CSE 431/531: Algorithm Analysis and Design (Fall 2022)
Graph Algorithms

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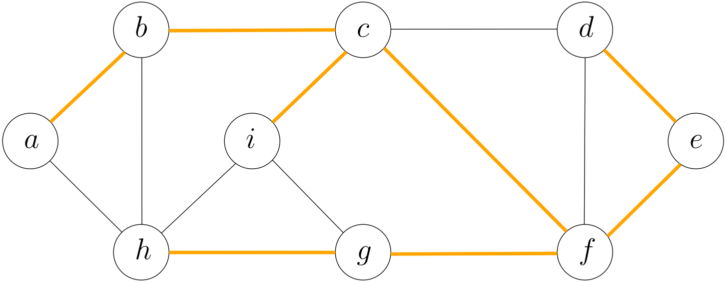
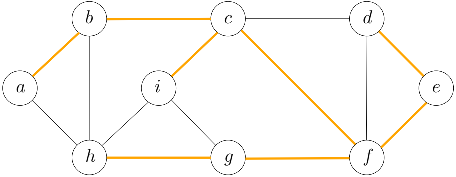
Outline

1. Minimum Spanning Tree
   - Kruskal's Algorithm
   - Reverse-Kruskal's Algorithm
   - Prim's Algorithm
2. Single Source Shortest Paths
   - Dijkstra's Algorithm
3. Shortest Paths in Graphs with Negative Weights
4. All-Pair Shortest Paths and Floyd-Warshall

Spanning Tree

Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of G is a sub-graph of G that is a tree including all vertices V.





Lemma Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- T is a spanning tree of G;
- T is acyclic and connected;
- T is connected and has $n - 1$ edges;
- T is acyclic and has $n - 1$ edges;
- T is minimally connected: removal of any edge disconnects it;
- T is maximally acyclic: addition of any edge creates a cycle;
- T has a unique simple path between every pair of nodes.

Minimum Spanning Tree (MST) Problem

Input: Graph $G = (V, E)$ and edge weights $w : E \to \mathbb{R}$

Output: the spanning tree T of G with the minimum total weight

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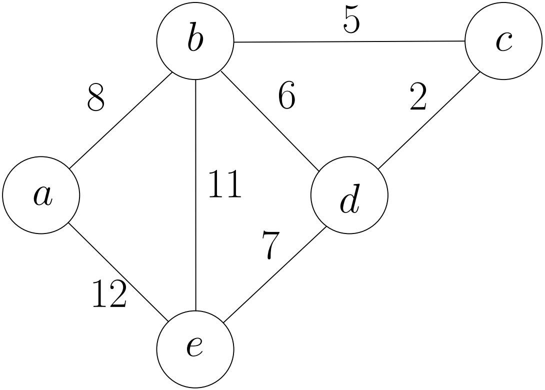

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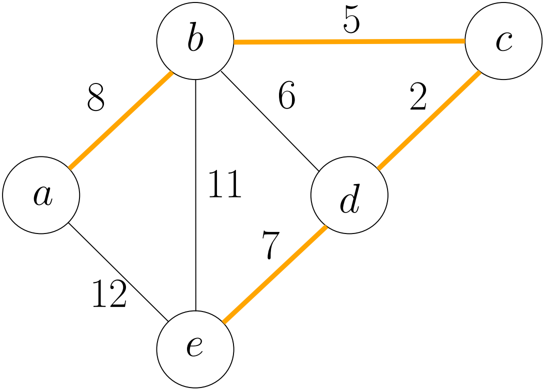

Recall: Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

Def. A choice is “safe” if there is an optimum solution that is “consistent” with the choice

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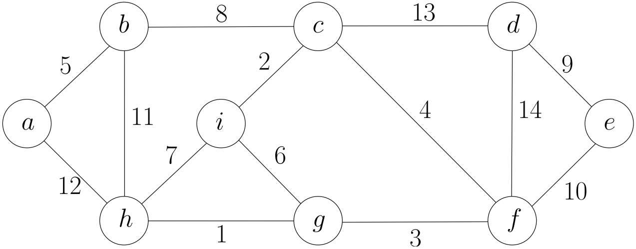
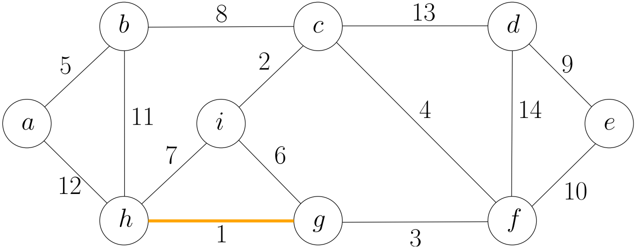
Two Classic Greedy Algorithms for MST

- Kruskal's Algorithm
- Prim's Algorithm

Outline



Q: Which edge can be safely included in the MST?



Q: Which edge can be safely included in the MST?

A: The edge with the smallest weight (lightest edge).

Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

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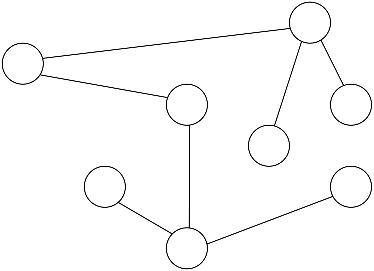
Proof.

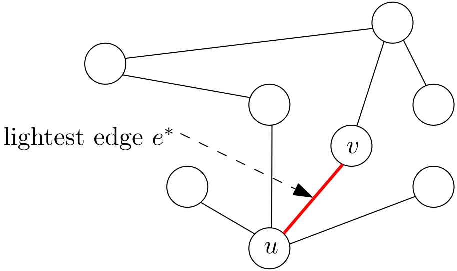
- Take a minimum spanning tree T



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
- Assume the lightest edge e^* is not in T



Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree T
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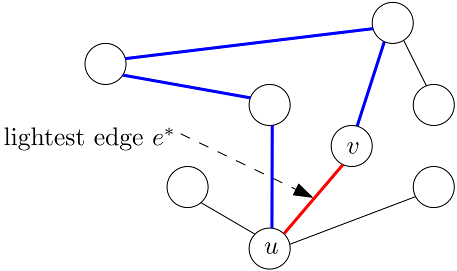

Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

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- Take a minimum spanning tree T
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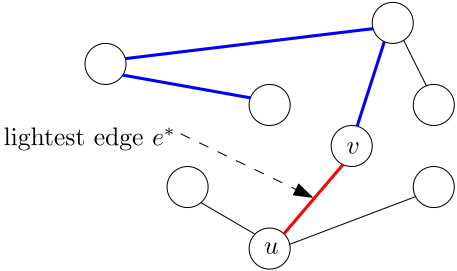
- Remove any edge e in the path to obtain tree T'



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- Assume the lightest edge e^* is not in T
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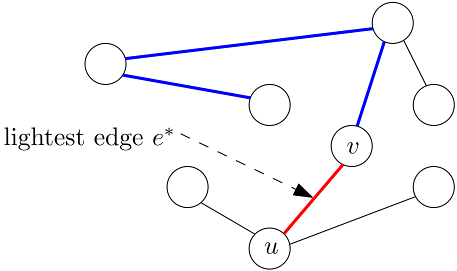
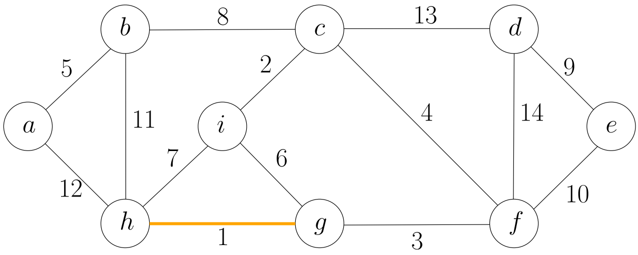
- $w(e^*) \le w(e) \implies w(T') \le w(T)$: T' is also a MST $\square$



Is the Residual Problem Still a MST Problem?



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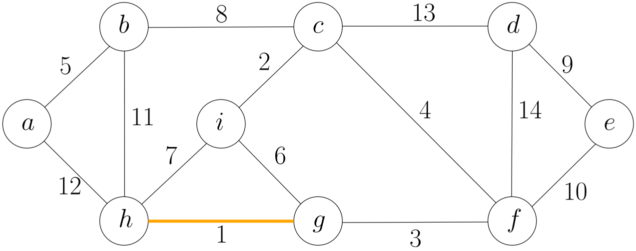

- Residual problem: find the minimum spanning tree that contains edge (g, h)

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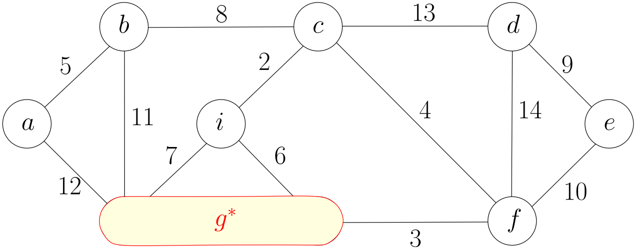

- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)

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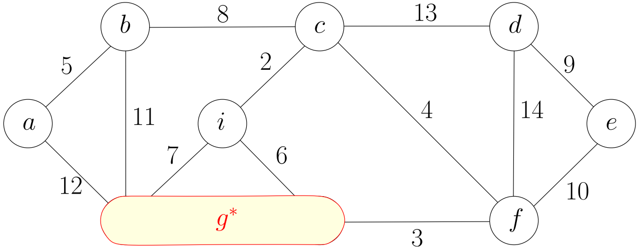
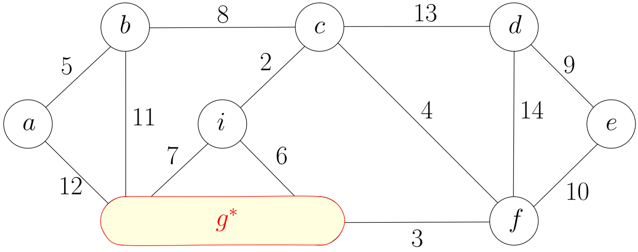


- Residual problem: find the minimum spanning tree that contains edge (g, h)
- Contract the edge (g, h)
- Residual problem: find the minimum spanning tree in the contracted graph

Contraction of an Edge (u, v)



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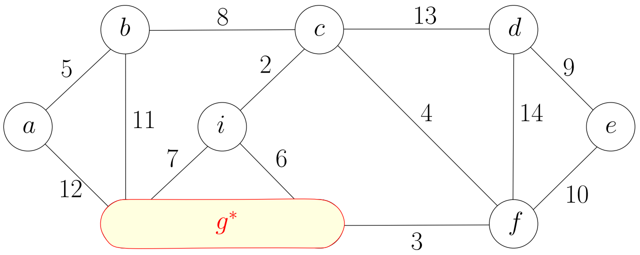
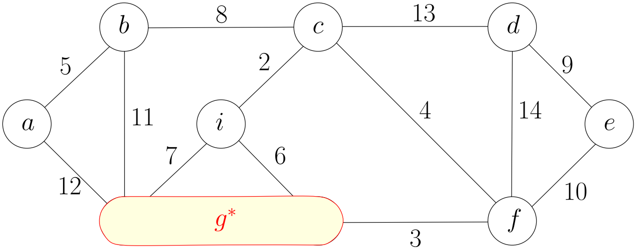


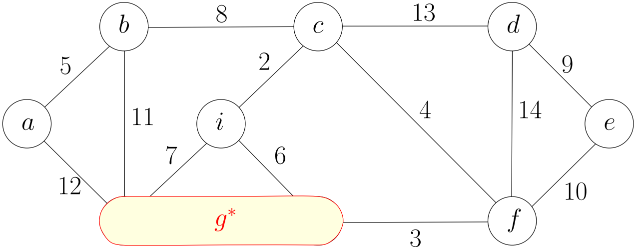
- Remove u and v from the graph, and add a new vertex u^*

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E

Contraction of an Edge (u, v)



- Remove u and v from the graph, and add a new vertex u^*
- Remove all edges (u, v) from E
- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)

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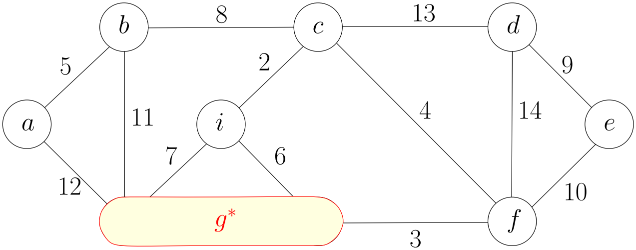

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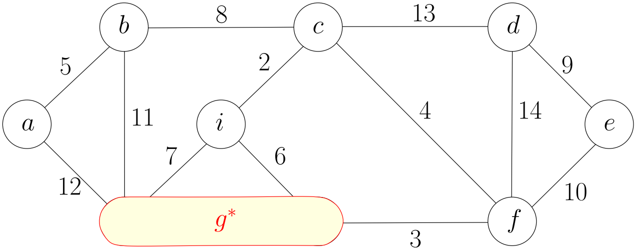

- Remove u and v from the graph, and add a new vertex u^*
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- For every edge $(u, w) \in E, w \neq v$, change it to (u^*, w)
- For every edge $(v, w) \in E, w \neq u$, change it to (u^*, w)
- May create parallel edges! E.g. : two edges (i, g^*)

Greedy Algorithm

Repeat the following step until G contains only one vertex:

1. Choose the lightest edge e^*, add e^* to the spanning tree
2. Contract e^* and update G be the contracted graph

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Q: What edges are removed due to contractions?

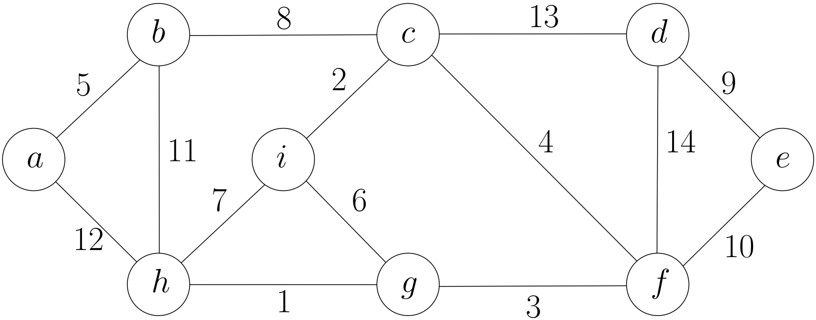
A: Edge (u, v) is removed if and only if there is a path connecting u and v formed by edges we selected

Greedy Algorithm

MST-Greedy(G, w)

```
F ← ∅
sort edges in E in non-decreasing order of weights w
for each edge (u, v) in the order do
    if u and v are not connected by a path of edges in F then
        F ← F ∪ {(u, v)}
return (V, F)
```

Kruskal's Algorithm: Example



Sets: $\{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}$

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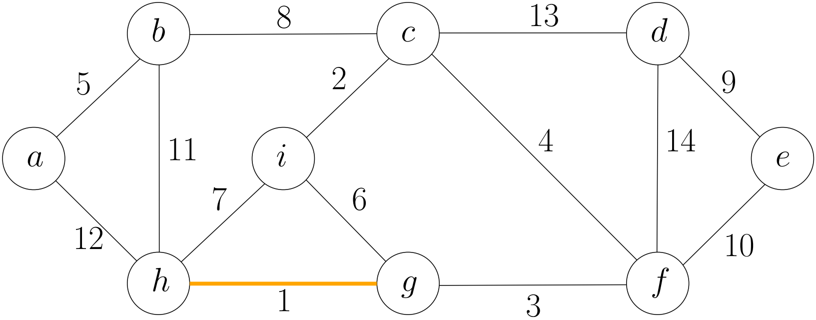

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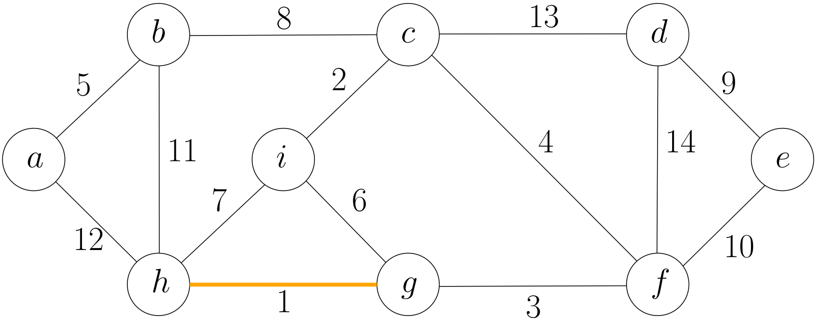

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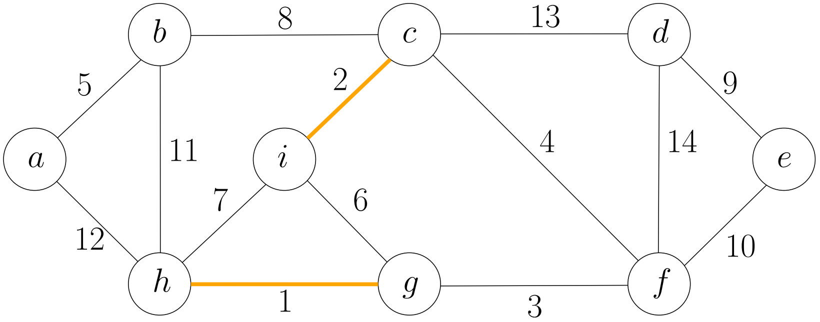

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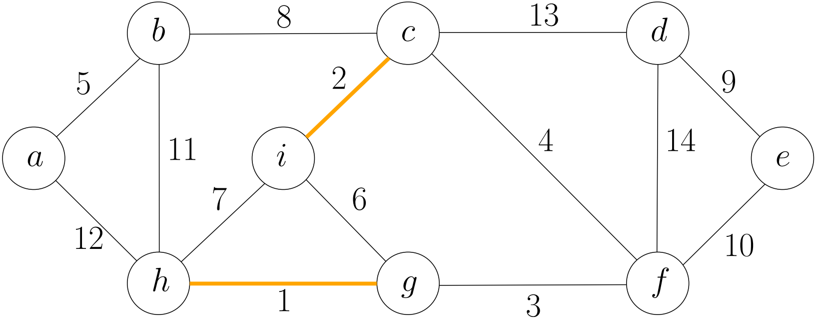

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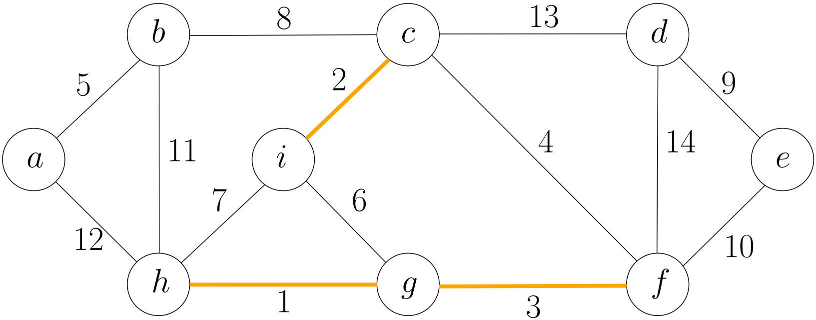

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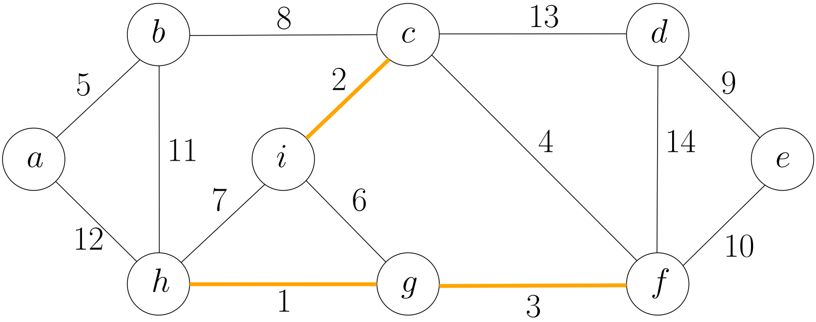

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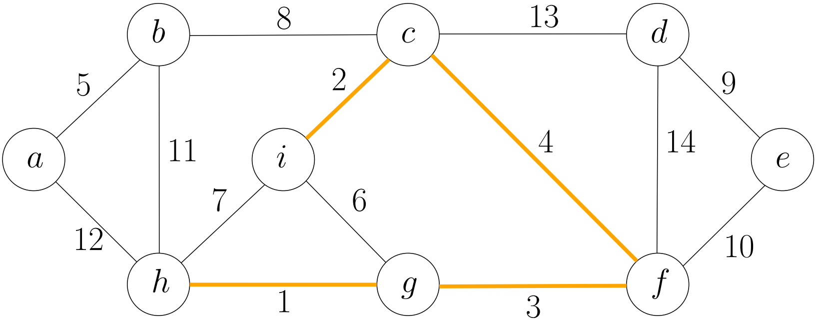

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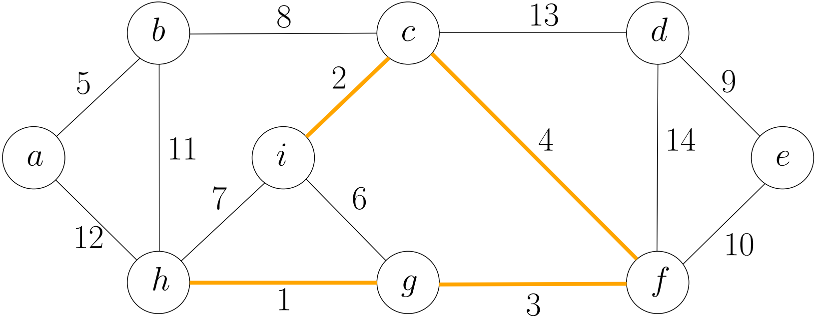

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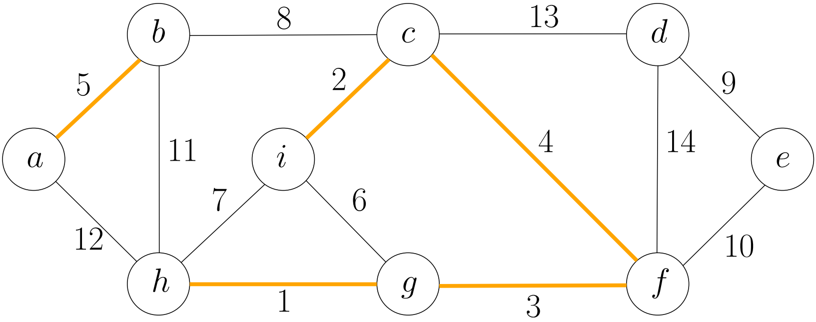

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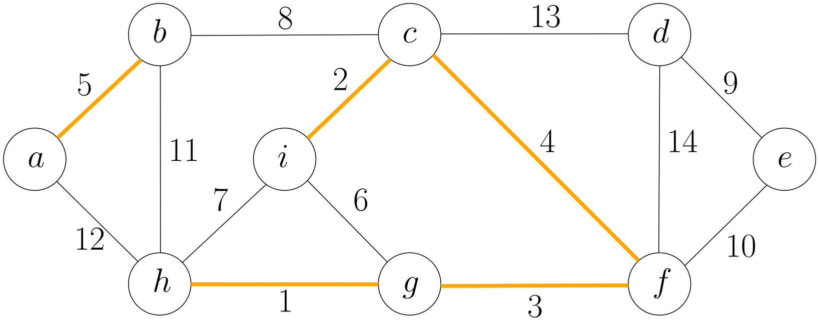

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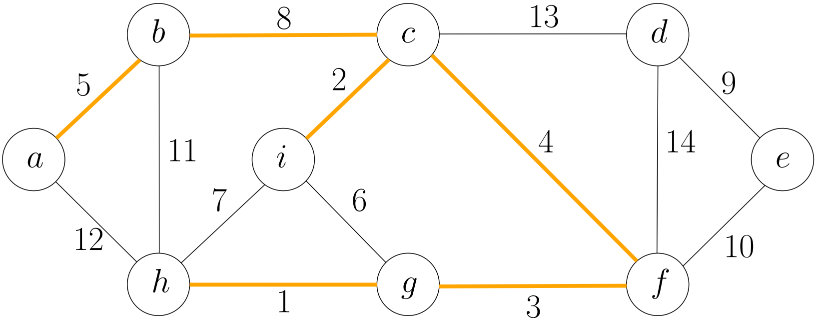

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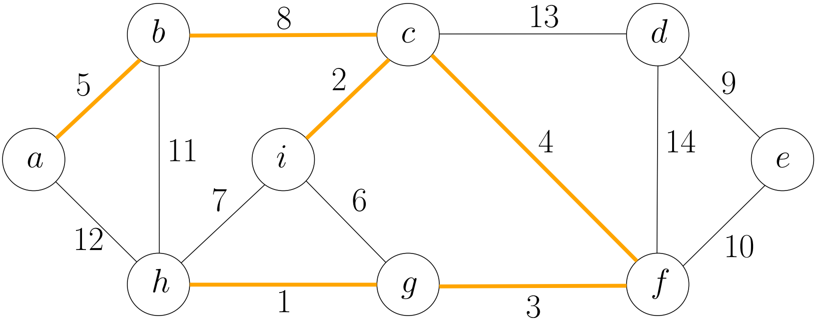

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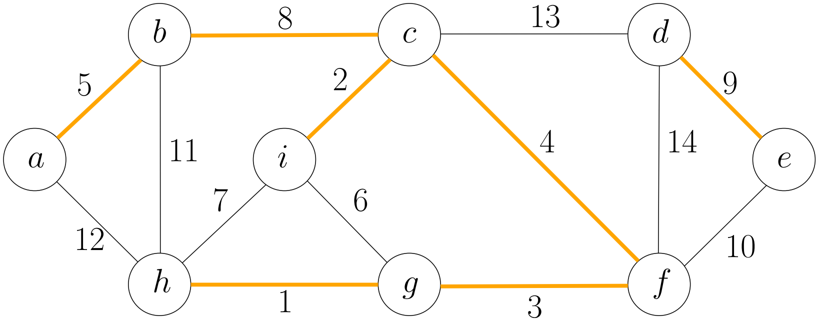

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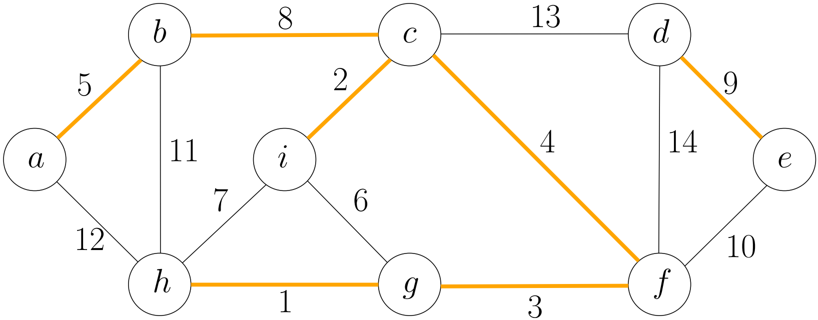

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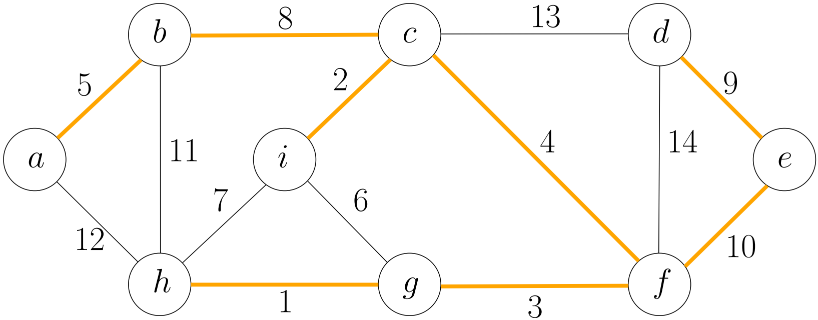

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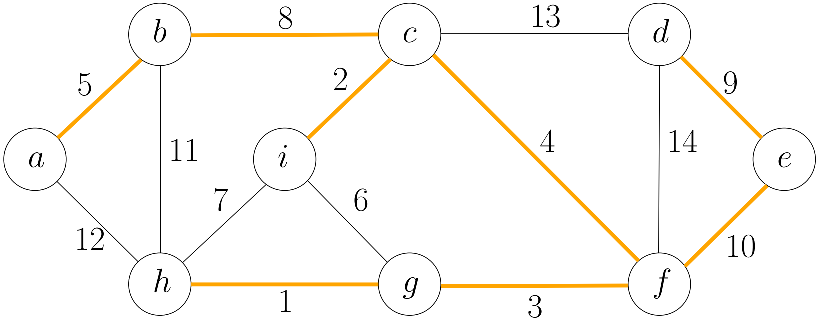


Sets: $\{a, b, c, i, f, g, h, d, e\}$

Kruskal's Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal(G, w)

$F \leftarrow \emptyset$
$\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
sort the edges of E in non-decreasing order of weights w
for each edge $(u, v) \in E$ in the order **do**
    $S_u \leftarrow$ the set in $\mathcal{S}$ containing u
    $S_v \leftarrow$ the set in $\mathcal{S}$ containing v
    if $S_u \neq S_v$ **then**
        $F \leftarrow F \cup \{(u, v)\}$
        $\mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
return (V, F)

Running Time of Kruskal's Algorithm

MST-Kruskal(G, w)

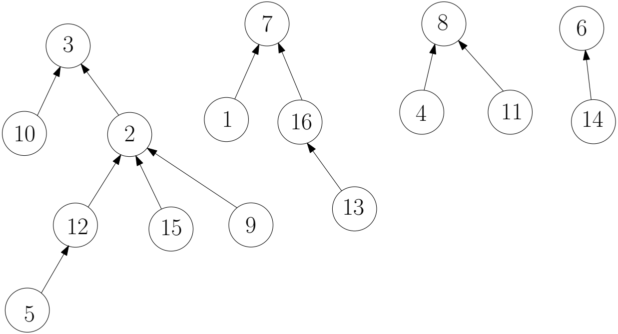
1: $F \leftarrow \emptyset$
2: $\mathcal{S} \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of E in non-decreasing order of weights w
4: **for** each edge $(u, v) \in E$ in the order **do**
5: $\quad S_u \leftarrow$ the set in $\mathcal{S}$ containing u
6: $\quad S_v \leftarrow$ the set in $\mathcal{S}$ containing v
7: $\quad$ **if** $S_u \neq S_v$ **then**
8: $\qquad F \leftarrow F \cup \{(u, v)\}$
9: $\qquad \mathcal{S} \leftarrow \mathcal{S} \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: **return** (V, F)

Use union-find data structure to support ❷, ❺, ❻, ❼, ❾.

Union-Find Data Structure

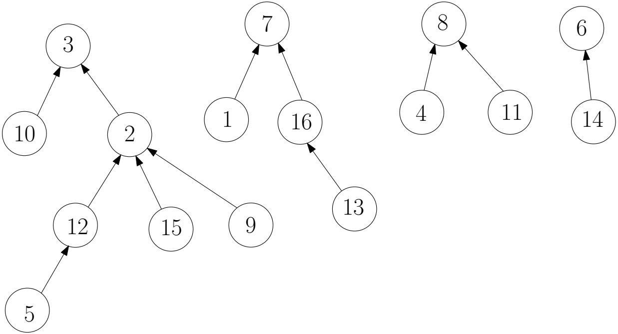
- V: ground set
- We need to maintain a partition of V and support following operations:
  - Check if u and v are in the same set of the partition
  - Merge two sets in partition

- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}, \{1, 7, 13, 16\}, \{4, 8, 11\}, \{6, 14\}$

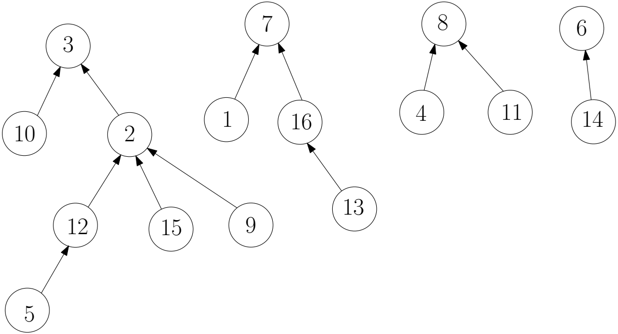


- $par[i]$: parent of i, ($par[i] = \perp$ if i is a root).

Union-Find Data Structure

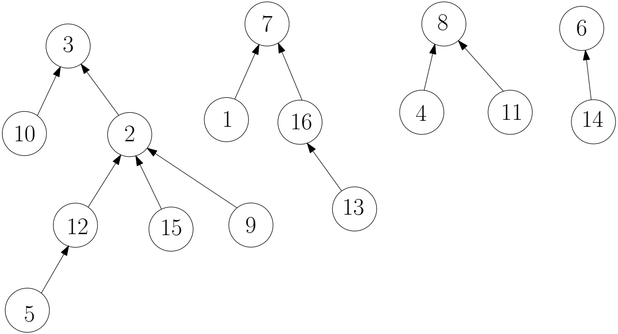


Union-Find Data Structure



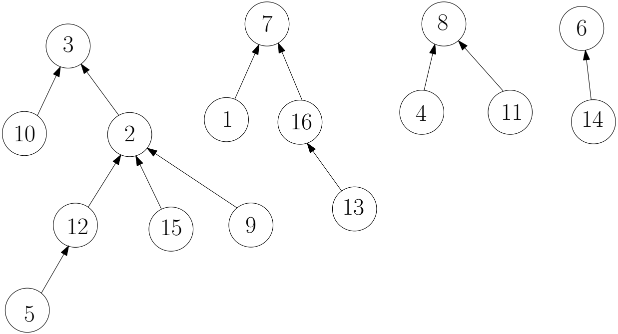
- Q: how can we check if u and v are in the same set?

Union-Find Data Structure



- Q: how can we check if u and v are in the same set?
- A: Check if root(u) = root(v).

Union-Find Data Structure



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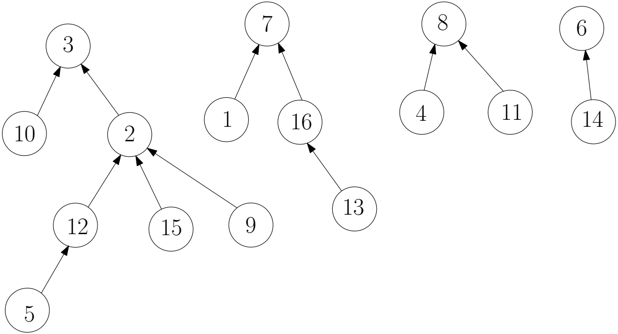
- root(u): the root of the tree containing u

Union-Find Data Structure



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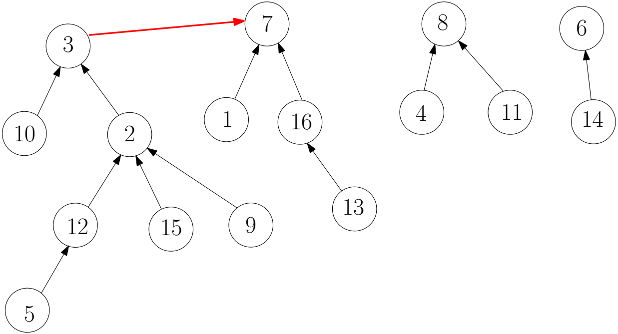
- Merge the trees with root r and r': $par[r] \leftarrow r'$.

Union-Find Data Structure



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Union-Find Data Structure

root(v)

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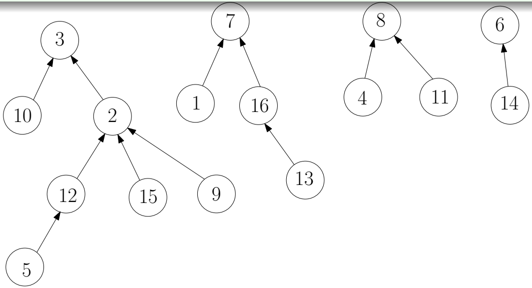

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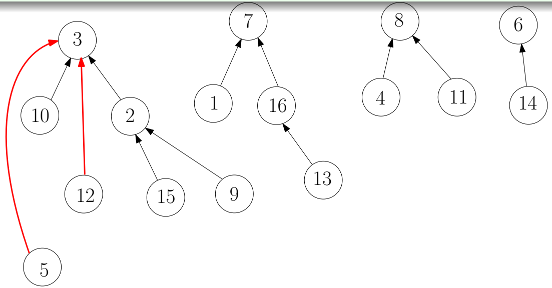

MST-Kruskal(G, w)

```
F ← ∅
𝒮 ← {{v} : v ∈ V}
sort the edges of E in non-decreasing order of weights w
for each edge (u, v) ∈ E in the order do
    S_u ← the set in 𝒮 containing u
    S_v ← the set in 𝒮 containing v
    if S_u ≠ S_v then
        F ← F ∪ {(u, v)}
        𝒮 ← 𝒮 \ {S_u} \ {S_v} ∪ {S_u ∪ S_v}
return (V, F)
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MST-Kruskal(G, w)

$F \leftarrow \emptyset$

for every $v \in V$ **do**: $par[v] \leftarrow \bot$

sort the edges of E in non-decreasing order of weights w

for each edge $(u, v) \in E$ in the order **do**

    $u' \leftarrow \mathrm{root}(u)$

    $v' \leftarrow \mathrm{root}(v)$

    if $u' \neq v'$ **then**

        $F \leftarrow F \cup \{(u, v)\}$

        $par[u'] \leftarrow v'$

return (V, F)

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7:     **if** $u' \neq v'$ **then**
8:         $F \leftarrow F \cup \{(u, v)\}$
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10: **return** (V, F)

- 2,5,6,7,9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.

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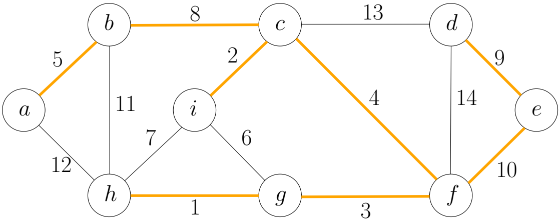
- Running time = time for 3 $= O(m \lg n)$.

Assumption Assume all edge weights are different.

Lemma An edge $e \in E$ is not in the MST, if and only if there is cycle C in G in which e is the heaviest edge.



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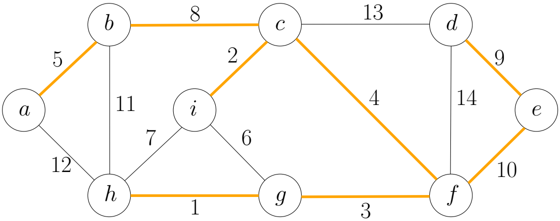

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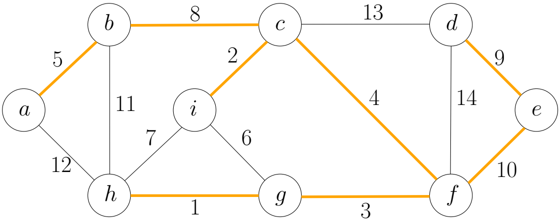

- (i, g) is not in the MST because of cycle (i, c, f, g)
- (e, f) is in the MST because no such cycle exists

Outline

Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to F one by one until we obtain a spanning tree

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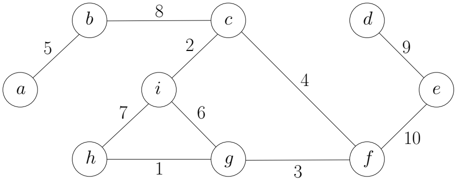

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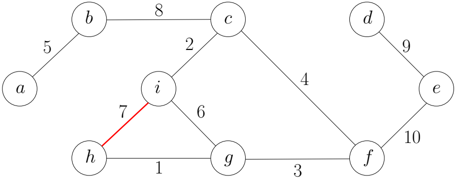

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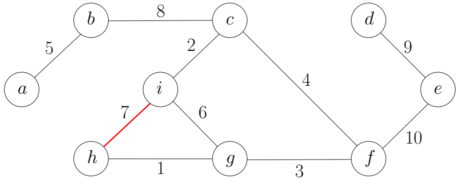

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A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.

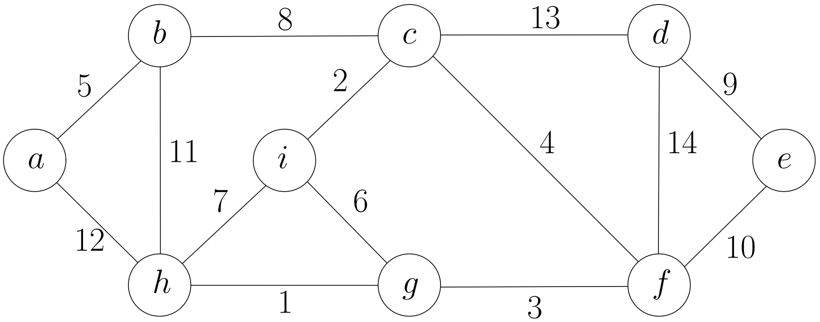
Lemma It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.

Reverse Kruskal's Algorithm

MST-Greedy(G, w)

$F \leftarrow E$
sort E in non-increasing order of weights
for every e in this order **do**
    if $(V, F \setminus \{e\})$ is connected **then**
        $F \leftarrow F \setminus \{e\}$
return (V, F)

Reverse Kruskal's Algorithm: Example



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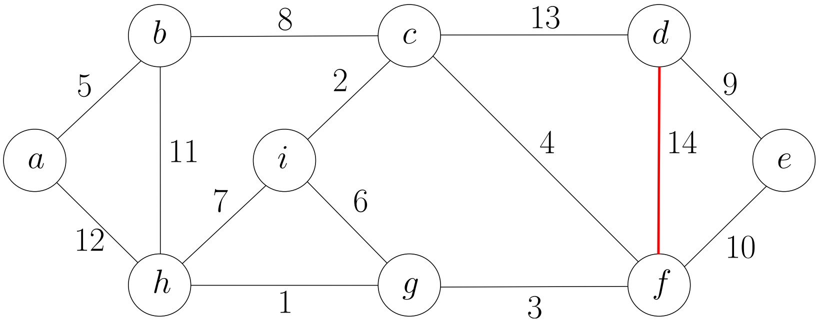

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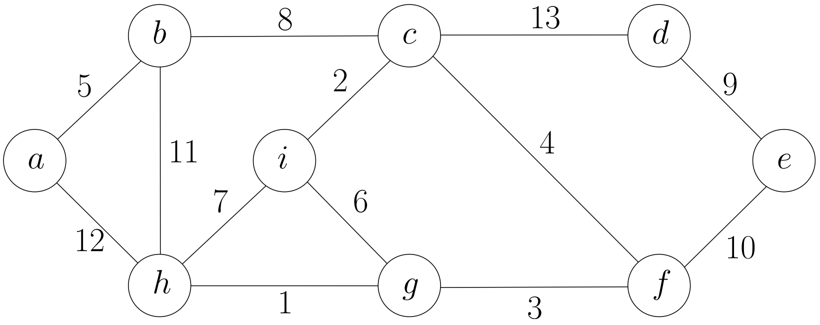

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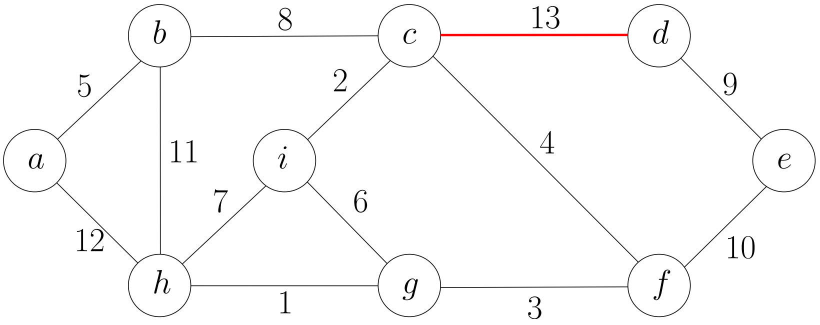

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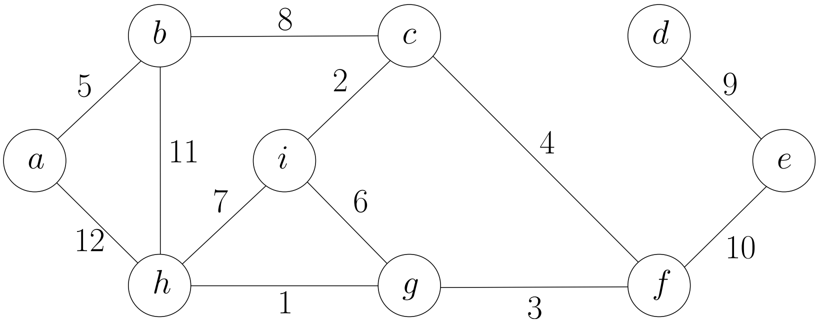

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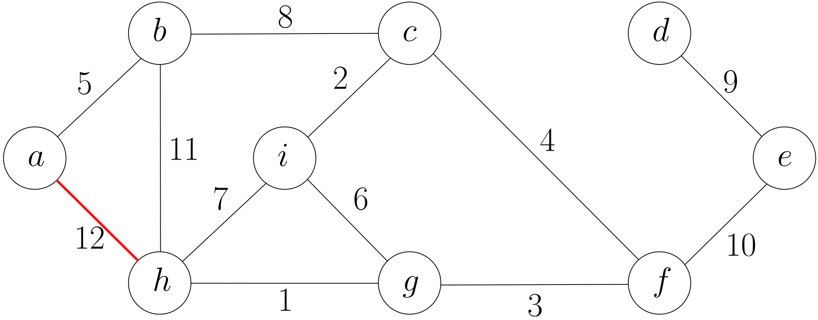

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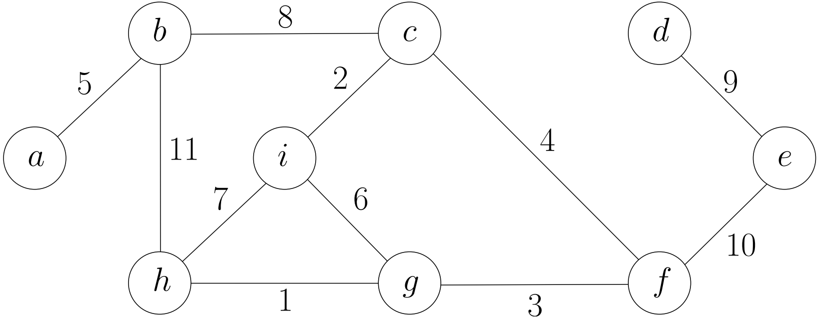

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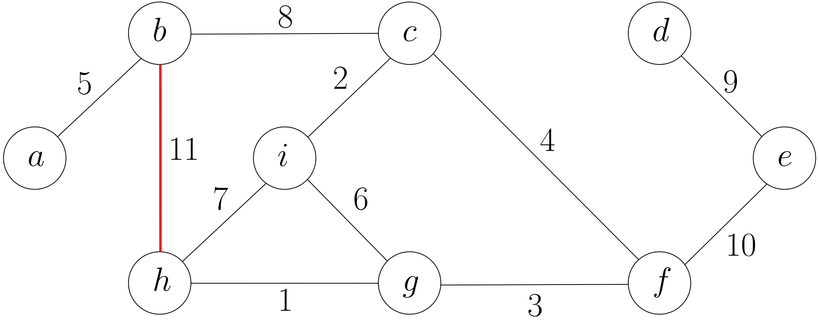

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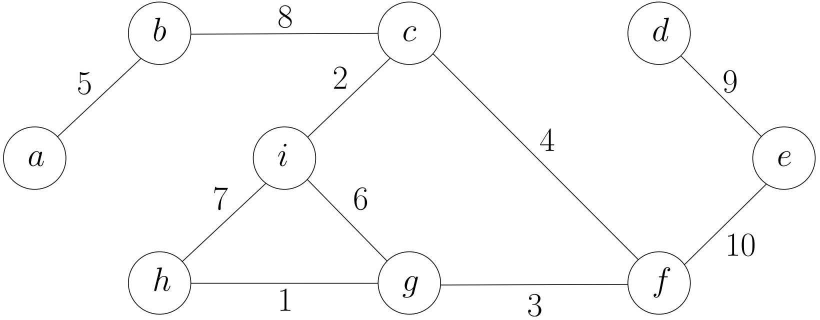

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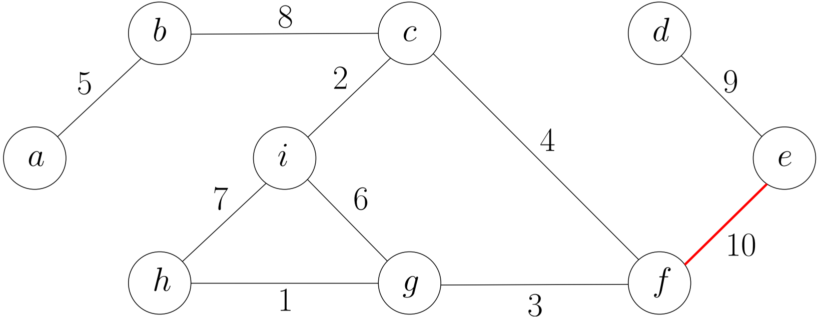

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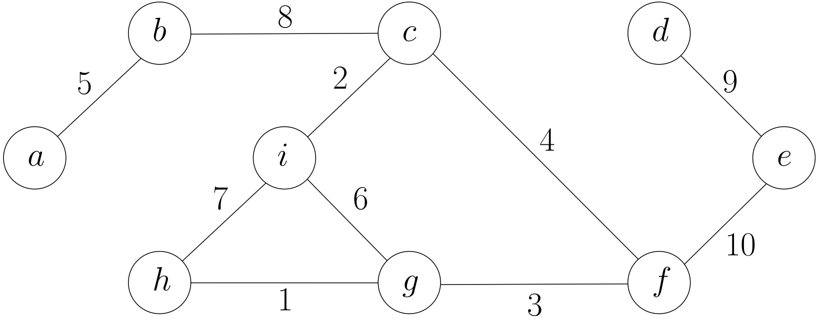

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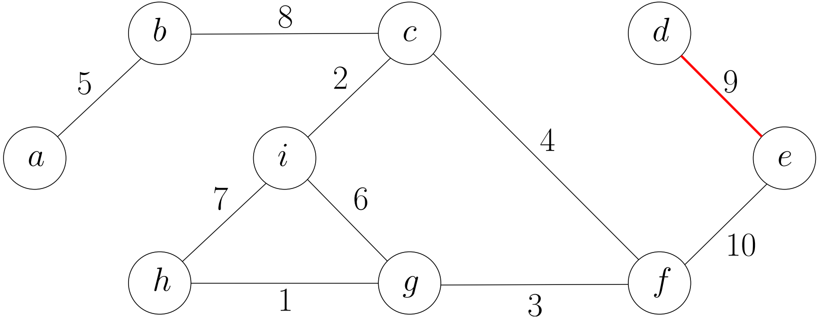

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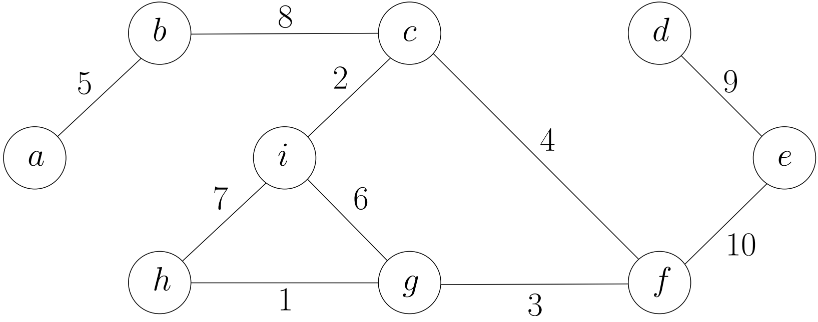

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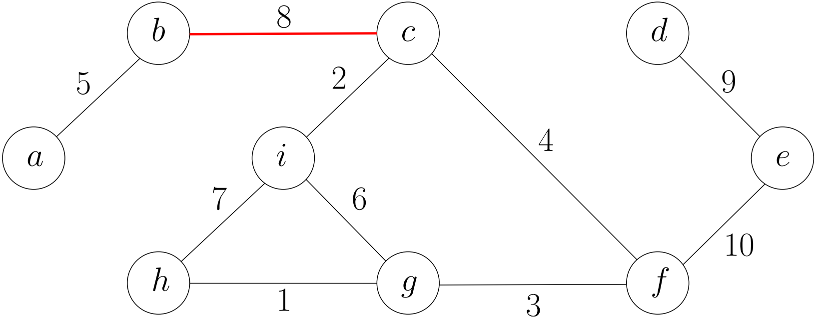

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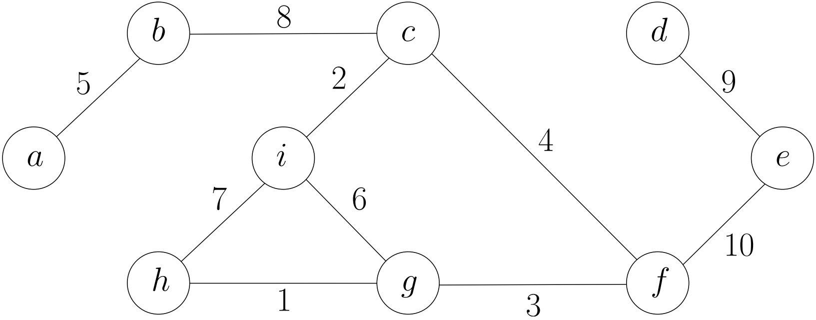

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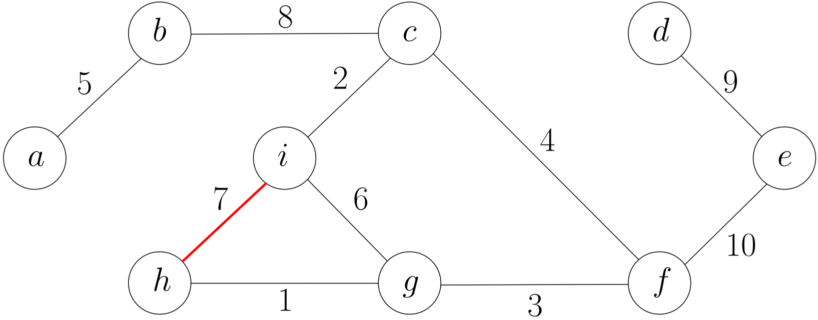

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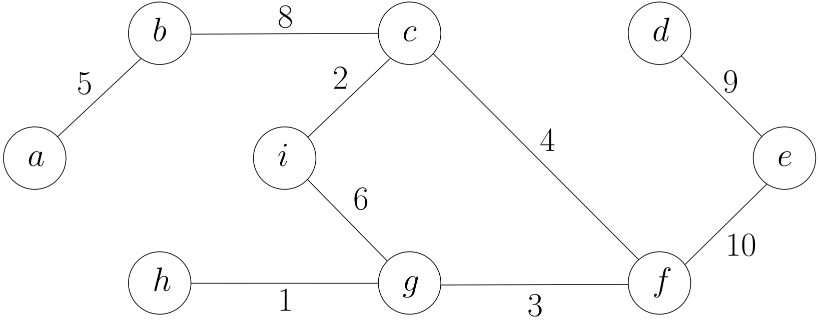

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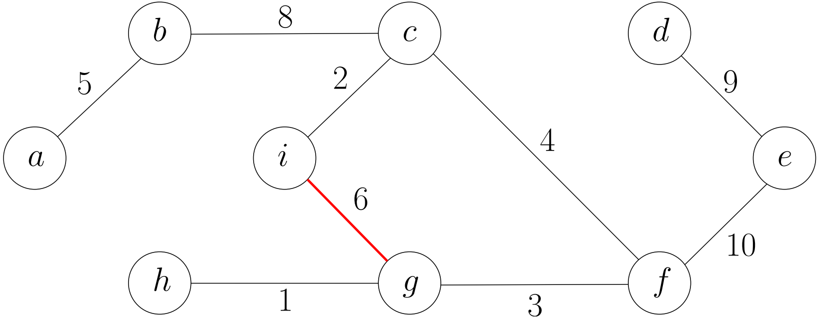

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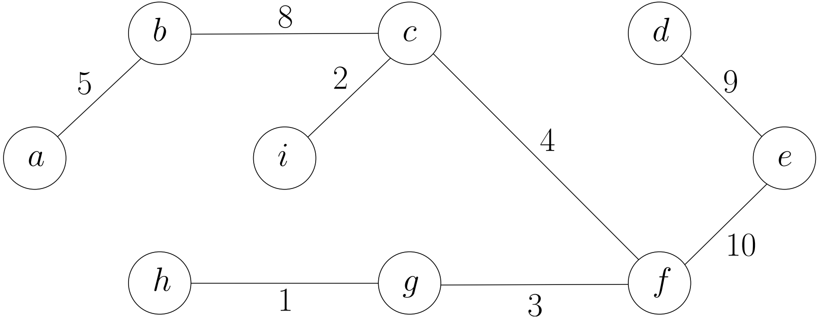
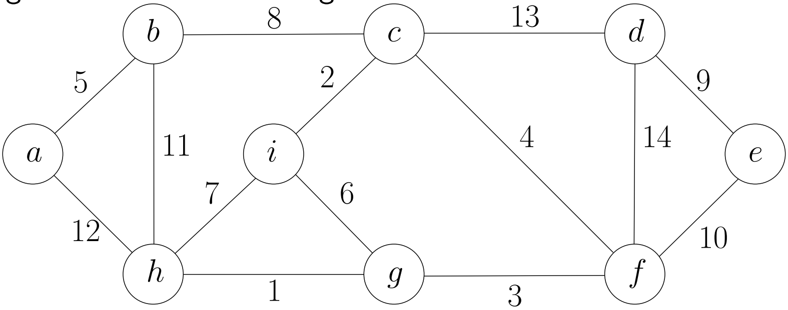


Outline

Design Greedy Strategy for MST

- Recall the greedy strategy for Kruskal's algorithm: choose the edge with the smallest weight.



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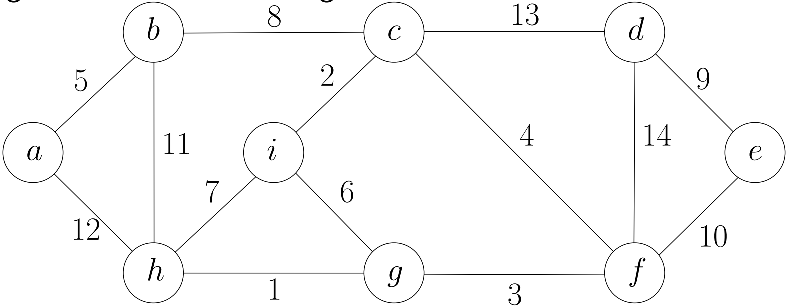

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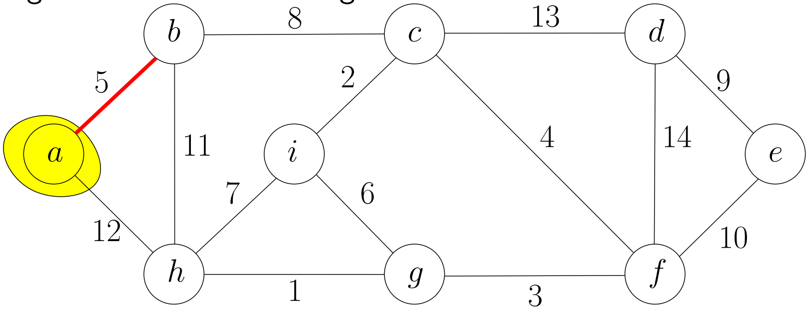

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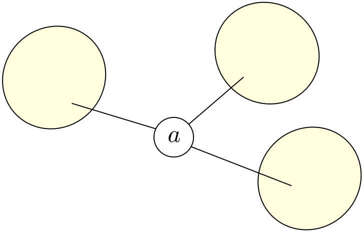

Proof.

- Let T be a MST
- Consider all components obtained by removing a from T

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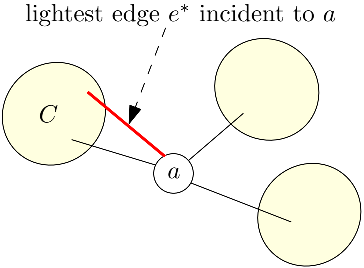

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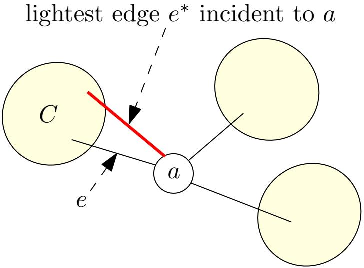

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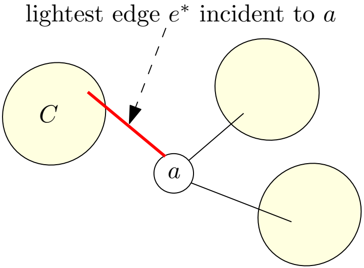

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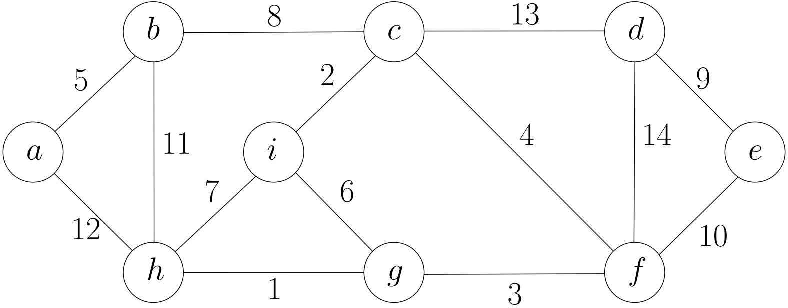
- Let e^* be the lightest edge incident to a and e^* connects a to component C
- Let e be the edge in T connecting a to C
- $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \le w(T)$ $\square$

Prim's Algorithm: Example



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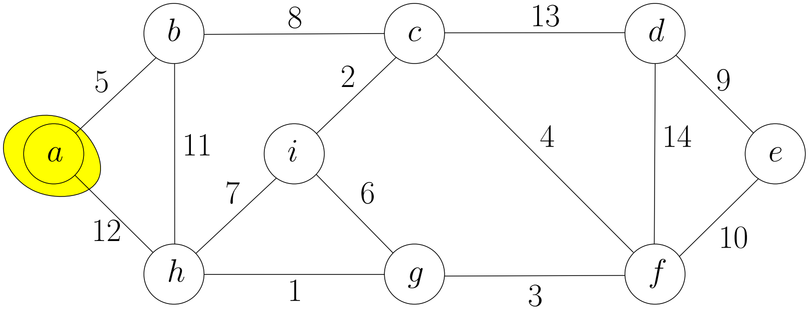

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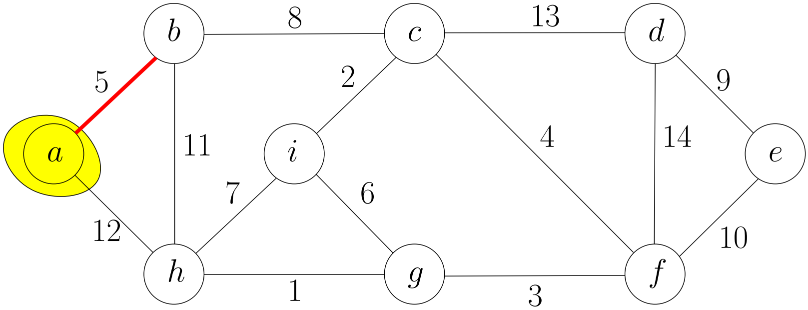

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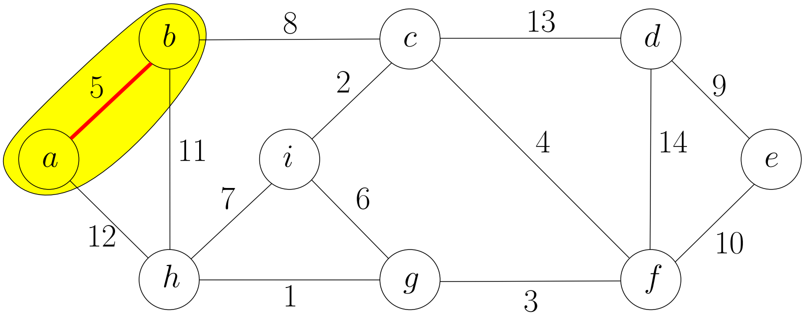

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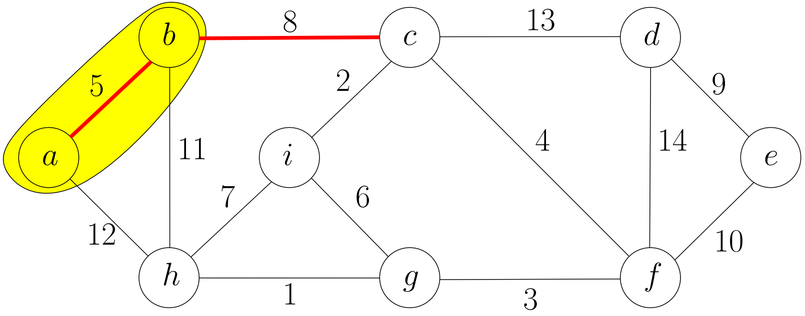

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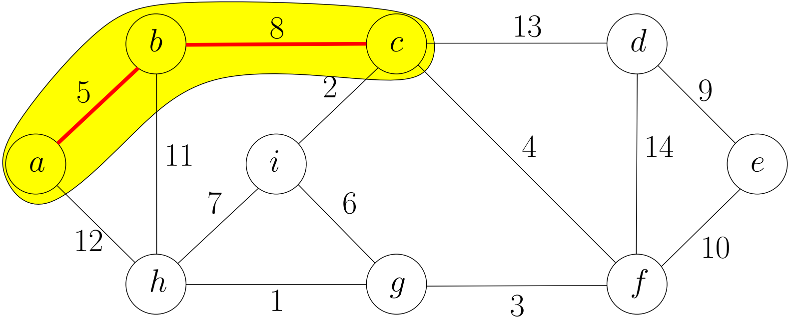

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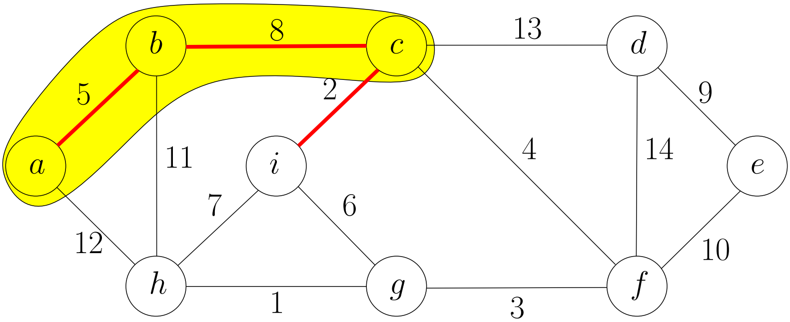

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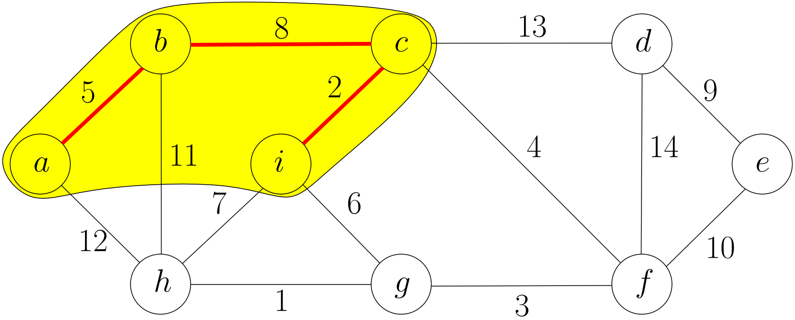

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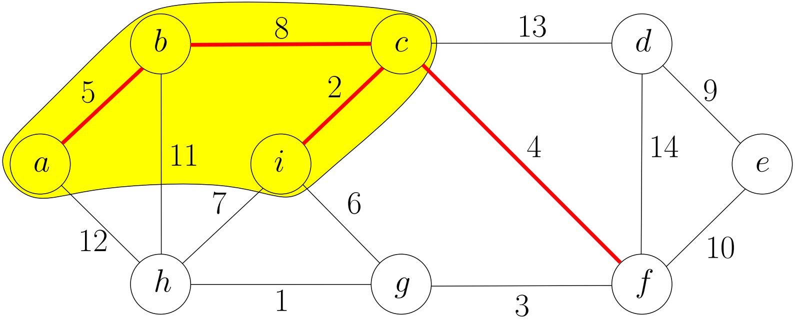

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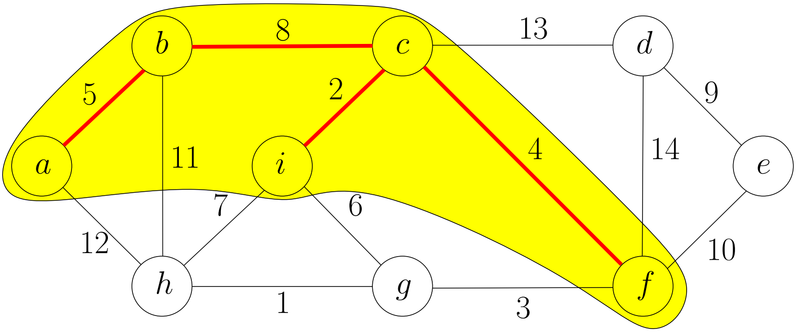

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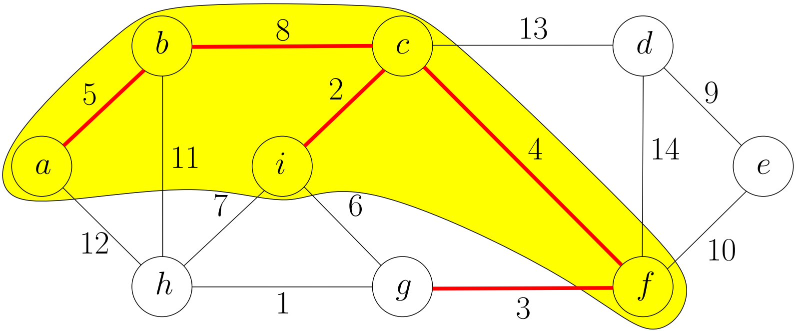

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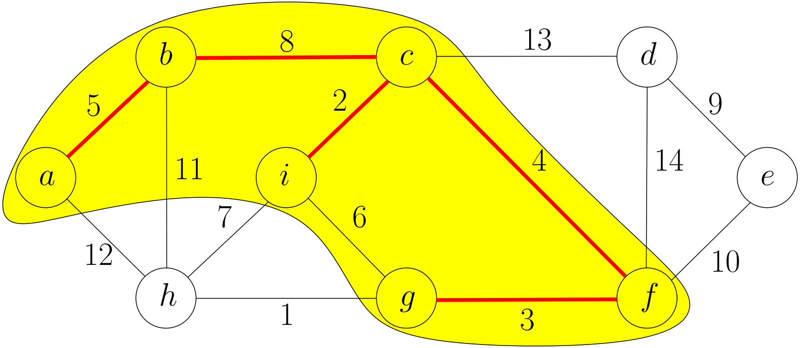

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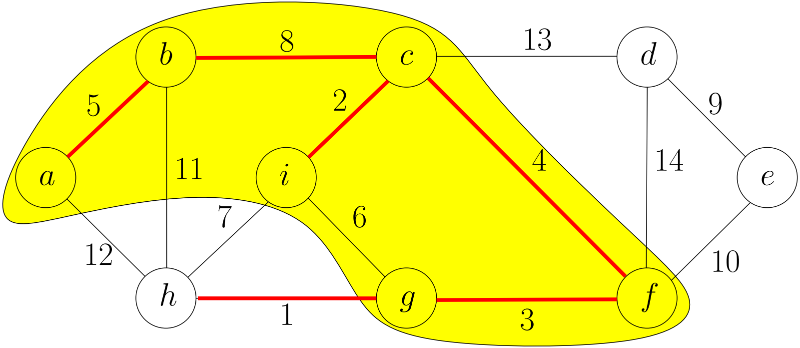

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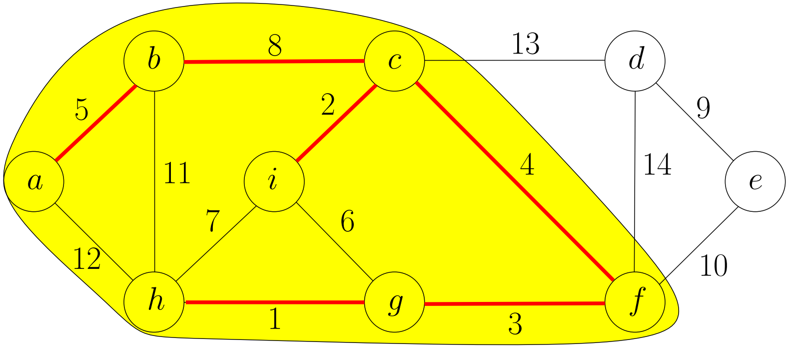

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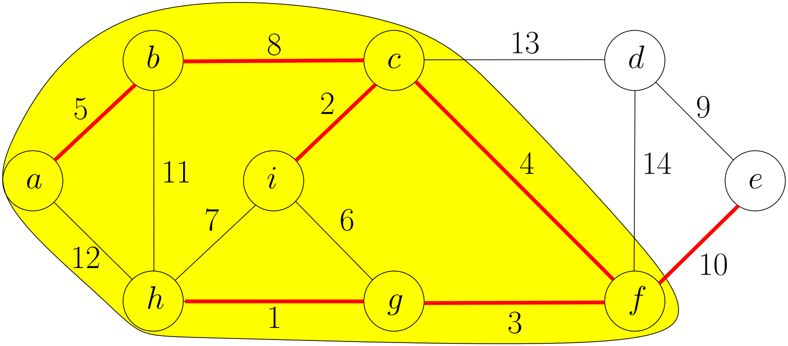

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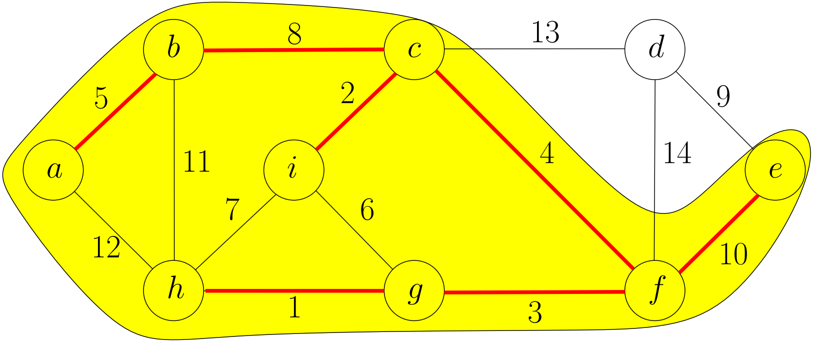

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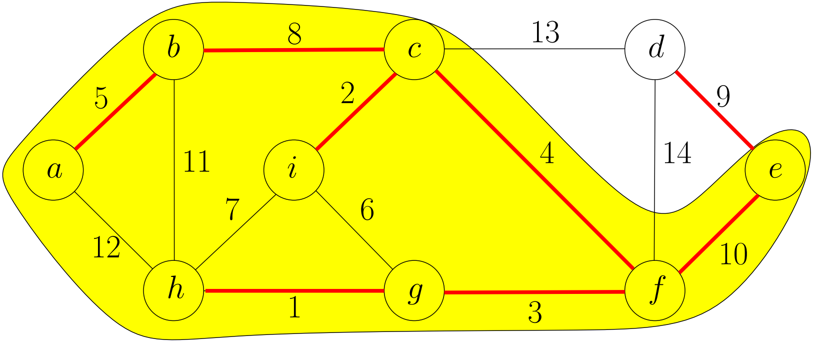
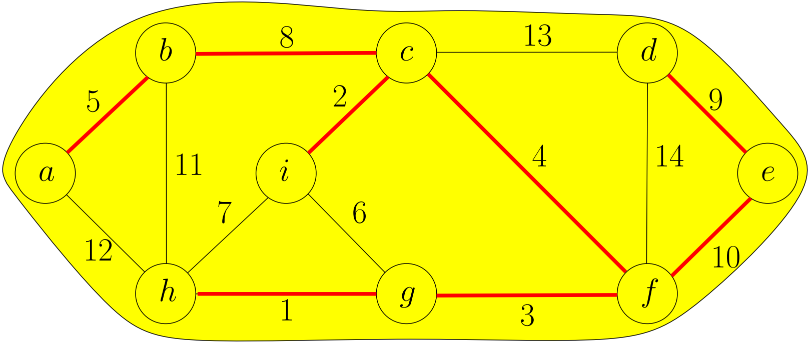


Prim's Algorithm: Example



Greedy Algorithm

MST-Greedy1(G, w)

```
S ← {s}, where s is arbitrary vertex in V
F ← ∅
while S ≠ V do
    (u, v) ← lightest edge between S and V \ S,
                                  where u ∈ S and v ∈ V \ S
    S ← S ∪ {v}
    F ← F ∪ {(u, v)}
return (V, F)
```

Greedy Algorithm

MST-Greedy1(G, w)

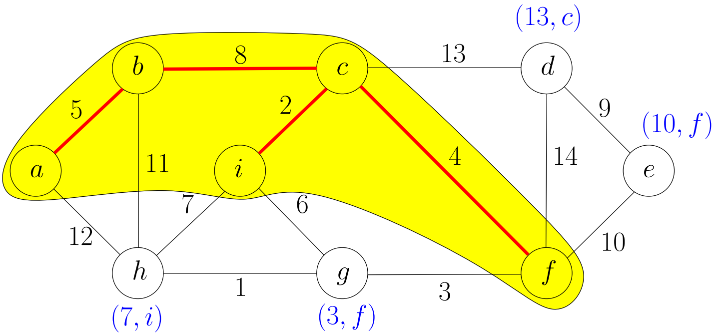
1: $S \leftarrow \{s\}$, where s is arbitrary vertex in V
2: $F \leftarrow \emptyset$
3: **while** $S \neq V$ **do**
4: $\quad (u, v) \leftarrow$ lightest edge between S and $V \setminus S$, where $u \in S$ and $v \in V \setminus S$
5: $\quad S \leftarrow S \cup \{v\}$
6: $\quad F \leftarrow F \cup \{(u, v)\}$
7: **return** (V, F)

- Running time of naive implementation: $O(nm)$

Prim's Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S:(u,v) \in E} w(u,v)$:
  the weight of the lightest edge between v and S
- $\pi[v] = \arg\min_{u \in S:(u,v) \in E} w(u,v)$:
  $(\pi[v], v)$ is the lightest edge between v and S



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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values.

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$
            $\pi[v] \leftarrow u$
return $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$

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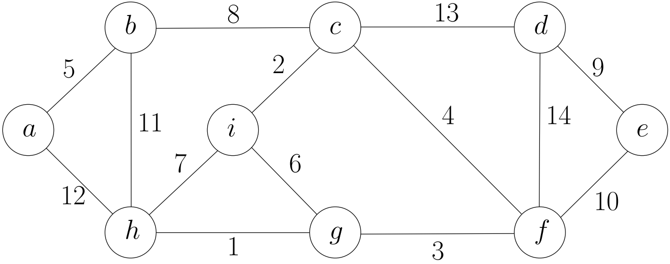

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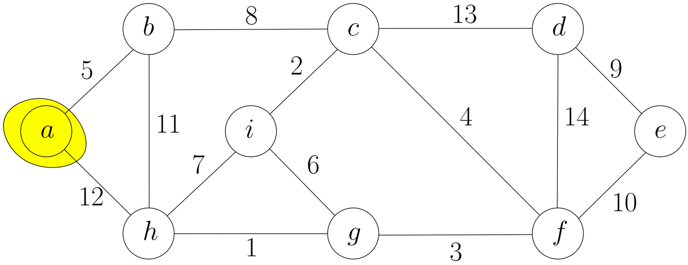

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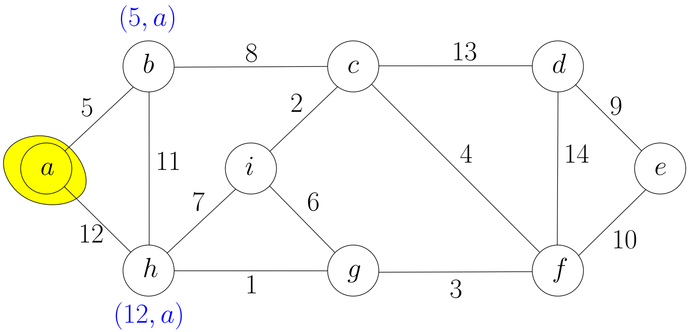

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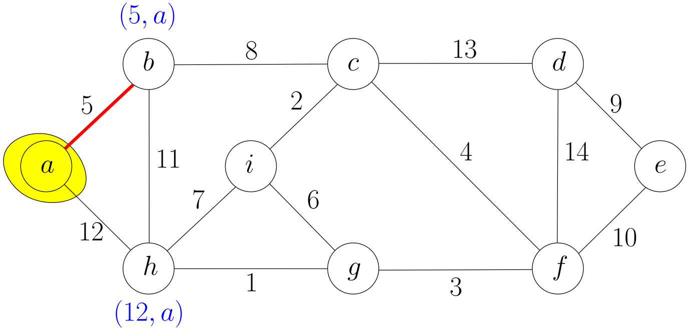

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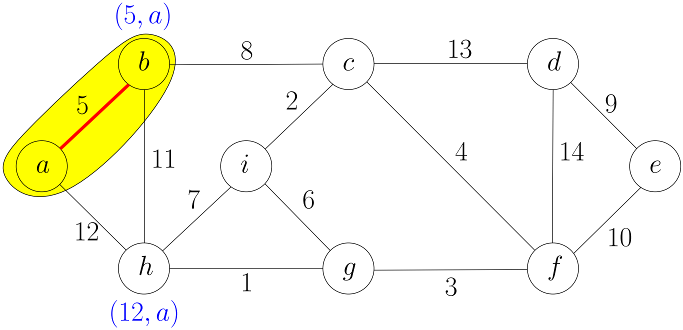

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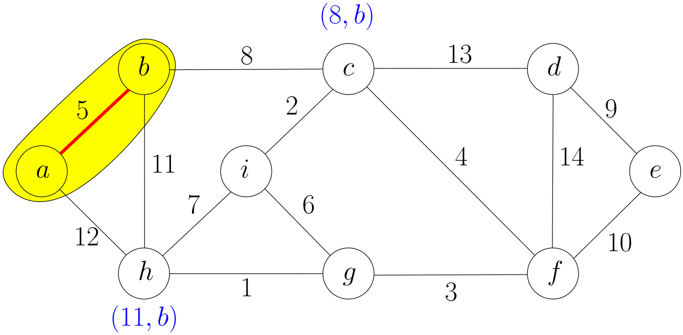

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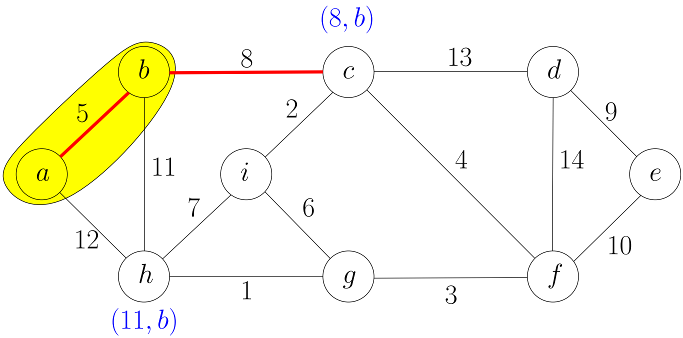

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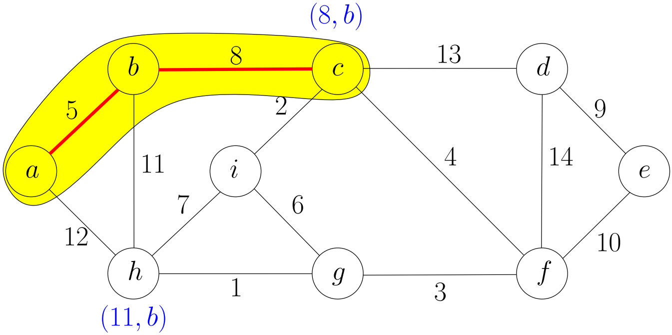

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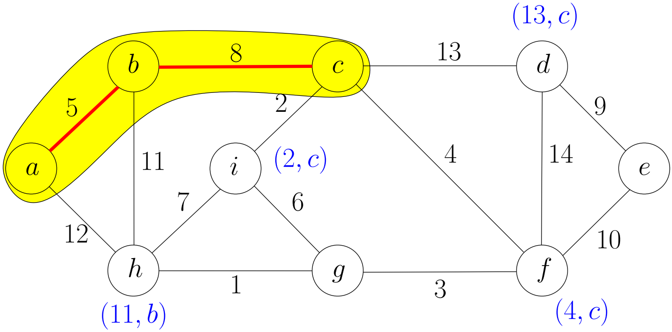

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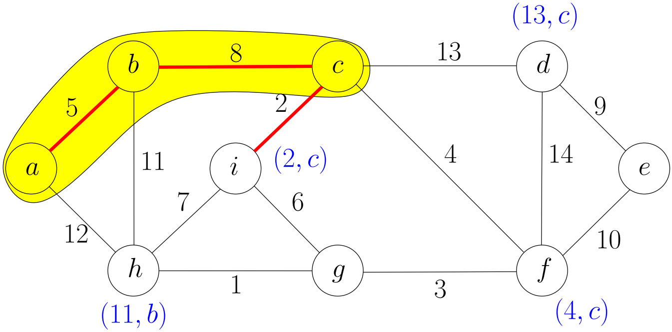

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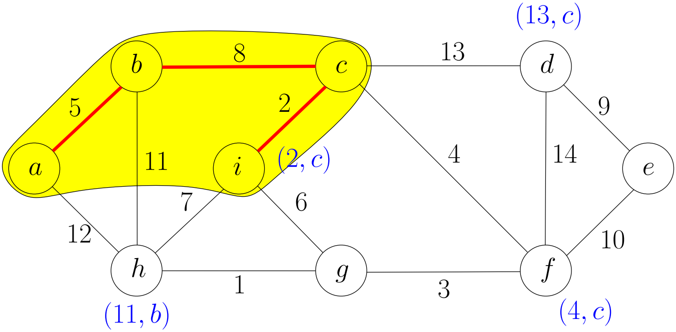

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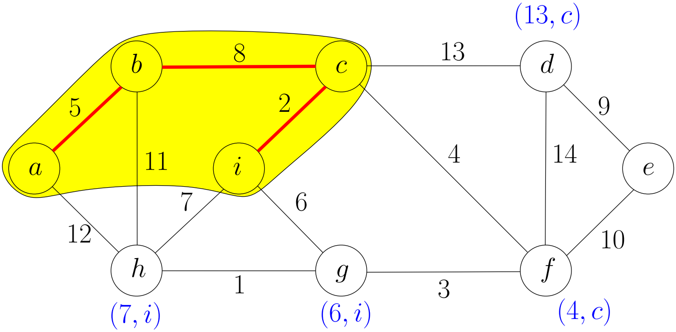

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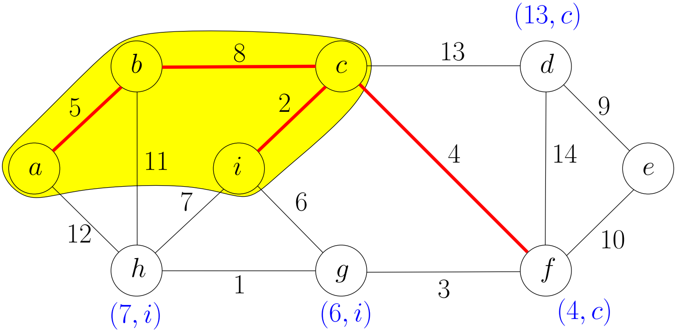

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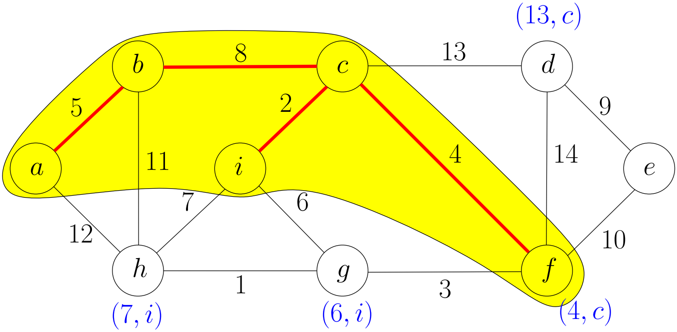

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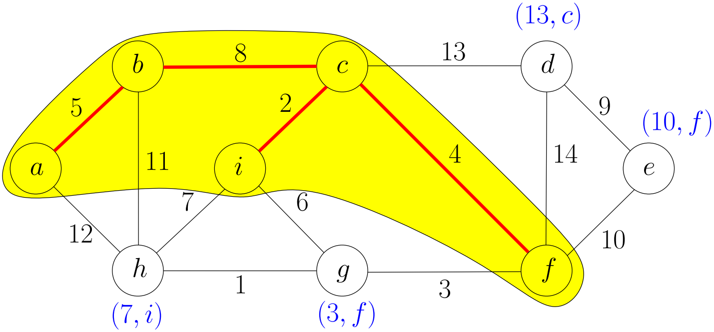

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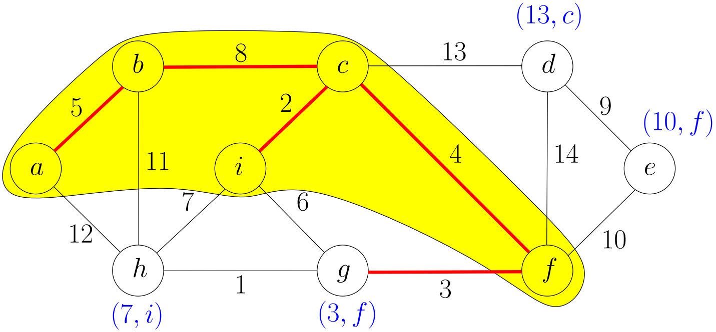

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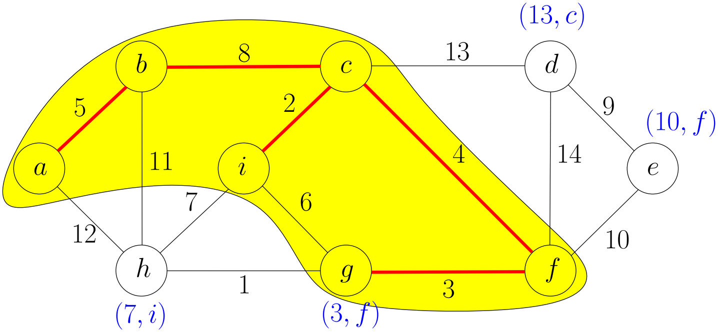

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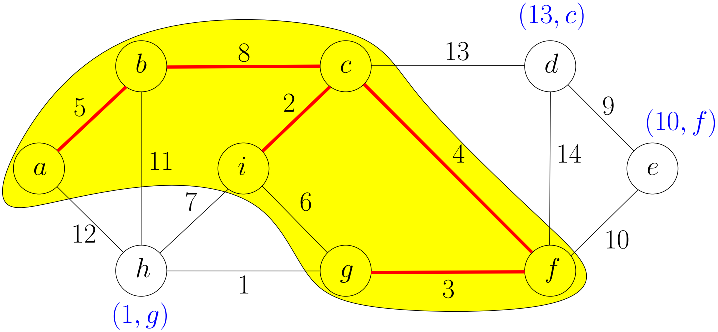

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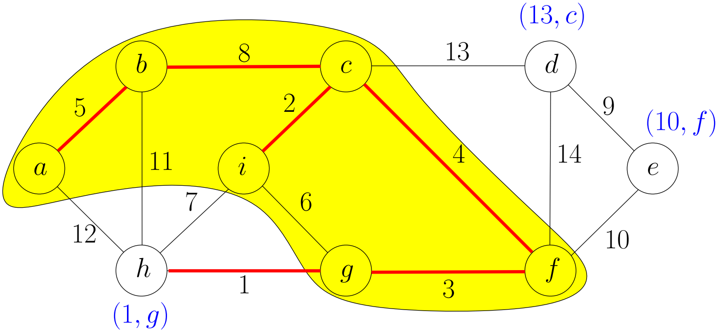

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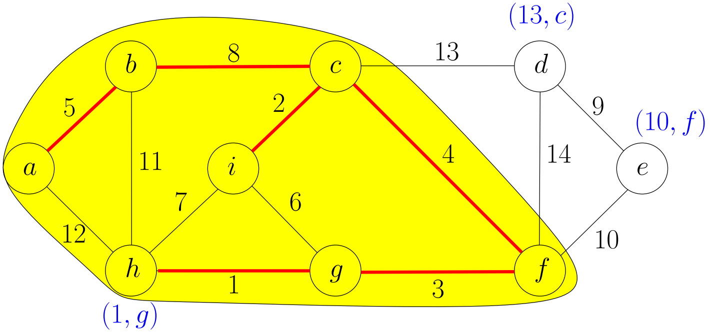

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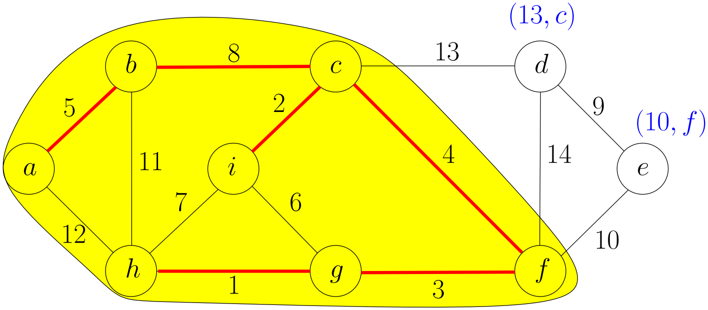

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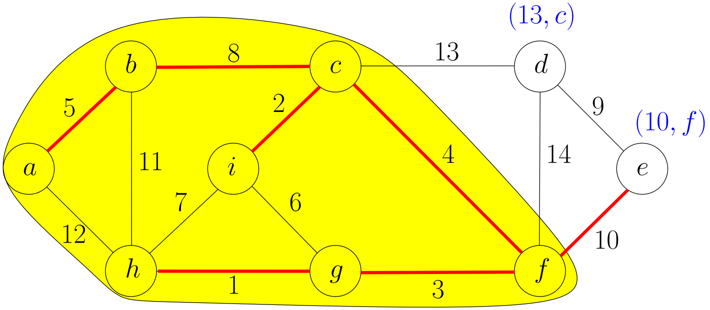

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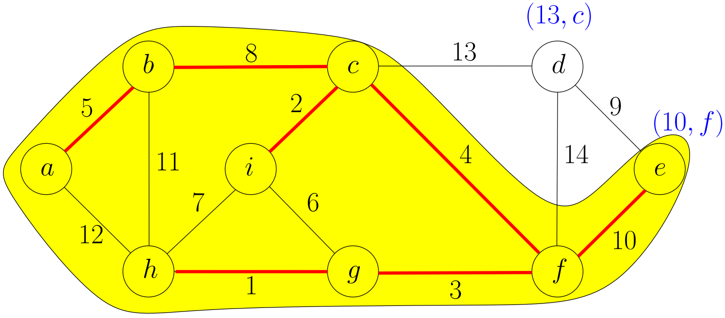

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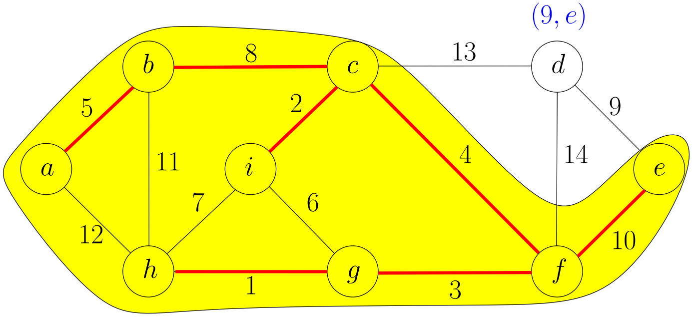

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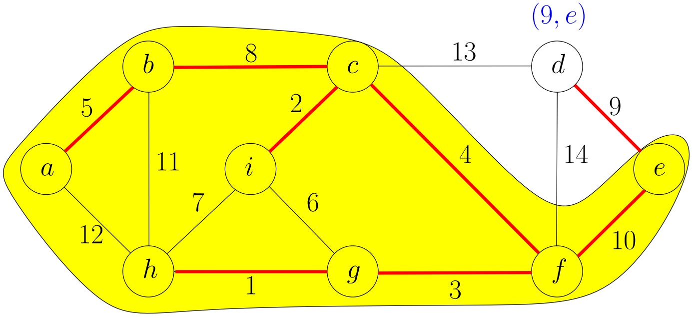

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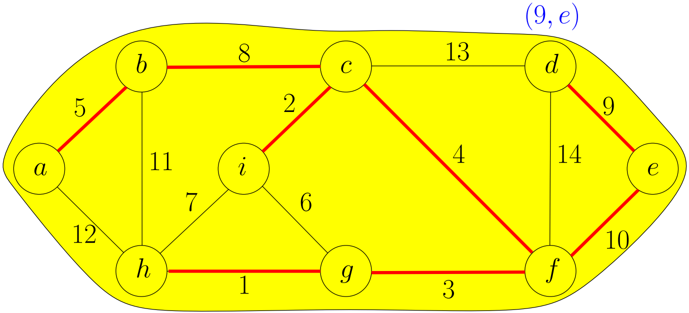

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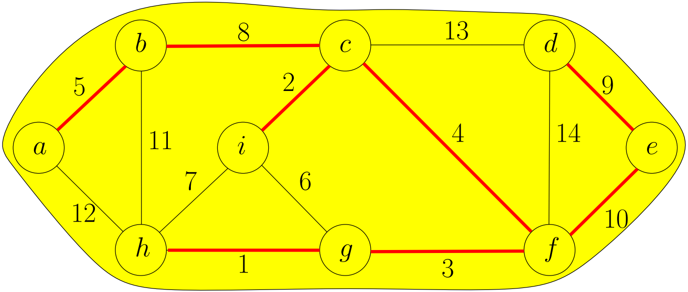

Prim's Algorithm

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In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value — extract_min
- Add $(\pi[u], u)$ to F
- Add u to S, update d and π values. — decrease_key

Use a priority queue to support the operations

Def. A priority queue is an abstract data structure that maintains a set U of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element v, whose associated key value is key_value.
- decrease_key(v, new_key_value): decrease the key value of an element v in queue to new_key_value
- extract_min$()$: return and remove the element in queue with the smallest key value
- $\cdots$

Prim's Algorithm

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
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Prim's Algorithm Using Priority Queue

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $w(u, v) < d[v]$ **then**
            $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$
            $\pi[v] \leftarrow u$
return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Running Time of Prim's Algorithm Using Priority Queue

$O(n)\times$ (time for extract_min) $+ O(m)\times$ (time for decrease_key)

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concrete DS	extract_min	decrease_key	overall time
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Fibonacci heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

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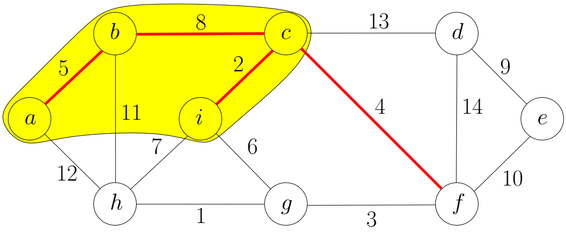

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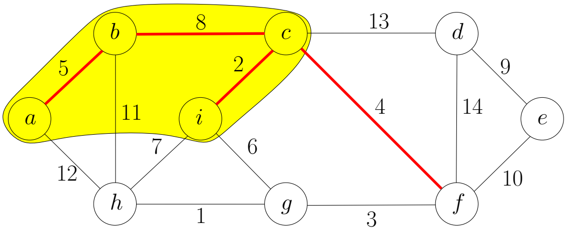

- (c, f) is in MST because of cut $\big(\{a, b, c, i\}, V \setminus \{a, b, c, i\}\big)$
- (i, g) is not in MST because no such cut exists

“Evidence” for $e \in$ MST or $e \notin$ MST

Assumption Assume all edge weights are different.

- $e \in$ MST $\leftrightarrow$ there is a cut in which e is the lightest edge
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Thus, the minimum spanning tree is unique with assumption.

Outline

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs

s-t Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \to \mathbb{R}_{\geq 0}$

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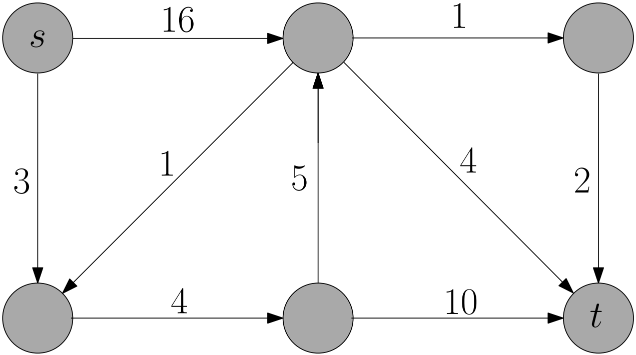

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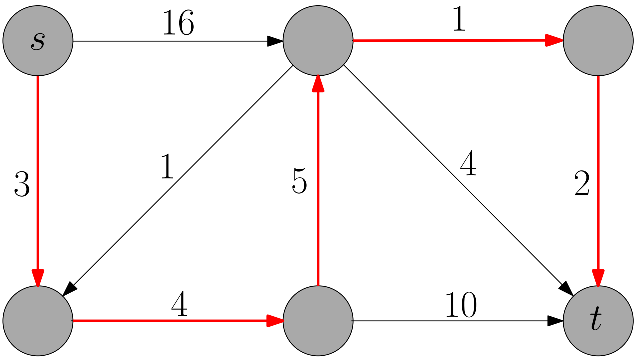

Single Source Shortest Paths

Input: (directed or undirected) graph $G = (V, E)$, $s \in V$
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Single Source Shortest Paths

Input: directed graph $G = (V, E)$, $s \in V$

$w : E \to \mathbb{R}_{\geq 0}$

Output: $\pi[v], v \in V \setminus s$: the parent of v in shortest path tree

$d[v], v \in V \setminus s$: the length of shortest path from s to v

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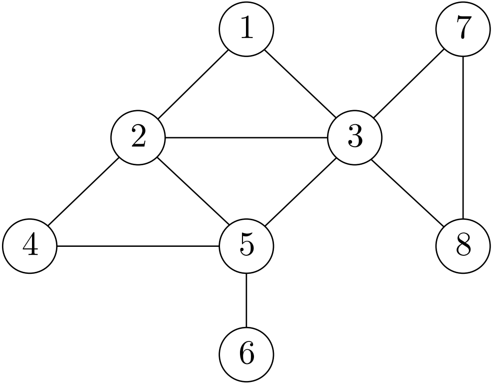

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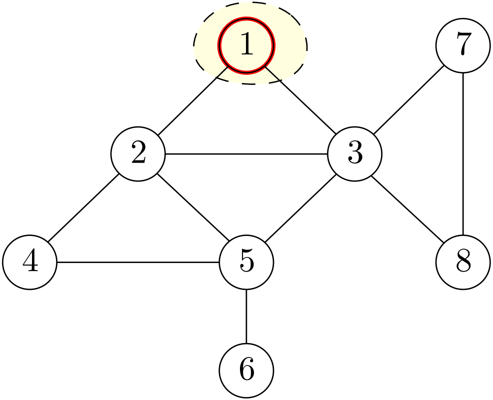

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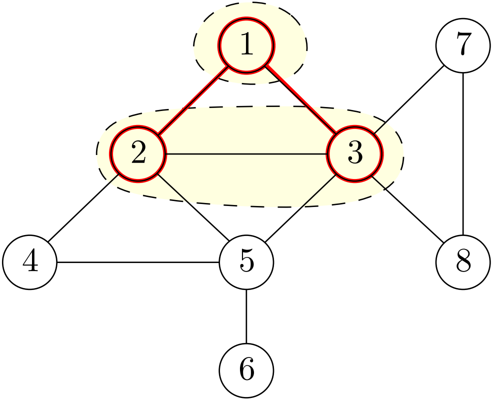

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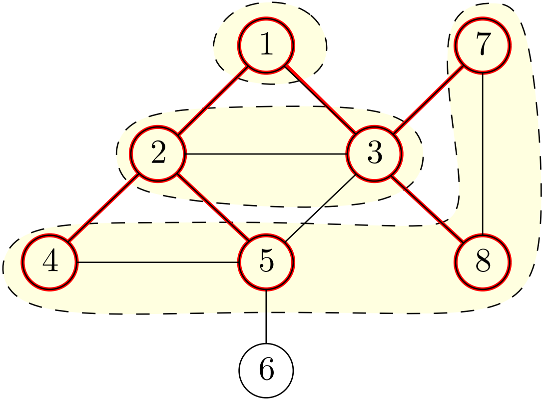

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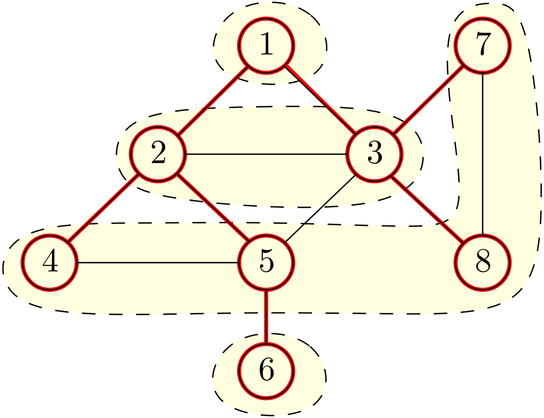

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Shortest Path Algorithm by Running BFS

1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2: run BFS
3: $\pi[v] \leftarrow$ vertex from which v is visited
4: $d[v] \leftarrow$ index of the level containing v

Assumption Weights $w(u, v)$ are integers (w.l.o.g).

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Shortest Path Algorithm by Running BFS

1: replace (u, v) of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2: run BFS virtually
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4: $d[v] \leftarrow$ index of the level containing v

- Problem: $w(u, v)$ may be too large!

Shortest Path Algorithm by Running BFS Virtually

```
S ← {s}, d(s) ← 0
while |S| ≤ n do
    find a v ∉ S that minimizes min_{u∈S:(u,v)∈E} {d[u] + w(u,v)}
    S ← S ∪ {v}
    d[v] ← min_{u∈S:(u,v)∈E} {d[u] + w(u,v)}
```

Equivalently:

$S \leftarrow \{s\}, d(s) \leftarrow 0$

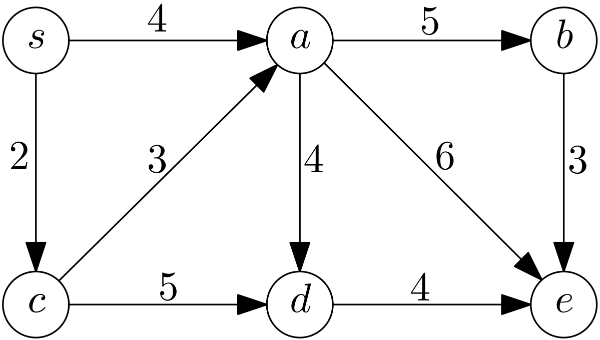
while $|S| \leq n$ **do**

find a $v \notin S$ that minimizes $\min\limits_{u \in S:(u,v)\in E} \{d[u] + w(u,v)\}$

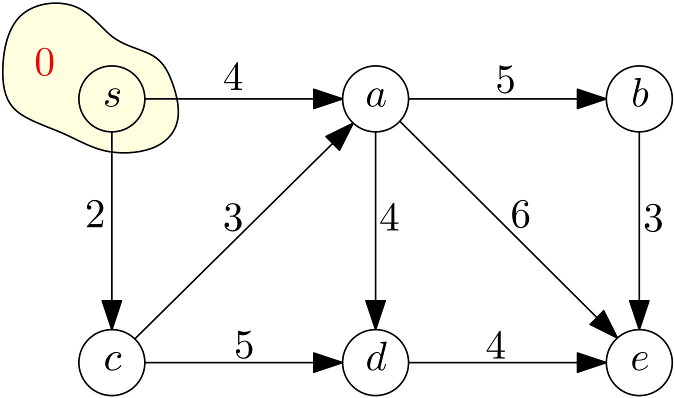
$S \leftarrow S \cup \{v\}$

$d[v] \leftarrow \min_{u \in S:(u,v)\in E}\{d[u] + w(u,v)\}$

Virtual BFS: Example

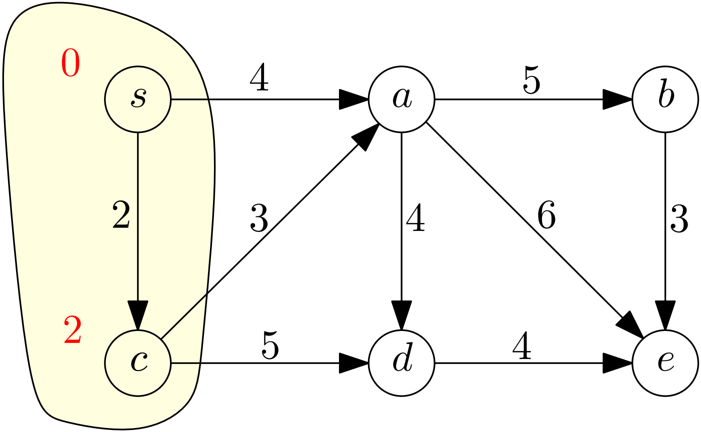


Virtual BFS: Example



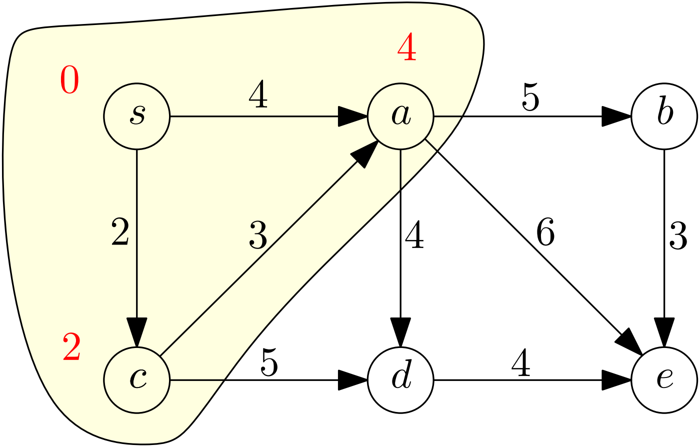
Time 0

Virtual BFS: Example



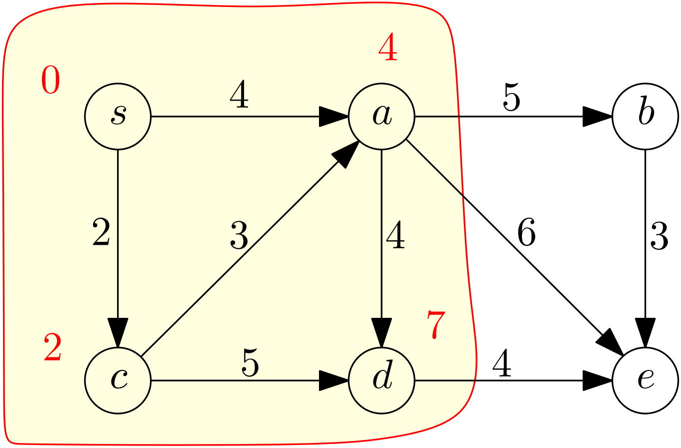
Time 2

Virtual BFS: Example



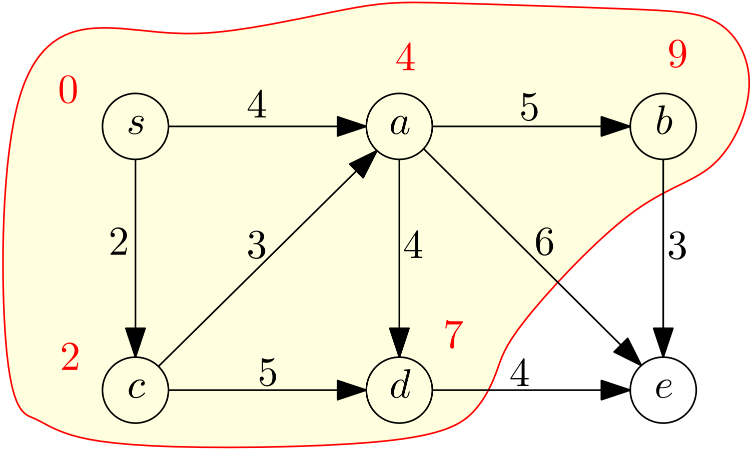
Time 4

Virtual BFS: Example



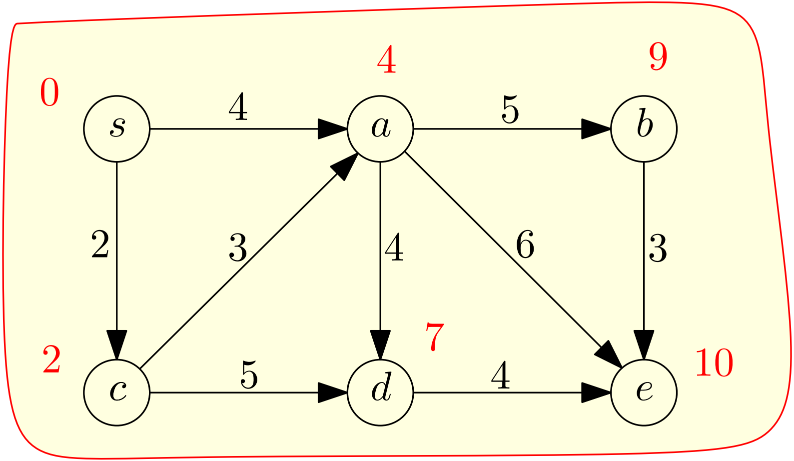
Time 7

Virtual BFS: Example



Time 9

Virtual BFS: Example



Time 10

Outline

Dijkstra's Algorithm

Dijkstra(G, w, s)

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
while $S \neq V$ **do**
    $u \leftarrow$ vertex in $V \setminus S$ with the minimum $d[u]$
    add u to S
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
return (d, π)

Dijkstra's Algorithm

Dijkstra(G, w, s)

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while $S \neq V$ **do**
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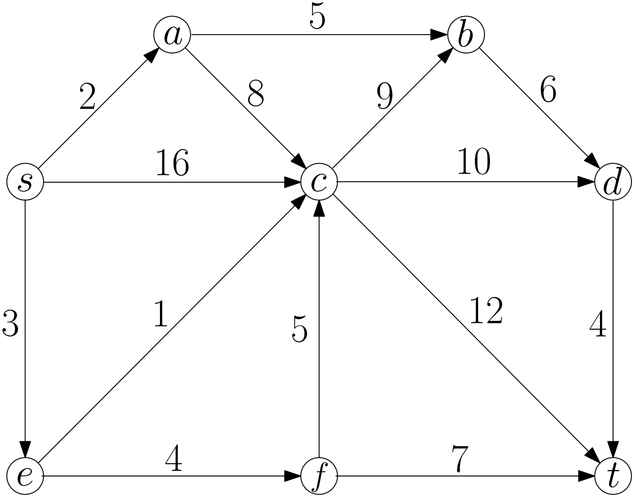
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$
            $\pi[v] \leftarrow u$
return (d, π)

- Running time $= O(n^2)$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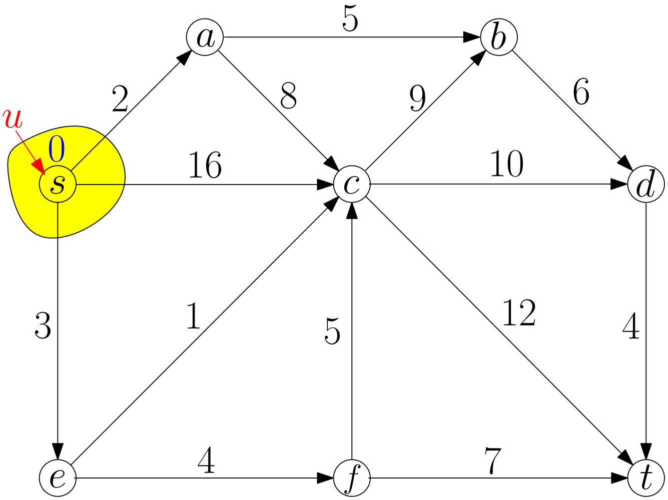

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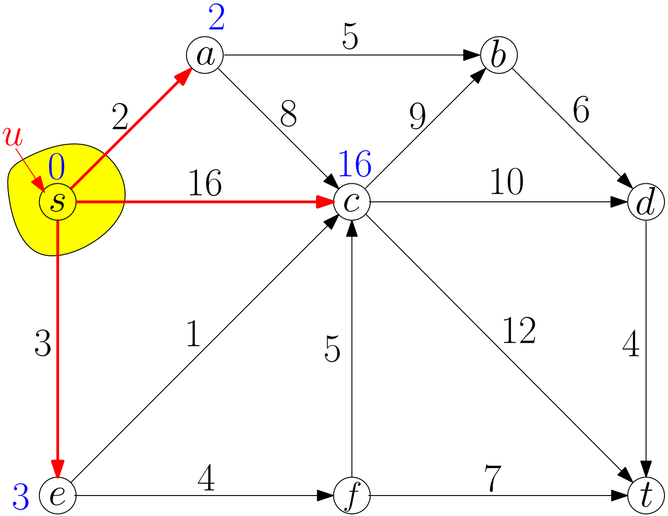
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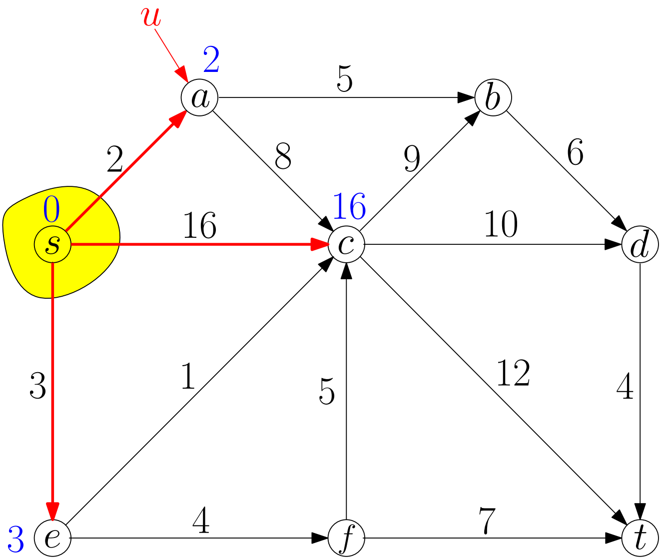
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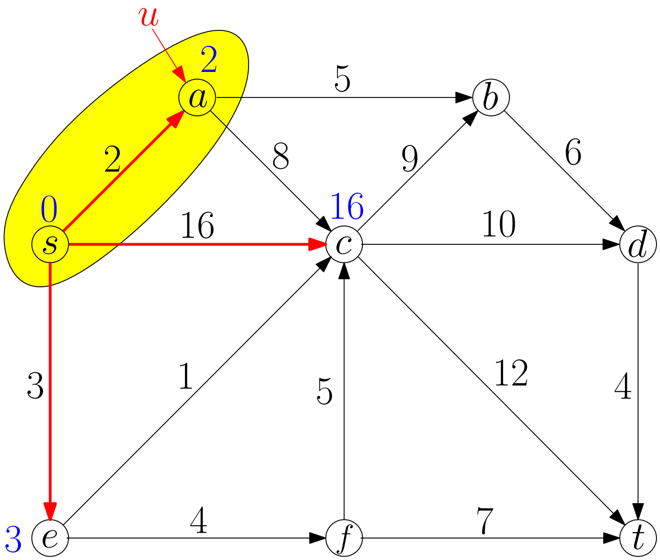
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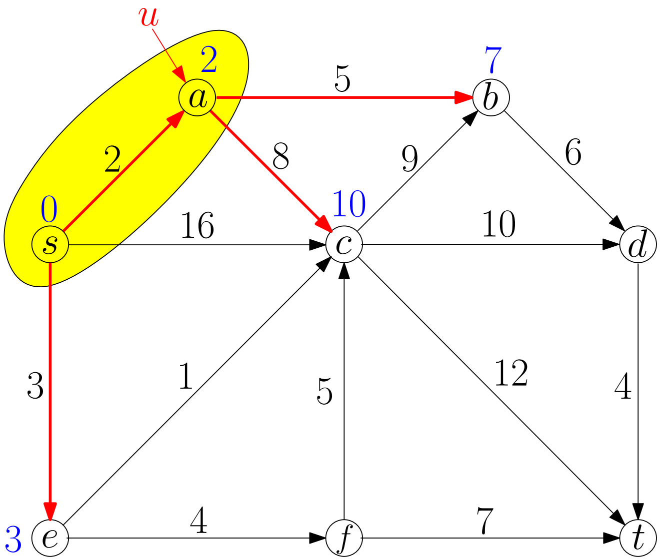
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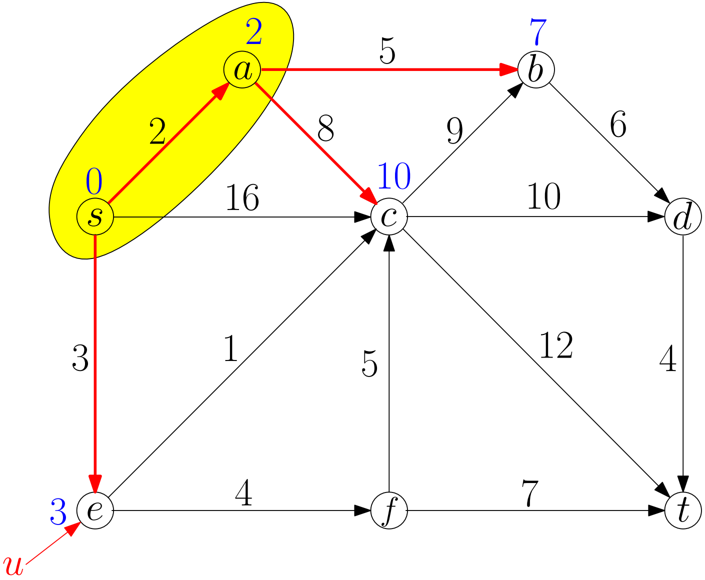
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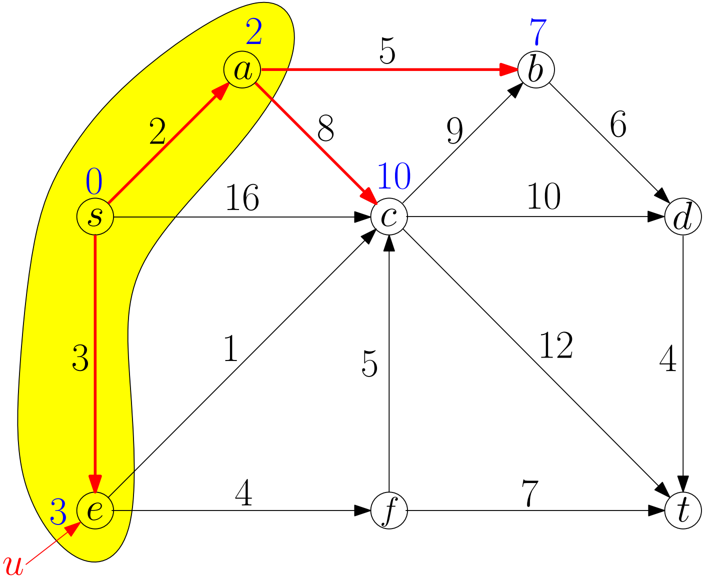
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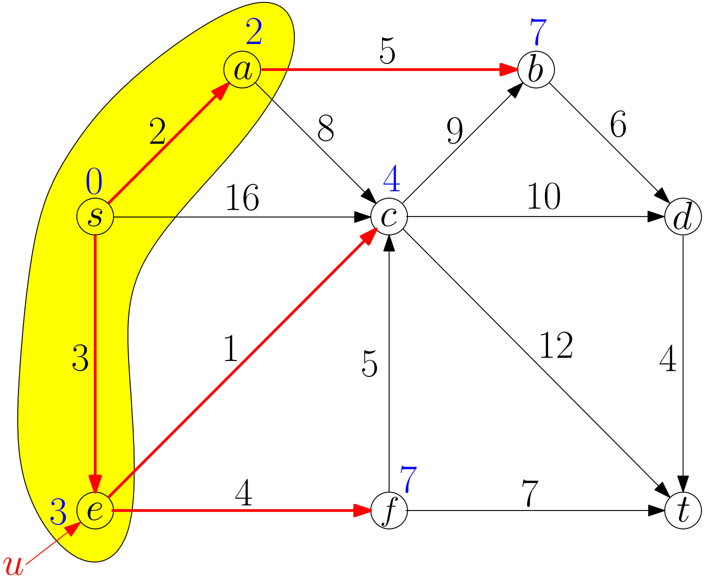

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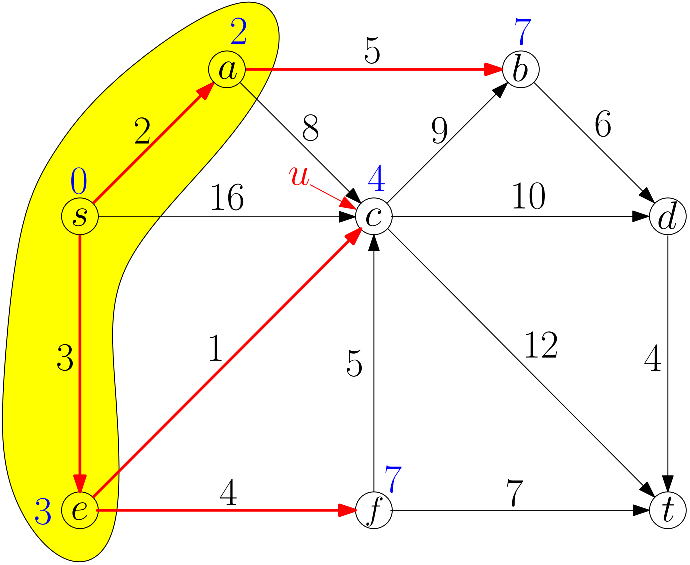
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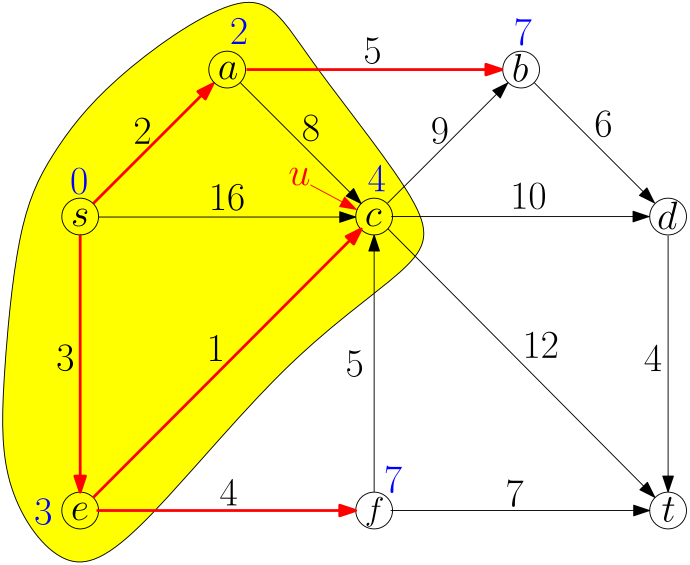

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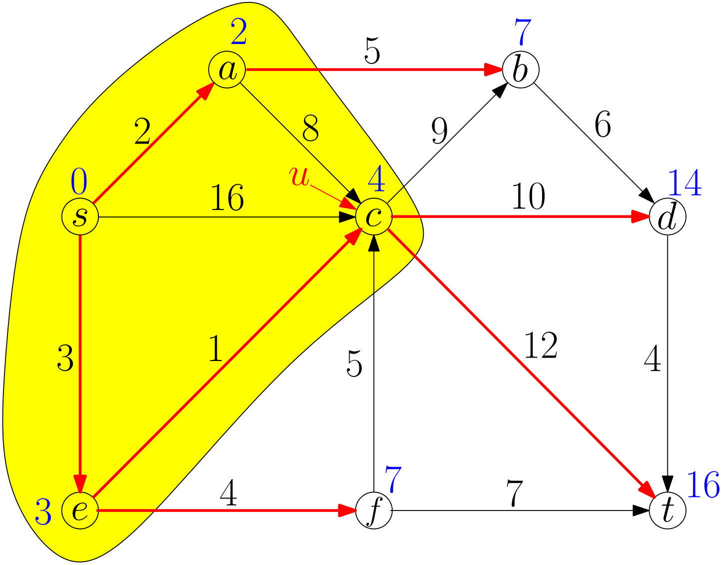

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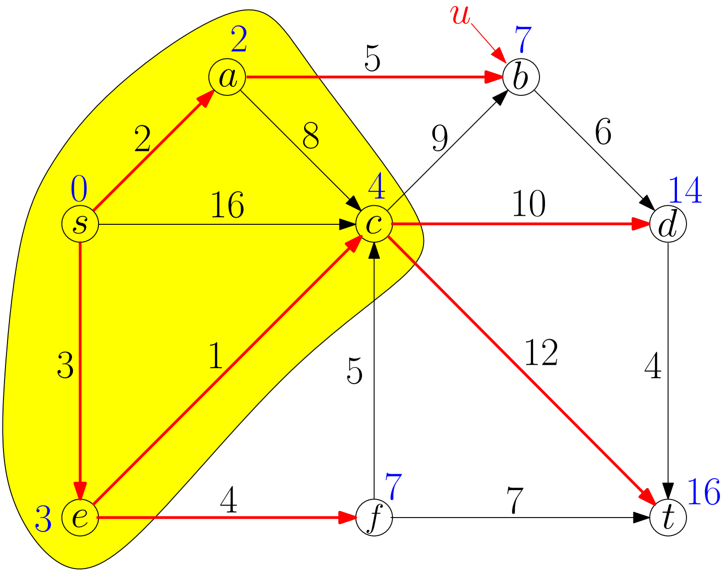

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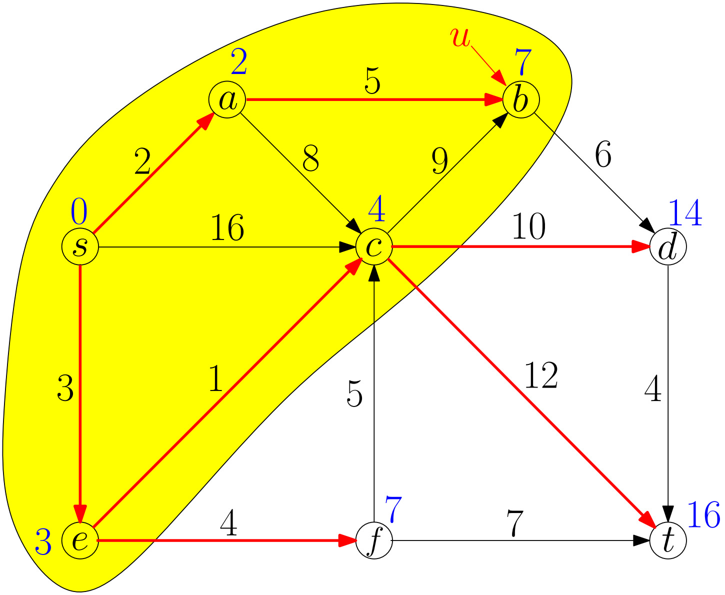

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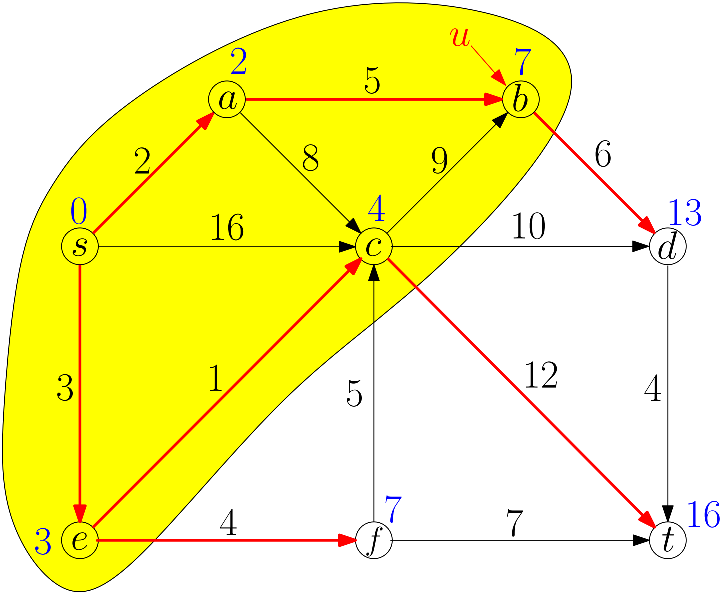
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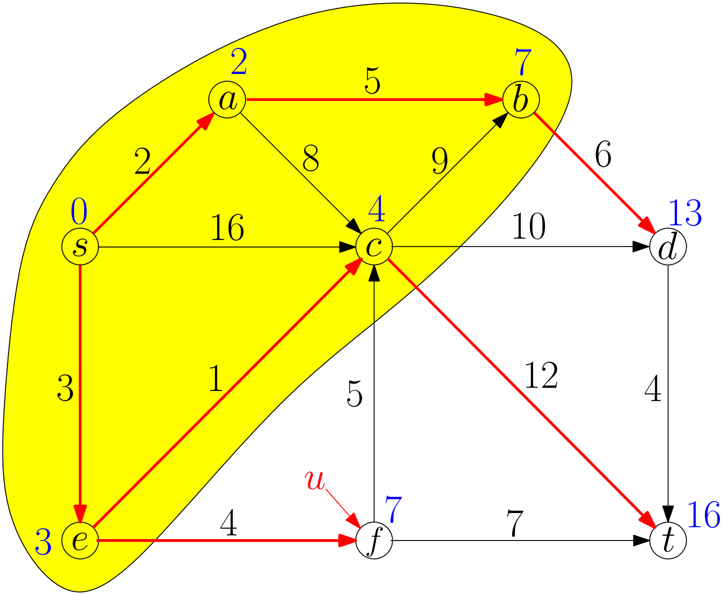

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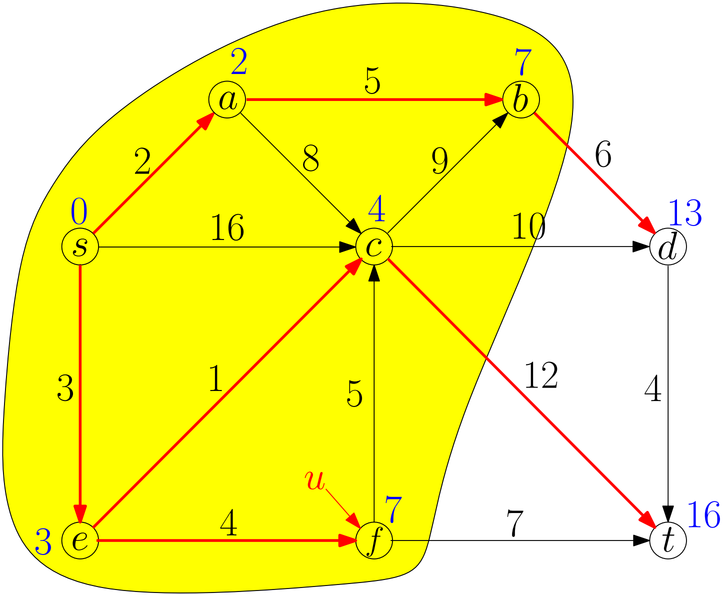

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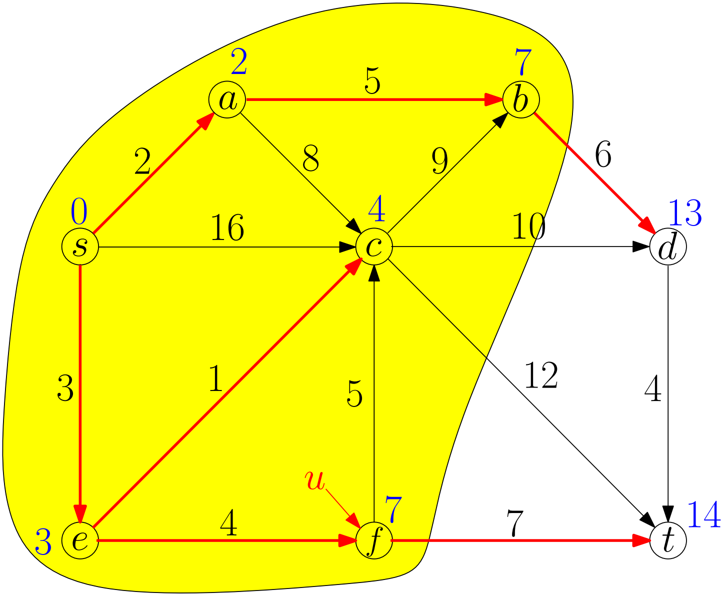

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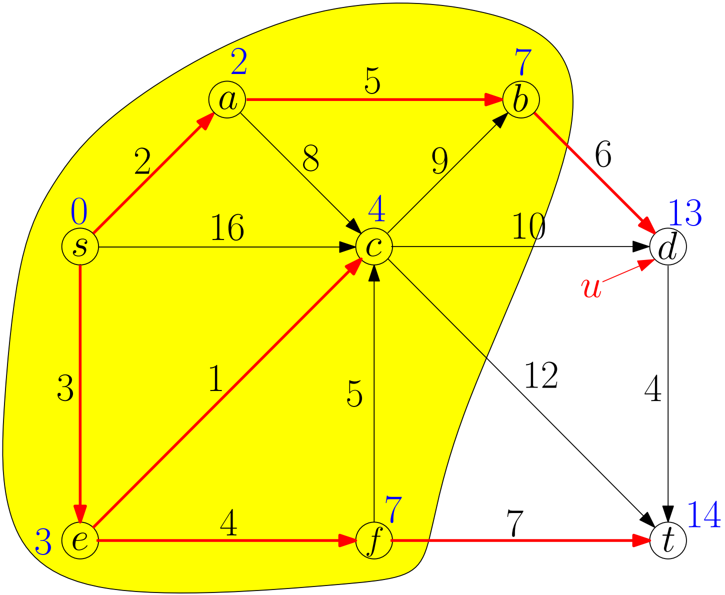

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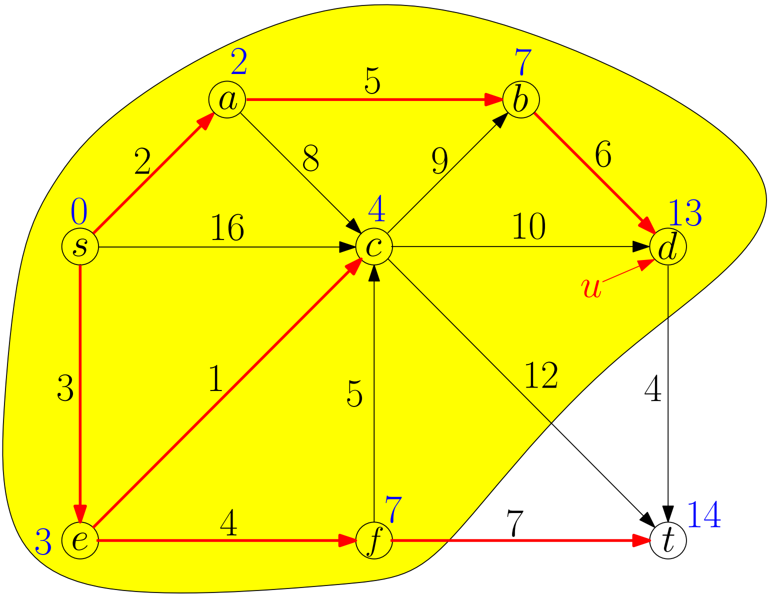

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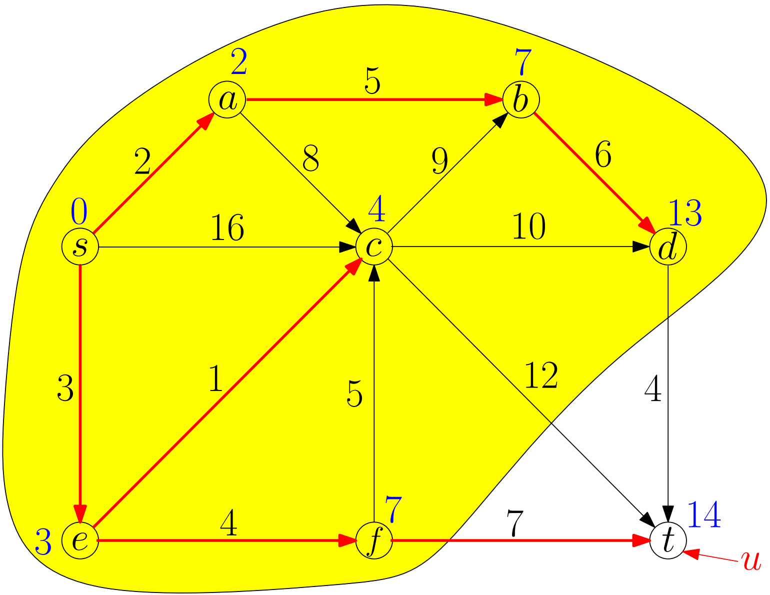

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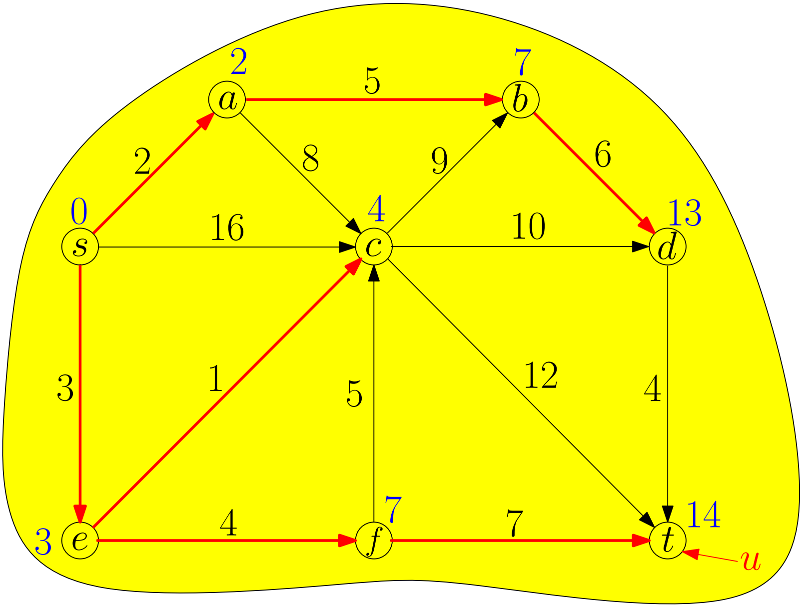

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Improved Running Time using Priority Queue

Dijkstra(G, w, s)

$s \leftarrow$ arbitrary vertex in G
$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$
while $S \neq V$ **do**
    $u \leftarrow Q$.extract_min()
    $S \leftarrow S \cup \{u\}$
    for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
        if $d[u] + w(u, v) < d[v]$ **then**
            $d[v] \leftarrow d[u] + w(u, v)$, Q.decrease_key$(v, d[v])$
            $\pi[v] \leftarrow u$
return (π, d)

Recall: Prim's Algorithm for MST

MST-Prim(G, w)

$s \leftarrow$ arbitrary vertex in G

$S \leftarrow \emptyset, d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$

$Q \leftarrow$ empty queue, for each $v \in V$: Q.insert$(v, d[v])$

while $S \neq V$ **do**

  $u \leftarrow Q$.extract_min()

  $S \leftarrow S \cup \{u\}$

  for each $v \in V \setminus S$ such that $(u, v) \in E$ **do**

    if $w(u, v) < d[v]$ **then**

      $d[v] \leftarrow w(u, v)$, Q.decrease_key$(v, d[v])$

      $\pi[v] \leftarrow u$

return $\big\{(u, \pi[u]) | u \in V \setminus \{s\}\big\}$

Improved Running Time

Running time:

$O(n) \times$ (time for extract_min) $+ O(m) \times$ (time for decrease_key)

Priority-Queue	extract_min	decrease_key	Time
Heap	$O(\log n)$	$O(\log n)$	$O(m \log n)$
Fibonacci Heap	$O(\log n)$	$O(1)$	$O(n \log n + m)$

Outline

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

Single Source Shortest Paths, Weights May be Negative

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- In transition graphs, negative weights make sense

Single Source Shortest Paths, Weights May be Negative

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- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)

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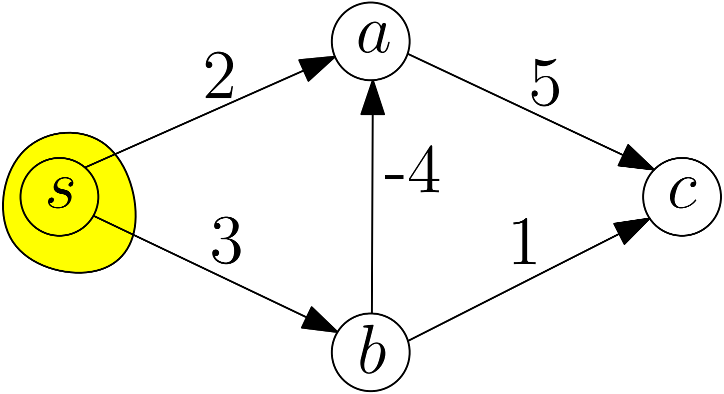
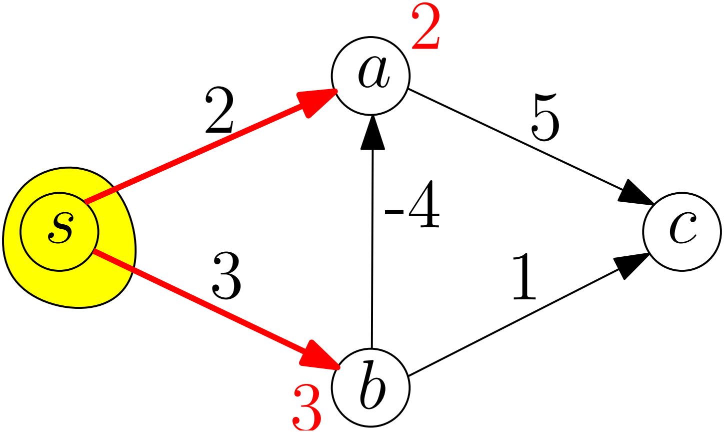
- In transition graphs, negative weights make sense
- If we sell a item: 'having the item' $\to$ 'not having the item', weight is negative (we gain money)
- Dijkstra's algorithm does not work any more!

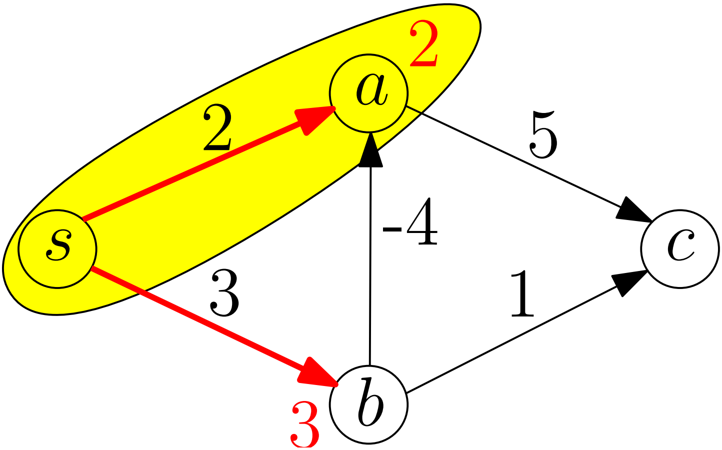
Dijkstra's Algorithm Fails if We Have Negative Weights



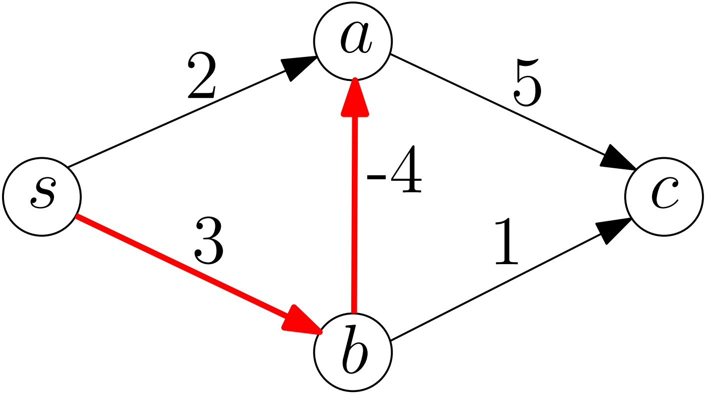
Dijkstra's Algorithm Fails if We Have Negative Weights

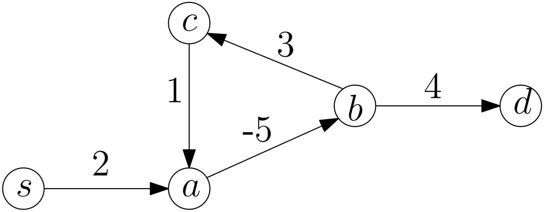


Dijkstra's Algorithm Fails if We Have Negative Weights



Dijkstra's Algorithm Fails if We Have Negative Weights





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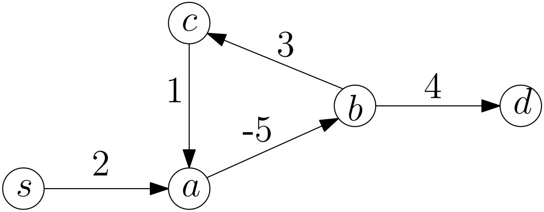
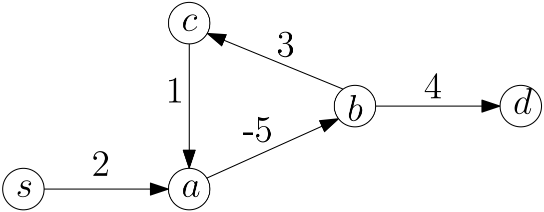
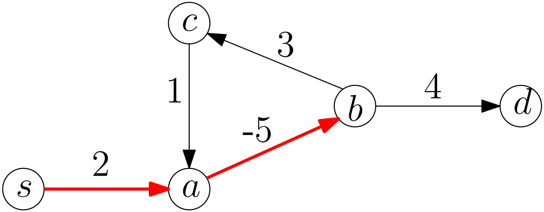


Q: What is the length of the shortest path from s to d?



Q: What is the length of the shortest path from s to d?

A: $-\infty$



Q: What is the length of the shortest path from s to d?

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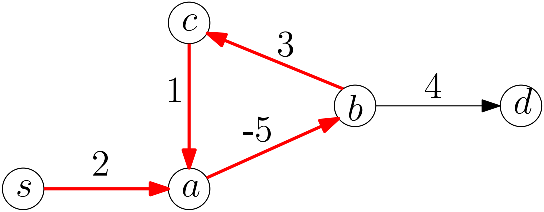

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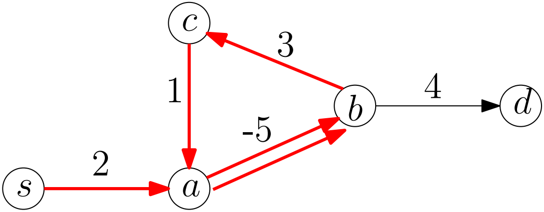

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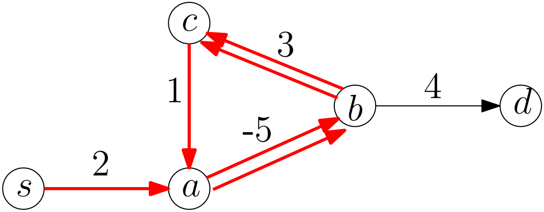

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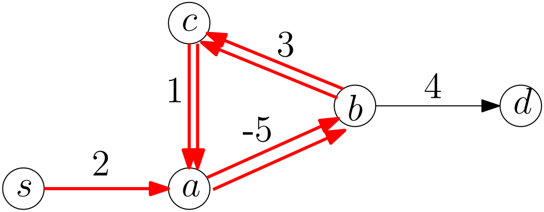

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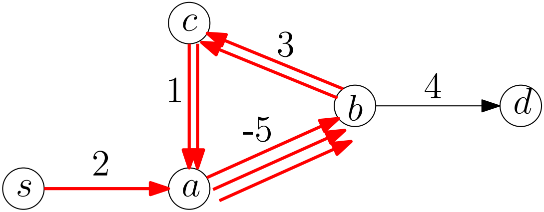

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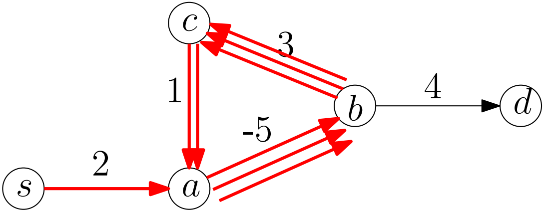

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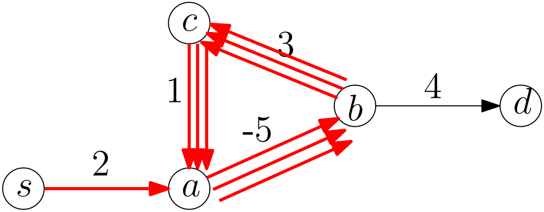

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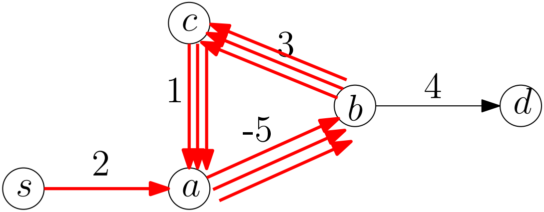

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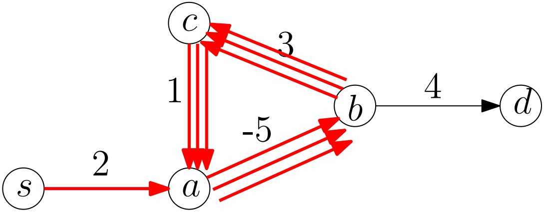
Def. A negative cycle is a cycle in which the total weight of edges is negative.



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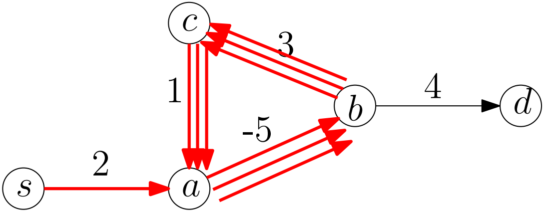
Dealing with Negative Cycles



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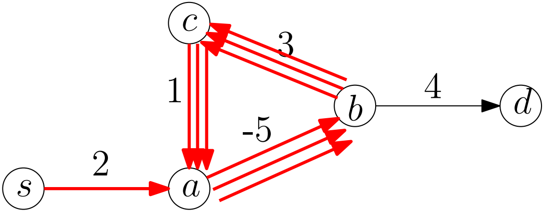
- assume the input graph does not contain negative cycles, or



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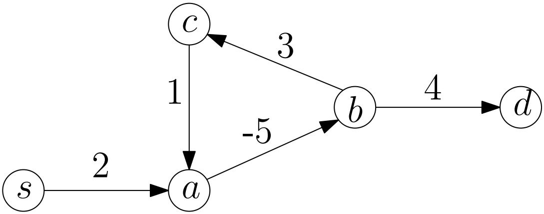
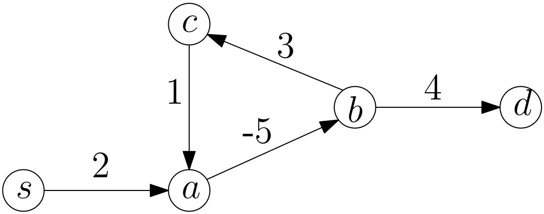
Dealing with Negative Cycles

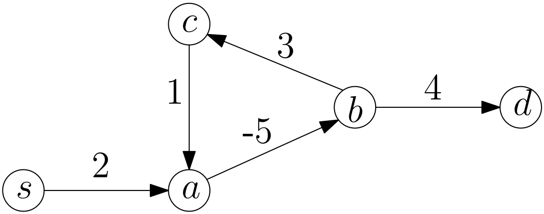
- assume the input graph does not contain negative cycles, or
- allow algorithm to report "negative cycle exists"



c
3
1
b
4
d
-5
2
s
a

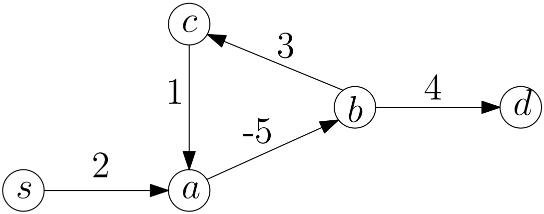


Q: What is the length of the shortest simple path from s to d?



Q: What is the length of the shortest simple path from s to d?

A: 1



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- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

Defining Cells of Table

Single Source Shortest Paths, Weights May be Negative

Input: directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from s

$w : E \to \mathbb{R}$

Output: shortest paths from s to all other vertices $v \in V$

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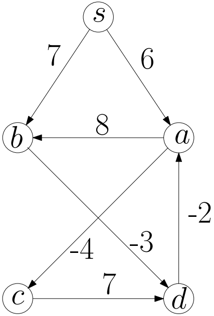

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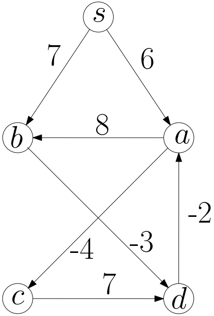

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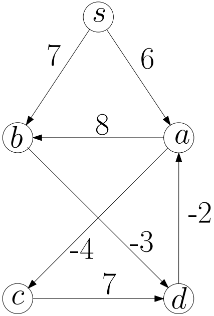

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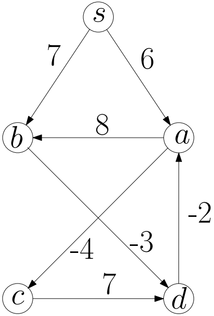

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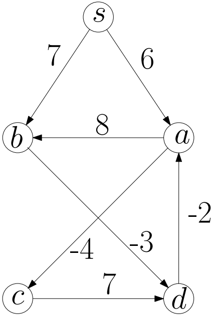

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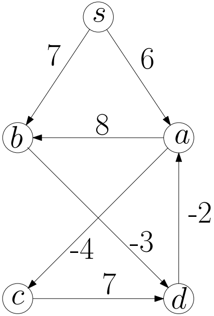

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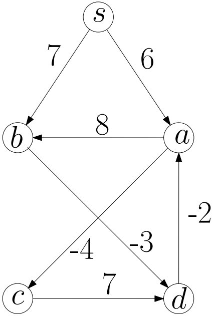

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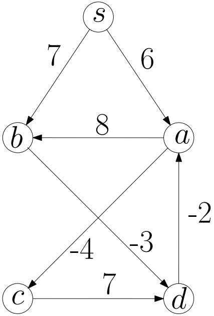

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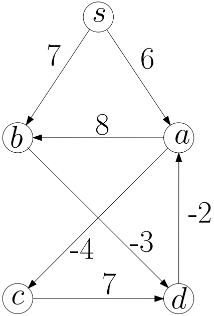

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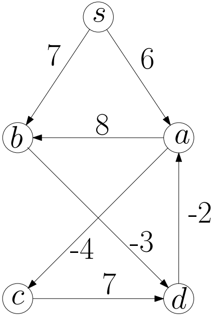

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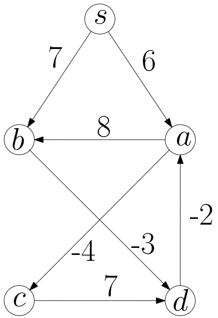

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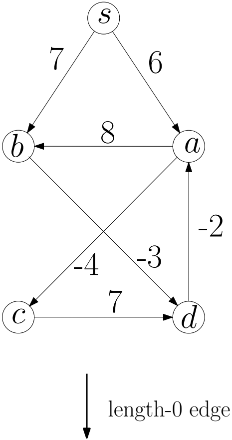
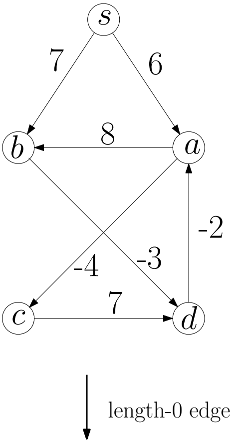
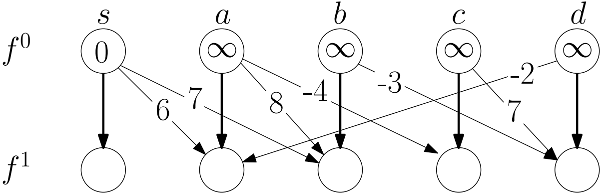
Dynamic Programming: Example



	s	a	b	c	d
f^0	0	∞	∞	∞	∞

Dynamic Programming: Example





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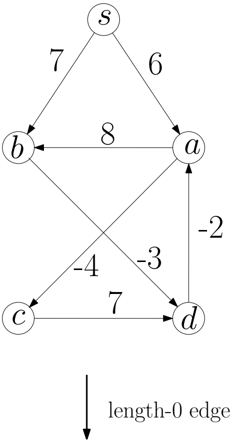

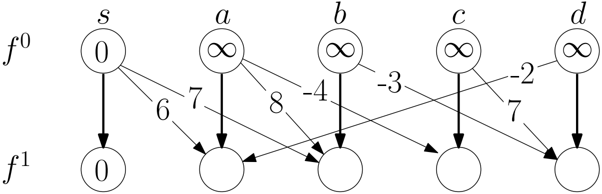

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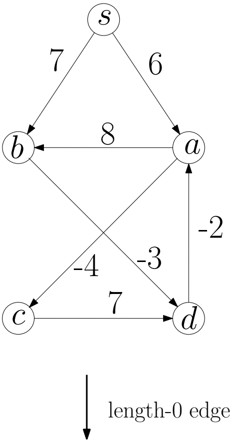

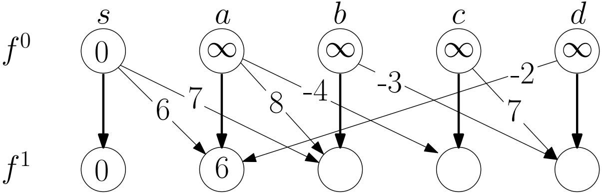

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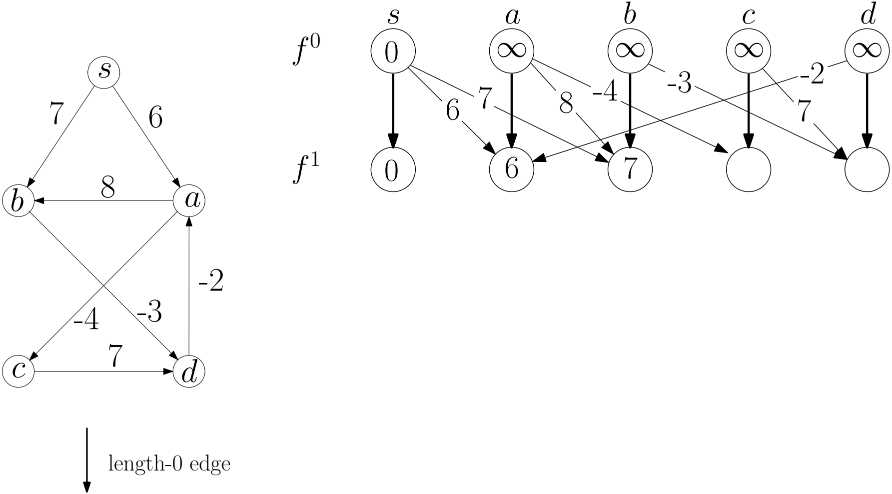

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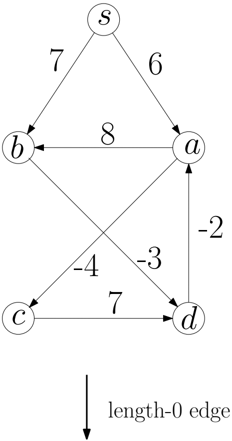

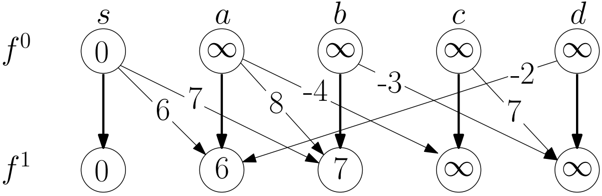

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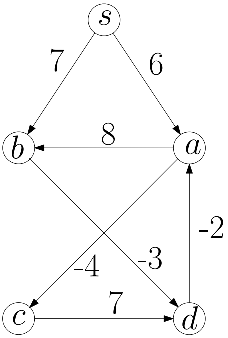
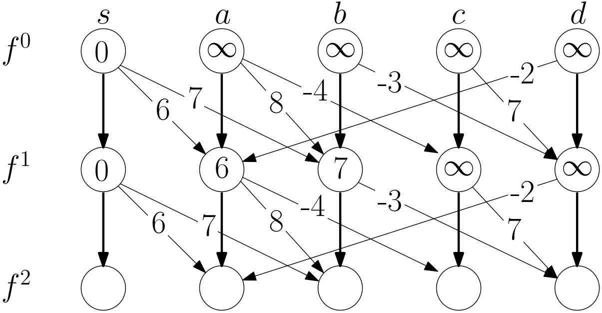
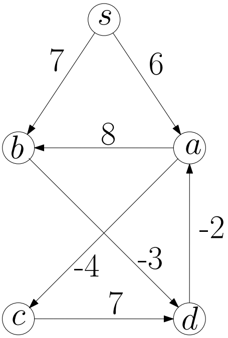
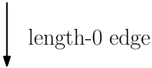


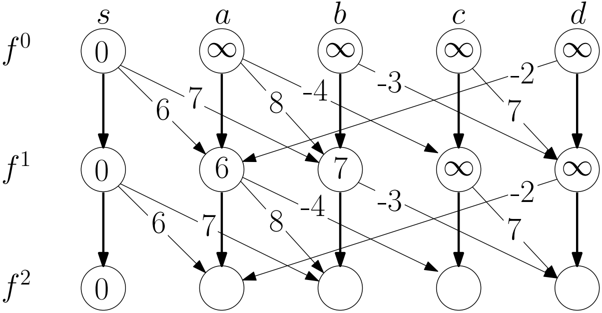


length-0 edge

Dynamic Programming: Example







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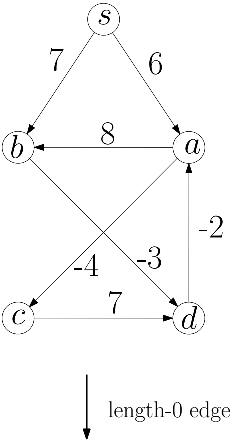

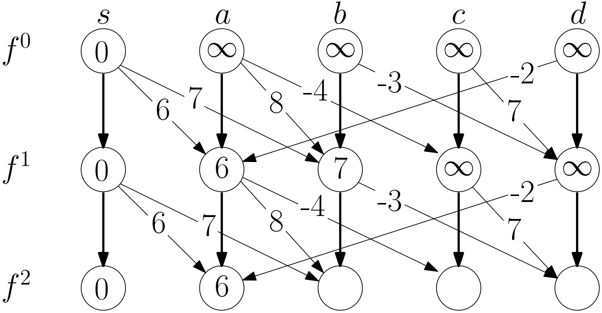

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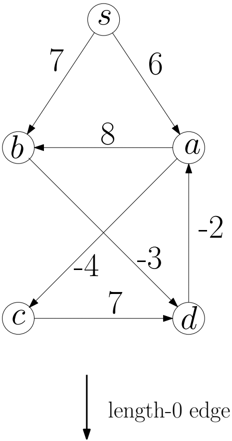

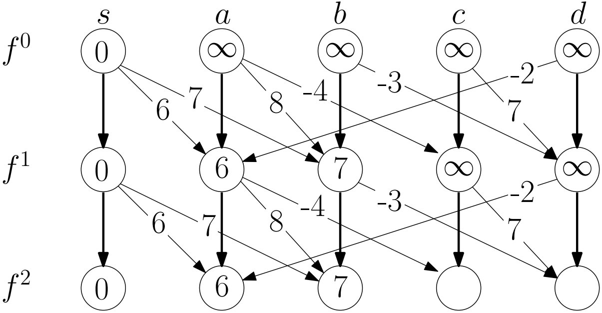

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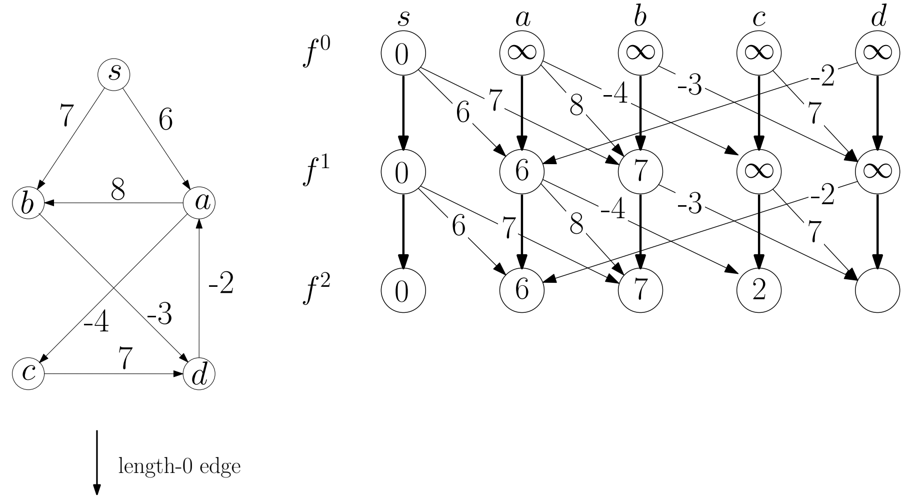

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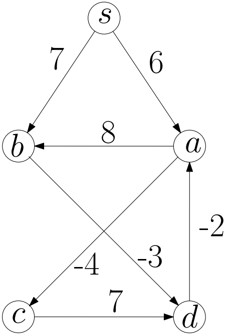
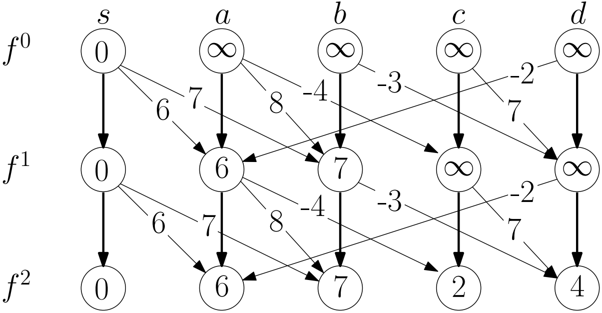
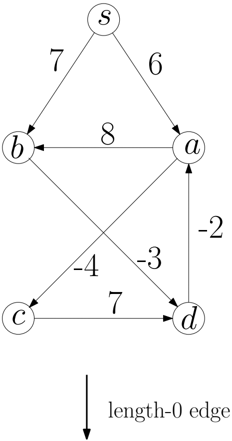
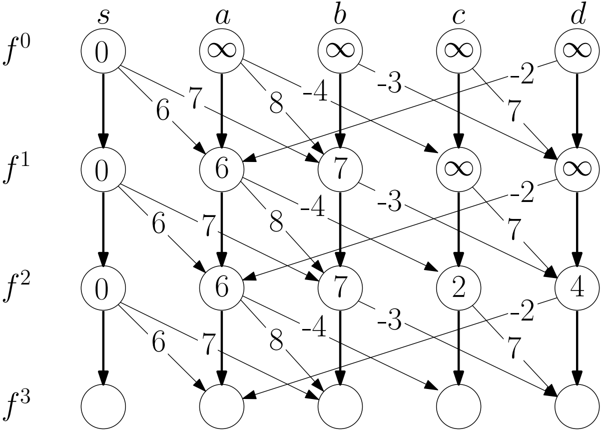




length-0 edge

Dynamic Programming: Example





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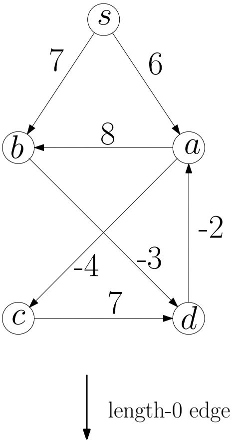

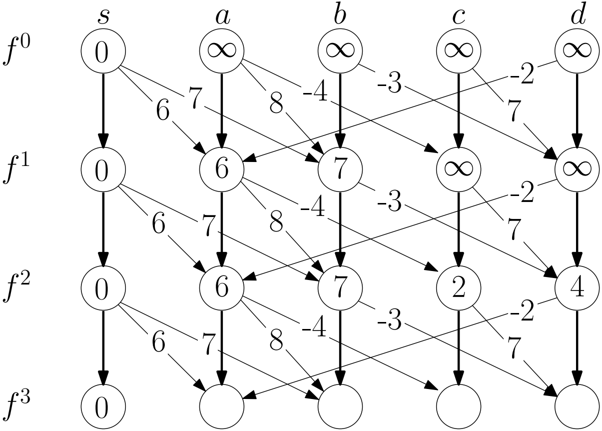

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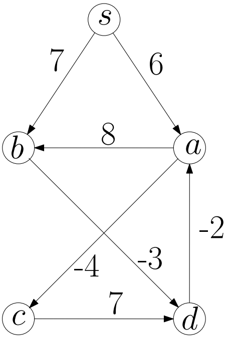

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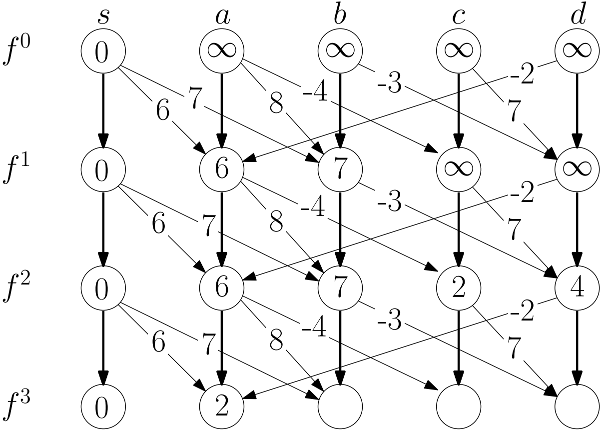
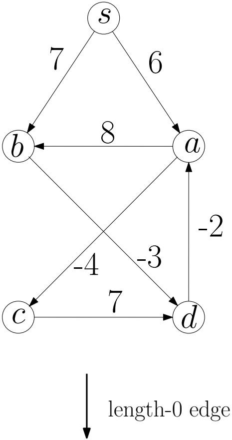
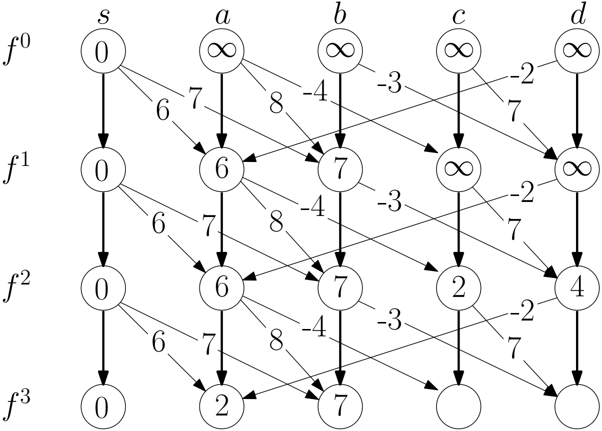
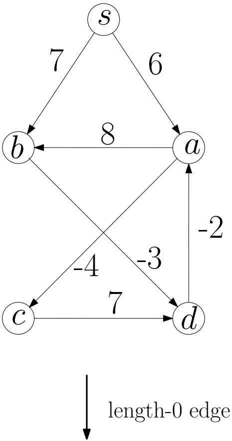


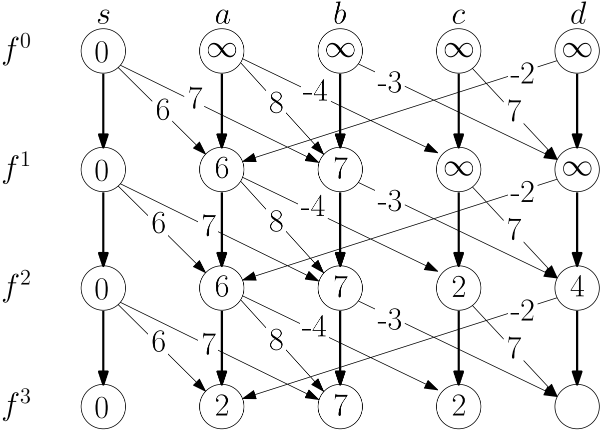
Dynamic Programming: Example





Dynamic Programming: Example





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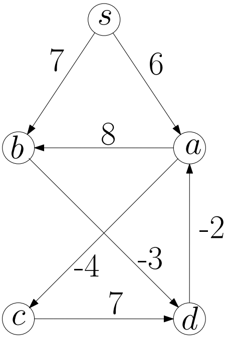
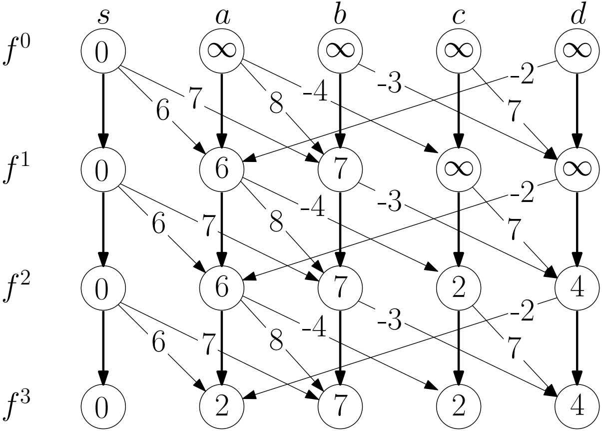


length-0 edge



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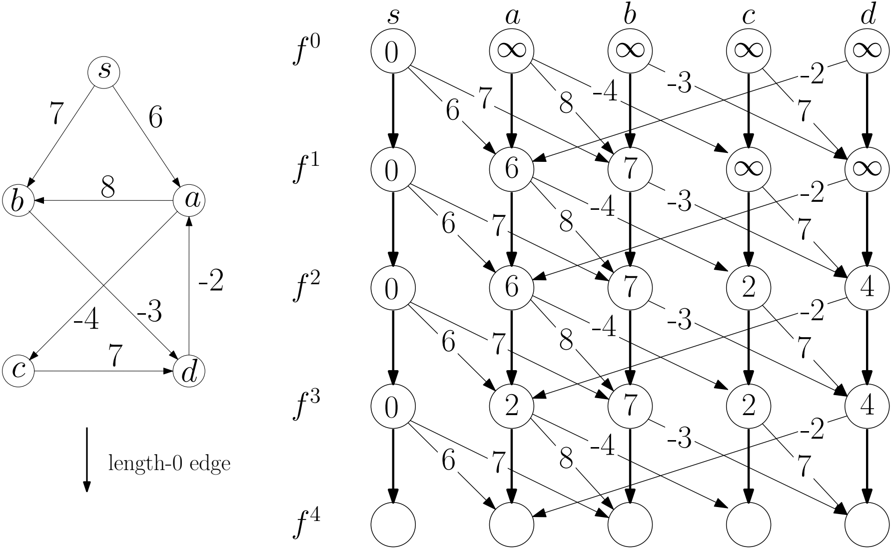

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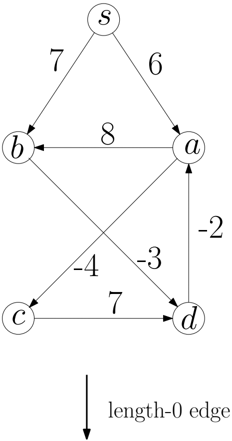

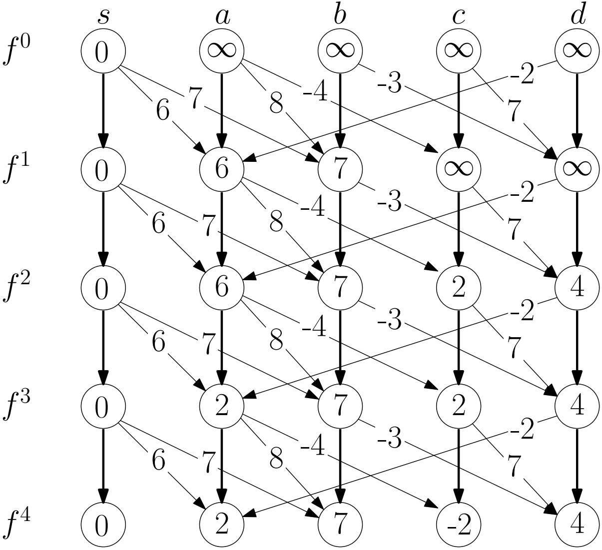

dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \rightarrow f^{\ell}$
    for each $(u,v) \in E$ **do**
        if $f^{\ell-1}[u] + w(u,v) < f^{\ell}[v]$ **then**
            $f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u,v)$
return $(f^{n-1}[v])_{v \in V}$

dynamic-programming(G, w, s)

```
f^0[s] ← 0 and f^0[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^{ℓ−1} → f^ℓ
    for each (u, v) ∈ E do
        if f^{ℓ−1}[u] + w(u, v) < f^ℓ[v] then
            f^ℓ[v] ← f^{ℓ−1}[u] + w(u, v)
return (f^{n−1}[v])_{v∈V}
```

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

dynamic-programming(G, w, s)

$f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f^{\ell-1} \rightarrow f^\ell$
    for each $(u, v) \in E$ **do**
        if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ **then**
            $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n-1$ edges

Proof.

If there is a path containing at least n edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. □

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

```
f^old[s] ← 0 and f^old[v] ← ∞ for any v ∈ V \ {s}
for ℓ ← 1 to n − 1 do
    copy f^old → f^new
    for each (u, v) ∈ E do
        if f^old[u] + w(u, v) < f^new[v] then
            f^new[v] ← f^old[u] + w(u, v)
    copy f^new → f^old
return f^old
```

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors

Dynamic Programming with Better Space Usage

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    copy f^new → f^old
return f^old
```

- f^ℓ only depends on $f^{\ell-1}$: only need 2 vectors
- only need 1 vector!

Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to $n-1$ **do**
    copy $f \rightarrow f$
    for each $(u,v) \in E$ **do**
        if $f[u] + w(u,v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u,v)$
    copy $f \rightarrow f$
return f

- f^{ℓ} only depends on $f^{\ell-1}$: only need 2 vectors
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Dynamic Programming with Better Space Usage

dynamic-programming(G, w, s)

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Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

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f[s] ← 0 and f[v] ← ∞ for any v ∈ V \ {s}
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            f[v] ← f[u] + w(u, v)
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```

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```

- Issue: when we compute $f[u] + w(u,v)$, $f[u]$ may be changed since the end of last iteration

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Bellman-Ford(G, w, s)

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```

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- This is OK: it can only "accelerate" the process!

Bellman-Ford Algorithm

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f[s] ← 0 and f[v] ← ∞ for any v ∈ V \ {s}
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return f
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- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: **for** $\ell \leftarrow 1$ to $n-1$ **do**
3:     **for** each $(u, v) \in E$ **do**
4:         **if** $f[u] + w(u, v) < f[v]$ **then**
5:             $f[v] \leftarrow f[u] + w(u, v)$
6: **return** f

- Issue: when we compute $f[u] + w(u, v)$, $f[u]$ may be changed since the end of last iteration
- This is OK: it can only "accelerate" the process!
- After iteration ℓ, $f[v]$ is at most the length of the shortest path from s to v that uses at most ℓ edges
- $f[v]$ is always the length of some path from s to v

Bellman-Ford Algorithm

- After iteration ℓ:

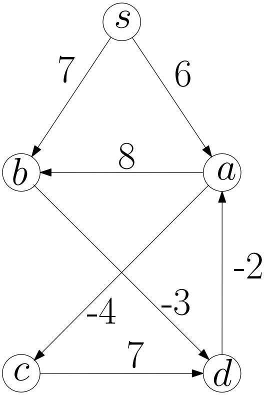
$$\begin{aligned} & \text{length of shortest } s\text{-}v \text{ path} \\ \leq\ & f[v] \\ \leq\ & \text{length of shortest } s\text{-}v \text{ path using at most } \ell \text{ edges} \end{aligned}$$

- Assuming there are no negative cycles:

$$\begin{aligned} & \text{length of shortest } s\text{-}v \text{ path} \\ =\ & \text{length of shortest } s\text{-}v \text{ path using at most } n-1 \text{ edges} \end{aligned}$$

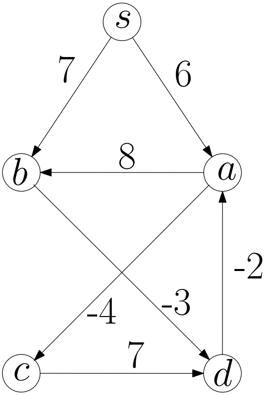
- So, assuming there are no negative cycles, after iteration $n-1$:

$$f[v] = \text{length of shortest } s\text{-}v \text{ path}$$



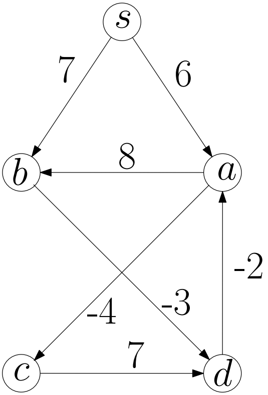
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



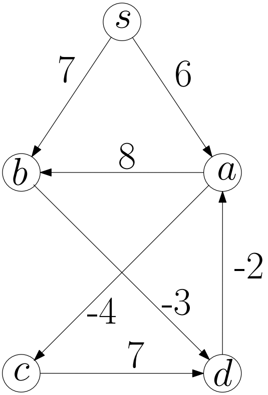
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	∞	∞	∞	∞



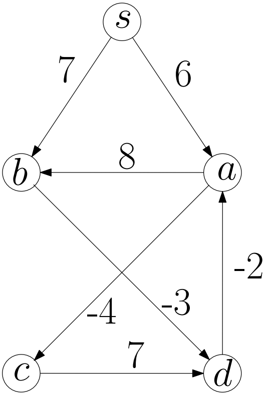
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	∞	∞	∞



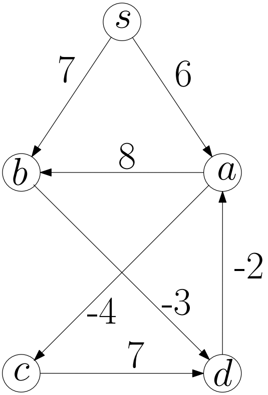
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	∞	∞	∞



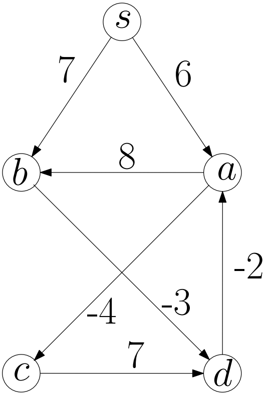
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	∞	∞



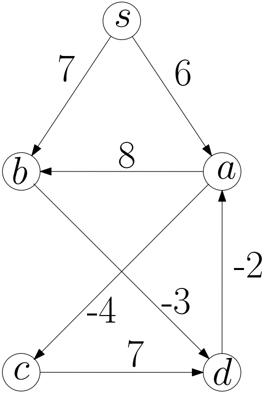
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	∞	∞



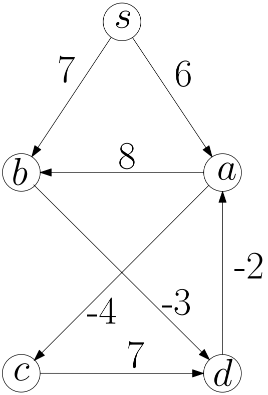
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	∞	∞



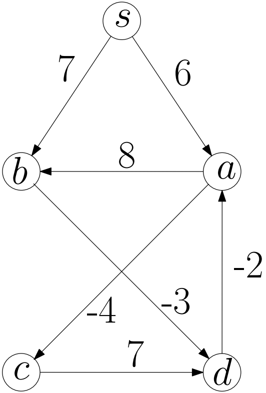
- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	2	∞



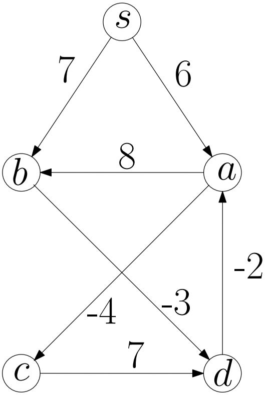
- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	6	7	2	∞



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	6	7	2	4



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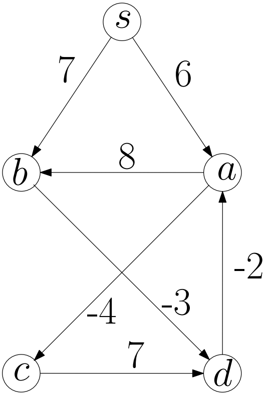

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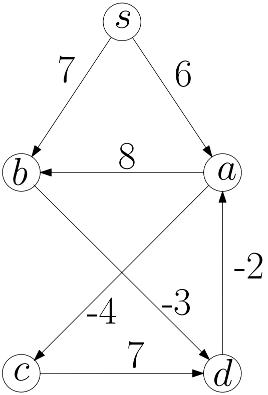

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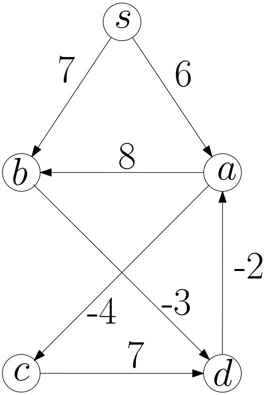

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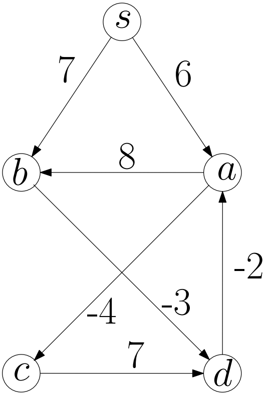
vertices	s	a	b	c	d
f	0	2	7	2	4

- end of iteration 1: 0, 2, 7, 2, 4



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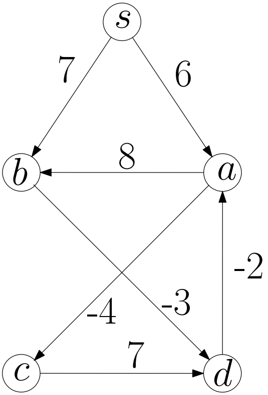
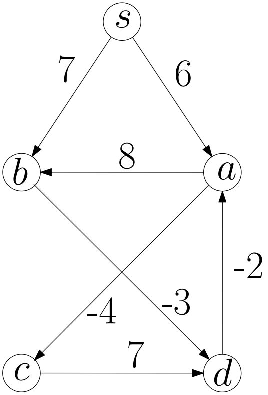


- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	2	7	2	4

- end of iteration 1: 0, 2, 7, 2, 4



- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
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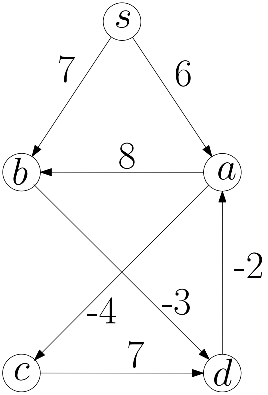

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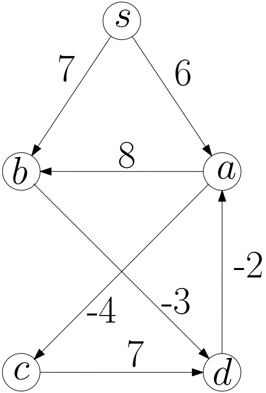

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vertices	s	a	b	c	d
f	0	2	7	-2	4

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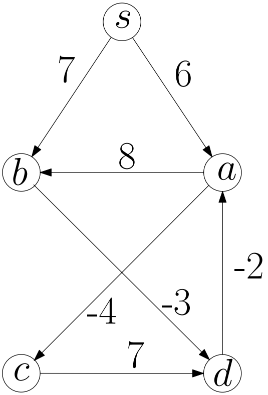

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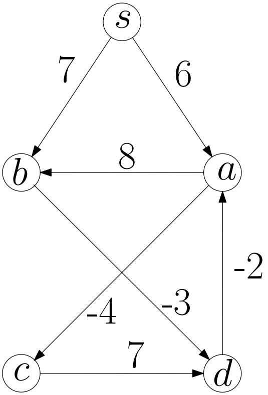

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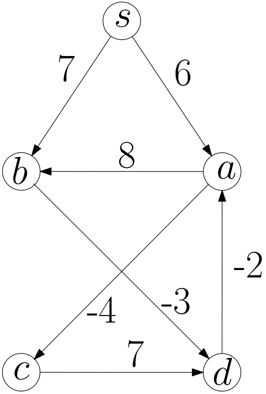

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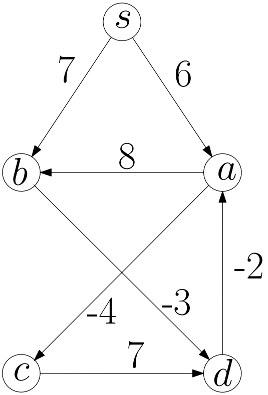

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vertices	s	a	b	c	d
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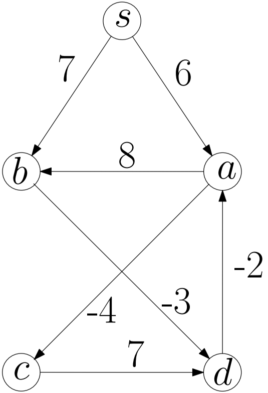
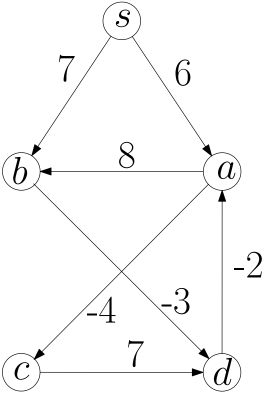
- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4



- order in which we consider edges: (s,a), (s,b), (a,b), (a,c), (b,d), (c,d), (d,a)

vertices	s	a	b	c	d
f	0	2	7	-2	4

- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4



- order in which we consider edges: (s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices	s	a	b	c	d
f	0	2	7	-2	4

- end of iteration 1: 0, 2, 7, 2, 4
- end of iteration 2: 0, 2, 7, -2, 4
- end of iteration 3: 0, 2, 7, -2, 4
- Algorithm terminates in 3 iterations, instead of 4.

Bellman-Ford Algorithm

Bellman-Ford(G, w, s)

$f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
for $\ell \leftarrow 1$ to n **do**
    $\mathit{updated} \leftarrow$ false
    for each $(u, v) \in E$ **do**
        if $f[u] + w(u, v) < f[v]$ **then**
            $f[v] \leftarrow f[u] + w(u, v)$
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    if not $\mathit{updated}$, then return f
output "negative cycle exists"

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- $\pi[v]$: the parent of v in the shortest path tree
- Running time $= O(nm)$

Outline

All-Pair Shortest Paths

All Pair Shortest Paths

Input: directed graph $G = (V, E)$,
$w : E \to \mathbb{R}$ (can be negative)

Output: shortest path from u to v for every $u, v \in V$

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```
1: for every starting point s ∈ V do
2:     run Bellman-Ford(G, w, s)
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for every starting point s ∈ V do
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```

- Running time $= O(n^2 m)$

Summary of Shortest Path Algorithms we learned

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
Bellman-Ford	U/D	$\mathbb{R}$	SS	$O(nm)$
Floyd-Warshall	U/D	$\mathbb{R}$	AP	$O(n^3)$

- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs

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$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

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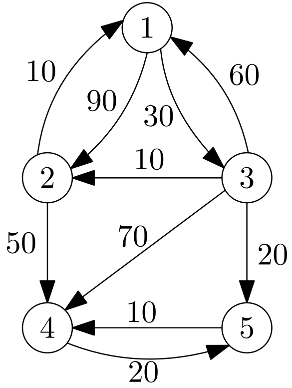
- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

Example for Definition of $f^k[i,j]$'s



$$f^0[1,4] = \infty$$
$$f^1[1,4] = \infty$$
$$f^2[1,4] = 140 \qquad (1 \to 2 \to 4)$$
$$f^3[1,4] = 90 \qquad (1 \to 3 \to 2 \to 4)$$
$$f^4[1,4] = 90 \qquad (1 \to 3 \to 2 \to 4)$$
$$f^5[1,4] = 60 \qquad (1 \to 3 \to 5 \to 4)$$

$$w(i,j) = \begin{cases} 0 & i = j \\ \text{weight of edge } (i,j) & i \neq j, (i,j) \in E \\ \infty & i \neq j, (i,j) \notin E \end{cases}$$

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f^k[i,j] = \begin{cases} & k = 0 \\ & k = 1, 2, \cdots, n \end{cases}
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$$f^k[i,j] = \begin{cases} w(i,j) & k = 0 \\ \min \left\{ \begin{array}{l} f^{k-1}[i,j] \\ \\ \end{array} \right. & k = 1, 2, \cdots, n \end{cases}$$

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- $f^k[i,j]$: length of shortest path from i to j that only uses vertices $\{1,2,3,\cdots,k\}$ as intermediate vertices

$$f^k[i,j] = \begin{cases} w(i,j) & k = 0 \\ \min \begin{cases} f^{k-1}[i,j] \\ f^{k-1}[i,k] + f^{k-1}[k,j] \end{cases} & k = 1,2,\cdots,n \end{cases}$$

Floyd-Warshall(G, w)

$f^0 \leftarrow w$

for $k \leftarrow 1$ to n **do**

  copy $f^{k-1} \rightarrow f^k$

  for $i \leftarrow 1$ to n **do**

    for $j \leftarrow 1$ to n **do**

      if $f^{k-1}[i,k] + f^{k-1}[k,j] < f^k[i,j]$ **then**

        $f^k[i,j] \leftarrow f^{k-1}[i,k] + f^{k-1}[k,j]$

Floyd-Warshall(G, w)

$f^{\text{old}} \leftarrow w$
for $k \leftarrow 1$ to n **do**
    copy $f^{\text{old}} \rightarrow f^{\text{new}}$
    for $i \leftarrow 1$ to n **do**
        for $j \leftarrow 1$ to n **do**
            if $f^{\text{old}}[i,k] + f^{\text{old}}[k,j] < f^{\text{new}}[i,j]$ **then**
                $f^{\text{new}}[i,j] \leftarrow f^{\text{old}}[i,k] + f^{\text{old}}[k,j]$

Floyd-Warshall(G, w)

$f^{\text{old}} \leftarrow w$
for $k \leftarrow 1$ to n **do**
  copy $f^{\text{old}} \to f^{\text{new}}$
  for $i \leftarrow 1$ to n **do**
    for $j \leftarrow 1$ to n **do**
      if $f^{\text{old}}[i,k] + f^{\text{old}}[k,j] < f^{\text{new}}[i,j]$ **then**
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$f \leftarrow w$
for $k \leftarrow 1$ to n **do**
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```
f ← w
for k ← 1 to n do
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            if f[i,k] + f[k,j] < f[i,j] then
                f[i,j] ← f[i,k] + f[k,j]
```

Lemma Assume there are no negative cycles in G. After iteration k, for $i, j \in V$, $f[i,j]$ is exactly the length of shortest path from i to j that only uses vertices in $\{1, 2, 3, \cdots, k\}$ as intermediate vertices.

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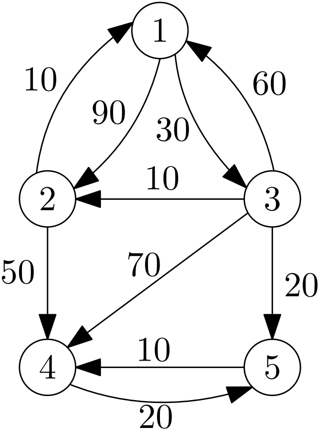
- Running time $= O(n^3)$.



	1	2	3	4	5
1	0	90	30	∞	∞
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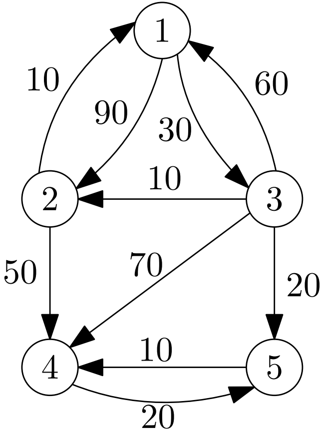
-



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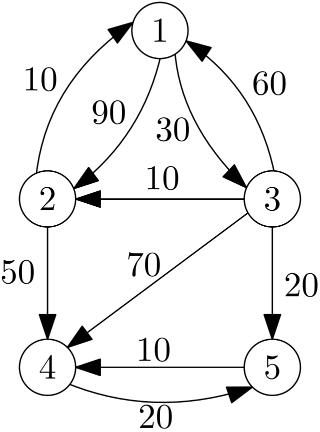
- $i = 2,\ k = 1,\ j = 3$



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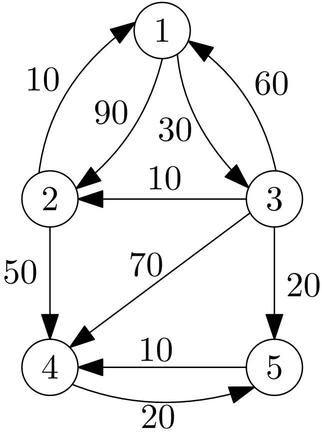

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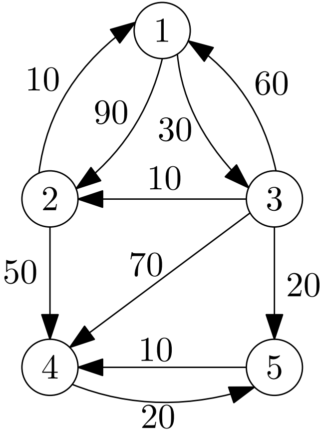

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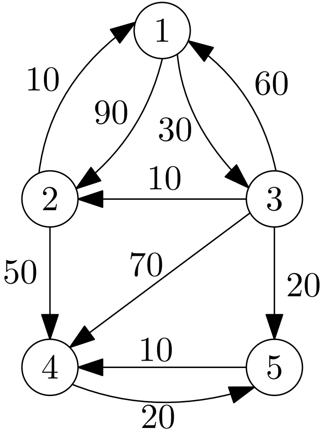

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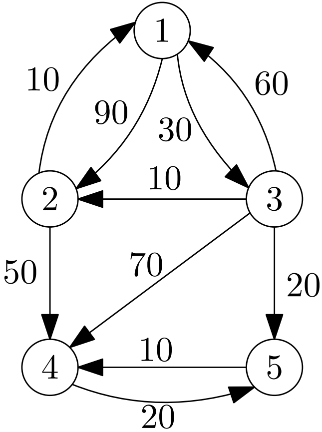

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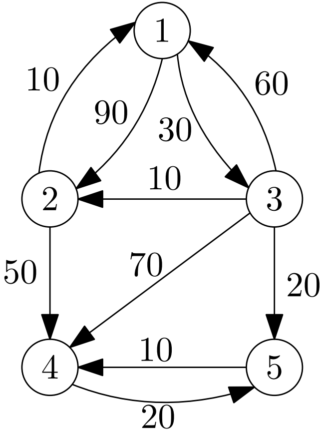

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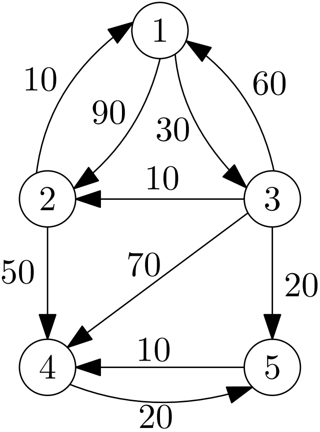

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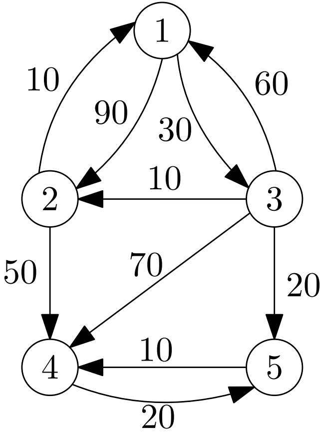

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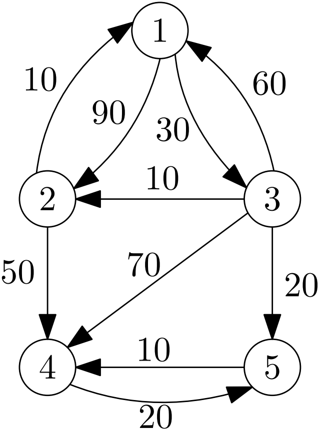

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Recovering Shortest Paths

Floyd-Warshall(G, w)

$f \leftarrow w$, $\pi[i,j] \leftarrow \perp$ for every $i, j \in V$
for $k \leftarrow 1$ to n **do**
    for $i \leftarrow 1$ to n **do**
        for $j \leftarrow 1$ to n **do**
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print-path(i, j)

if $\pi[i, j] = \perp$ **then** then
    if $i \neq j$ **then** print$(i, \text{“,”})$
else
    print-path$(i, \pi[i, j])$, print-path$(\pi[i, j], j)$

Detecting Negative Cycles

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        for $j \leftarrow 1$ to n **do**
            if $f[i,k] + f[k,j] < f[i,j]$ **then**
                report "negative cycle exists" and exit

Summary of Shortest Path Algorithms

algorithm	graph	weights	SS?	running time
Simple DP	DAG	$\mathbb{R}$	SS	$O(n+m)$
Dijkstra	U/D	$\mathbb{R}_{\geq 0}$	SS	$O(n \log n + m)$
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- DAG = directed acyclic graph U = undirected D = directed
- SS = single source AP = all pairs