$ do $j \leftarrow j - 1$
    if $i = j$ then break
    swap $A[i]$ and $A[j]$; $i \leftarrow i + 1$
    while $i < j$ and $A[i] < A[j]$ do $i \leftarrow i + 1$
    if $i = j$ then break
    swap $A[i]$ and $A[j]$; $j \leftarrow j - 1$
return $i$
```

# In-Place Implementation of Quick-Sort

quicksort$(A, \ell, r)$

```
if $\ell \geq r$ then return
$m \leftarrow$ patition$(A, \ell, r)$
quicksort$(A, \ell, m-1)$
quicksort$(A, m+1, r)$
```

- To sort an array $A$ of size $n$, call quicksort$(A, 1, n)$.

**Note:** We pass the array $A$ by reference, instead of by copying.

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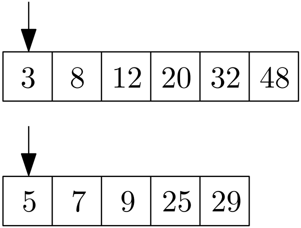
| | | | | | |
|---|---|---|---|---|---|
| 3 | 8 | 12 | 20 | 32 | 48 |

| | | | | |
|---|---|---|---|---|
| 5 | 7 | 9 | 25 | 29 |

# Merge-Sort is Not In-Place

- To merge two arrays, we need a third array with size equaling the total size of two arrays



| | | | | | |
|---|---|---|---|---|---|
| 3 | 8 | 12 | 20 | 32 | 48 |

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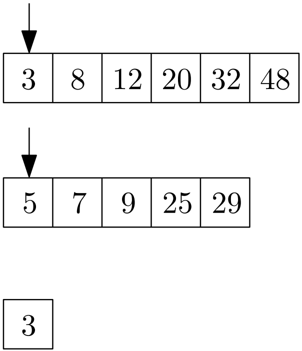

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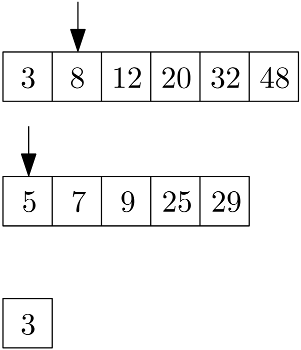

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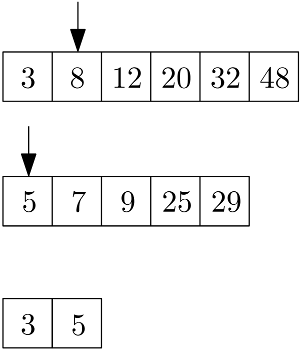

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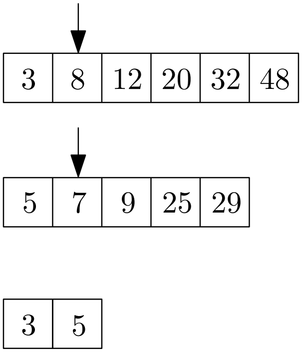

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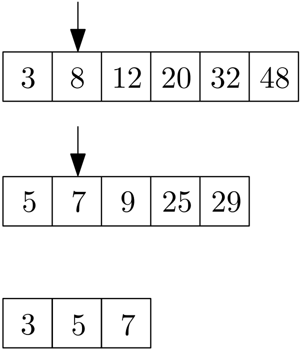

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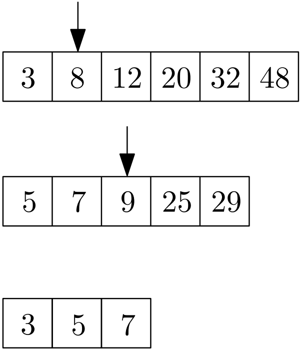

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|---|---|---|
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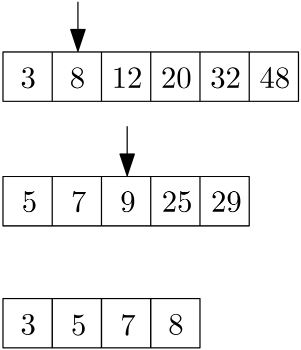

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| | | | | |
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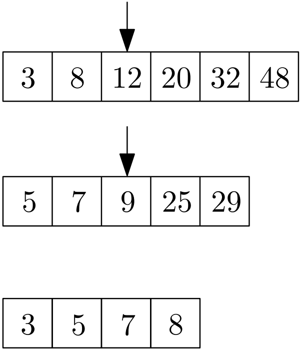

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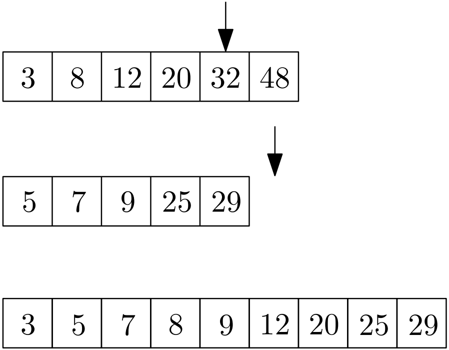

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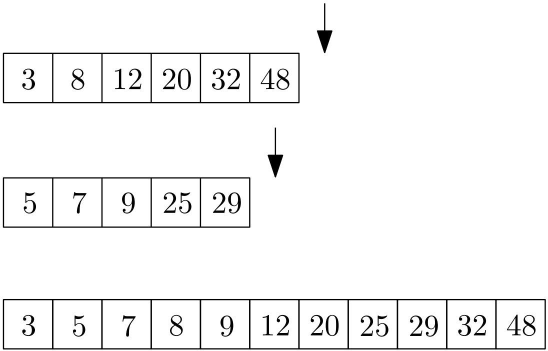

# Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
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## Comparison-Based Sorting Algorithms

- To sort, we are only allowed to compare two elements
- We can not use “internal structures” of the elements

**Lemma** The (worst-case) running time of any comparison-based sorting algorithm is $\Omega(n \lg n)$.

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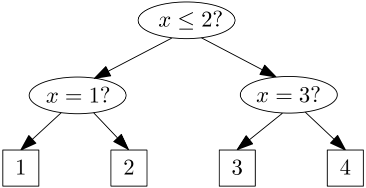

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# Comparison-Based Sorting Algorithms

**Q:** Can we do better than $O(n\log n)$ for sorting?

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- You can ask Bob questions of the form "does $i$ appear before $j$ in $\pi$?"

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### Selection Problem

**Input:** a set $A$ of $n$ numbers, and $1 \le i \le n$

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- Our goal: $O(n)$ running time

# Recall: Quicksort with Median Finder

**quicksort$(A, n)$**

```
if $n \leq 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ elements in $A$ that are less than $x$          ▷ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$       ▷ Divide
$B_L \leftarrow$ quicksort$(A_L, A_L$.size$)$                    ▷ Conquer
$B_R \leftarrow$ quicksort$(A_R, A_R$.size$)$                    ▷ Conquer
$t \leftarrow$ number of times $x$ appear $A$
return the array obtained by concatenating $B_L$, the array containing $t$ copies of $x$, and $B_R$
```

# Selection Algorithm with Median Finder

**selection$(A, n, i)$**

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if $n = 1$ then return $A$
$x \leftarrow$ lower median of $A$
$A_L \leftarrow$ elements in $A$ that are less than $x$                      ▷ Divide
$A_R \leftarrow$ elements in $A$ that are greater than $x$                   ▷ Divide
if $i \leq A_L$.size then
    return selection($A_L, A_L$.size$, i$)                                   ▷ Conquer
else if $i > n - A_R$.size then
    return selection($A_R, A_R$.size$, i - (n - A_R$.size$)$)               ▷ Conquer
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# Randomized Selection Algorithm

**selection$(A, n, i)$**

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if $n = 1$ thenreturn $A$
$x \leftarrow$ random element of $A$ (called pivot)
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- expected running time $= O(n)$

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**Input:** two polynomials of degree $n-1$

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\end{aligned}
$$

- **Input**: $(4,-5,2,3),(-5,6,-3,2)$
- **Output**: $(-20,49,-52,20,2,-5,6)$

# Naïve Algorithm

**polynomial-multiplication$(A, B, n)$**

```
let $C[k] \leftarrow 0$ for every $k = 0, 1, 2, \cdots, 2n-2$
for $i \leftarrow 0$ to $n-1$ do
    for $j \leftarrow 0$ to $n-1$ do
        $C[i+j] \leftarrow C[i+j] + A[i] \times B[j]$
return $C$
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Running time: $O(n^2)$

# Divide-and-Conquer for Polynomial Multiplication

$$p(x) = 3x^3 + 2x^2 - 5x + 4 = (3x + 2)x^2 + (-5x + 4)$$
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- $p(x)$: degree of $n - 1$ (assume $n$ is even)
- $p(x) = p_H(x)x^{n/2} + p_L(x)$,
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- $p_H q_L + p_L q_H = (p_H + p_L)(q_H + q_L) - p_H q_H - p_L q_L$

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$$\begin{aligned}\mathsf{multiply}(p, q) = {} & r_H \times x^n \\ & + \big(\mathsf{multiply}(p_H + p_L, q_H + q_L) - r_H - r_L\big) \times x^{n/2} \\ & + r_L\end{aligned}$$

# Divide-and-Conquer for Polynomial Multiplication

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- Solving Recurrence: $T(n) = 3T(n/2) + O(n)$
- $T(n) = O(n^{\lg_2 3}) = O(n^{1.585})$

**Assumption** $n$ is a power of $2$. Arrays are $0$-indexed.

### multiply($A, B, n$)

```
if $n = 1$ then return $(A[0]B[0])$
$A_L \leftarrow A[0 \,..\, n/2 - 1], A_H \leftarrow A[n/2 \,..\, n - 1]$
$B_L \leftarrow B[0 \,..\, n/2 - 1], B_H \leftarrow B[n/2 \,..\, n - 1]$
$C_L \leftarrow$ multiply$(A_L, B_L, n/2)$
$C_H \leftarrow$ multiply$(A_H, B_H, n/2)$
$C_M \leftarrow$ multiply$(A_L + A_H, B_L + B_H, n/2)$
$C \leftarrow$ array of $(2n - 1)$ 0's
for $i \leftarrow 0$ to $n - 2$ do
    $C[i] \leftarrow C[i] + C_L[i]$
    $C[i + n] \leftarrow C[i + n] + C_H[i]$
    $C[i + n/2] \leftarrow C[i + n/2] + C_M[i] - C_L[i] - C_H[i]$
return $C$
```

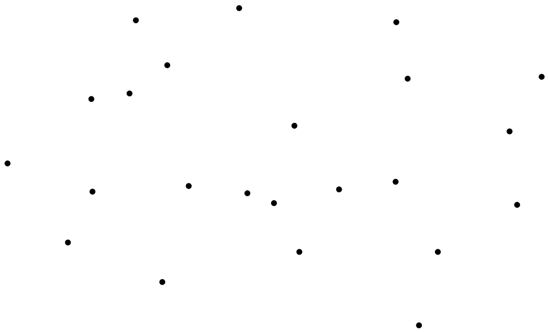
# Outline

- Closest pair
- Convex hull
- Matrix multiplication
- FFT(Fast Fourier Transform): polynomial multiplication in $O(n \lg n)$ time

# Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

**Output:** the pair of points that are closest



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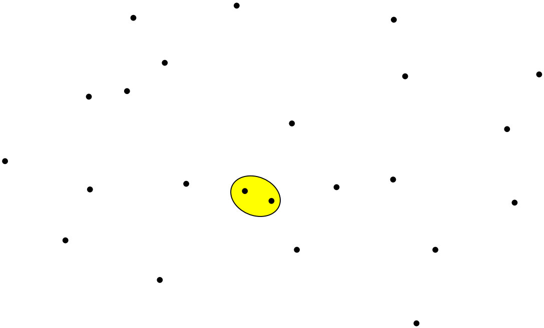

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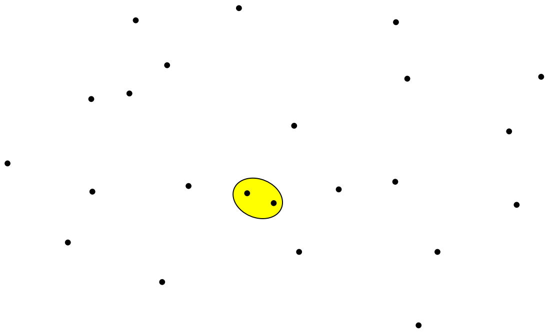
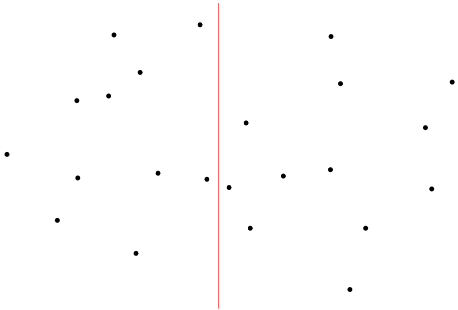


- Trivial algorithm: $O(n^2)$ running time

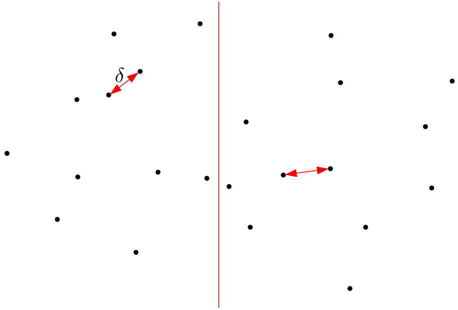
# Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line



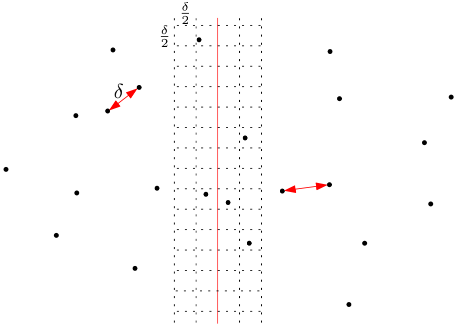
# Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively



# Divide-and-Conquer Algorithm for Closest Pair

- **Divide**: Divide the points into two halves via a vertical line
- **Conquer**: Solve two sub-instances recursively
- **Combine**: Check if there is a closer pair between left-half and right-half



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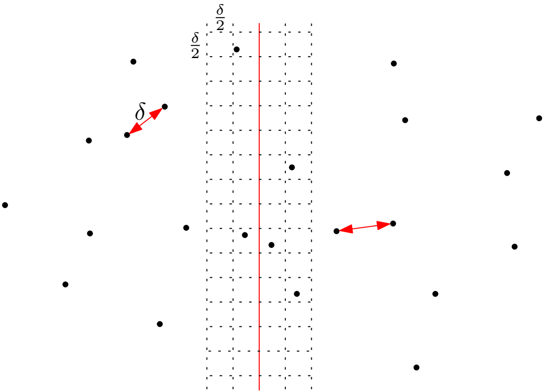

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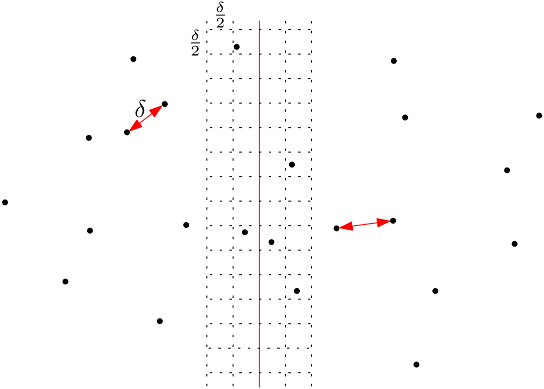
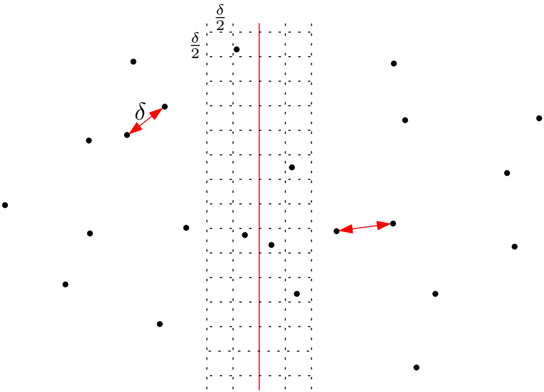


- Each box contains at most one pair

# Divide-and-Conquer Algorithm for Closest Pair



- Each box contains at most one pair
- For each point, only need to consider $O(1)$ boxes nearby

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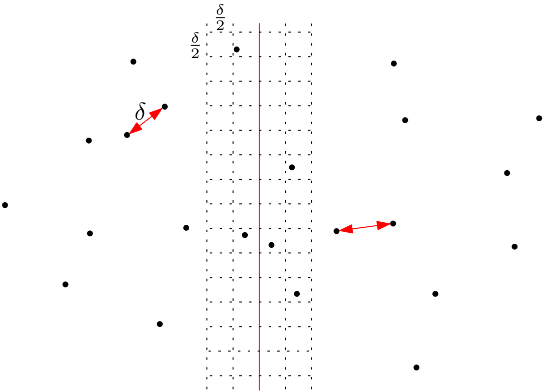

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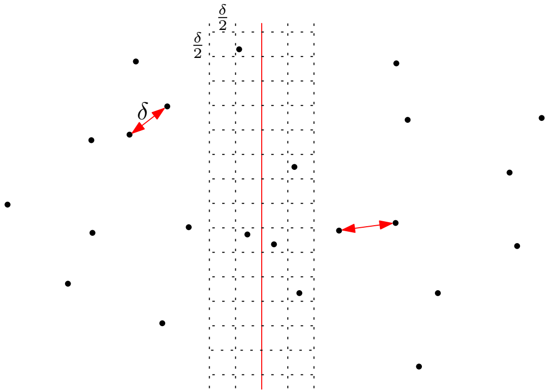

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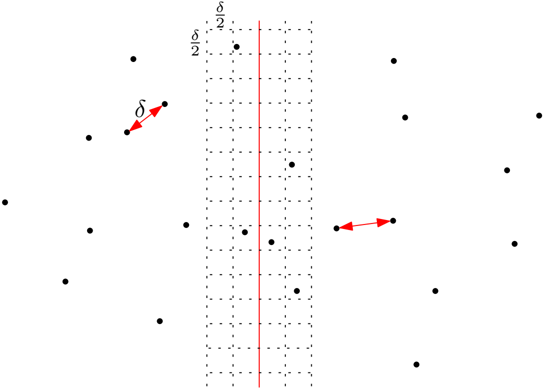

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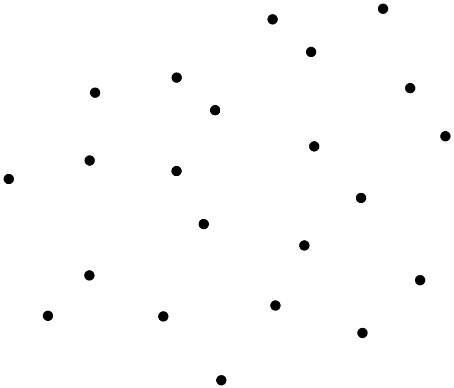
- Running time: $O(n \lg n)$

# $O(n \lg n)$-Time Algorithm for Convex Hull



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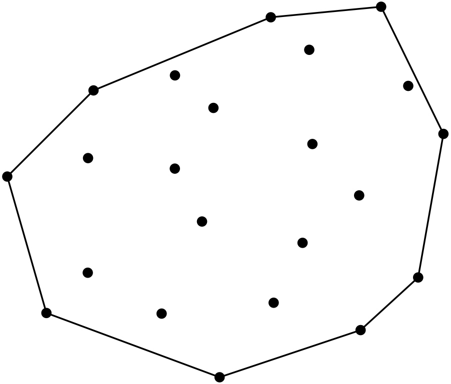

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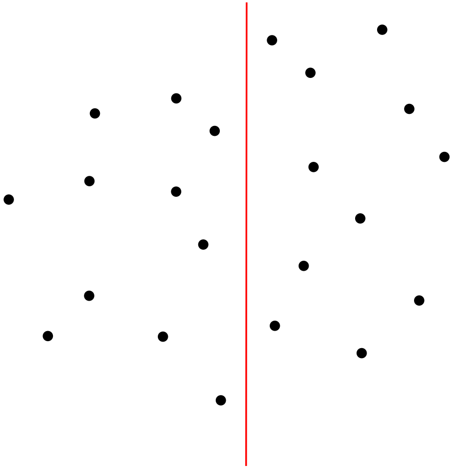

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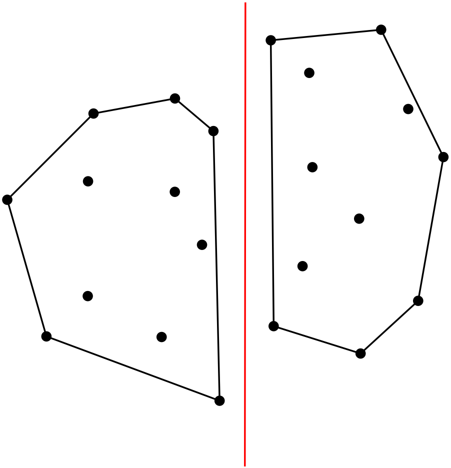

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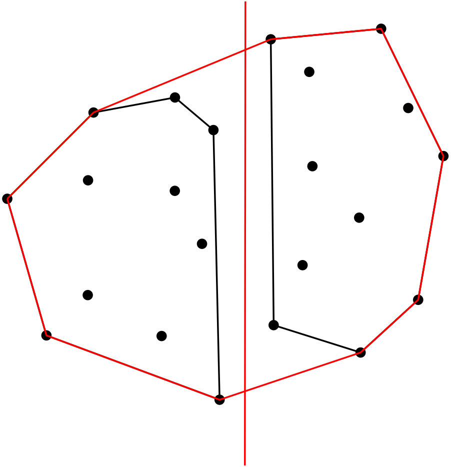

# Strassen's Algorithm for Matrix Multiplication

### Matrix Multiplication

**Input:** two $n \times n$ matrices $A$ and $B$

**Output:** $C = AB$

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## Naive Algorithm: matrix-multiplication$(A, B, n)$

```
for $i \leftarrow 1$ to $n$ do
    for $j \leftarrow 1$ to $n$ do
        $C[i, j] \leftarrow 0$
        for $k \leftarrow 1$ to $n$ do
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return $C$
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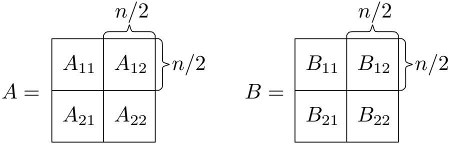
- running time $= O(n^3)$

# Try to Use Divide-and-Conquer



- $C = \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix}$
- matrix_multiplication$(A, B)$ recursively calls matrix_multiplication$(A_{11}, B_{11})$, matrix_multiplication$(A_{12}, B_{21})$, $\cdots$

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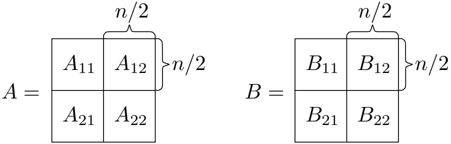

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- matrix_multiplication$(A, B)$ recursively calls matrix_multiplication$(A_{11}, B_{11})$, matrix_multiplication$(A_{12}, B_{21})$, $\cdots$
- Recurrence for running time: $T(n) = 8T(n/2) + O(n^2)$
- $T(n) = O(n^3)$

# Strassen's Algorithm

- $T(n) = 8T(n/2) + O(n^2)$
- Strassen's Algorithm: improve the number of multiplications from $8$ to 7!
- New recurrence: $T(n) = \color{red}{7}T(n/2) + O(n^2)$

# Strassen's Algorithm

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- Solving Recurrence $T(n) = O(n^{\log_2 7}) = O(n^{2.808})$

# Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
6. **Solving Recurrences**
7. Computing $n$-th Fibonacci Number

# Methods for Solving Recurrences

- The recursion-tree method
- The master theorem

# Recursion-Tree Method

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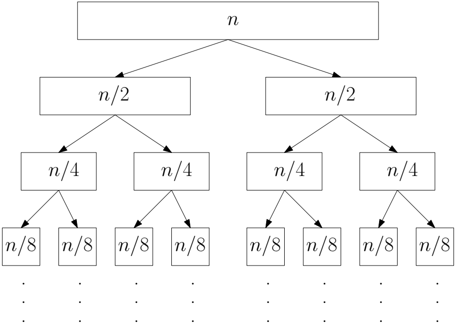

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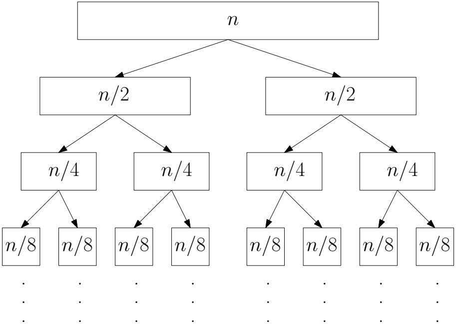

- Each level takes running time $O(n)$

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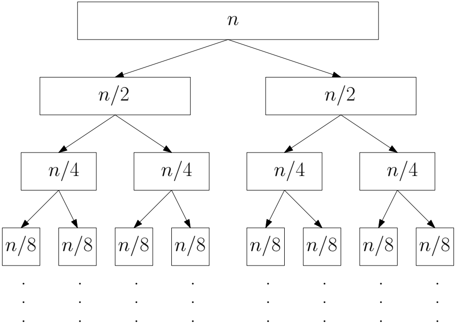

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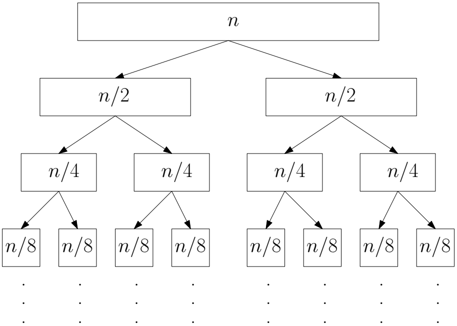

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- Running time $= O(n \lg n)$

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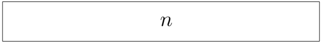

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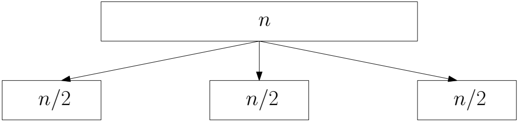

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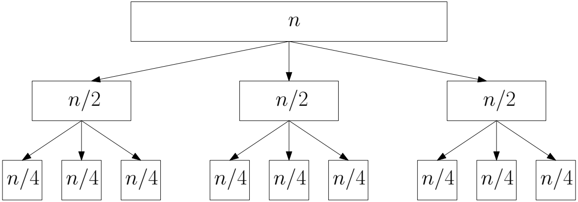

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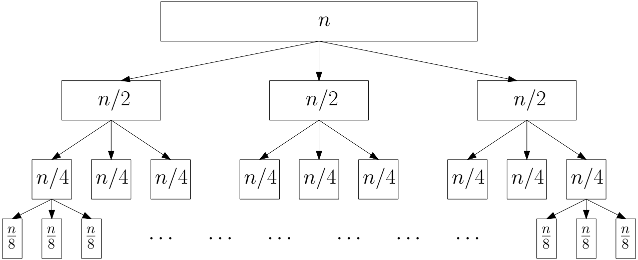

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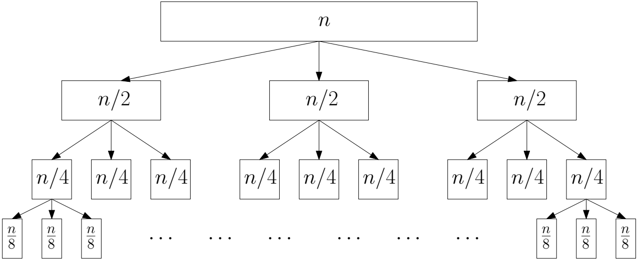

- Total running time at level $i$?

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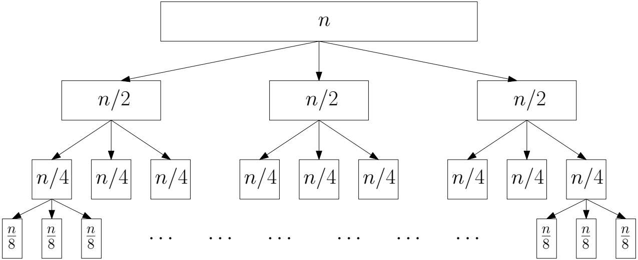

- Total running time at level $i$? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$

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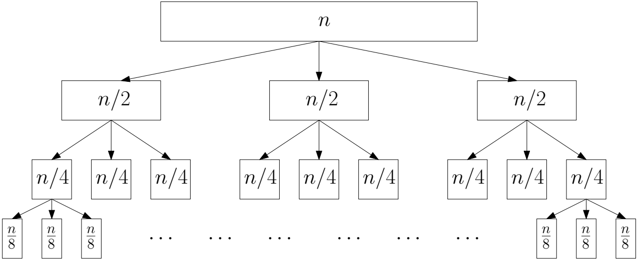

- Total running time at level $i$? $\frac{n}{2^i} \times 3^i = \left(\frac{3}{2}\right)^i n$
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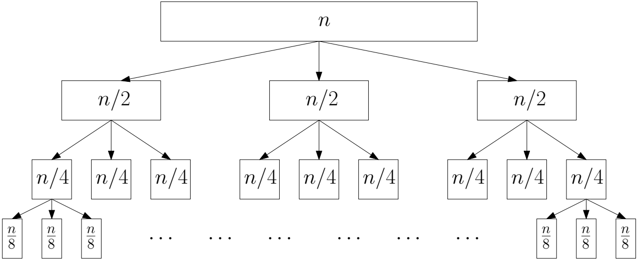

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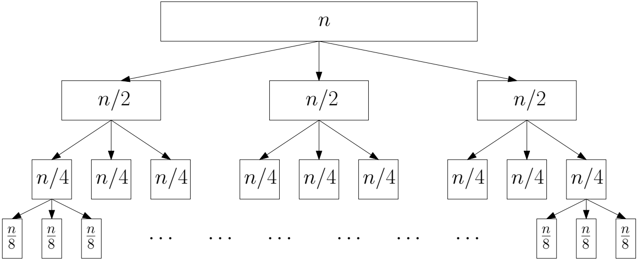

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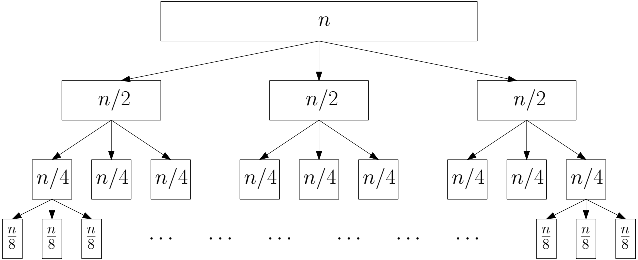

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- Index of last level? $\lg_2 n$
- Total running time?

$$\sum_{i=0}^{\lg_2 n} \left(\frac{3}{2}\right)^i n = O\left(n \left(\frac{3}{2}\right)^{\lg_2 n}\right) = O(3^{\lg_2 n}) = O(n^{\lg_2 3}).$$

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$$\boxed{n^2}$$

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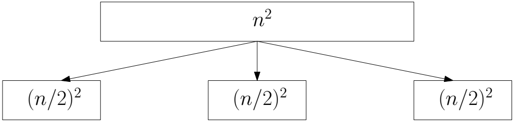

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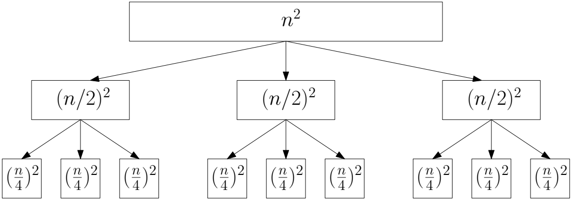

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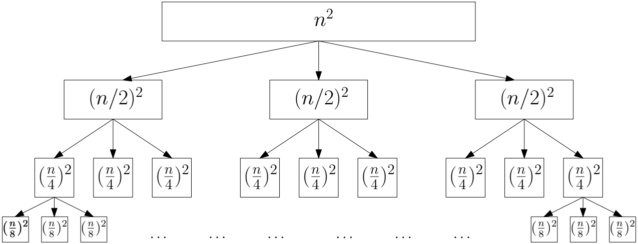

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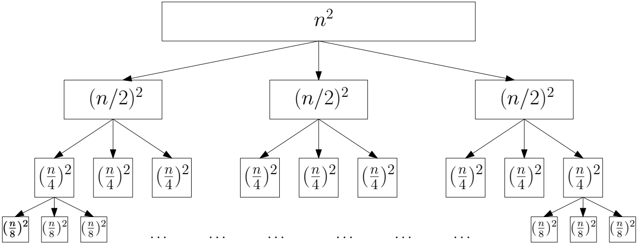

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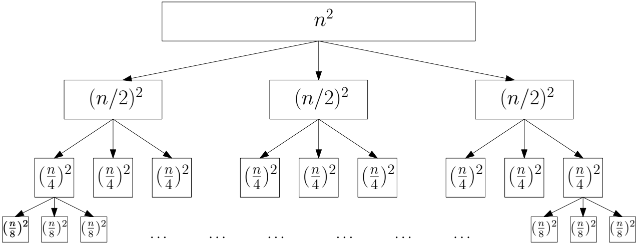

- Total running time at level $i$? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$

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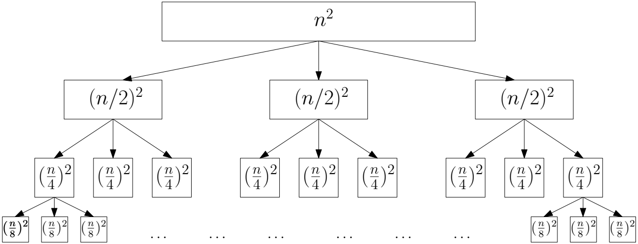

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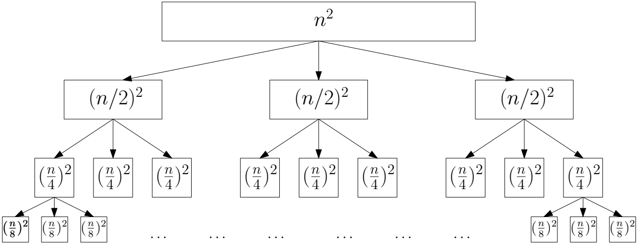
# Recursion-Tree Method

- $T(n) = 3T(n/2) + O(n^2)$



- Total running time at level $i$? $\left(\frac{n}{2^i}\right)^2 \times 3^i = \left(\frac{3}{4}\right)^i n^2$
- Index of last level? $\lg_2 n$

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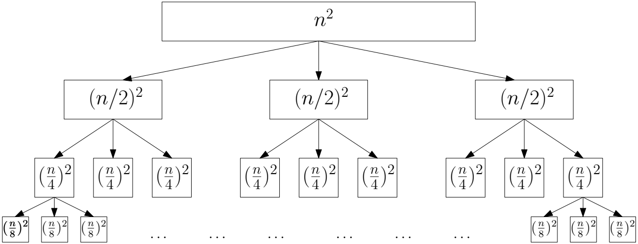

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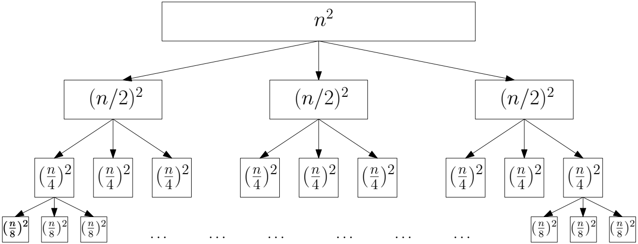

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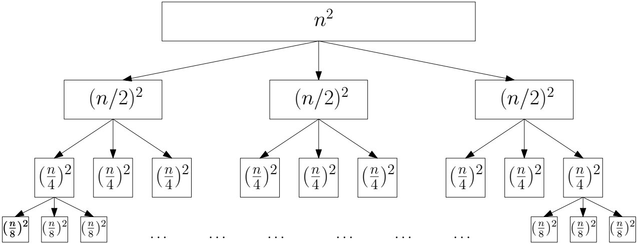

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# Master Theorem

| Recurrences | $a$ | $b$ | $c$ | time |
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| $T(n) = 2T(n/2) + O(n)$ | | | | $O(n \lg n)$ |
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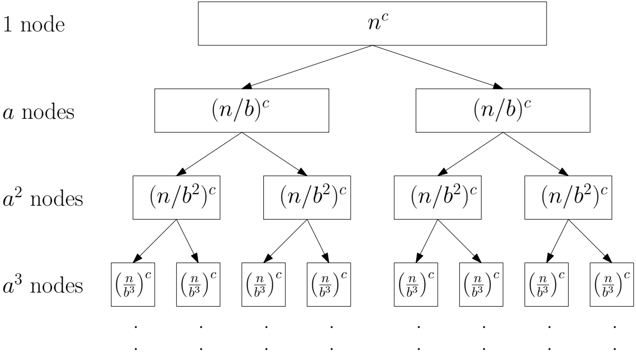
# Proof of Master Theorem Using Recursion Tree

$$T(n) = aT(n/b) + O(n^c)$$



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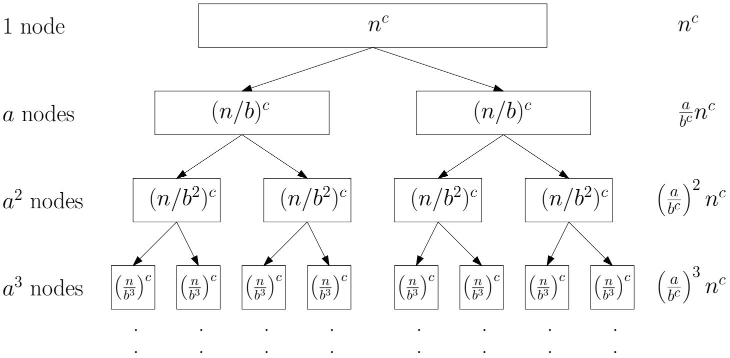

1 node $\quad n^c \quad n^c$

$a$ nodes $\quad (n/b)^c \quad (n/b)^c \quad \frac{a}{b^c} n^c$

$a^2$ nodes $\quad (n/b^2)^c \quad (n/b^2)^c \quad (n/b^2)^c \quad (n/b^2)^c \quad \left(\frac{a}{b^c}\right)^2 n^c$

$a^3$ nodes $\quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{n}{b^3}\right)^c \quad \left(\frac{a}{b^c}\right)^3 n^c$

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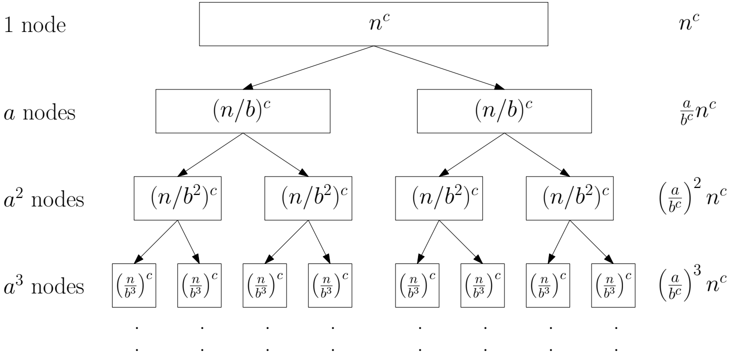

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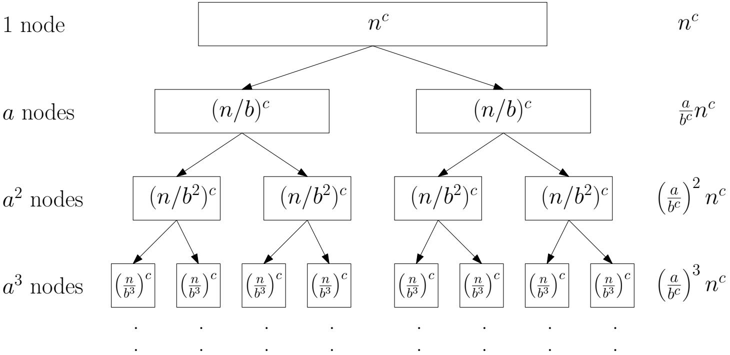

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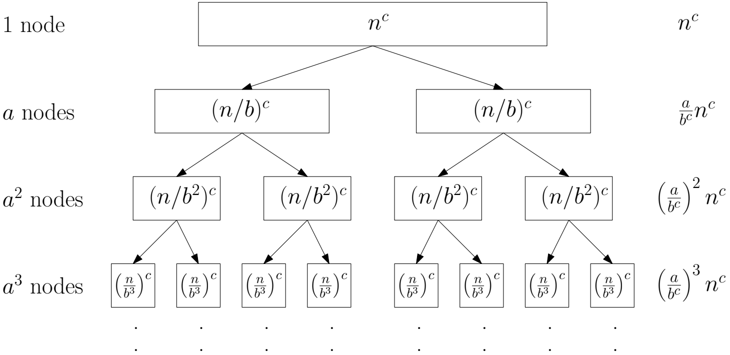

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- $c = \lg_b a$ : all levels have same time: $n^c \lg_b n = O(n^c \lg n)$
- $c > \lg_b a$ : top-level dominates: $O(n^c)$

# Outline

1. Divide-and-Conquer
2. Counting Inversions
3. Quicksort and Selection
    - Quicksort
    - Lower Bound for Comparison-Based Sorting Algorithms
    - Selection Problem
4. Polynomial Multiplication
5. Other Classic Algorithms using Divide-and-Conquer
6. Solving Recurrences
7. Computing $n$-th Fibonacci Number

# Fibonacci Numbers

- $F_0 = 0, F_1 = 1$
- $F_n = F_{n-1} + F_{n-2}, \forall n \geq 2$
- Fibonacci sequence: $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$

### $n$-th Fibonacci Number

**Input:** integer $n > 0$

**Output:** $F_n$

# Computing $F_n$ : Stupid Divide-and-Conquer Algorithm

**Fib($n$)**

```
if $n = 0$ return 0
if $n = 1$ return 1
return Fib($n-1$) + Fib($n-2$)
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**Q:** Is the running time of the algorithm polynomial or exponential in $n$?

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**A:** Exponential

- Running time is at least $\Omega(F_n)$

# Computing $F_n$ : Stupid Divide-and-Conquer Algorithm

**Fib($n$)**

```
if $n = 0$ return 0
if $n = 1$ return 1
return Fib($n - 1$) + Fib($n - 2$)
```

**Q:** Is the running time of the algorithm polynomial or exponential in $n$?

**A:** Exponential

- Running time is at least $\Omega(F_n)$
- $F_n$ is exponential in $n$

# Computing $F_n$: Reasonable Algorithm

**Fib($n$)**

```
F[0] ← 0
F[1] ← 1
for i ← 2 to n do
    F[i] ← F[i − 1] + F[i − 2]
return F[n]
```

- Dynamic Programming

# Computing $F_n$: Reasonable Algorithm

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for $i \leftarrow 2$ to $n$ do
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- Dynamic Programming
- Running time = ?

# Computing $F_n$: Reasonable Algorithm

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- Dynamic Programming
- Running time $= O(n)$

# Computing $F_n$: Even Better Algorithm

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\cdots$$

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### power$(n)$

```
if $n = 0$ then return $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
$R \leftarrow$ power$(\lfloor n/2 \rfloor)$
$R \leftarrow R \times R$
if $n$ is odd then $R \leftarrow R \times \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$
**return** $R$
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## Fixing the Problem

To compute $F_n$, we need $O(\lg n)$ basic arithmetic operations on integers

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- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
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- **Divide**: Divide instance into many smaller instances
- **Conquer**: Solve each of smaller instances recursively and separately
- **Combine**: Combine solutions to small instances to obtain a solution for the original big instance
- Write down recurrence for running time
- Solve recurrence using master theorem

# Summary: Divide-and-Conquer

- Merge sort, quicksort, count-inversions, closest pair, $\cdots$:
  $T(n) = 2T(n/2) + O(n) \Rightarrow T(n) = O(n \lg n)$

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- To improve running time, design better algorithm for "combine" step, or reduce number of recursions, ...