CSE 431/531: Algorithm Analysis and Design (Spring 2022)

Graph Algorithms

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Outline

1 Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2 Single Source Shortest Paths
   - Dijkstra’s Algorithm

3 Shortest Paths in Graphs with Negative Weights

4 All-Pair Shortest Paths and Floyd-Warshall
Def. Given a connected graph \( G = (V, E) \), a spanning tree \( T = (V, F) \) of \( G \) is a sub-graph of \( G \) that is a tree including all vertices \( V \).
Lemma  Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- $T$ is a spanning tree of $G$;
- $T$ is acyclic and connected;
- $T$ is connected and has $n - 1$ edges;
- $T$ is acyclic and has $n - 1$ edges;
- $T$ is minimally connected: removal of any edge disconnects it;
- $T$ is maximally acyclic: addition of any edge creates a cycle;
- $T$ has a unique simple path between every pair of nodes.
Minimum Spanning Tree (MST) Problem

**Input:** Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

**Output:** the spanning tree $T$ of $G$ with the minimum total weight
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Recall: Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

**Def.** A choice is “safe” if there is an optimum solution that is “consistent” with the choice
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Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

Two Classic Greedy Algorithms for MST

- Kruskal’s Algorithm
- Prim’s Algorithm
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2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall
Q: Which edge can be safely included in the MST?
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A: The edge with the smallest weight (lightest edge).
**Lemma** It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.
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Proof.
- Take a minimum spanning tree $T$
**Lemma**  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**
- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$

![Diagram](image-url)
**Lemma**  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**
- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$
- There is a unique path in $T$ connecting $u$ and $v$
Lemma  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$
- There is a unique path in $T$ connecting $u$ and $v$
- Remove any edge $e$ in the path to obtain tree $T'$
Lemma  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree $T$
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$$w(e^*) \leq w(e) \implies w(T') \leq w(T): T' \text{ is also a MST}$$
Residual problem: find the minimum spanning tree that contains edge \((g, h)\)

Contract the edge \((g, h)\)

Residual problem: find the minimum spanning tree in the contracted graph.
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Residual problem: find the minimum spanning tree that contains edge \((g, h)\)

**Contract** the edge \((g, h)\)

Residual problem: find the minimum spanning tree in the contracted graph
Contraction of an Edge \((u, v)\)

Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\).

Remove all edges \((u, v)\) from \(E\).

For every edge \((u, w)\) \(\in E\), if \(w \neq v\), change it to \((u^*, w)\).

For every edge \((v, w)\) \(\in E\), if \(w \neq u\), change it to \((u^*, w)\).

May create parallel edges! E.g.: two edges \((i, g^*)\)
Contraction of an Edge \((u, v)\)

- Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\)
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- May create parallel edges! E.g. : two edges \((i, g^*)\)
Greedy Algorithm

Repeat the following step until \( G \) contains only one vertex:

1. Choose the lightest edge \( e^* \), add \( e^* \) to the spanning tree
2. Contract \( e^* \) and update \( G \) be the contracted graph

Q: What edges are removed due to contractions?
A: Edge \((u,v)\) is removed if and only if there is a path connecting \( u \) and \( v \) formed by edges we selected
Greedy Algorithm

Repeat the following step until $G$ contains only one vertex:
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Greedy Algorithm

MST-Greedy\((G, w)\)

1: \(F \leftarrow \emptyset\)
2: sort edges in \(E\) in non-decreasing order of weights \(w\)
3: for each edge \((u, v)\) in the order do
4: if \(u\) and \(v\) are not connected by a path of edges in \(F\) then
5: \(F \leftarrow F \cup \{(u, v)\}\)
6: return \((V, F)\)
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g, h\}, \{i\}
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Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f\}, \{g, h\}
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Sets: \( \{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f, g, h\} \)
Kruskal’s Algorithm: Example

Sets: \(\{a\}, \{b\}, \{c, i, f, g, h\}, \{d\}, \{e\}\)
Kruskal’s Algorithm: Example

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Kruskal’s Algorithm: Example

Sets: \{a, b, c, i, f, g, h, d, e\}
Kruskal’s Algorithm: Efficient Implementation of Greedy Algorithm

\textbf{MST-Kruskal}(G, w)

1: \( F \leftarrow \emptyset \)
2: \( S \leftarrow \{\{v\} : v \in V\} \)
3: sort the edges of \( E \) in non-decreasing order of weights \( w \)
4: \textbf{for} each edge \((u, v) \in E\) in the order \textbf{do}
5: \( S_u \leftarrow \) the set in \( S \) containing \( u \)
6: \( S_v \leftarrow \) the set in \( S \) containing \( v \)
7: \textbf{if} \( S_u \neq S_v \) \textbf{then}
8: \( F \leftarrow F \cup \{(u, v)\} \)
9: \( S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\} \)
10: \textbf{return} \((V, F)\)
Running Time of Kruskal’s Algorithm

**MST-Kruskal(G, w)**

1. \( F \leftarrow \emptyset \)
2. \( S \leftarrow \{\{v\} : v \in V\} \)
3. sort the edges of \( E \) in non-decreasing order of weights \( w \)
4. for each edge \((u, v) \in E\) in the order do
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10. return \((V, F)\)

Use **union-find** data structure to support 2, 5, 6, 7, 9.
Union-Find Data Structure

- $\mathcal{V}$: ground set
- We need to maintain a partition of $\mathcal{V}$ and support following operations:
  - Check if $u$ and $v$ are in the same set of the partition
  - Merge two sets in partition
- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$

- $par[i]$: parent of $i$, ($par[i] = \bot$ if $i$ is a root).
Q: how can we check if $u$ and $v$ are in the same set?
A: Check if root($u$) = root($v$).

root($u$): the root of the tree containing $u$.

Merge the trees with root $r$ and $r'$:
par[$r$] ← $r'$. 
Q: how can we check if \( u \) and \( v \) are in the same set?
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Q: how can we check if $u$ and $v$ are in the same set?
A: Check if $\text{root}(u) = \text{root}(v)$.

- $\text{root}(u)$: the root of the tree containing $u$
- Merge the trees with root $r$ and $r'$: $\text{par}[r] \leftarrow r'$.
Q: how can we check if $u$ and $v$ are in the same set?
A: Check if $\text{root}(u) = \text{root}(v)$.

$\text{root}(u)$: the root of the tree containing $u$

Merge the trees with root $r$ and $r'$: $\text{par}[r] \leftarrow r'$.
Union-Find Data Structure

\[
\text{root}(v) \\
1: \text{ if } par[v] = \bot \text{ then} \\
2: \quad \text{return } v \\
3: \text{ else} \\
4: \quad \text{return } \text{root}(par[v])
\]
Union-Find Data Structure

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\text{root}(v) \\
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- Problem: the tree might be too deep; running time might be large
Union-Find Data Structure

root(v)

1: if \( par[v] = \bot \) then
2: return \( v \)
3: else
4: return root(\( par[v] \))

- Problem: the tree might be too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.
**Problem:** the tree might too deep; running time might be large

**Improvement:** all vertices in the path directly point to the root, saving time in the future.
root(v)

1: if \( \text{par}[v] = \bot \) then
2: return \( v \)
3: else
4: \( \text{par}[v] \leftarrow \text{root}(\text{par}[v]) \)
5: return \( \text{par}[v] \)
**Union-Find Data Structure**

\[ \text{root}(v) \]

1. if \( \text{par}[v] = \bot \) then
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MST-Kruskal($G, w$)

1: $F \leftarrow \emptyset$
2: $S \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: for each edge $(u, v) \in E$ in the order do
5: $S_u \leftarrow$ the set in $S$ containing $u$
6: $S_v \leftarrow$ the set in $S$ containing $v$
7: if $S_u \neq S_v$ then
8: $F \leftarrow F \cup \{(u, v)\}$
9: $S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: return $(V, F)$
MST-Kruskal($G, w$)

1: $F \leftarrow \emptyset$
2: for every $v \in V$ do: $par[v] \leftarrow \perp$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: for each edge $(u, v) \in E$ in the order do
5: $u' \leftarrow \text{root}(u)$
6: $v' \leftarrow \text{root}(v)$
7: if $u' \neq v'$ then
8: $F \leftarrow F \cup \{(u, v)\}$
9: $par[u'] \leftarrow v'$
10: return $(V, F)$
MST-Kruskal($G$, $w$)

1: $F \leftarrow \emptyset$
2: for every $v \in V$ do: $\text{par}[v] \leftarrow \perp$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: for each edge $(u, v) \in E$ in the order do
5: $u' \leftarrow \text{root}(u)$
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7: if $u' \neq v'$ then
8: $F \leftarrow F \cup \{(u, v)\}$
9: $\text{par}[u'] \leftarrow v'$
10: return $(V, F)$

- 2, 5, 6, 7, 9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$. 
MST-Kruskal($G, w$)

1. $F \leftarrow \emptyset$
2. for every $v \in V$ do: $\text{par}[v] \leftarrow \perp$
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- $2, 5, 6, 7, 9$ takes time $O(m \alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time $= \text{time for } 3 = O(m \lg n)$. 
**Assumption**  Assume all edge weights are different.

**Lemma**  An edge \( e \in E \) is not in the MST, if and only if there is cycle \( C \) in \( G \) in which \( e \) is the heaviest edge.
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**Lemma**  An edge \( e \in E \) is **not** in the MST, if and only if there is cycle \( C \) in \( G \) in which \( e \) is the heaviest edge.

\[(i, g)\] is not in the MST because of cycle \((i, c, f, g)\)
**Assumption**  Assume all edge weights are different.

**Lemma**  An edge \( e \in E \) is **not** in the MST, if and only if there is cycle \( C \) in \( G \) in which \( e \) is the heaviest edge.

- \((i, g)\) is not in the MST because of cycle \((i, c, f, g)\)
- \((e, f)\) is in the MST because no such cycle exists
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Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.
2. Start from $F \leftarrow E$, and remove edges from $F$ one by one until we obtain a spanning tree.

Q: Which edge can be safely excluded from the MST?
A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.
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Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.

2. Start from $F \leftarrow E$, and remove edges from $F$ one by one until we obtain a spanning tree.

Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.
Lemma  It is safe to exclude the heaviest non-bridge edge: there is an MST that does not contain the heaviest non-bridge edge.
Reverse Kruskal’s Algorithm

MST-Greedy\((G, w)\)

1: \(F \leftarrow E\)
2: sort \(E\) in non-increasing order of weights
3: for every \(e\) in this order do
4: \[\text{if } (V, F \setminus \{e\}) \text{ is connected then}\]
5: \(F \leftarrow F \setminus \{e\}\)
6: return \((V, F)\)
Reverse Kruskal’s Algorithm: Example

Diagram of a graph with nodes labeled a, b, c, d, e, f, g, h, i and edges labeled with weights 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14.
Reverse Kruskal’s Algorithm: Example

```
      8
    /   
   b----c
   |    |
   5    2
   |    |
   a----i
   |    |
   11   1
   |    |
   h----g
   |    |
   7    6
   |    |
   f----e
   |    |
   14   10
   |    |
   d----
   9
```
Reverse Kruskal’s Algorithm: Example
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Reverse Kruskal’s Algorithm: Example

The diagram represents a graph with labeled edges and vertices. The graph includes the following vertices: a, b, c, d, e, f, g, h, i. The edges are labeled with the following weights: 5, 8, 2, 4, 7, 6, 3, 9, 10.
Reverse Kruskal’s Algorithm: Example
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4. All-Pair Shortest Paths and Floyd-Warshall
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.

Greedy strategy for Prim’s algorithm: choose the lightest edge incident to \( a \).
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.

Greedy strategy for Prim’s algorithm: choose the lightest edge incident to $a$. 
Lemma It is safe to include the lightest edge incident to $a$.
**Lemma**  It is safe to include the lightest edge incident to $a$.

**Proof.**
- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
**Lemma** It is safe to include the lightest edge incident to $a$.

Proof.
- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
- Let $e^*$ be the lightest edge incident to $a$ and $e^*$ connects $a$ to component $C$
**Lemma**  It is safe to include the lightest edge incident to $a$.

Proof.

- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
- Let $e^*$ be the lightest edge incident to $a$ and $e^*$ connects $a$ to component $C$
- Let $e$ be the edge in $T$ connecting $a$ to $C$
Lemma  It is safe to include the lightest edge incident to $a$.

Proof.

- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
- Let $e^*$ be the lightest edge incident to $a$ and $e^*$ connects $a$ to component $C$
- Let $e$ be the edge in $T$ connecting $a$ to $C$
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$
Prim’s Algorithm: Example
Prim’s Algorithm: Example
Prim’s Algorithm: Example
Prim’s Algorithm: Example
Prim’s Algorithm: Example
Prim’s Algorithm: Example
Prim’s Algorithm: Example

![Graph](image-url)
Prim’s Algorithm: Example
Prim’s Algorithm: Example
Prim’s Algorithm: Example

The image shows a graph with labeled vertices and edges. The graph is used to illustrate Prim’s algorithm, which is a method for finding the minimum spanning tree of a weighted graph.
Prim’s Algorithm: Example
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\begin{itemize}
  \item \textbf{a} i
  \item \textbf{b}
  \item \textbf{h} g
  \item \textbf{c} d
  \item \textbf{f}
  \item \textbf{e}
\end{itemize}
Greedy Algorithm

MST-Greedy1(\(G, w\))

1: \(S \leftarrow \{s\}\), where \(s\) is arbitrary vertex in \(V\)
2: \(F \leftarrow \emptyset\)
3: \textbf{while} \(S \neq V\) \textbf{do}
4: \((u, v) \leftarrow \text{lightest edge between } S \text{ and } V \setminus S, \text{ where } u \in S \text{ and } v \in V \setminus S\)
5: \(S \leftarrow S \cup \{v\}\)
6: \(F \leftarrow F \cup \{(u, v)\}\)
7: \textbf{return} \((V, F)\)
Greedy Algorithm

MST-Greedy1($G, w$)

1: $S \leftarrow \{s\}$, where $s$ is arbitrary vertex in $V$
2: $F \leftarrow \emptyset$
3: while $S \neq V$ do
4: $(u, v) \leftarrow$ lightest edge between $S$ and $V \setminus S$, where $u \in S$ and $v \in V \setminus S$
5: $S \leftarrow S \cup \{v\}$
6: $F \leftarrow F \cup \{(u, v)\}$
7: return $(V, F)$

- Running time of naive implementation: $O(nm)$
Prim’s Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S: (u, v) \in E} w(u, v)$: the weight of the lightest edge between $v$ and $S$
- $\pi(v) = \arg \min_{u \in S: (u, v) \in E} w(u, v)$: $(\pi(v), v)$ is the lightest edge between $v$ and $S$
Prim’s Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S: (u, v) \in E} w(u, v)$:
  the weight of the lightest edge between $v$ and $S$

- $\pi(v) = \arg \min_{u \in S: (u, v) \in E} w(u, v)$:
  $(\pi(v), v)$ is the lightest edge between $v$ and $S$

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values.
Prim’s Algorithm

MST-Prim($G, w$)

1: $s \leftarrow$ arbitrary vertex in $G$
2: $S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3: while $S \neq V$ do
4: $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d(u)$
5: $S \leftarrow S \cup \{u\}$
6: for each $v \in V \setminus S$ such that $(u, v) \in E$ do
7: if $w(u, v) < d(v)$ then
8: $d(v) \leftarrow w(u, v)$
9: $\pi(v) \leftarrow u$
10: return $\{(u, \pi(u)) \mid u \in V \setminus \{s\}\}$
Example
Example
Example

(a, b)

(5, a)

(12, a)
Example

\[ (5, a) \]

\[ (12, a) \]
Example

\[
\begin{align*}
&\text{Example} \\
&\text{Diagram of a graph with nodes labeled } a, b, c, d, e, f, g, h, i, j, k, \text{ and edges labeled with weights.}
\end{align*}
\]
Example

```
(8, b)
(11, b)
```
Example
Example
Example
Example
Example

(13, c)
(11, b)
(2, c)
(4, c)
Example
Example
Example
Example
Example
Example
Example
Example

(a, i)
(b, h)
(c, g)
(d, f)
(e, (10, f))

(13, c)
(1, g)
(10, f)
Example

(13, c)
Example

\[
\begin{align*}
(13, c) & \quad (10, f)
\end{align*}
\]
Example

- Node a
- Node b
- Node c
- Node d
- Node e
- Node f
- Node g
- Node h

Edges and Weights:
- a to b (5)
- b to c (8)
- c to d (13)
- d to e (9)
- a to h (11)
- h to i (7)
- i to c (2)
- i to g (6)
- g to f (3)
- f to e (10)

Nodes with Labels:
- (13, c)
- (10, f)
Example
Example

(9, e)
Example
Example

![Graph with labeled vertices and edges]

- Vertices: a, b, c, d, e, f, g, h, i
- Edges with labels:
  - a to b: 5
  - b to c: 8
  - c to d: 13
  - d to e: 9
  - d to i: 14
  - b to g: 2
  - g to b: 6
  - g to i: 4
  - i to c: 13
  - i to h: 7
  - h to g: 1
  - g to f: 3
  - f to i: 14
  - f to h: 10
For every $v \in V \setminus S$ maintain

- $d(v) = \min_{u \in S: (u,v) \in E} w(u, v)$: the weight of the lightest edge between $v$ and $S$
- $\pi(v) = \arg\min_{u \in S: (u,v) \in E} w(u, v)$: $(\pi(v), v)$ is the lightest edge between $v$ and $S$

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values.
Prim’s Algorithm

For every $v \in V \setminus S$ maintain

1. $d(v) = \min_{u \in S: (u, v) \in E} w(u, v)$: the weight of the lightest edge between $v$ and $S$
2. $\pi(v) = \arg \min_{u \in S: (u, v) \in E} w(u, v)$: $(\pi(v), v)$ is the lightest edge between $v$ and $S$

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d(u)$ value
- Add $(\pi(u), u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values.

Use a priority queue to support the operations
Def. A **priority queue** is an abstract data structure that maintains a set \( U \) of elements, each with an associated key value, and supports the following operations:

- \( \text{insert}(v, \text{key}\_\text{value}) \): insert an element \( v \), whose associated key value is \( \text{key}\_\text{value} \).
- \( \text{decrease}\_\text{key}(v, \text{new}\_\text{key}\_\text{value}) \): decrease the key value of an element \( v \) in queue to \( \text{new}\_\text{key}\_\text{value} \)
- \( \text{extract}\_\text{min}() \): return and remove the element in queue with the smallest key value
- \( \ldots \)
Prim’s Algorithm

**MST-Prim**\((G, w)\)

1: \(s \leftarrow \text{arbitrary vertex in } G\)

2: \(S \leftarrow \emptyset, d(s) \leftarrow 0 \text{ and } d(v) \leftarrow \infty \text{ for every } v \in V \setminus \{s\}\)

3: 

4: while \(S \neq V\) do

5: \(u \leftarrow \text{vertex in } V \setminus S \text{ with the minimum } d(u)\)

6: \(S \leftarrow S \cup \{u\}\)

7: for each \(v \in V \setminus S \text{ such that } (u, v) \in E\) do

8: if \(w(u, v) < d(v)\) then

9: \(d(v) \leftarrow w(u, v)\)

10: \(\pi(v) \leftarrow u\)

11: return \(\{(u, \pi(u)) | u \in V \setminus \{s\}\}\)
Prim’s Algorithm Using Priority Queue

MST-Prim\((G, w)\)

1. \(s \leftarrow \text{arbitrary vertex in } G\)
2. \(S \leftarrow \emptyset, d(s) \leftarrow 0 \text{ and } d(v) \leftarrow \infty \text{ for every } v \in V \setminus \{s\}\)
3. \(Q \leftarrow \text{empty queue, for each } v \in V: Q.\text{insert}(v, d(v))\)
4. \(\text{while } S \neq V \text{ do}\)
5. \(u \leftarrow Q.\text{extract\_min}()\)
6. \(S \leftarrow S \cup \{u\}\)
7. \(\text{for each } v \in V \setminus S \text{ such that } (u, v) \in E \text{ do}\)
8. \(\text{if } w(u, v) < d(v) \text{ then}\)
9. \(d(v) \leftarrow w(u, v), Q.\text{decrease\_key}(v, d(v))\)
10. \(\pi(v) \leftarrow u\)
11. \(\text{return } \{(u, \pi(u))|u \in V \setminus \{s\}\}\)
Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times \text{(time for extract\_min)} + O(m) \times \text{(time for decrease\_key)} \]
Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times (\text{time for } \text{extract\_min}) + O(m) \times (\text{time for } \text{decrease\_key}) \]

<table>
<thead>
<tr>
<th>concrete DS</th>
<th>extract_min</th>
<th>decrease_key</th>
<th>overall time</th>
</tr>
</thead>
<tbody>
<tr>
<td>heap</td>
<td>( O(\log n) )</td>
<td>( O(\log n) )</td>
<td>( O(m \log n) )</td>
</tr>
<tr>
<td>Fibonacci heap</td>
<td>( O(\log n) )</td>
<td>( O(1) )</td>
<td>( O(n \log n + m) )</td>
</tr>
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Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times (\text{time for extract\_min}) + O(m) \times (\text{time for decrease\_key}) \]

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**Assumption**  Assume all edge weights are different.

**Lemma**  
\( (u, v) \) is in MST, if and only if there exists a cut \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).
Assumption  Assume all edge weights are different.

Lemma  \((u, v)\) is in MST, if and only if there exists a cut \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).

\((c, f)\) is in MST because of cut \((\{a, b, c, i\}, V \setminus \{a, b, c, i\})\)
Assumption  Assume all edge weights are different.

Lemma  \((u, v)\) is in MST, if and only if there exists a cut \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).

\[
\begin{align*}
\text{\((c, f)\) is in MST because of cut \((\{a, b, c, i\}, V \setminus \{a, b, c, i\})\)}
\text{\((i, g)\) is not in MST because no such cut exists}
\end{align*}
\]
“Evidence” for $e \in \text{MST}$ or $e \not\in \text{MST}$

**Assumption**  Assume all edge weights are different.

- $e \in \text{MST} \iff$ there is a cut in which $e$ is the lightest edge
- $e \not\in \text{MST} \iff$ there is a cycle in which $e$ is the heaviest edge
“Evidence” for $e \in \text{MST}$ or $e \notin \text{MST}$

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**Exactly one** of the following is true:
- There is a cut in which $e$ is the lightest edge
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Exactly one of the following is true:

- There is a cut in which $e$ is the lightest edge
- There is a cycle in which $e$ is the heaviest edge

Thus, the minimum spanning tree is unique with assumption.
Outline

1. Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall
<table>
<thead>
<tr>
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<th>graph</th>
<th>weights</th>
<th>SS?</th>
<th>running time</th>
</tr>
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<tbody>
<tr>
<td>Simple DP</td>
<td>DAG</td>
<td>R</td>
<td>SS</td>
<td>$O(n + m)$</td>
</tr>
<tr>
<td>Dijkstra</td>
<td>U/D</td>
<td>$\mathbb{R}_{\geq 0}$</td>
<td>SS</td>
<td>$O(n \log n + m)$</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>U/D</td>
<td>R</td>
<td>SS</td>
<td>$O(nm)$</td>
</tr>
<tr>
<td>Floyd-Warshall</td>
<td>U/D</td>
<td>R</td>
<td>AP</td>
<td>$O(n^3)$</td>
</tr>
</tbody>
</table>

- DAG = directed acyclic graph    U = undirected    D = directed
- SS = single source       AP = all pairs
\( s-t \) Shortest Paths

**Input:** (directed or undirected) graph \( G = (V, E), \ s, t \in V \)

\[ w : E \rightarrow \mathbb{R}_{\geq 0} \]

**Output:** shortest path from \( s \) to \( t \)
$s$-$t$ Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** shortest path from $s$ to $t$
**s-t Shortest Paths**

**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

**Output:** shortest path from $s$ to $t$
Single Source Shortest Paths

**Input:** (directed or undirected) graph \( G = (V, E) \), \( s \in V \)

\[ w : E \rightarrow \mathbb{R}_{\geq 0} \]

**Output:** shortest paths from \( s \) to all other vertices \( v \in V \)
**Single Source Shortest Paths**

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

---

**Reason for Considering Single Source Shortest Paths Problem**

- We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem
Single Source Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight
### Single Source Shortest Paths

**Input:** (directed or undirected) graph \( G = (V, E) \), \( s \in V \)

\[ w : E \to \mathbb{R}_{\geq 0} \]

**Output:** shortest paths from \( s \) to all other vertices \( v \in V \)

### Reason for Considering Single Source Shortest Paths Problem

- We do not know how to solve \( s-t \) shortest path problem more efficiently than solving single source shortest path problem

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**Single Source Shortest Paths**

**Input:** directed graph $G = (V, E)$, $s \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

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- We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem.

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight.
Single Source Shortest Paths

**Input:** directed graph $G = (V, E)$, $s \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** $\pi(v), v \in V \setminus s$: the parent of $v$ in shortest path tree

$d(v), v \in V \setminus s$: the length of shortest path from $s$ to $v$
Q: How to compute shortest paths from $s$ when all edges have weight 1?
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A: Breadth first search (BFS) from source $s$
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- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges.
**Assumption**  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges

\[ u \quad 4 \quad v \]

\[ u \quad 1 \quad 1 \quad 1 \quad 1 \quad v \]

**Shortest Path Algorithm by Running BFS**

1. replace $(u, v)$ of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2. run BFS
3. $\pi(v) \leftarrow$ vertex from which $v$ is visited
4. $d(v) \leftarrow$ index of the level containing $v$
Assumption  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges

\[
\begin{array}{c}
\text{Shortest Path Algorithm by Running BFS} \\
1: \text{replace } (u, v) \text{ of length } w(u, v) \text{ with a path of } w(u, v) \\
\text{unit-weight edges, for every } (u, v) \in E \\
2: \text{run BFS} \\
3: \pi(v) \leftarrow \text{vertex from which } v \text{ is visited} \\
4: d(v) \leftarrow \text{index of the level containing } v
\end{array}
\]

- Problem: $w(u, v)$ may be too large!
**Assumption**  Weights \( w(u, v) \) are integers (w.l.o.g).

- An edge of weight \( w(u, v) \) is equivalent to a path of \( w(u, v) \) unit-weight edges

![Graph](image)

**Shortest Path Algorithm by Running BFS**

1: replace \((u, v)\) of length \( w(u, v) \) with a path of \( w(u, v) \) unit-weight edges, for every \((u, v) \in E\)
2: run BFS virtually
3: \( \pi(v) \leftarrow \) vertex from which \( v \) is visited
4: \( d(v) \leftarrow \) index of the level containing \( v \)

- Problem: \( w(u, v) \) may be too large!
Shortest Path Algorithm by Running BFS Virtually

1: $S \leftarrow \{s\}$, $d(s) \leftarrow 0$
2: while $|S| \leq n$ do
3: find a $v \notin S$ that minimizes $\min_{u \in S: (u,v) \in E} \{d(u) + w(u, v)\}$
4: $S \leftarrow S \cup \{v\}$
5: $d(v) \leftarrow \min_{u \in S: (u,v) \in E}\{d(u) + w(u, v)\}$
Virtual BFS: Example
Virtual BFS: Example

Time 0
Virtual BFS: Example

Time 2
Virtual BFS: Example

Time 4
Virtual BFS: Example

Time 7
Virtual BFS: Example

Time 9
Virtual BFS: Example

Time 10
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Dijkstra’s Algorithm

\[ \text{Dijkstra}(G, w, s) \]

1: \( S \leftarrow \emptyset, d(s) \leftarrow 0 \text{ and } d(v) \leftarrow \infty \text{ for every } v \in V \setminus \{s\} \)
2: \textbf{while } S \neq V \textbf{ do}
3: \hspace{1em} u \leftarrow \text{vertex in } V \setminus S \text{ with the minimum } d(u) \)
4: \hspace{1em} add \ u \text{ to } S
5: \hspace{1em} \textbf{for each } v \in V \setminus S \text{ such that } (u, v) \in E \textbf{ do}
6: \hspace{2em} \textbf{if } d(u) + w(u, v) < d(v) \textbf{ then}
7: \hspace{3em} d(v) \leftarrow d(u) + w(u, v)
8: \hspace{3em} \pi(v) \leftarrow u
9: \textbf{return } (d, \pi) \]

Running time = \( O(n^2) \)
Dijkstra’s Algorithm

Dijkstra\( (G, w, s) \)

1: \( S \leftarrow \emptyset, d(s) \leftarrow 0 \) and \( d(v) \leftarrow \infty \) for every \( v \in V \setminus \{s\} \)
2: \textbf{while} \( S \neq V \) \textbf{do}
3: \( u \leftarrow \) vertex in \( V \setminus S \) with the minimum \( d(u) \)
4: \textbf{add} \( u \) to \( S \)
5: \textbf{for} each \( v \in V \setminus S \) such that \( (u, v) \in E \) \textbf{do}
6: \textbf{if} \( d(u) + w(u, v) < d(v) \) \textbf{then}
7: \( d(v) \leftarrow d(u) + w(u, v) \)
8: \( \pi(v) \leftarrow u \)
9: \textbf{return} \( (d, \pi) \)

- Running time = \( O(n^2) \)
Improved Running Time using Priority Queue

Dijkstra($G$, $w$, $s$)

1: $s \leftarrow$ arbitrary vertex in $G$
2: $S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3: $Q \leftarrow$ empty queue, for each $v \in V$: $Q.insert(v, d(v))$
4: while $S \neq V$ do
5: \hspace{0.5cm} $u \leftarrow Q.extract\_min()$
6: \hspace{0.5cm} $S \leftarrow S \cup \{u\}$
7: \hspace{1cm} for each $v \in V \setminus S$ such that $(u, v) \in E$ do
8: \hspace{1.5cm} if $d(u) + w(u, v) < d(v)$ then
9: \hspace{2cm} $d(v) \leftarrow d(u) + w(u, v)$, $Q.decrease\_key(v, d(v))$
10: \hspace{1cm} $\pi(v) \leftarrow u$
11: return $(\pi, d)$
Recall: Prim’s Algorithm for MST

MST-Prim($G, w$)

1: $s \leftarrow$ arbitrary vertex in $G$
2: $S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d(v) \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3: $Q \leftarrow$ empty queue, for each $v \in V$: $Q.insert(v, d(v))$
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5: $u \leftarrow Q.extract\_min()$
6: $S \leftarrow S \cup \{u\}$
7: for each $v \in V \setminus S$ such that $(u, v) \in E$ do
8: if $w(u, v) < d(v)$ then
9: $d(v) \leftarrow w(u, v)$, $Q.decrease\_key(v, d(v))$
10: $\pi(v) \leftarrow u$
11: return $\{(u, \pi(u)) | u \in V \setminus \{s\}\}$
Improved Running Time

Running time:
\(O(n) \times (\text{time for extract\_min}) + O(m) \times (\text{time for decrease\_key})\)

<table>
<thead>
<tr>
<th>Priority-Queue</th>
<th>extract_min</th>
<th>decrease_key</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heap</td>
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Single Source Shortest Paths, Weights May be Negative

**Input:** directed graph \( G = (V, E) \), \( s \in V \)

assume all vertices are reachable from \( s \)

\( w : E \rightarrow \mathbb{R} \)

**Output:** shortest paths from \( s \) to all other vertices \( v \in V \)
Single Source Shortest Paths, Weights May be Negative

**Input:** directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from $s$

$w : E \rightarrow \mathbb{R}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

- In transition graphs, negative weights make sense
Single Source Shortest Paths, Weights May be Negative

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- Dijkstra’s algorithm does not work any more!
Dijkstra’s Algorithm Fails if We Have Negative Weights
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Q: What is the length of the shortest path from $s$ to $d$?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Dealing with Negative Cycles
What is the length of the shortest path from \( s \) to \( d \)?
Q: What is the length of the shortest path from $s$ to $d$?

A: $-\infty$
Q: What is the length of the shortest path from $s$ to $d$?

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**Q:** What is the length of the shortest path from \( s \) to \( d \)?

**A:** \(-\infty\)
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**Dealing with Negative Cycles**

- assume the input graph does not contain negative cycles, or
Q: What is the length of the shortest path from $s$ to $d$?

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Dealing with Negative Cycles
- assume the input graph does not contain negative cycles, or
- allow algorithm to report “negative cycle exists”
Q: What is the length of the shortest simple path from $s$ to $d$?

A: Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.
Q: What is the length of the shortest simple path from \( s \) to \( d \)?
Q: What is the length of the shortest simple path from $s$ to $d$?

A: 1
Q: What is the length of the shortest simple path from $s$ to $d$?

A: 1

Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.
<table>
<thead>
<tr>
<th>algorithm</th>
<th>graph</th>
<th>weights</th>
<th>SS?</th>
<th>running time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple DP</td>
<td>DAG</td>
<td>(\mathbb{R})</td>
<td>SS</td>
<td>(O(n + m))</td>
</tr>
<tr>
<td>Dijkstra</td>
<td>U/D</td>
<td>(\mathbb{R}_{\geq 0})</td>
<td>SS</td>
<td>(O(n \log n + m))</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>U/D</td>
<td>(\mathbb{R})</td>
<td>SS</td>
<td>(O(nm))</td>
</tr>
<tr>
<td>Floyd-Warshall</td>
<td>U/D</td>
<td>(\mathbb{R})</td>
<td>AP</td>
<td>(O(n^3))</td>
</tr>
</tbody>
</table>

- DAG = directed acyclic graph    U = undirected    D = directed
- SS = single source    AP = all pairs
**Single Source Shortest Paths, Weights May be Negative**

**Input:** directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from $s$

$w : E \rightarrow \mathbb{R}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$
**Defining Cells of Table**

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- first try: $f[v]$: length of shortest path from $s$ to $v$
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### Single Source Shortest Paths, Weights May be Negative

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- $f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V$ : length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
$f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V$:

length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
\( f^{\ell}[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V: \) length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

\( f^2[a] = \)
\( f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V : \) length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

\( f^2[a] = 6 \)
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \ldots, n - 1\}$, $v \in V$: length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
- $f^2[a] = 6$
- $f^3[a] =$
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \ldots, n - 1\}, v \in V : \]
length of shortest path from \( s \) to \( v \) that uses
at most \( \ell \) edges

- \( f^2[a] = 6 \)
- \( f^3[a] = 2 \)
\( f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, v \in V: \) length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

- \( f^2[a] = 6 \)
- \( f^3[a] = 2 \)

\[
f^\ell[v] = \begin{cases} 
0 & \ell = 0, v = s \\
\infty & \ell = 0, v \neq s \\
\min \{ f^{\ell - 1}[v] + w(u,v) \} & \ell > 0 \end{cases}
\]
$f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n - 1\}$, $v \in V$:
length of shortest path from $s$ to $v$ that uses at most $\ell$ edges

- $f^2[a] = 6$
- $f^3[a] = 2$

$$f^\ell[v] = \begin{cases} 0 & \ell = 0, v = s \\ 0 & \ell = 0, v \neq s \\ \min \{f^{\ell - 1}[v] + w(u,v) : (u,v) \in E\} & \ell > 0 \end{cases}$$
- $f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n-1\}, v \in V$:
  - length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
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- $f^3[a] = 2$

\[
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\infty & \ell = 0, v \neq s \\
& \ell > 0 
\end{cases}
\]
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- \(f^2[a] = 6\)
- \(f^3[a] = 2\)

\[ f^\ell[v] = \begin{cases} 
0 & \text{if } \ell = 0, v = s \\
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\end{cases} \]
\( f^{\ell}[v], \ell \in \{0, 1, 2, 3 \cdots , n-1\}, v \in V : \)
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f^\ell[v] = \begin{cases} 
0 & \ell = 0, v = s \\
\infty & \ell = 0, v \neq s \\
\min \left\{ \min_{u:(u,v)\in E} \left( f^{\ell-1}[u] + w(u,v) \right) \right\} & \ell > 0 
\end{cases}
\]
Dynamic Programming: Example

\[ f^0 \quad s \quad a \quad b \quad c \quad d \]

\[ f^0 \quad s \quad a \quad b \quad c \quad d \]

\[ s \quad 6 \quad 7 \]

\[ b \quad 8 \quad a \quad -2 \]

\[ c \quad -4 \quad -3 \quad 7 \quad d \]

length-0 edge
Dynamic Programming: Example

![Graph](image)

- Graph with nodes labeled $s$, $a$, $b$, $c$, $d$.
- Edges with weights: $7$, $6$, $8$, $-3$, $-4$, $-2$, $7$.
- Length-0 edge.

- Dynamic programming functions $f^0$ and $f^1$.

- Initial state: $f^0(s) = 0$.
- Transition function: $f^{i+1}(v) = \min\{f^i(u) + w(u,v)\}$ for all edges $(u,v)$.

- Example:
  - $f^0(s) = 0$.
  - $f^1(s)$ updated iteratively.

- Conclusion:
  - Optimal path can be found using dynamic programming techniques.
Dynamic Programming: Example

**Diagram:**
- **Nodes:** s, a, b, c, d
- **Edges:**
  - (s, a) with weight 6
  - (s, b) with weight 7
  - (s, c) with weight -3
  - (s, d) with weight -4
  - (b, a) with weight 8
  - (c, d) with weight 7
- **Label:** length-0 edge

**Tables:**
- **f^0:**
  - s: 0
  - a: ∞
  - b: ∞
  - c: ∞
  - d: ∞
- **f^1:**
  - s: 0
  - a: 6, 7
  - b: 8, -4, -3, -2
  - c: ∞, -2
  - d: ∞, 7
Dynamic Programming: Example

length-0 edge
Dynamic Programming: Example

\[
\begin{align*}
\text{length-0 edge} & \\
\end{align*}
\]
Dynamic Programming: Example

Length-0 edge
Dynamic Programming: Example

length-0 edge
Dynamic Programming: Example

\[ \begin{align*}
  f^0 & : s & a & b & c & d \\
  0 & \infty & \infty & \infty & \infty \\
  f^1 & : s & a & b & c & d \\
  0 & 6 & 7 & 8 & -4 & -3 \\
  f^2 & : s & a & b & c & d \\
  0 & 6 & 7 & 8 & -4 & -3 \\
\end{align*} \]

length-0 edge
Dynamic Programming: Example

\[ f^0 \]

\[ f^1 \]

\[ f^2 \]

length-0 edge
Dynamic Programming: Example

\[
\begin{array}{c}
 s \\
 b \\
 c \\
 a \\
 d \\
\end{array}
\]

\[
\begin{array}{c}
 7 \\
 6 \\
 8 \\
-2 \\
-3 \\
\end{array}
\]

\[
\begin{array}{c}
 f^0 \\
 f^1 \\
 f^2 \\
\end{array}
\]

\[
\begin{array}{c}
 0 \\
 6 \\
 7 \\
\end{array}
\]

length-0 edge

\[
\begin{array}{c}
 \infty \\
 \infty \\
 \infty \\
 \infty \\
\end{array}
\]

\[
\begin{array}{c}
 6 \\
 7 \\
\end{array}
\]
Dynamic Programming: Example

\[
\begin{array}{c}
s \
\downarrow \\
 7 \quad 6 \\
\downarrow \\
b \quad a \\
 8 \\
\downarrow \\
c \quad d \\
-4 \quad 7 \\
\end{array}
\]

-2

\[
\begin{array}{c}
f^0 \\
0 \\
\downarrow \\
s \\
\infty \\
\downarrow \\
a \\
6 \\
\downarrow \\
b \\
7 \\
\downarrow \\
c \\
\infty \\
\downarrow \\
d \\
\infty \\
\end{array}
\]

\[
\begin{array}{c}
f^1 \\
0 \\
\downarrow \\
6 \\
\downarrow \\
7 \\
\downarrow \\
\infty \\
\downarrow \\
a \\
\infty \\
\downarrow \\
b \\
\infty \\
\downarrow \\
c \\
\infty \\
\downarrow \\
d \\
\infty \\
\end{array}
\]

\[
\begin{array}{c}
f^2 \\
0 \\
\downarrow \\
6 \\
\downarrow \\
7 \\
\downarrow \\
\infty \\
\downarrow \\
a \\
6 \\
\downarrow \\
b \\
-4 \\
\downarrow \\
c \\
-3 \\
\downarrow \\
d \\
-2 \\
\end{array}
\]

length-0 edge

\[
\begin{array}{c}
\infty \\
\downarrow \\
2 \\
\end{array}
\]
Dynamic Programming: Example

$$\begin{align*}
\text{length-0 edge} \\
\end{align*}$$
Dynamic Programming: Example

\[ s \rightarrow a \rightarrow b \rightarrow c \rightarrow d \]

\[ f^0 \]
\[ s \rightarrow 0 \]
\[ a \rightarrow 6 \]
\[ b \rightarrow 7 \]
\[ c \rightarrow \infty \]
\[ d \rightarrow \infty \]

\[ f^1 \]
\[ s \rightarrow 0 \]
\[ a \rightarrow 6 \]
\[ b \rightarrow 8 \]
\[ c \rightarrow \infty \]
\[ d \rightarrow \infty \]

\[ f^2 \]
\[ s \rightarrow 0 \]
\[ a \rightarrow 6 \]
\[ b \rightarrow 7 \]
\[ c \rightarrow 2 \]
\[ d \rightarrow 4 \]

\[ f^3 \]
\[ s \rightarrow 0 \]
\[ a \rightarrow 6 \]
\[ b \rightarrow 7 \]
\[ c \rightarrow 2 \]
\[ d \rightarrow 4 \]

length-0 edge
Dynamic Programming: Example

length-0 edge

\[ f^0 \]

\[ f^1 \]

\[ f^2 \]

\[ f^3 \]
Dynamic Programming: Example
Dynamic Programming: Example

![Graph with nodes and edges labeled with weights.](image)

- $s$: Source
- $a$, $b$, $c$, $d$: Targets

**Weights:**
- Source to $a$: 7
- Source to $b$: 6
- Source to $c$: 8
- Source to $d$: 6
- $a$ to $d$: 7
- $b$ to $c$: 8
- $c$ to $d$: 7

**F-values (length-0 edge):**
- $f^0$: Initial values
- $f^1$: First iteration
- $f^2$: Second iteration
- $f^3$: Third iteration

Length-0 edge:
- From source to itself

Values:
- $f^0(s) = 0$
- $f^0(a) = \infty$
- $f^0(b) = \infty$
- $f^0(c) = \infty$
- $f^0(d) = \infty$

- $f^1(s) = 0$
- $f^1(a) = 6$
- $f^1(b) = \infty$
- $f^1(c) = \infty$
- $f^1(d) = \infty$

- $f^2(s) = 0$
- $f^2(a) = 6$
- $f^2(b) = 7$
- $f^2(c) = \infty$
- $f^2(d) = \infty$

- $f^3(s) = 0$
- $f^3(a) = 2$
- $f^3(b) = 7$
- $f^3(c) = 2$
- $f^3(d) = 4$
Dynamic Programming: Example

\[ f^0 \]
\[ f^1 \]
\[ f^2 \]
\[ f^3 \]

length-0 edge
Dynamic Programming: Example

$$\begin{array}{c}
\begin{array}{c}
s \quad 7 \\
b \quad 8 \\
c \quad 6 \\
d \quad 8
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
s \quad 7 \\
b \quad 8 \\
c \quad 6 \\
d \quad 8
\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
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b \quad 8 \\
c \quad 6 \\
d \quad 8
\end{array}
\end{array}$$

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b \quad 8 \\
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d \quad 8
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$$\begin{array}{c}
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b \quad 8 \\
c \quad 6 \\
d \quad 8
\end{array}
\end{array}$$

$$\begin{array}{c}
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\end{array}
\end{array}$$

$$\begin{array}{c}
\begin{array}{c}
s \quad 7 \\
b \quad 8 \\
c \quad 6 \\
d \quad 8
\end{array}
\end{array}$$

length-0 edge
Dynamic Programming: Example

- Diagram of a graph with nodes and edges labeled with numbers.
- Calculation of values for each node using dynamic programming:
  - $f^0$: Initial values
  - $f^1$: First iteration
  - $f^2$: Second iteration
  - $f^3$: Third iteration
  - $f^4$: Fourth iteration

- The values are calculated by taking the minimum of the values from adjacent nodes and adding the edge weight.

- The length-0 edge is highlighted, showing the minimum path from the source node to each other node.
Dynamic Programming: Example

- Diagram of a graph with labeled edges and vertices.
- Dynamic programming algorithm representation:
  - Initial value: $f^0(0) = 0$, $f^0(1) = \infty$, $f^0(2) = \infty$, $f^0(3) = \infty$.
  - Iterative values:
    - $f^1(0) = 6$, $f^1(1) = 6$, $f^1(2) = 7$, $f^1(3) = \infty$.
    - $f^2(0) = 6$, $f^2(1) = 7$, $f^2(2) = 7$, $f^2(3) = \infty$.
    - $f^3(0) = 6$, $f^3(1) = 7$, $f^3(2) = 7$, $f^3(3) = 4$.
    - $f^4(0) = 6$, $f^4(1) = 7$, $f^4(2) = 7$, $f^4(3) = 4$.

- The graph shows the progression of the dynamic programming algorithm with each iteration.
dynamic-programming$(G, w, s)$

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n - 1$ do
3.     copy $f^{\ell-1} \rightarrow f^\ell$
4.     for each $(u, v) \in E$ do
5.         if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ then
6.             $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
7.     return $(f^{n-1}[v])_{v \in V}$

Obs.
Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges.

Proof.
If there is a path containing at least $n$ edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length.
**dynamic-programming**($G, w, s$)

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. **for** $\ell \leftarrow 1$ **to** $n - 1$ **do**
3. copy $f^{\ell-1} \rightarrow f^\ell$
4. **for** each $(u, v) \in E$ **do**
5. if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ **then**
6. $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
7. **return** $(f^{n-1}[v])_{v \in V}$

**Obs.** Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges
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If there is a path containing at least $n$ edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length.
Dynamic Programming with Better Space Usage

**dynamic-programming**($G, w, s$)

1. $f^{old}[s] \leftarrow 0$ and $f^{old}[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n - 1$ do
3. copy $f^{old} \rightarrow f^{new}$
4. for each $(u, v) \in E$ do
5. if $f^{old}[u] + w(u, v) < f^{new}[v]$ then
6. $f^{new}[v] \leftarrow f^{old}[u] + w(u, v)$
7. copy $f^{new} \rightarrow f^{old}$
8. return $f^{old}$

- $f^{\ell}$ only depends on $f^{\ell-1}$: only need 2 vectors
Dynamic Programming with Better Space Usage

dynamic-programming\( (G, w, s) \)

1: \( f^{\text{old}}[s] \leftarrow 0 \) and \( f^{\text{old}}[v] \leftarrow \infty \) for any \( v \in V \setminus \{s\} \)
2: for \( \ell \leftarrow 1 \) to \( n - 1 \) do
3: copy \( f^{\text{old}} \rightarrow f^{\text{new}} \)
4: for each \( (u, v) \in E \) do
5: if \( f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v] \) then
6: \( f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v) \)
7: copy \( f^{\text{new}} \rightarrow f^{\text{old}} \)
8: return \( f^{\text{old}} \)

- \( f^{\ell} \) only depends on \( f^{\ell-1} \): only need 2 vectors
- only need 1 vector!
Dynamic Programming with Better Space Usage

dynamic-programming\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: for \(\ell \leftarrow 1\) to \(n - 1\) do
3:   copy \(f \rightarrow f\)
4:   for each \((u, v) \in E\) do
5:     if \(f[u] + w(u, v) < f[v]\) then
6:       \(f[v] \leftarrow f[u] + w(u, v)\)
7:   copy \(f \rightarrow f\)
8: return \(f\)

- \(f^{\ell}\) only depends on \(f^{\ell-1}\): only need 2 vectors
- only need 1 vector!
Dynamic Programming with Better Space Usage

dynamic-programming($G, w, s$)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n - 1$ do
3:   for each $(u,v) \in E$ do
4:     if $f[u] + w(u,v) < f[v]$ then
5:       $f[v] \leftarrow f[u] + w(u,v)$
6: return $f$

- $f^\ell$ only depends on $f^{\ell-1}$: only need 2 vectors
- only need 1 vector!
### Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1. \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2. \textbf{for} \(\ell \leftarrow 1\) to \(n - 1\) \textbf{do}
3. \textbf{for each} \((u, v) \in E\) \textbf{do}
4. \quad \textbf{if} \(f[u] + w(u, v) < f[v]\) \textbf{then}
5. \quad \quad \(f[v] \leftarrow f[u] + w(u, v)\)
6. \textbf{return} \(f\)

- \(f^\ell\) only depends on \(f^{\ell-1}\): only need 2 vectors
- only need 1 vector!
Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: \textbf{for} \(\ell \leftarrow 1\) to \(n - 1\) \textbf{do}
3: \hspace{1em} \textbf{for} each \((u, v) \in E\) \textbf{do}
4: \hspace{2em} \textbf{if} \(f[u] + w(u, v) < f[v]\) \textbf{then}
5: \hspace{3em} \(f[v] \leftarrow f[u] + w(u, v)\)
6: \textbf{return} \(f\)

- Issue: when we compute \(f[u] + w(u, v)\), \(f[u]\) may be changed since the end of last iteration
Bellman-Ford Algorithm

Bellman-Ford($G, w, s$)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n - 1$ do
3:   for each $(u, v) \in E$ do
4:     if $f[u] + w(u, v) < f[v]$ then
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- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
Bellman-Ford Algorithm

**Bellman-Ford**(*G*, *w*, *s*)

1. \( f[s] \leftarrow 0 \) and \( f[v] \leftarrow \infty \) for any \( v \in V \setminus \{s\} \)
2. for \( \ell \leftarrow 1 \) to \( n - 1 \) do
3.     for each \((u, v) \in E\) do
4.        if \( f[u] + w(u, v) < f[v] \) then
5.            \( f[v] \leftarrow f[u] + w(u, v) \)
6.     return \( f \)

- Issue: when we compute \( f[u] + w(u, v) \), \( f[u] \) may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
- After iteration \( \ell \), \( f[v] \) is at most the length of the shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges
Bellman-Ford Algorithm

Bellman-Ford$(G, w, s)$

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}
2: \textbf{for } \ell \leftarrow 1 \textbf{ to } n - 1 \textbf{ do}
3: \hspace{1em} \textbf{for each } (u, v) \in E \textbf{ do}
4: \hspace{2em} \textbf{if } f[u] + w(u, v) < f[v] \textbf{ then}
5: \hspace{3em} f[v] \leftarrow f[u] + w(u, v)
6: \textbf{return } f

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
- After iteration $\ell$, $f[v]$ is at most the length of the shortest path from $s$ to $v$ that uses at most $\ell$ edges
- $f[v]$ is always the length of some path from $s$ to $v$
Bellman-Ford Algorithm

- After iteration \( \ell \):
  
  \[
  \text{length of shortest } s-v \text{ path} \\
  \leq f[v] \\
  \leq \text{length of shortest } s-v \text{ path using at most } \ell \text{ edges}
  \]

- Assuming there are no negative cycles:
  
  \[
  \text{length of shortest } s-v \text{ path} \\
  = \text{length of shortest } s-v \text{ path using at most } n-1 \text{ edges}
  \]

- So, assuming there are no negative cycles, after iteration \( n-1 \):
  
  \[
  f[v] = \text{length of shortest } s-v \text{ path}
  \]
order in which we consider edges:

\((s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)\)

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(s, a), (s, b), (a, b), (a, c), (b, d),
(c, d), (d, a)

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order in which we consider edges: 
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vertices

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\[
\begin{array}{c|c|c|c|c|c}
\text{vertices} & s & a & b & c & d \\
\hline
f & 0 & 2 & 7 & 2 & 4 \\
\end{array}
\]

end of iteration 1: 0, 2, 7, 2, 4

Algorithm terminates in 3 iterations, instead of 4.
order in which we consider edges:

\((s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)\)

end of iteration 1: 0, 2, 7, 2, 4

end of iteration 2: 0, 2, 7, -2, 4

end of iteration 3: 0, 2, 7, -2, 4

Algorithm terminates in 3 iterations, instead of 4.
- order in which we consider edges: 
  \((s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)\)

- vertices

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- end of iteration 1: 0, 2, 7, 2, 4
order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
(c, d), (d, a)

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end of iteration 1: 0, 2, 7, 2, 4
order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
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end of iteration 1: 0, 2, 7, 2, 4
order in which we consider edges:

- $(s, a)$,
- $(s, b)$,
- $(a, b)$,
- $(a, c)$,
- $(b, d)$,
- $(c, d)$,
- $(d, a)$

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end of iteration 1: 0, 2, 7, 2, 4
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$(s, a), (s, b), (a, b), (a, c), (b, d),
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$$(s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)$$

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end of iteration 1: 0, 2, 7, 2, 4
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end of iteration 1: 0, 2, 7, 2, 4

Algorithm terminates in 3 iterations, instead of 4.
order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
(c, d), (d, a)

vertices

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end of iteration 1: 0, 2, 7, 2, 4
end of iteration 2: 0, 2, 7, -2, 4
order in which we consider edges:
(s, a), (s, b), (a, b), (a, c), (b, d),
(c, d), (d, a)

end of iteration 1: 0, 2, 7, 2, 4
end of iteration 2: 0, 2, 7, -2, 4
end of iteration 3: 0, 2, 7, -2, 4
order in which we consider edges:
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Algorithm terminates in 3 iterations, instead of 4.
Bellman-Ford Algorithm

Bellman-Ford\( (G, w, s) \)

1: \( f[s] \leftarrow 0 \) and \( f[v] \leftarrow \infty \) for any \( v \in V \setminus \{s\} \)
2: \textbf{for } \ell \leftarrow 1 \text{ to } n \textbf{ do}
3: \quad \text{updated } \leftarrow \text{ false}
4: \quad \textbf{for each } (u, v) \in E \textbf{ do}
5: \quad \quad \textbf{if } f[u] + w(u, v) < f[v] \textbf{ then}
6: \quad \quad \quad f[v] \leftarrow f[u] + w(u, v)
7: \quad \quad \quad \text{updated } \leftarrow \text{ true}
8: \quad \quad \textbf{if not updated, then return } f
9: \quad \text{output “negative cycle exists”}
Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: for \(\ell \leftarrow 1\) to \(n\) do
3: \(\text{updated } \leftarrow \text{false}\)
4: for each \((u, v) \in E\) do
5: if \(f[u] + w(u, v) < f[v]\) then
6: \(f[v] \leftarrow f[u] + w(u, v), \pi[v] \leftarrow u\)
7: \(\text{updated } \leftarrow \text{true}\)
8: if not \(\text{updated}\), then return \(f\)
9: output “negative cycle exists”

- \(\pi[v]\): the parent of \(v\) in the shortest path tree
Bellman-Ford Algorithm

Bellman-Ford($G, w, s$)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n$ do
3:     updated $\leftarrow$ false
4:     for each $(u, v) \in E$ do
5:         if $f[u] + w(u, v) < f[v]$ then
6:             $f[v] \leftarrow f[u] + w(u, v)$, $\pi[v] \leftarrow u$
7:             updated $\leftarrow$ true
8:     if not updated, then return $f$
9: output “negative cycle exists”

- $\pi[v]$: the parent of $v$ in the shortest path tree
- Running time $= O(nm)$
Outline

1. Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall
All-Pair Shortest Paths

Input: directed graph \( G = (V, E) \),
\[ w : E \rightarrow \mathbb{R} \] (can be negative)

Output: shortest path from \( u \) to \( v \) for every \( u, v \in V \)

1: for every starting point \( s \in V \) do
2: run Bellman-Ford \((G, w, s)\)

Running time = \( O(n^2 m) \)
All-Pair Shortest Paths

**Input:** directed graph $G = (V, E)$,

$w : E \rightarrow \mathbb{R}$ (can be negative)

**Output:** shortest path from $u$ to $v$ for every $u, v \in V$

1. for every starting point $s \in V$ do
2. run Bellman-Ford($G, w, s$)
All-Pair Shortest Paths

**Input:** directed graph \( G = (V, E) \),
\[ w : E \to \mathbb{R} \text{ (can be negative)} \]

**Output:** shortest path from \( u \) to \( v \) for every \( u, v \in V \)

1. **for** every starting point \( s \in V \) **do**
2. run Bellman-Ford\((G, w, s)\)

\[ \text{Running time} = O(n^2m) \]
# Summary of Shortest Path Algorithms we learned

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- DAG = directed acyclic graph
- U = undirected
- D = directed
- SS = single source
- AP = all pairs
Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$
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- For simplicity, extend the $w$ values to non-edges:

$$w(i, j) = \begin{cases} 
0 & i = j \\
\text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\
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$$f^k[i, j]: \text{length of shortest path from } i \text{ to } j \text{ that only uses vertices } \{1, 2, 3, \ldots, k\} \text{ as intermediate vertices}$$
Example for Definition of $f^k[i, j]$’s

- $f^0[1, 4] = \infty$
- $f^1[1, 4] = \infty$
- $f^2[1, 4] = 140$ (1 → 2 → 4)
- $f^3[1, 4] = 90$ (1 → 3 → 2 → 4)
- $f^4[1, 4] = 90$ (1 → 3 → 2 → 4)
- $f^5[1, 4] = 60$ (1 → 3 → 5 → 4)
\[ w(i, j) = \begin{cases} 
0 & i = j \\
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\[ f^k[i, j] = \begin{cases} 
\text{weight of edge } (i, j) & k = 0 \\
\min\{ f^{k-1}[i, j], f^{k-1}[i, k] + f^{k-1}[k, j] \} & k = 1, 2, \ldots, n 
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\end{cases} \]
Floyd-Warshall($G, w$)

1: $f^0 \leftarrow w$
2: \textbf{for} $k \leftarrow 1$ to $n$ \textbf{do}
3: \hspace{1em} copy $f^{k-1} \rightarrow f^k$
4: \textbf{for} $i \leftarrow 1$ to $n$ \textbf{do}
5: \hspace{1em} \textbf{for} $j \leftarrow 1$ to $n$ \textbf{do}
6: \hspace{2em} \textbf{if} $f^{k-1}[i, k] + f^{k-1}[k, j] < f^k[i, j]$ \textbf{then}
7: \hspace{3em} $f^k[i, j] \leftarrow f^{k-1}[i, k] + f^{k-1}[k, j]$
Floyd-Warshall($G, w$)

1: $f^{\text{old}} \leftarrow w$
2: for $k \leftarrow 1$ to $n$ do
3: copy $f^{\text{old}} \rightarrow f^{\text{new}}$
4: for $i \leftarrow 1$ to $n$ do
5: for $j \leftarrow 1$ to $n$ do
6: if $f^{\text{old}}[i, k] + f^{\text{old}}[k, j] < f^{\text{new}}[i, j]$ then
7: $f^{\text{new}}[i, j] \leftarrow f^{\text{old}}[i, k] + f^{\text{old}}[k, j]$

Lemma
Assume there are no negative cycles in $G$. After iteration $k$, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from $i$ to $j$ that only uses vertices in \{1, 2, 3, ..., $k$\} as intermediate vertices.

Running time = $O(n^3)$. 


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Floyd-Warshall\((G, w)\)

1: \( f \leftarrow w \)
2: \( \text{for } k \leftarrow 1 \text{ to } n \text{ do} \)
3: \( \text{copy } f \rightarrow f \)
4: \( \text{for } i \leftarrow 1 \text{ to } n \text{ do} \)
5: \( \quad \text{for } j \leftarrow 1 \text{ to } n \text{ do} \)
6: \( \quad \text{if } f[i, k] + f[k, j] < f[i, j] \text{ then} \)
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- Running time $= O(n^3)$. 
\[
\begin{array}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 90 & 30 & \infty & \infty \\
2 & 10 & 0 & \infty & 50 & \infty \\
3 & 60 & 10 & 0 & 70 & 20 \\
4 & \infty & \infty & \infty & 0 & 20 \\
5 & \infty & \infty & \infty & 10 & 0 \\
\end{array}
\]
\[ i = 2, \quad k = 1, \quad j = 3 \]
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\[
\begin{array}{ccccc}
   & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 90 & 30 & 140 & \infty \\
2 & 10 & 0 & 40 & 50 & \infty \\
3 & 60 & 10 & 0 & 70 & 20 \\
4 & \infty & \infty & \infty & 0 & 20 \\
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\[i = 3, \ k = 2, \ j = 1,\]
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Floyd-Warshall\((G, w)\)

1: \( f \leftarrow w, \pi[i, j] \leftarrow \bot \) for every \( i, j \in V \)
2: \textbf{for} \( k \leftarrow 1 \) to \( n \) \textbf{do}
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Recovering Shortest Paths

Floyd-Warshall\((G, w)\)

1: \(f \leftarrow w, \pi[i, j] \leftarrow \perp\) for every \(i, j \in V\)
2: for \(k \leftarrow 1\) to \(n\) do
3: for \(i \leftarrow 1\) to \(n\) do
4: for \(j \leftarrow 1\) to \(n\) do
5: if \(f[i, k] + f[k, j] < f[i, j]\) then
6: \(f[i, j] \leftarrow f[i, k] + f[k, j], \pi[i, j] \leftarrow k\)

print-path\((i, j)\)

1: if \(\pi[i, j] = \perp\) then then
2: if \(i \neq j\) then print\((i, \text{"","})\)
3: else
4: print-path\((i, \pi[i, j])\), print-path\((\pi[i, j], j)\)
Detecting Negative Cycles

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2: for $k \leftarrow 1$ to $n$ do
3:    for $i \leftarrow 1$ to $n$ do
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8: \hspace{1em} \textbf{for} \( i \leftarrow 1 \) to \( n \) \textbf{do}
9: \hspace{2em} \textbf{for} \( j \leftarrow 1 \) to \( n \) \textbf{do}
10: \hspace{3em} \textbf{if} \( f[i, k] + f[k, j] < f[i, j] \) \textbf{then}
11: \hspace{3em} \textbf{report} “negative cycle exists” and \textbf{exit}
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