CSE 431/531: Algorithm Analysis and Design (Fall 2022)
Graph Algorithms

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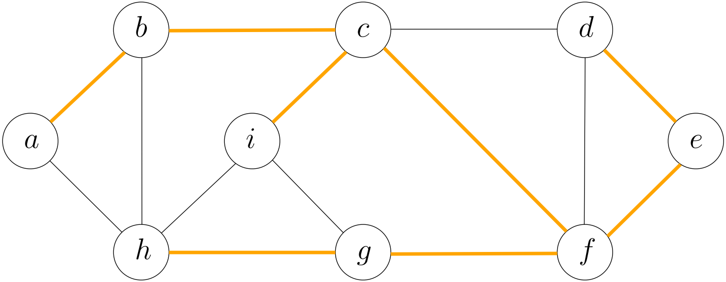
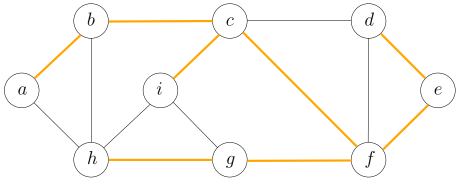
Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
4. All-Pair Shortest Paths and Floyd-Warshall

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





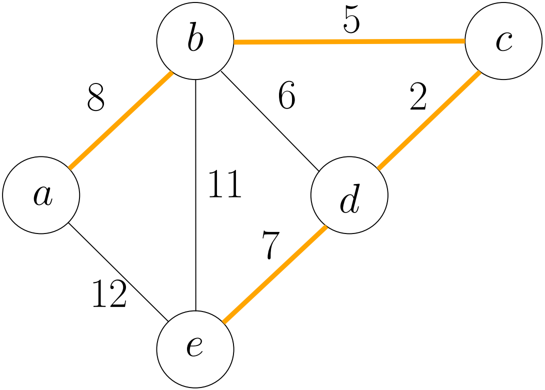
Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight



Recall: Steps of Designing A Greedy Algorithm

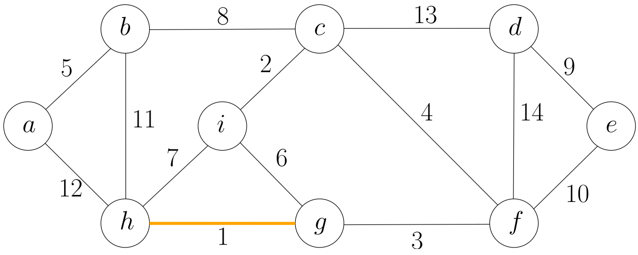
- Design a "reasonable" strategy
- Prove that the reasonable strategy is "safe" (key, usually done by "exchanging argument")
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is "safe" if there is an optimum solution that is "consistent" with the choice

Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

Outline



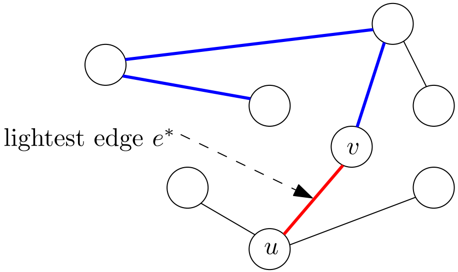
Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

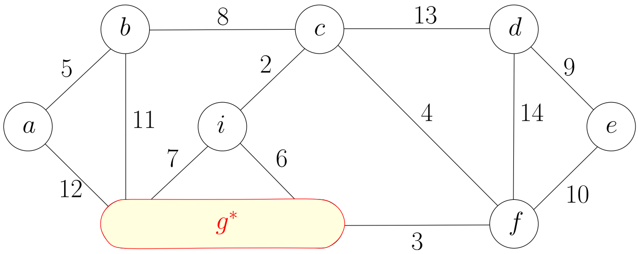
Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T
- There is a unique path in T connecting u and v
- Remove any edge e in the path to obtain tree T'
- $w(e^*) \le w(e) \implies w(T') \le w(T)$: T' is also a MST □

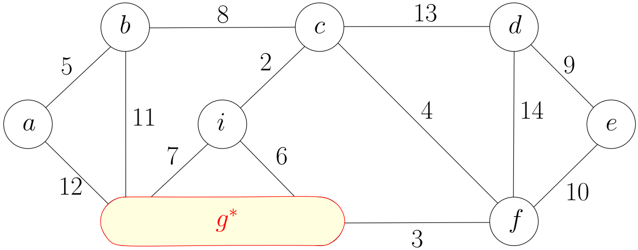


Is the Residual Problem Still a MST Problem?



- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

Q: What edges are removed due to contractions?

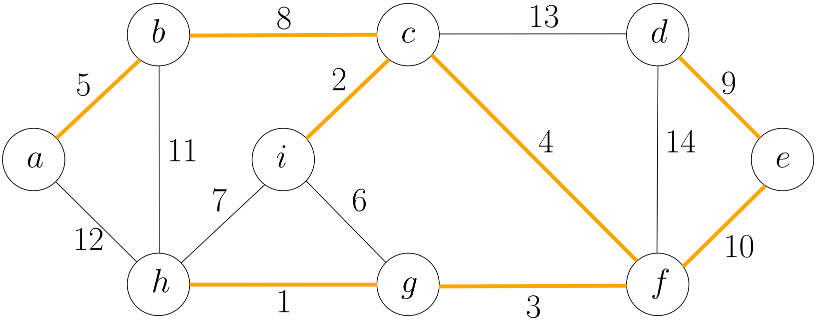
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

```
F ← ∅
sort edges in E in non-decreasing order of weights w
for each edge (u, v) in the order do
    if u and v are not connected by a path of edges in F then
        F ← F ∪ {(u, v)}
return (V, F)
```

Kruskal's Algorithm: Example



Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

$F \leftarrow \emptyset$
$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
sort the edges of E in non-decreasing order of weights w
for each edge $(u, v) \in E$ in the order **do**
    $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
    $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
    if $S_u \neq S_v$ **then**
        $F \leftarrow F \cup \{(u, v)\}$
        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
return (V, F)

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

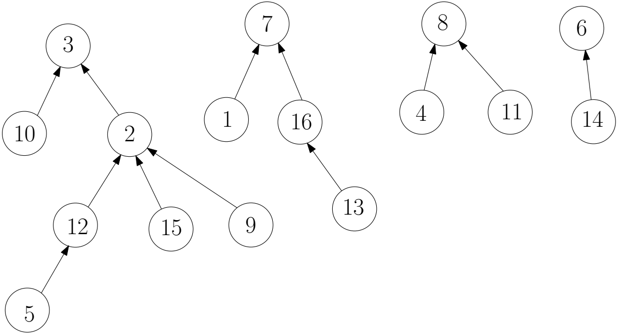
1: $F \leftarrow \emptyset$
2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of E in non-decreasing order of weights w
4: **for** each edge $(u, v) \in E$ in the order **do**
5:     $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6:     $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7:     **if** $S_u \neq S_v$ **then**
8:         $F \leftarrow F \cup \{(u, v)\}$
9:         $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: **return** (V, F)

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

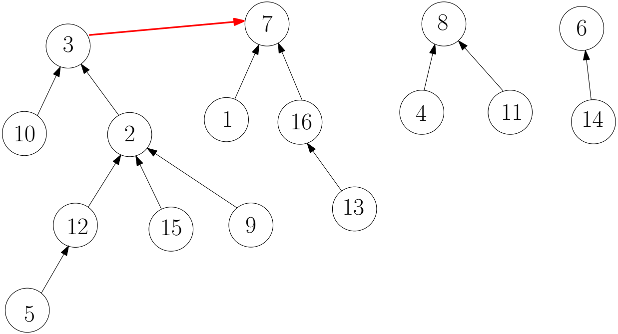
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$



- $par[i]$: parent of i, ($par[i] = \perp$ if i is a root).

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if $\text{root}(u) = \text{root}(v)$.
- $\text{root}(u)$: the root of the tree containing u
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure

root(v)

```
if par[v] = ⊥ then
    return v
else
    return root(par[v])
```

root(v)

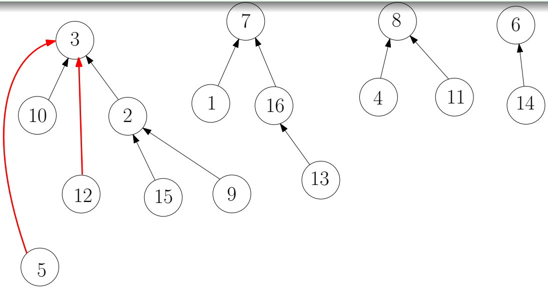
```
if par[v] = ⊥ then
    return v
else
    par[v] ← root(par[v])
return par[v]
```

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.

Union-Find Data Structure

root(v)

if $par[v] = \bot$ **then**
  return v
else
  $par[v] \leftarrow \text{root}(par[v])$
  return $par[v]$



MST-Kruskal(G, w)

$F \leftarrow \emptyset$
$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
sort the edges of E in non-decreasing order of weights w
for each edge $(u, v) \in E$ in the order **do**
    $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
    $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
    if $S_u \neq S_v$ **then**
        $F \leftarrow F \cup \{(u, v)\}$
        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
return (V, F)

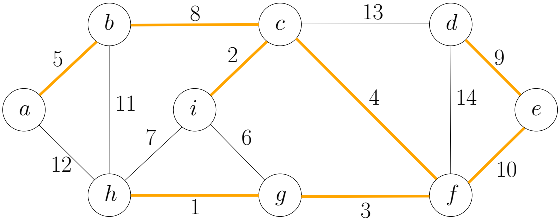
MST-Kruskal(G, w)

1: $F \leftarrow \emptyset$
2: **for** every $v \in V$ **do**: $par[v] \leftarrow \perp$
3: sort the edges of E in non-decreasing order of weights w
4: **for** each edge $(u, v) \in E$ in the order **do**
5:     $u' \leftarrow \text{root}(u)$
6:     $v' \leftarrow \text{root}(v)$
7:     **if** $u' \neq v'$ **then**
8:         $F \leftarrow F \cup \{(u, v)\}$
9:         $par[u'] \leftarrow v'$
10: **return** (V, F)

- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $=$ time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

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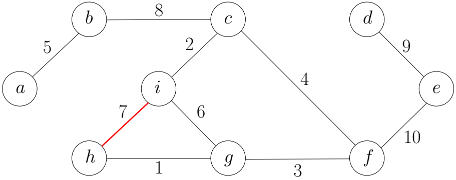
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree
2. Start from $F \leftarrow E$, and remove edges from F one by one until we obtain a spanning tree



Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

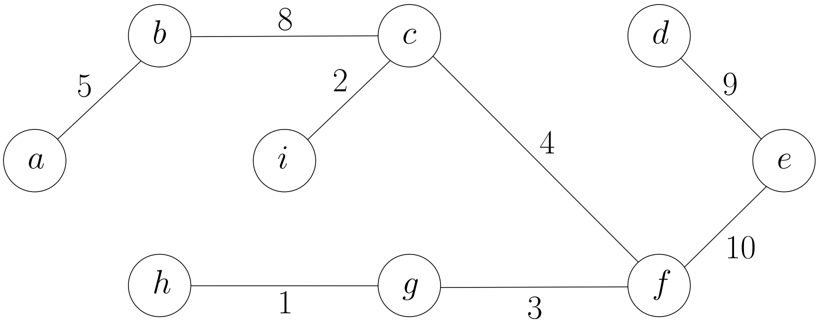
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

$F \leftarrow E$
sort E in non-increasing order of weights
for every e in this order **do**
    if $(V, F \setminus \{e\})$ is connected **then**
        $F \leftarrow F \setminus \{e\}$
return (V, F)

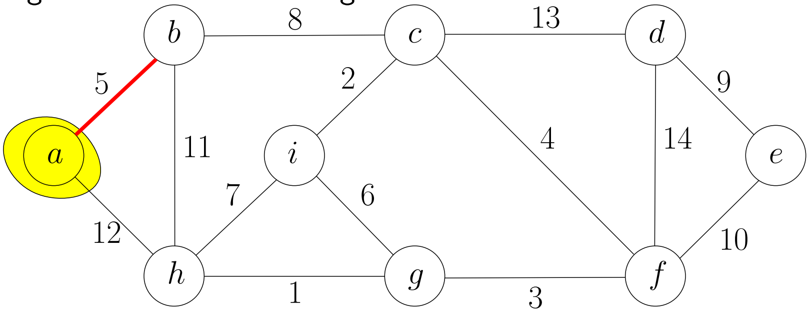
Reverse Kruskal's Algorithm: Example



Outline

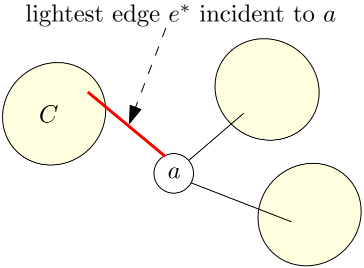
Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



- Greedy strategy for Prim's algorithm: choose the lightest edge incident to a.

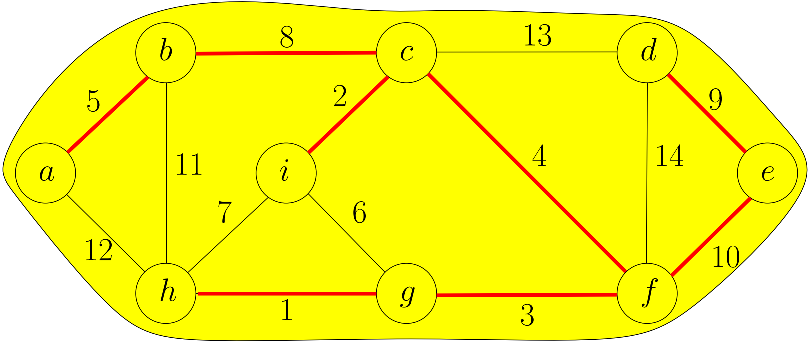
Lemma It is safe to include the lightest edge incident to a.



Proof.

- Let T be a MST
- Consider all components obtained by removing a from T
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$ $\square$

Prim's Algorithm: Example



Greedy Algorithm

MST-Greedy1(G, w)

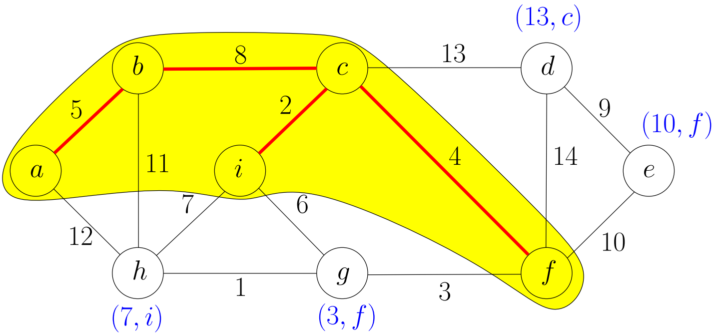
$S \leftarrow \{s\}$, where s is arbitrary vertex in V
$F \leftarrow \emptyset$
while $S \neq V$ **do**
  $(u, v) \leftarrow$ lightest edge between S and $V \setminus S$,
   where $u \in S$ and $v \in V \setminus S$
  $S \leftarrow S \cup \{v\}$
  $F \leftarrow F \cup \{(u, v)\}$
return (V, F)

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi[v], v)$ is the lightest edge between v and S



Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi[v], v)$ is the lightest edge between v and S

In every iteration

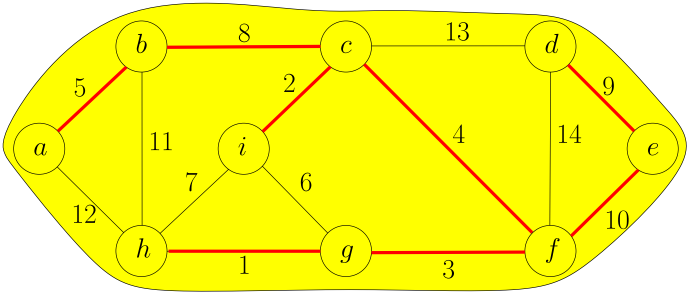
- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

```
s ← arbitrary vertex in G
S ← ∅, d(s) ← 0 and d[v] ← ∞ for every v ∈ V \ {s}
while S ≠ V do
    u ← vertex in V \ S with the minimum d[u]
    S ← S ∪ {u}
    for each v ∈ V \ S such that (u, v) ∈ E do
        if w(u, v) < d[v] then
            d[v] ← w(u, v)
            π[v] ← u
return {(u, π[u]) | u ∈ V \ {s}}
```

Example



Prim's Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi[v], v)$ is the lightest edge between v and S

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value — extract_min
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values. — decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$

while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$
            $\pi[v] \leftarrow u$
return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

1. $s \leftarrow$ arbitrary vertex in G
2. $S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3. $Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
4. **while** $S \neq V$ **do**
5. $u \leftarrow Q$.extract_min()
6. $S \leftarrow S \cup \{u\}$
7.   **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
8.   **if** $w(u, v) < d[v]$ **then**
9. $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$
10. $\pi[v] \leftarrow u$
11. **return** $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$

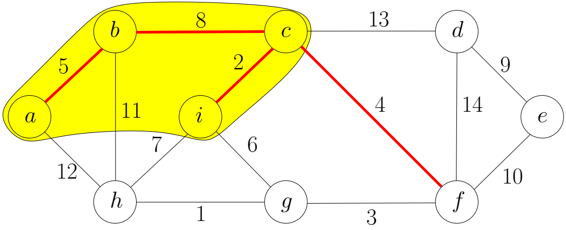
Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+ O(m)\times$ (time for decrease_key)

concrete DS	extract_min	decrease_key	overall time
heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Assumption Assume all edge weights are different.

Lemma (u, v) is in MST, if and only if there exists a cut $(U, V \setminus U)$, such that (u, v) is the lightest edge between U and $V \setminus U$.



- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

“Evidence” for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
- $e \notin$ MST $\leftrightarrow$ there is a cycle in which e is the heaviest edge

Exactly one of the following is true:

- There is a cut in which e is the lightest edge
- There is a cycle in which e is the heaviest edge

Thus, the minimum spanning tree is unique with assumption.

Outline

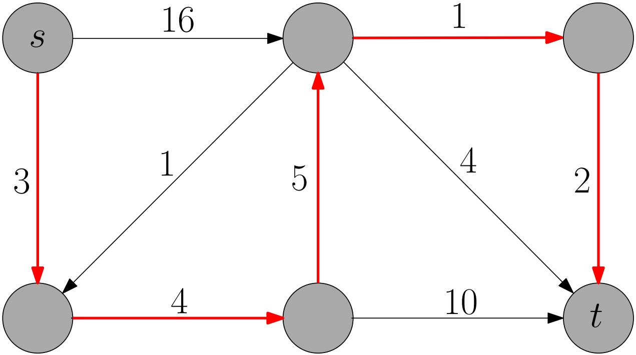
algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG $=$ directed acyclic graph U $=$ undirected D $=$ directed
- SS $=$ single source AP $=$ all pairs

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$
$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest path from s to t



Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: shortest paths from s to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve s-t shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight

Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

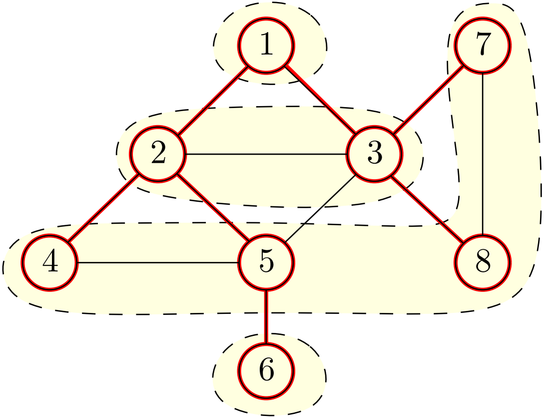
$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi[v], v \in V \setminus s$: the parent of v in shortest path tree

$d[v], v \in V \setminus s$: the length of shortest path from s to v

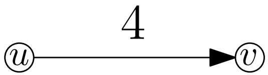
Q: How to compute shortest paths from s when all edges have weight 1?

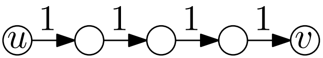
A: Breadth first search (BFS) from source s



Assumption Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a pah of $w(u, v)$ unit-weight edges





Shortest Path Algorithm by Running BFS

1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2: run BFS virtually
3: $\pi[v] \leftarrow$ vertex from which v is visited
4: $d[v] \leftarrow$ index of the level containing v

- Problem: $w(u, v)$ may be too large!

Shortest Path Algorithm by Running BFS Virtually

$S \leftarrow \{s\}, d(s) \leftarrow 0$

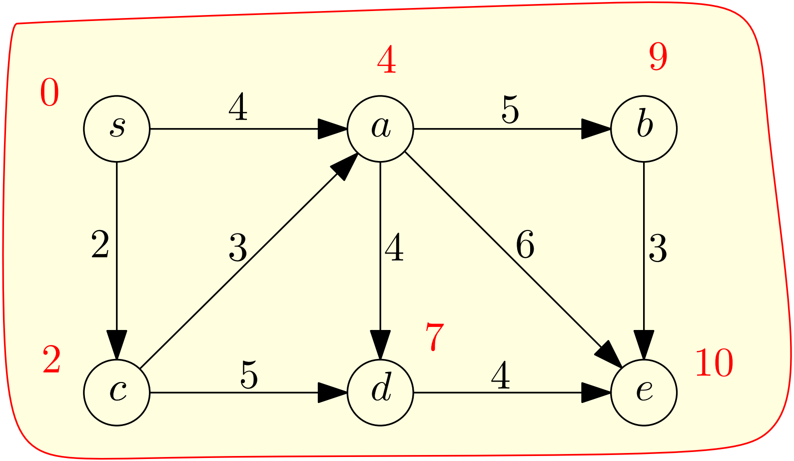
while $|S| \leq n$ **do**

  find a $v \notin S$ that minimizes $\min\limits_{u \in S:(u,v) \in E} \{d[u] + w(u,v)\}$

  $S \leftarrow S \cup \{v\}$

  $d[v] \leftarrow \min_{u \in S:(u,v) \in E} \{d[u] + w(u,v)\}$

Virtual BFS: Example



Time 10

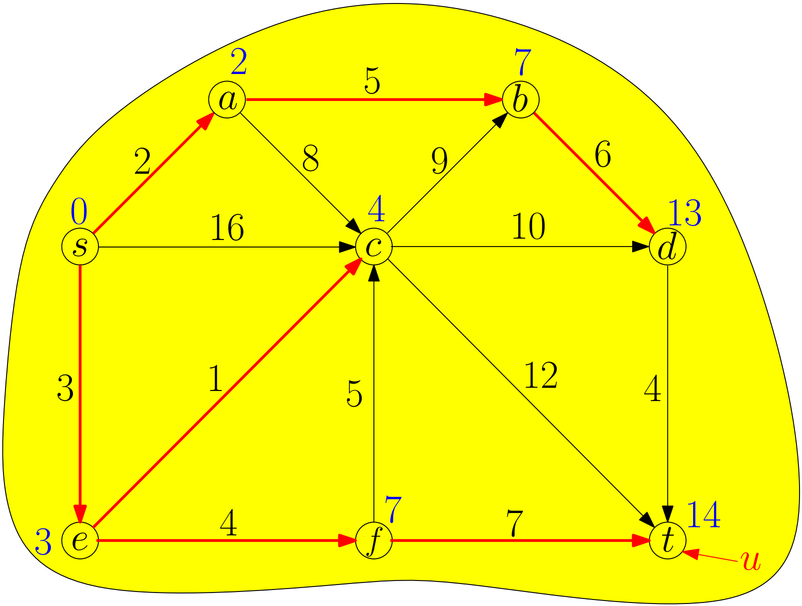
Outline

Dijkstra's Algorithm

Dijkstra(G, w, s)

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    add u to S
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
return (d, π)

- Running time $= O(n^2)$


2
7
a
5
b
2
8
9
6
0
4
13
s
16
c
10
d
3
1
5
12
4
7
3
e
4
f
7
t
14
u

Improved Running Time using Priority Queue

Dijkstra(G, w, s)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$, Q.decrease_key$(v, d[v])$
            $\pi[v] \leftarrow u$
return (π, d)

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$
            $\pi[v] \leftarrow u$
return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Improved Running Time

Running time:
$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

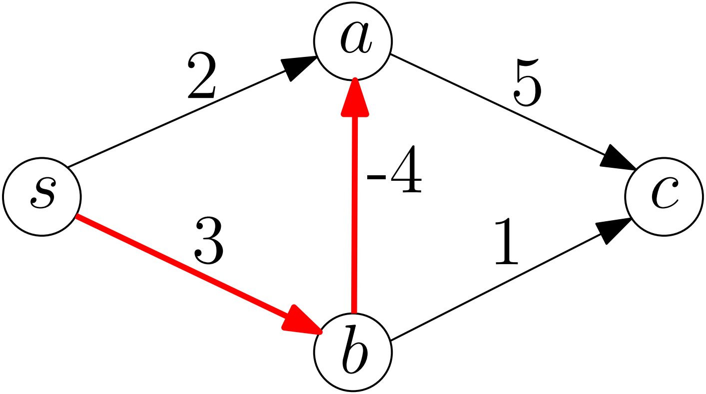
assume all vertices are reachable from s

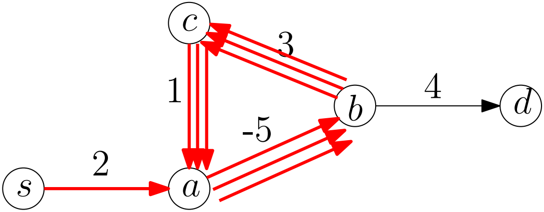
$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

Dijkstra's Algorithm Fails if We Have Negative Weights





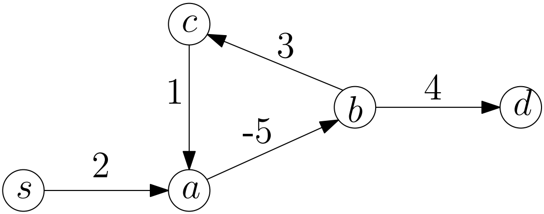
Q: What is the length of the shortest path from s to d?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Dealing with Negative Cycles

- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"



Q: What is the length of the shortest simple path from s to d?

A: 1

- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG $=$ directed acyclic graph   U $=$ undirected   D $=$ directed
- SS $=$ single source   AP $=$ all pairs

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

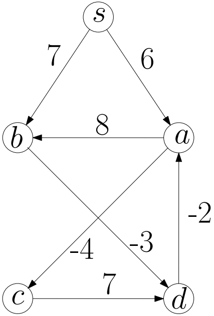
Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

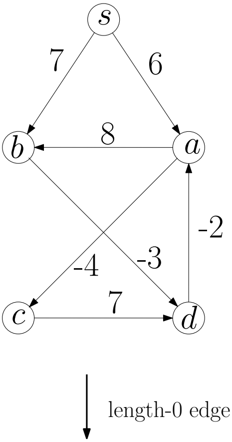
- first try: $f[v]$: length of shortest path from s to v
- issue: do not know in which order we compute $f[v]$'s
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges

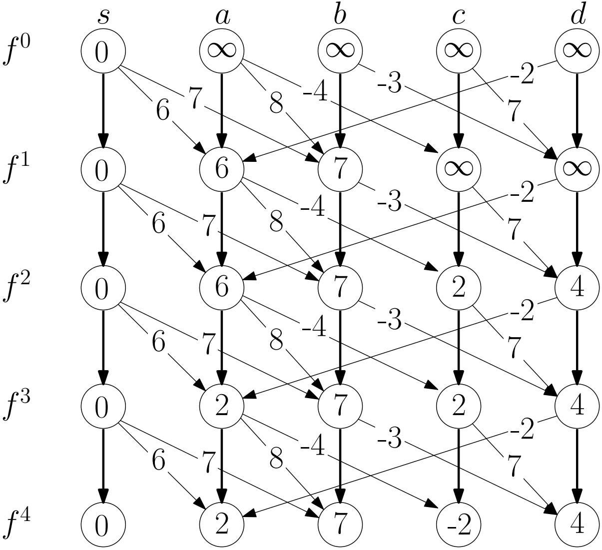


- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from s to v that uses at most ℓ edges
- $f^2[a] = 6$
- $f^3[a] = 2$

$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ \infty & \ell = 0, v \neq s \\ \min \begin{cases} f^{\ell-1}[v] \\ \min_{u:(u,v)\in E} \left(f^{\ell-1}[u] + w(u,v) \right) \end{cases} & \ell > 0 \end{cases}$$

Dynamic Programming: Example





dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \rightarrow f^{\ell}$
    for each $(u, v) \in E$ **do**
        if $f^{\ell-1}[u] + w(u, v) < f^{\ell}[v]$ **then**
            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Proof.

If there is a path containing at least n edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. $\square$

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n - 1$ **do**
  for each $(u, v) \in E$ **do**
    if $f[u] + w(u, v) < f[v]$ **then**
      $f[v] \leftarrow f[u] + w(u, v)$
return f

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!
- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v

Bellman-Ford Algorithm

- After iteration ℓ:

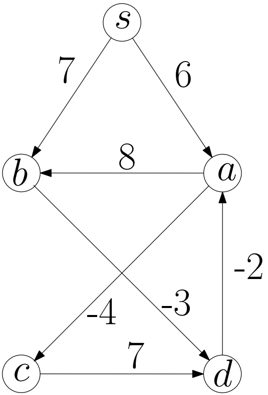
$$\begin{aligned} &\text{length of shortest } s\text{-}v \text{ path} \\ \leq\ & f[v] \\ \leq\ & \text{length of shortest } s\text{-}v \text{ path using at most } \ell \text{ edges} \end{aligned}$$

- Assuming there are no negative cycles:

$$\begin{aligned} &\text{length of shortest } s\text{-}v \text{ path} \\ =\ & \text{length of shortest } s\text{-}v \text{ path using at most } n-1 \text{ edges} \end{aligned}$$

- So, assuming there are no negative cycles, after iteration $n-1$:

$$f[v] = \text{length of shortest } s\text{-}v \text{ path}$$



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	∞62	∞7	∞2-2	∞4

- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4
- Algorithm terminates in 3 iterations, instead of 4.

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to n **do**
    $updated \leftarrow$ false
    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
            $updated \leftarrow$ true
    if not $updated$, then return f
output "negative cycle exists"

- $\pi[v]$: the parent of v in the shortest path tree
- Running time $= O(nm)$

Outline

All-Pair Shortest Paths

All Pair Shortest Paths

Input: directed graph $G = (V, E)$,
$w : E \to \mathbb{R}$ (can be negative)

Output: shortest path from u to v for every $u, v \in V$

```
for every starting point s ∈ V do
    run Bellman-Ford(G, w, s)
```

- Running time $= O(n^2 m)$

Summary of Shortest Path Algorithms we learned

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$
- For simplicity, extend the w values to non-edges:

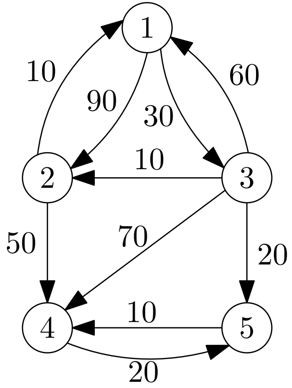
$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

- For now assume there are no negative cycles

Cells for Floyd-Warshall Algorithm

- First try: $f[i,j]$ is length of shortest path from i to j
- Issue: do not know in which order we compute $f[i,j]$'s
- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

Example for Definition of $f^k[i,j]$'s



$$f^0[1,4] = \infty$$
$$f^1[1,4] = \infty$$
$$f^2[1,4] = 140 \qquad (1 \to 2 \to 4)$$
$$f^3[1,4] = 90 \qquad (1 \to 3 \to 2 \to 4)$$
$$f^4[1,4] = 90 \qquad (1 \to 3 \to 2 \to 4)$$
$$f^5[1,4] = 60 \qquad (1 \to 3 \to 5 \to 4)$$

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

$$f^k[i,j] = \begin{cases} w(i,j) & k = 0 \\ \min \begin{cases} f^{k-1}[i,j] \\ f^{k-1}[i,k] + f^{k-1}[k,j] \end{cases} & k = 1, 2, \cdots, n \end{cases}$$

Floyd-Warshall(G, w)

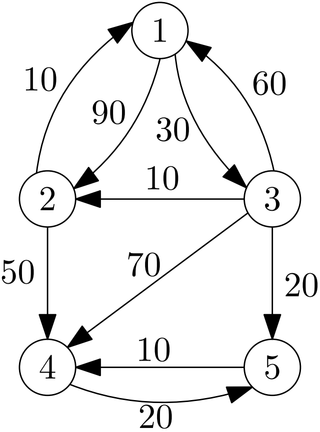
$f^0 \leftarrow w$
for $k \leftarrow 1$ to n **do**
    copy $f^{k-1} \rightarrow f^k$
    for $i \leftarrow 1$ to n **do**
        for $j \leftarrow 1$ to n **do**
            if $f^{k-1}[i,k] + f^{k-1}[k,j] < f^k[i,j]$ **then**
                $f^k[i,j] \leftarrow f^{k-1}[i,k] + f^{k-1}[k,j]$

Floyd-Warshall(G, w)

```
f^old ← w
for k ← 1 to n do
    copy f^old → f^new
    for i ← 1 to n do
        for j ← 1 to n do
            if f^old[i,k] + f^old[k,j] < f^new[i,j] then
                f^new[i,j] ← f^old[i,k] + f^old[k,j]
```

Lemma Assume there are no negative cycles in G. After iteration k, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from i to j that only uses vertices in $\{1, 2, 3, \cdots, k\}$ as intermediate vertices.

- Running time $= O(n^3)$.



	1	2	3	4	5
1	0	9040	30	∞140	∞
2	10	0	∞40	50	∞
3	6020	10	0	7060	20
4	∞	∞	∞	0	20
5	∞	∞	∞	10	0

- $i = 1$, $i = 2$, $i = 3$, $k = 1$, $k = 2$, $k = 3$, $j = 1$, $j = 2j = 3j = 4$

Recovering Shortest Paths

Floyd-Warshall(G, w)

$f \leftarrow w$, $\pi[i,j] \leftarrow \perp$ for every $i, j \in V$
for $k \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to n **do**
        for $j \leftarrow 1$ to n **do**
            if $f[i,k] + f[k,j] < f[i,j]$ **then**
                $f[i,j] \leftarrow f[i,k] + f[k,j]$, $\pi[i,j] \leftarrow k$

print-path(i, j)

if $\pi[i,j] = \perp$ **then** then
    if $i \neq j$ **then** print$(i, \text{“,”})$
else
    print-path$(i, \pi[i,j])$, print-path$(\pi[i,j], j)$

Detecting Negative Cycles

Floyd-Warshall(G, w)

$f \leftarrow w$, $\pi[i,j] \leftarrow \perp$ for every $i, j \in V$
for $k \leftarrow 1$ to n **do**
  for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to n **do**
      if $f[i,k] + f[k,j] < f[i,j]$ **then**
        $f[i,j] \leftarrow f[i,k] + f[k,j]$, $\pi[i,j] \leftarrow k$
for $k \leftarrow 1$ to n **do**
  for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to n **do**
      if $f[i,k] + f[k,j] < f[i,j]$ **then**
        report "negative cycle exists" and exit

Summary of Shortest Path Algorithms

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