CSE 431/531: Algorithm Analysis and Design (Spring 2022)
Graph Algorithms

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Outline

1 Minimum Spanning Tree
   • Kruskal’s Algorithm
   • Reverse-Kruskal’s Algorithm
   • Prim’s Algorithm

2 Single Source Shortest Paths
   • Dijkstra’s Algorithm

3 Shortest Paths in Graphs with Negative Weights

4 All-Pair Shortest Paths and Floyd-Warshall
Def. Given a connected graph $G = (V, E)$, a spanning tree $T = (V, F)$ of $G$ is a sub-graph of $G$ that is a tree including all vertices $V$. 
**Lemma**  Let $T = (V, F)$ be a subgraph of $G = (V, E)$. The following statements are equivalent:

- $T$ is a spanning tree of $G$;
- $T$ is acyclic and connected;
- $T$ is connected and has $n - 1$ edges;
- $T$ is acyclic and has $n - 1$ edges;
- $T$ is minimally connected: removal of any edge disconnects it;
- $T$ is maximally acyclic: addition of any edge creates a cycle;
- $T$ has a unique simple path between every pair of nodes.
Minimum Spanning Tree (MST) Problem

**Input:** Graph $G = (V, E)$ and edge weights $w : E \rightarrow \mathbb{R}$

**Output:** the spanning tree $T$ of $G$ with the minimum total weight
Minimum Spanning Tree (MST) Problem

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Recall: Steps of Designing A Greedy Algorithm

- Design a “reasonable” strategy
- Prove that the reasonable strategy is “safe” (key, usually done by “exchanging argument”)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually trivial)

**Def.** A choice is “safe” if there is an optimum solution that is “consistent” with the choice
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Two Classic Greedy Algorithms for MST

- Kruskal’s Algorithm
- Prim’s Algorithm
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4 All-Pair Shortest Paths and Floyd-Warshall
Q: Which edge can be safely included in the MST?
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A: The edge with the smallest weight (lightest edge).
Lemma  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof. Take a minimum spanning tree $T$. Assume the lightest edge $e^*$ is not in $T$. There is a unique path in $T$ connecting $u$ and $v$. Remove any edge $e$ in the path to obtain tree $T'$.

$w(e^*) \leq w(e) \Rightarrow w(T') \leq w(T)$: $T'$ is also a MST.
**Lemma**  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**

- Take a minimum spanning tree $T$

\[
\begin{align*}
\text{Proof.} & \quad \text{Take a minimum spanning tree } T \\
\end{align*}
\]
**Lemma**  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**

- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$

![Diagram showing a minimum spanning tree with a lightest edge $e^*$ highlighted between nodes $u$ and $v$.]
**Lemma** It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**
- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$
- There is a unique path in $T$ connecting $u$ and $v$
Lemma It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

Proof.

- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$
- There is a unique path in $T$ connecting $u$ and $v$
- Remove any edge $e$ in the path to obtain tree $T'$
**Lemma**  It is safe to include the lightest edge: there is a minimum spanning tree, that contains the lightest edge.

**Proof.**

- Take a minimum spanning tree $T$
- Assume the lightest edge $e^*$ is not in $T$
- There is a unique path in $T$ connecting $u$ and $v$
- Remove any edge $e$ in the path to obtain tree $T'$
- $w(e^*) \leq w(e) \implies w(T') \leq w(T)$: $T'$ is also a MST
Is the Residual Problem Still a MST Problem?

Residual problem: find the minimum spanning tree that contains edge (g, h)

Contract the edge (g, h)

Residual problem: find the minimum spanning tree in the contracted graph
Residual problem: find the minimum spanning tree that contains edge $(g, h)$
Residual problem: find the minimum spanning tree that contains edge \((g, h)\)

- **Contract** the edge \((g, h)\)
Residual problem: find the minimum spanning tree that contains edge \((g, h)\)

Contract the edge \((g, h)\)

Residual problem: find the minimum spanning tree in the contracted graph
Contraction of an Edge \((u, v)\)

Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\).

Remove all edges \((u, v)\) from \(E\).

For every edge \((u, w)\) \(\in E, w \neq v\), change it to \((u^*, w)\).

For every edge \((v, w)\) \(\in E, w \neq u\), change it to \((u^*, w)\).

May create parallel edges! E.g.: two edges \((i, g^*)\).
Contraction of an Edge \((u, v)\)

- Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\)
Contraction of an Edge \((u, v)\)

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Contraction of an Edge \((u, v)\)

- Remove \(u\) and \(v\) from the graph, and add a new vertex \(u^*\)
- Remove all edges \((u, v)\) from \(E\)
- For every edge \((u, w)\) \(\in E\), \(w \neq v\), change it to \((u^*, w)\)
- For every edge \((v, w)\) \(\in E\), \(w \neq u\), change it to \((u^*, w)\)
- May create parallel edges! E.g. : two edges \((i, g^*)\)
Greedy Algorithm

Repeat the following step until $G$ contains only one vertex:

1. Choose the lightest edge $e^*$, add $e^*$ to the spanning tree
2. Contract $e^*$ and update $G$ be the contracted graph
Greedy Algorithm

Repeat the following step until $G$ contains only one vertex:

1. Choose the lightest edge $e^*$, add $e^*$ to the spanning tree
2. Contract $e^*$ and update $G$ be the contracted graph

Q: What edges are removed due to contractions?
Greedy Algorithm

Repeat the following step until \( G \) contains only one vertex:

1. Choose the lightest edge \( e^* \), add \( e^* \) to the spanning tree
2. Contract \( e^* \) and update \( G \) be the contracted graph

**Q:** What edges are removed due to contractions?

**A:** Edge \((u, v)\) is removed if and only if there is a path connecting \( u \) and \( v \) formed by edges we selected
Greedy Algorithm

\[ \text{MST-Greedy}(G, w) \]

1: \[ F \leftarrow \emptyset \]
2: sort edges in \( E \) in non-decreasing order of weights \( w \)
3: for each edge \((u, v)\) in the order do
4: \hspace{1em} if \( u \) and \( v \) are not connected by a path of edges in \( F \) then
5: \hspace{2em} \[ F \leftarrow F \cup \{(u, v)\} \]
6: return \((V, F)\)
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g\}, \{h\}, \{i\}
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g, h\}, \{i\}
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{f\}, \{g, h\}, \{i\}
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Sets: \{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f, g, h\}
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c, i\}, \{d\}, \{e\}, \{f, g, h\}
Sets: \{a\}, \{b\}, \{c, i, f, g, h\}, \{d\}, \{e\}
Kruskal’s Algorithm: Example

Sets: \{a\}, \{b\}, \{c, i, f, g, h\}, \{d\}, \{e\}
Kruskal’s Algorithm: Example

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Sets: \{a, b, c, i, f, g, h\}, \{d\}, \{e\}
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Sets: \{a, b, c, i, f, g, h\}, \{d\}, \{e\}
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Kruskal’s Algorithm: Example

Sets: \{a, b, c, i, f, g, h, d, e\}
Kruskal’s Algorithm: Efficient Implementation of Greedy Algorithm

MST-Kruskal($G$, $w$)

1: $F \leftarrow \emptyset$
2: $S \leftarrow \{\{v\}: v \in V\}$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: for each edge $(u, v) \in E$ in the order do
5: $S_u \leftarrow$ the set in $S$ containing $u$
6: $S_v \leftarrow$ the set in $S$ containing $v$
7: if $S_u \neq S_v$ then
8: $F \leftarrow F \cup \{(u, v)\}$
9: $S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: return $(V, F)$
**Running Time of Kruskal’s Algorithm**

**MST-Kruskal**($G, w$)

1. $F \leftarrow \emptyset$
2. $S \leftarrow \{ \{v\} : v \in V \}$
3. sort the edges of $E$ in non-decreasing order of weights $w$
4. for each edge $(u, v) \in E$ in the order do
5. \hspace{1em} $S_u \leftarrow$ the set in $S$ containing $u$
6. \hspace{1em} $S_v \leftarrow$ the set in $S$ containing $v$
7. \hspace{1em} if $S_u \neq S_v$ then
8. \hspace{2em} $F \leftarrow F \cup \{(u, v)\}$
9. \hspace{2em} $S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10. return $(V, F)$

Use **union-find** data structure to support \(2, 5, 6, 7, 9\).
Union-Find Data Structure

- $V$: ground set
- We need to maintain a partition of $V$ and support following operations:
  - Check if $u$ and $v$ are in the same set of the partition
  - Merge two sets in partition
- $V = \{1, 2, 3, \cdots, 16\}$
- Partition: $\{2, 3, 5, 9, 10, 12, 15\}$, $\{1, 7, 13, 16\}$, $\{4, 8, 11\}$, $\{6, 14\}$

$par[i]$: parent of $i$, ($par[i] = \bot$ if $i$ is a root).
Union-Find Data Structure

Q: how can we check if \( u \) and \( v \) are in the same set?

A: Check if \( \text{root}(u) = \text{root}(v) \).

\[ \text{root}(u) \] is the root of the tree containing \( u \).

Merge the trees with root \( r \) and \( r' \):

\[ \text{par}[r] \leftarrow r' \]
Q: how can we check if $u$ and $v$ are in the same set?
Union-Find Data Structure

Q: how can we check if $u$ and $v$ are in the same set?
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Q: how can we check if \( u \) and \( v \) are in the same set?
A: Check if \( \text{root}(u) = \text{root}(v) \).

\( \text{root}(u) \): the root of the tree containing \( u \)
Q: how can we check if $u$ and $v$ are in the same set?

A: Check if $\text{root}(u) = \text{root}(v)$.

root($u$): the root of the tree containing $u$

Merge the trees with root $r$ and $r'$: $\text{par}[r] \leftarrow r'$. 
Q: how can we check if $u$ and $v$ are in the same set?

A: Check if $\text{root}(u) = \text{root}(v)$.

$\text{root}(u)$: the root of the tree containing $u$

Merge the trees with root $r$ and $r'$: $\text{par}[r] \leftarrow r'$. 
Union-Find Data Structure

\[ \text{root}(v) \]

1: \textbf{if } \text{par}[v] = \perp \textbf{ then}
2: \quad \textbf{return } v
3: \quad \textbf{else}
4: \quad \textbf{return } \text{root}(\text{par}[v])

Problem: the tree might be too deep; running time might be large

Improvement: all vertices in the path directly point to the root, saving time in the future.
Union-Find Data Structure

\textbf{root}\( (v) \)

1: \textbf{if} \( \text{par}[v] = \bot \) \textbf{then}
2: \hspace{1em} \textbf{return} \( v \)
3: \hspace{1em} \textbf{else}
4: \hspace{2em} \textbf{return} \text{root}\( (\text{par}[v]) \)

- Problem: the tree might too deep; running time might be large
Union-Find Data Structure

\[
\text{root}(v)
\]

1: if \( \text{par}[v] = \bot \) then
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\[
\text{root}(v)
\]

1. \textbf{if } \text{par}[v] = \bot \textbf{ then}
2. \textbf{return } v
3. \textbf{else}
4. \textbf{par}[v] \leftarrow \text{root}(\text{par}[v])
5. \textbf{return } \text{par}[v]

- Problem: the tree might too deep; running time might be large
- Improvement: all vertices in the path directly point to the root, saving time in the future.
Union-Find Data Structure

root(v)

1: if par[v] = ⊥ then
2: return v
3: else
4: par[v] ← root(par[v])
5: return par[v]

```
3
  10
  12
   15
    9
     5

2
  1
  16
13
6
  8
   4
    11
      14
```
root(v)

1: if \( \text{par}[v] = \bot \) then
2: return \( v \)
3: else
4: \( \text{par}[v] \leftarrow \text{root}(\text{par}[v]) \)
5: return \( \text{par}[v] \)
MST-Kruskal($G$, $w$)

1: $F \leftarrow \emptyset$
2: $S \leftarrow \{\{v\} : v \in V\}$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: for each edge $(u, v) \in E$ in the order do
5: $S_u \leftarrow$ the set in $S$ containing $u$
6: $S_v \leftarrow$ the set in $S$ containing $v$
7: if $S_u \neq S_v$ then
8: $F \leftarrow F \cup \{(u, v)\}$
9: $S \leftarrow S \setminus \{S_u\} \setminus \{S_v\} \cup \{S_u \cup S_v\}$
10: return $(V, F)$
MST-Kruskal(\(G, w\))

1: \(F \leftarrow \emptyset\)
2: for every \(v \in V\) do: \(par[v] \leftarrow \perp\)
3: sort the edges of \(E\) in non-decreasing order of weights \(w\)
4: for each edge \((u, v) \in E\) in the order do
5: \(u' \leftarrow \text{root}(u)\)
6: \(v' \leftarrow \text{root}(v)\)
7: if \(u' \neq v'\) then
8: \(F \leftarrow F \cup \{(u, v)\}\)
9: \(par[u'] \leftarrow v'\)
10: return \((V, F)\)
MST-Kruskal\((G, w)\)

1. \(F \leftarrow \emptyset\)
2. \(\text{for every } v \in V \text{ do: } \text{par}[v] \leftarrow \perp\)
3. sort the edges of \(E\) in non-decreasing order of weights \(w\)
4. \(\text{for each edge } (u, v) \in E \text{ in the order do}\)
5. \(u' \leftarrow \text{root}(u)\)
6. \(v' \leftarrow \text{root}(v)\)
7. \(\text{if } u' \neq v' \text{ then}\)
8. \(F \leftarrow F \cup \{(u, v)\}\)
9. \(\text{par}[u'] \leftarrow v'\)
10. \(\text{return } (V, F)\)

- \(2, 5, 6, 7, 9\) takes time \(O(m\alpha(n))\)
- \(\alpha(n)\) is very slow-growing: \(\alpha(n) \leq 4\) for \(n \leq 10^{80}\).
MST-Kruskal($G$, $w$)

1: $F \leftarrow \emptyset$
2: for every $v \in V$ do: $\text{par}[v] \leftarrow \bot$
3: sort the edges of $E$ in non-decreasing order of weights $w$
4: for each edge $(u, v) \in E$ in the order do
5: \hspace{1em} $u' \leftarrow \text{root}(u)$
6: \hspace{1em} $v' \leftarrow \text{root}(v)$
7: \hspace{1em} if $u' \neq v'$ then
8: \hspace{2em} $F \leftarrow F \cup \{(u, v)\}$
9: \hspace{2em} $\text{par}[u'] \leftarrow v'$
10: return $(V, F)$

- 2, 5, 6, 7, 9 takes time $O(m\alpha(n))$
- $\alpha(n)$ is very slow-growing: $\alpha(n) \leq 4$ for $n \leq 10^{80}$.
- Running time = time for 3 = $O(m\lg n)$. 
**Assumption**  Assume all edge weights are different.

**Lemma**  An edge $e \in E$ is not in the MST, if and only if there is cycle $C$ in $G$ in which $e$ is the heaviest edge.
Assumption  Assume all edge weights are different.

Lemma  An edge $e \in E$ is not in the MST, if and only if there is cycle $C$ in $G$ in which $e$ is the heaviest edge.

$(i, g)$ is not in the MST because of cycle $(i, c, f, g)$
**Assumption** Assume all edge weights are different.

**Lemma** An edge $e \in E$ is not in the MST, if and only if there is cycle $C$ in $G$ in which $e$ is the heaviest edge.

- $(i, g)$ is not in the MST because of cycle $(i, c, f, g)$
- $(e, f)$ is in the MST because no such cycle exists
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   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.

Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def.: A bridge is an edge whose removal disconnects the graph.
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.

2. Start from $F \leftarrow E$, and remove edges from $F$ one by one until we obtain a spanning tree.

Q: Which edge can be safely excluded from the MST?

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Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.

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1. Start from \( F \leftarrow \emptyset \), and add edges to \( F \) one by one until we obtain a spanning tree.

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A: The heaviest non-bridge edge.
Two Methods to Build a MST

1. Start from $F \leftarrow \emptyset$, and add edges to $F$ one by one until we obtain a spanning tree.

2. Start from $F \leftarrow E$, and remove edges from $F$ one by one until we obtain a spanning tree.

Q: Which edge can be safely excluded from the MST?

A: The heaviest non-bridge edge.

Def. A bridge is an edge whose removal disconnects the graph.
Lemma  It is safe to exclude the heaviest non-bridge edge: there is a MST that does not contain the heaviest non-bridge edge.
Reverse Kruskal’s Algorithm

**MST-Greedy**(\(G, w\))

1. \(F \leftarrow E\)
2. sort \(E\) in non-increasing order of weights
3. for every \(e\) in this order do
4. \(\text{if } (V, F \setminus \{e\}) \text{ is connected then}\)
5. \(F \leftarrow F \setminus \{e\}\)
6. return \((V, F)\)
Reverse Kruskal’s Algorithm: Example
Reverse Kruskal’s Algorithm: Example
Reverse Kruskal’s Algorithm: Example

Diagram of a graph with nodes labeled from 'a' to 'h' and edges with weights labeled from 1 to 13.
Reverse Kruskal’s Algorithm: Example

![Graph with labeled edges and nodes](image)
Reverse Kruskal’s Algorithm: Example
Reverse Kruskal’s Algorithm: Example

```
a
5
b
8
i
11
c
2
h
g
6
f
4
d
9
e
10
```
Reverse Kruskal’s Algorithm: Example
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Diagram:
- Nodes: a, b, c, d, e, f, g, h, i
- Edges with weights:
  - a to b: 5
  - b to c: 8
  - c to f: 4
  - c to i: 2
  - i to g: 6
  - g to f: 3
  - f to e: 10
  - e to d: 9

Explanation:
Reverse Kruskal’s Algorithm involves finding the minimum spanning tree of a graph and then removing the edges one by one, starting from the edge with the highest weight. The remaining edges form the maximum spanning tree.
Reverse Kruskal’s Algorithm: Example
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4. All-Pair Shortest Paths and Floyd-Warshall
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.

Greedy strategy for Prim’s algorithm: choose the lightest edge incident to $a$. 
Recall the greedy strategy for Kruskal’s algorithm: choose the edge with the smallest weight.

Greedy strategy for Prim’s algorithm: choose the lightest edge incident to a.
Lemma  It is safe to include the lightest edge incident to $a$.  

Proof. Let $T$ be a MST. Consider all components obtained by removing $a$ from $T$. Let $e^*$ be the lightest edge incident to $a$, and $e^*_{\text{connects}}$ connects $a$ to component $C$. Let $e$ be the edge in $T$ connecting $a$ to $C$. $T' = T \setminus \{e\} \cup \{e^*\}$ is a spanning tree with $w(T') \leq w(T)$.  

Lemma It is safe to include the lightest edge incident to $a$.

Proof.

- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
Lemma. It is safe to include the lightest edge incident to \( a \).

Proof.

- Let \( T \) be a MST.
- Consider all components obtained by removing \( a \) from \( T \).
- Let \( e^* \) be the lightest edge incident to \( a \) and \( e^* \) connects \( a \) to component \( C \).
Lemma  It is safe to include the lightest edge incident to $a$.

**Proof.**

- Let $T$ be a MST
- Consider all components obtained by removing $a$ from $T$
- Let $e^*$ be the lightest edge incident to $a$ and $e^*$ connects $a$ to component $C$
- Let $e$ be the edge in $T$ connecting $a$ to $C$
**Lemma**  It is safe to include the lightest edge incident to \( a \).

\[
\text{lightest edge } e^* \text{ incident to } a
\]

**Proof.**

- Let \( T \) be a MST
- Consider all components obtained by removing \( a \) from \( T \)
- Let \( e^* \) be the lightest edge incident to \( a \) and \( e^* \) connects \( a \) to component \( C \)
- Let \( e \) be the edge in \( T \) connecting \( a \) to \( C \)
- \( T' = T \setminus \{e\} \cup \{e^*\} \) is a spanning tree with \( w(T') \leq w(T) \)
Prim’s Algorithm: Example
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Greedy Algorithm

MST-Greedy1\((G, w)\)

1: \( S \leftarrow \{ s \} \), where \( s \) is arbitrary vertex in \( V \)
2: \( F \leftarrow \emptyset \)
3: \( \textbf{while} \ S \neq V \ \textbf{do} \)
4: \((u, v) \leftarrow \text{lightest edge between } S \text{ and } V \setminus S,\)
\( \quad \text{where } u \in S \text{ and } v \in V \setminus S \)
5: \( S \leftarrow S \cup \{ v \} \)
6: \( F \leftarrow F \cup \{(u, v)\} \)
7: \( \textbf{return} \ (V, F) \)
Greedy Algorithm

MST-Greedy1\((G, w)\)

1: \(S \leftarrow \{s\}\), where \(s\) is arbitrary vertex in \(V\)
2: \(F \leftarrow \emptyset\)
3: while \(S \neq V\) do
4: \((u, v) \leftarrow\) lightest edge between \(S\) and \(V \setminus S\),
   where \(u \in S\) and \(v \in V \setminus S\)
5: \(S \leftarrow S \cup \{v\}\)
6: \(F \leftarrow F \cup \{(u, v)\}\)
7: return \((V, F)\)

- Running time of naive implementation: \(O(nm)\)
Prim’s Algorithm: Efficient Implementation of Greedy Algorithm

For every \( v \in V \setminus S \) maintain
- \( d[v] = \min_{u \in S: (u,v) \in E} w(u, v) \): the weight of the lightest edge between \( v \) and \( S \)
- \( \pi[v] = \arg \min_{u \in S: (u,v) \in E} w(u, v) \): \((\pi[v], v)\) is the lightest edge between \( v \) and \( S \)
Prim’s Algorithm: Efficient Implementation of Greedy Algorithm

For every $v \in V \setminus S$ maintain

- $d[v] = \min_{u \in S : (u,v) \in E} w(u, v)$: the weight of the lightest edge between $v$ and $S$
- $\pi[v] = \arg \min_{u \in S : (u,v) \in E} w(u, v)$: $(\pi[v], v)$ is the lightest edge between $v$ and $S$

In every iteration

- Pick $u \in V \setminus S$ with the smallest $d[u]$ value
- Add $(\pi[u], u)$ to $F$
- Add $u$ to $S$, update $d$ and $\pi$ values.
Prim’s Algorithm

MST-Prim\((G, w)\)

1: \(s \leftarrow\) arbitrary vertex in \(G\)
2: \(S \leftarrow \emptyset, d(s) \leftarrow 0\) and \(d[v] \leftarrow \infty\) for every \(v \in V \setminus \{s\}\)
3: \textbf{while} \(S \neq V\) \textbf{do}
4: \(u \leftarrow\) vertex in \(V \setminus S\) with the minimum \(d[u]\)
5: \(S \leftarrow S \cup \{u\}\)
6: \textbf{for each} \(v \in V \setminus S\) such that \((u, v) \in E\) \textbf{do}
7: \textbf{if} \(w(u, v) < d[v]\) \textbf{then}
8: \(d[v] \leftarrow w(u, v)\)
9: \(\pi[v] \leftarrow u\)
10: \textbf{return} \(\{(u, \pi[u]) | u \in V \setminus \{s\}\}\)
Example
Example

A diagram of a network graph is shown, with nodes labeled from 'a' to 'h' and edges labeled with numbers from 1 to 14. The nodes are connected by weighted edges, indicating the cost or distance between them.
Example
Example

\begin{itemize}
\item \textbf{a}
\item \textbf{b}
\item \textbf{c}
\item \textbf{d}
\item \textbf{e}
\item \textbf{f}
\item \textbf{g}
\item \textbf{h}
\item \textbf{i}
\item \textbf{\textcolor{red}{a}}
\item \textbf{\textcolor{red}{\textbf{\textcolor{red}{b}}}}
\item \textbf{\textcolor{red}{\textbf{\textcolor{red}{\textcolor{red}{c}}}}}
\item \textbf{\textcolor{red}{\textbf{\textcolor{red}{\textcolor{red}{d}}}}}
\item \textbf{\textcolor{red}{\textbf{\textcolor{red}{\textcolor{red}{e}}}}}
\end{itemize}

\begin{itemize}
\item (5, a)
\item (12, a)
\end{itemize}
Example
Example

\begin{itemize}
\item \text{(8, b)}
\item \text{(11, b)}
\end{itemize}
Example
Example

(8, b)

(11, b)
Example
Example

```
(13, c)
(11, b)
(2, c)
(4, c)
```
Example
Example
Example
Example
Example
Example
Example

```
(13, c)
(7, i)
(3, f)
(10, f)
```
Example
Example

\( (13, c) \)

\( (1, g) \)

\( (10, f) \)
Example

(13, c)
(1, g)
(10, f)
Example

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{Example graph with labeled edges.}
\end{figure}
Example
Example

```
(13, c)  
(10, f)  
```
Example

\[ (9, e) \]
Example
Example

(9, e)
Example
Prim’s Algorithm

For every \( v \in V \setminus S \) maintain

- \( d[v] = \min_{u \in S: (u,v) \in E} w(u,v) \):
  the weight of the lightest edge between \( v \) and \( S \)

- \( \pi[v] = \arg\min_{u \in S: (u,v) \in E} w(u,v) \):
  \((\pi[v], v)\) is the lightest edge between \( v \) and \( S \)

In every iteration

- Pick \( u \in V \setminus S \) with the smallest \( d[u] \) value
- Add \((\pi[u], u)\) to \( F \)
- Add \( u \) to \( S \), update \( d \) and \( \pi \) values.
Prim’s Algorithm

For every \( v \in V \setminus S \) maintain

- \( d[v] = \min_{u \in S: (u, v) \in E} w(u, v) \):
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  \((\pi[v], v)\) is the lightest edge between \( v \) and \( S \)

In every iteration

- Pick \( u \in V \setminus S \) with the smallest \( d[u] \) value \( \text{extract\_min} \)
- Add \((\pi[u], u)\) to \( F \)
- Add \( u \) to \( S \), update \( d \) and \( \pi \) values. \( \text{decrease\_key} \)

Use a priority queue to support the operations
**Def.** A *priority queue* is an abstract data structure that maintains a set $U$ of elements, each with an associated key value, and supports the following operations:

- $\text{insert}(v, \text{key\_value})$: insert an element $v$, whose associated key value is $\text{key\_value}$.

- $\text{decrease\_key}(v, \text{new\_key\_value})$: decrease the key value of an element $v$ in queue to $\text{new\_key\_value}$

- $\text{extract\_min}()$: return and remove the element in queue with the smallest key value

...
MST-Prim(\(G, w\))

1: \(s \leftarrow\) arbitrary vertex in \(G\)
2: \(S \leftarrow \emptyset\), \(d(s) \leftarrow 0\) and \(d[v] \leftarrow \infty\) for every \(v \in V \setminus \{s\}\)
3: 
4: while \(S \neq V\) do
5: \(u \leftarrow\) vertex in \(V \setminus S\) with the minimum \(d[u]\)
6: \(S \leftarrow S \cup \{u\}\)
7: for each \(v \in V \setminus S\) such that \((u, v) \in E\) do
8: \(\text{if } w(u, v) < d[v] \text{ then}\)
9: \(d[v] \leftarrow w(u, v)\)
10: \(\pi[v] \leftarrow u\)
11: return \(\{(u, \pi[u]) | u \in V \setminus \{s\}\}\)
Prim’s Algorithm Using Priority Queue

MST-Prim($G, w$)

1: $s \leftarrow$ arbitrary vertex in $G$
2: $S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3: $Q \leftarrow$ empty queue, for each $v \in V$: $Q.insert(v, d[v])$
4: while $S \neq V$ do
5: \hspace{1em} $u \leftarrow Q.extract\_min()$
6: \hspace{1em} $S \leftarrow S \cup \{u\}$
7: \hspace{1em} for each $v \in V \setminus S$ such that $(u, v) \in E$ do
8: \hspace{2em} if $w(u, v) < d[v]$ then
9: \hspace{3em} $d[v] \leftarrow w(u, v)$, $Q.decrease\_key(v, d[v])$
10: \hspace{3em} $\pi[v] \leftarrow u$
11: return $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$
Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times \text{(time for extract\_min)} + O(m) \times \text{(time for decrease\_key)} \]
Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times \text{(time for extract\_min)} + O(m) \times \text{(time for decrease\_key)} \]

<table>
<thead>
<tr>
<th>concrete DS</th>
<th>extract_min</th>
<th>decrease_key</th>
<th>overall time</th>
</tr>
</thead>
<tbody>
<tr>
<td>heap</td>
<td>(O(\log n))</td>
<td>(O(\log n))</td>
<td>(O(m \log n))</td>
</tr>
<tr>
<td>Fibonacci heap</td>
<td>(O(\log n))</td>
<td>(O(1))</td>
<td>(O(n \log n + m))</td>
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Running Time of Prim’s Algorithm Using Priority Queue

\[ O(n) \times \text{(time for extract\_min)} + O(m) \times \text{(time for decrease\_key)} \]

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</table>
**Assumption**  Assume all edge weights are different.

**Lemma**  \((u, v)\) is in MST, if and only if there exists a cut \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).
Assumption  Assume all edge weights are different.

Lemma  \((u, v)\) is in MST, if and only if there exists a cut \((U, V \setminus U)\), such that \((u, v)\) is the lightest edge between \(U\) and \(V \setminus U\).

\((c, f)\) is in MST because of cut \((\{a, b, c, i\}, V \setminus \{a, b, c, i\})\)
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- \((c, f)\) is in MST because of cut \((\{a, b, c, i\}, V \setminus \{a, b, c, i\})\)
- \((i, g)\) is not in MST because no such cut exists
“Evidence” for $e \in \text{MST}$ or $e \notin \text{MST}$

**Assumption** Assume all edge weights are different.

- $e \in \text{MST} \iff$ there is a cut in which $e$ is the lightest edge
- $e \notin \text{MST} \iff$ there is a cycle in which $e$ is the heaviest edge
“Evidence” for $e \in \text{MST}$ or $e \notin \text{MST}$

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Exactly one of the following is true:

- There is a cut in which $e$ is the lightest edge
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Thus, the minimum spanning tree is unique with assumption.
Outline

1. Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall
<table>
<thead>
<tr>
<th>algorithm</th>
<th>graph</th>
<th>weights</th>
<th>SS?</th>
<th>running time</th>
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</thead>
<tbody>
<tr>
<td>Simple DP</td>
<td>DAG</td>
<td>ℝ</td>
<td>SS</td>
<td>$O(n + m)$</td>
</tr>
<tr>
<td>Dijkstra</td>
<td>U/D</td>
<td>ℝ$_{≥0}$</td>
<td>SS</td>
<td>$O(n \log n + m)$</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>U/D</td>
<td>ℝ</td>
<td>SS</td>
<td>$O(nm)$</td>
</tr>
<tr>
<td>Floyd-Warshall</td>
<td>U/D</td>
<td>ℝ</td>
<td>AP</td>
<td>$O(n^3)$</td>
</tr>
</tbody>
</table>

- DAG = directed acyclic graph   U = undirected   D = directed
- SS = single source   AP = all pairs
**s-t Shortest Paths**

**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$$w : E \rightarrow \mathbb{R}_{\geq 0}$$

**Output:** shortest path from $s$ to $t$
**s-t Shortest Paths**

**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

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**Input:** (directed or undirected) graph $G = (V, E)$, $s, t \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** shortest path from $s$ to $t$
Single Source Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

$$w : E \to \mathbb{R}_{\geq 0}$$

**Output:** shortest paths from $s$ to all other vertices $v \in V$
Single Source Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

Reason for Considering Single Source Shortest Paths

Problem

- **We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem**
Single Source Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

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Reason for Considering Single Source Shortest Paths

- We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem

- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight
Single Source Shortest Paths

**Input:** (directed or undirected) graph $G = (V, E)$, $s \in V$

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Single Source Shortest Paths

**Input:** directed graph $G = (V, E)$, $s \in V$
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- We do not know how to solve $s$-$t$ shortest path problem more efficiently than solving single source shortest path problem
- Shortest paths in directed graphs is more general than in undirected graphs: we can replace every undirected edge with two anti-parallel edges of the same weight
Single Source Shortest Paths

**Input:** directed graph $G = (V, E)$, $s \in V$

$w : E \rightarrow \mathbb{R}_{\geq 0}$

**Output:** $\pi[v], v \in V \setminus s$: the parent of $v$ in shortest path tree

$d[v], v \in V \setminus s$: the length of shortest path from $s$ to $v$
Q: How to compute shortest paths from $s$ when all edges have weight 1?
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A: Breadth first search (BFS) from source $s$
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Assumption  Weights $w(u, v)$ are integers (w.l.o.g.).
**Assumption**  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges.

![Diagram showing equivalent paths](image)

---

Problem: $w(u, v)$ may be too large!
Assumption  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges

Shortest Path Algorithm by Running BFS

1. replace $(u, v)$ of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2. run BFS
3. $\pi[v] \leftarrow$ vertex from which $v$ is visited
4. $d[v] \leftarrow$ index of the level containing $v$
Assumption  Weights $w(u, v)$ are integers (w.l.o.g).

- An edge of weight $w(u, v)$ is equivalent to a path of $w(u, v)$ unit-weight edges.

![Diagram of an edge and a path](image)

**Shortest Path Algorithm by Running BFS**

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![Diagram](image)

Shortest Path Algorithm by Running BFS

1: replace $(u, v)$ of length $w(u, v)$ with a path of $w(u, v)$ unit-weight edges, for every $(u, v) \in E$
2: run BFS virtually
3: $\pi[v] \leftarrow$ vertex from which $v$ is visited
4: $d[v] \leftarrow$ index of the level containing $v$

- Problem: $w(u, v)$ may be too large!
Shortest Path Algorithm by Running BFS Virtually

1: \( S \leftarrow \{s\}, d(s) \leftarrow 0 \)
2: \( \textbf{while} \ |S| \leq n \ \textbf{do} \)
3: \( \text{find a } v \notin S \text{ that minimizes } \min_{u \in S : (u,v) \in E} \{d[u] + w(u, v)\} \)
4: \( S \leftarrow S \cup \{v\} \)
5: \( d[v] \leftarrow \min_{u \in S : (u,v) \in E} \{d[u] + w(u, v)\} \)
Virtual BFS: Example
Virtual BFS: Example

Time 0
Virtual BFS: Example

Time 2
Virtual BFS: Example

Time 4
Virtual BFS: Example
Virtual BFS: Example

Time 9
Virtual BFS: Example

Time 10
Outline

1 Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2 Single Source Shortest Paths
   - Dijkstra’s Algorithm

3 Shortest Paths in Graphs with Negative Weights

4 All-Pair Shortest Paths and Floyd-Warshall
# Dijkstra’s Algorithm

Dijkstra(G, w, s)

1: \( S \leftarrow \emptyset \), \( d(s) \leftarrow 0 \) and \( d[v] \leftarrow \infty \) for every \( v \in V \setminus \{s\} \)
2: while \( S \neq V \) do
3: \( u \leftarrow \text{vertex in } V \setminus S \text{ with the minimum } d[u] \)
4: add \( u \) to \( S \)
5: for each \( v \in V \setminus S \) such that \( (u, v) \in E \) do
6: if \( d[u] + w(u, v) < d[v] \) then
7: \( d[v] \leftarrow d[u] + w(u, v) \)
8: \( \pi[v] \leftarrow u \)
9: return \((d, \pi)\)
Dijkstra’s Algorithm

\[ \text{Dijkstra}(G, w, s) \]

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2. \textbf{while} \( S \neq V \) \textbf{do}
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4. \( \text{add } u \text{ to } S \)
5. \textbf{for} each \( v \in V \setminus S \text{ such that } (u, v) \in E \) \textbf{do}
6. \quad \textbf{if} \( d[u] + w(u, v) < d[v] \) \textbf{then}
7. \quad \quad \( d[v] \leftarrow d[u] + w(u, v) \)
8. \quad \( \pi[v] \leftarrow u \)
9. \textbf{return} \( (d, \pi) \)

* Running time = \( O(n^2) \)
Improved Running Time using Priority Queue

Dijkstra($G, w, s$)

1: $s \leftarrow$ arbitrary vertex in $G$
2: $S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3: $Q \leftarrow$ empty queue, for each $v \in V$: $Q.insert(v, d[v])$
4: while $S \neq V$ do
5:  $u \leftarrow Q.extract.min()$
6:  $S \leftarrow S \cup \{u\}$
7:  for each $v \in V \setminus S$ such that $(u, v) \in E$ do
8:     if $d[u] + w(u, v) < d[v]$ then
9:         $d[v] \leftarrow d[u] + w(u, v)$, $Q.decrease.key(v, d[v])$
10:    $\pi[v] \leftarrow u$
11: return $(\pi, d)$
Recall: Prim’s Algorithm for MST

**MST-Prim**($G, w$)

1. $s \leftarrow$ arbitrary vertex in $G$
2. $S \leftarrow \emptyset$, $d(s) \leftarrow 0$ and $d[v] \leftarrow \infty$ for every $v \in V \setminus \{s\}$
3. $Q \leftarrow$ empty queue, for each $v \in V$: $Q$.insert($v, d[v]$)
4. **while** $S \neq V$ **do**
   5. $u \leftarrow Q$.extract_min()
   6. $S \leftarrow S \cup \{u\}$
   7. **for** each $v \in V \setminus S$ such that $(u, v) \in E$ **do**
      8. **if** $w(u, v) < d[v]$ **then**
         9. $d[v] \leftarrow w(u, v)$, $Q$.decrease_key($v, d[v]$)
     10. $\pi[v] \leftarrow u$
11. **return** $\{(u, \pi[u]) | u \in V \setminus \{s\}\}$
Improved Running Time

Running time:
\[ O(n) \times \text{(time for extract\_min)} + O(m) \times \text{(time for decrease\_key)} \]

<table>
<thead>
<tr>
<th>Priority-Queue</th>
<th>extract_min</th>
<th>decrease_key</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heap</td>
<td>(O(\log n))</td>
<td>(O(\log n))</td>
<td>(O(m \log n))</td>
</tr>
<tr>
<td>Fibonacci Heap</td>
<td>(O(\log n))</td>
<td>(O(1))</td>
<td>(O(n \log n + m))</td>
</tr>
</tbody>
</table>
Outline

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   - Reverse-Kruskal’s Algorithm
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2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

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4. All-Pair Shortest Paths and Floyd-Warshall
Single Source Shortest Paths, Weights May be Negative

**Input:** directed graph $G = (V, E)$, $s \in V$

assume all vertices are reachable from $s$

$w : E \to \mathbb{R}$

**Output:** shortest paths from $s$ to all other vertices $v \in V$

In transition graphs, negative weights make sense

If we sell an item: 'having the item' $\rightarrow$ 'not having the item', weight is negative (we gain money)

Dijkstra's algorithm does not work any more!
**Input:** directed graph $G = (V, E)$, $s \in V$

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Dijkstra’s Algorithm Fails if We Have Negative Weights
Dijkstra’s Algorithm Fails if We Have Negative Weights
Dijkstra’s Algorithm Fails if We Have Negative Weights
Dijkstra’s Algorithm Fails if We Have Negative Weights

Graph:
- Nodes: s, a, b, c
- Edges:
  - s to a: 2
  - a to b: 5
  - s to b: 3
  - b to a: -4
  - b to c: 1
  - a to c: unknown
Q: What is the length of the shortest path from $s$ to $d$?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Dealing with Negative Cycles
Q: What is the length of the shortest path from $s$ to $d$?
Q: What is the length of the shortest path from \( s \) to \( d \)?

A: \(-\infty\)
Q: What is the length of the shortest path from $s$ to $d$?

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Dealing with Negative Cycles
- assume the input graph does not contain negative cycles, or
Q: What is the length of the shortest path from $s$ to $d$?

A: $-\infty$

Def. A negative cycle is a cycle in which the total weight of edges is negative.

Dealing with Negative Cycles
- assume the input graph does not contain negative cycles, or
- allow algorithm to report “negative cycle exists”
What is the length of the shortest simple path from $s$ to $d$?

Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.
Q: What is the length of the shortest simple path from $s$ to $d$?

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Q: What is the length of the shortest simple path from $s$ to $d$?

A: 1
Q: What is the length of the shortest simple path from $s$ to $d$?

A: 1

- Unfortunately, computing the shortest simple path between two vertices is an NP-hard problem.
<table>
<thead>
<tr>
<th>algorithm</th>
<th>graph</th>
<th>weights</th>
<th>SS?</th>
<th>running time</th>
</tr>
</thead>
<tbody>
<tr>
<td>Simple DP</td>
<td>DAG</td>
<td>(\mathbb{R})</td>
<td>SS</td>
<td>(O(n + m))</td>
</tr>
<tr>
<td>Dijkstra</td>
<td>U/D</td>
<td>(\mathbb{R}_{\geq 0})</td>
<td>SS</td>
<td>(O(n \log n + m))</td>
</tr>
<tr>
<td>Bellman-Ford</td>
<td>U/D</td>
<td>(\mathbb{R})</td>
<td>SS</td>
<td>(O(nm))</td>
</tr>
<tr>
<td>Floyd-Warshall</td>
<td>U/D</td>
<td>(\mathbb{R})</td>
<td>AP</td>
<td>(O(n^3))</td>
</tr>
</tbody>
</table>

- DAG = directed acyclic graph   
- U = undirected   
- D = directed   
- SS = single source   
- AP = all pairs
Defining Cells of Table

<table>
<thead>
<tr>
<th>Single Source Shortest Paths, Weights May be Negative</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong> directed graph $G = (V, E)$, $s \in V$</td>
</tr>
<tr>
<td>assume all vertices are reachable from $s$</td>
</tr>
<tr>
<td>$w : E \rightarrow \mathbb{R}$</td>
</tr>
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<td><strong>Output:</strong> shortest paths from $s$ to all other vertices $v \in V$</td>
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- first try: \( f[v] \): length of shortest path from \( s \) to \( v \)
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- issue: do not know in which order we compute $f[v]$’s
Defining Cells of Table

**Single Source Shortest Paths, Weights May be Negative**

**Input:** directed graph \( G = (V, E) \), \( s \in V \)

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- first try: \( f[v] \): length of shortest path from \( s \) to \( v \)
- issue: do not know in which order we compute \( f[v]'s \)
- \( f^\ell[v] \), \( \ell \in \{0, 1, 2, 3 \cdots , n - 1\} \), \( v \in V \): length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges
\( f^\ell[v], \ell \in \{0, 1, 2, 3 \ldots, n - 1\}, v \in V : \)

length of shortest path from \( s \) to \( v \) that uses

at most \( \ell \) edges
\[ f^\ell[v], \; \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, \; v \in V: \]
length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

\[ f^2[a] = \]
- $f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots, n-1\}, v \in V$: length of shortest path from $s$ to $v$ that uses at most $\ell$ edges

- $f^2[a] = 6$
• \( f^{\ell}[v] \), \( \ell \in \{0, 1, 2, 3 \cdots, n - 1\} \), \( v \in V \): length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

• \( f^2[a] = 6 \)

• \( f^3[a] = \)
\[ f^\ell[v], \, \ell \in \{0, 1, 2, 3 \cdots, n - 1\}, \, v \in V: \]
length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

- \( f^2[a] = 6 \)
- \( f^3[a] = 2 \)
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \ldots, n - 1\}, v \in V : \text{length of shortest path from } s \text{ to } v \text{ that uses at most } \ell \text{ edges} \]

- \( f^2[a] = 6 \)
- \( f^3[a] = 2 \)

\[
f^\ell[v] = \begin{cases} 
0 & \ell = 0, v = s \\
\infty & \ell = 0, v \neq s \\
\min \{ f^\ell - 1[v] \} & \ell > 0 
\end{cases}
\]
\begin{itemize}
  \item $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n-1\}$, $v \in V$: length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
  \item $f^2[a] = 6$
  \item $f^3[a] = 2$
\end{itemize}

\[
f^\ell[v] = \begin{cases}
0 & \text{if } \ell = 0, v = s \\
\min_{u \in V} \{ f^{\ell-1}[u] + w(u, v) \} & \text{if } \ell > 0, v \neq s \\
\infty & \text{if } \ell = 0, v \neq s
\end{cases}
\]
- $f^\ell[v]$, $\ell \in \{0, 1, 2, 3 \cdots, n - 1\}$, $v \in V$: length of shortest path from $s$ to $v$ that uses at most $\ell$ edges
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\end{cases}$$
\[ f^\ell[v], \ell \in \{0, 1, 2, 3 \cdots , n - 1\}, \ v \in V : \] length of shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges

- \( f^2[a] = 6 \)
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f^\ell[v] = \begin{cases} 
0 & \text{if } \ell = 0, v = s \\
\infty & \text{if } \ell = 0, v \neq s \\
\min \left\{ \min_{u: (u,v) \in E} \left( f^{\ell-1}[u] + w(u,v) \right) \right\} & \text{if } \ell > 0 
\end{cases}
\]
Dynamic Programming: Example

\[ \begin{align*}
& f^0 \\
& s \\
& a \\
& b \\
& c \\
& d \\
\end{align*} \]

\[
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

\[
\begin{array}{ccccccc}
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & & \\
\end{array}
\]

length-0 edge
Dynamic Programming: Example

![Graph](image)
Dynamic Programming: Example

- **Diagram**: A graph showing nodes labeled as 's', 'a', 'b', 'c', and 'd' with edges between them.
- **Equations**: Two functions, $f^0$ and $f^1$, are shown with values for each node and edge.
- **Values**: Various integer values are assigned to nodes and edges, such as 0, 6, 7, 8, -4, -3, -2, and ∞.
Dynamic Programming: Example

![Diagram of a weighted graph with nodes s, a, b, c, d and edges with weights 7, 6, 8, -2, -3, -4, 7, ∞, ∞, ∞, ∞, ∞, 0.]

- $f^0$: initial state
- $f^1$: intermediate state

Length-0 edge

$\text{length-0 edge}$
Dynamic Programming: Example

![Graph](image)

- Length-0 edge

- $f^0$
  - $s$: 0
  - $a$: ∞
  - $b$: ∞
  - $c$: ∞
  - $d$: ∞

- $f^1$
  - $s$: 0
  - $a$: 6
  - $b$: 7
  - $c$: ∞
  - $d$: ∞
Dynamic Programming: Example
Dynamic Programming: Example

- Length-0 edge

\[ f^0 \]
\[ f^1 \]
\[ f^2 \]
Dynamic Programming: Example

\[ \begin{align*}
\text{length-0 edge} & \\
\end{align*} \]
Dynamic Programming: Example

\[ \begin{array}{c}
\text{length-0 edge} \\
\end{array} \]
Dynamic Programming: Example

\[
\begin{align*}
&f^0 \\
&f^1 \\
&f^2
\end{align*}
\]

length-0 edge
Dynamic Programming: Example
Dynamic Programming: Example

- Graph representation of a network with nodes and edges labeled with weights.
- Dynamic programming table with stages labeled as $f^0, f^1, f^2, f^3$, showing the progression of calculations for finding the minimum cost path.
- The concept of a length-0 edge is illustrated, indicating special conditions for certain paths.

This diagram illustrates the application of dynamic programming to solve a pathfinding problem, where the goal is to find the minimum weight path from the source node $s$ to other nodes in the network.
Dynamic Programming: Example

\[ f^0 \]

\[ f^1 \]

\[ f^2 \]

\[ f^3 \]

length-0 edge
Dynamic Programming: Example

The image shows a network graph with nodes labeled as s, a, b, c, and d. The edges between these nodes are labeled with weights. The graph is used to demonstrate the concept of dynamic programming, with levels labeled as $f^0$, $f^1$, $f^2$, and $f^3$. Each level represents a step in the dynamic programming algorithm, with the goal of finding the shortest path from node s to node d.

The diagram also includes a length-0 edge, indicating a special case in the algorithm.

The weights on the edges are as follows:

- Edge s-a: 7
- Edge s-b: 6
- Edge s-c: 8
- Edge s-d: -2
- Edge a-b: 8
- Edge a-c: -4
- Edge a-d: -3
- Edge b-c: -2
- Edge b-d: 7
- Edge c-d: 2

The values at the nodes are:

- $s$: 0
- $a$: 6
- $b$: 7
- $c$: 2
- $d$: 4

These values are computed iteratively using dynamic programming principles.
Dynamic Programming: Example

\[
\begin{array}{c}
\text{length-0 edge} \\
\end{array}
\]

\[
\begin{array}{c}
\text{length-0 edge} \\
\end{array}
\]
Dynamic Programming: Example

- Diagram of a graph with nodes labeled as follows: s, a, b, c, d, and edges labeled with numbers such as 7, 6, 8, -4, -3, etc.
- Diagram showing the function values $f^0, f^1, f^2, f^3$ with values 0, 6, 7, 8, etc.
- Length-0 edge indicated by an arrow pointing downward from each node to itself.

The diagram illustrates the use of dynamic programming in solving a problem, with the graph representing the problem's structure and the function values indicating the solution at each step.
Dynamic Programming: Example

- Diagram of a graph with nodes labeled s, b, a, c, and d, and edges labeled with weights (7, 6, 8, -4, -3, 7, -2, 7, 6, 7, 8, -4, -3).

- Table with entries for $f^0$, $f^1$, $f^2$, and $f^3$ for nodes s, a, b, c, and d, with values $0$, $\infty$, $\infty$, $\infty$, $\infty$, $\infty$, $6$, $7$, $\infty$, $\infty$, $\infty$, $\infty$, $2$, $4$. Length-0 edge is indicated.

- Growth in the table as the $f^i$ values increase with each iteration, reflecting the dynamic programming approach to finding the shortest path.
Dynamic Programming: Example

- Diagram of a graph with nodes and edges labeled with weights.
- Diagram shows the computation of function values $f^0, f^1, f^2, f^3, f^4$.
- Node $s$ connects to $a$ and $b$ with weights 7 and 8, respectively.
- Node $a$ connects to $b$ and $c$ with weights 6 and 8.
- Node $b$ connects to $c$ with a weight of 2.
- Node $c$ connects to $d$ with a weight of 7.
- Diagram illustrates the recursive computation of function values.
Dynamic Programming: Example

![Network Diagram]

**Graph Representation:**
- **Nodes:** s, a, b, c, d
- **Edges:** (s, a) with length 7, (a, c) with length 8, (c, d) with length 7, (b, a) with length -2

**Dynamic Programming Table:**

<table>
<thead>
<tr>
<th></th>
<th>f^0</th>
<th>f^1</th>
<th>f^2</th>
<th>f^3</th>
<th>f^4</th>
</tr>
</thead>
<tbody>
<tr>
<td>s</td>
<td>0</td>
<td>0</td>
<td>6</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>a</td>
<td>∞</td>
<td>6</td>
<td>7</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>b</td>
<td>∞</td>
<td>8</td>
<td>-4</td>
<td>-3</td>
<td>-2</td>
</tr>
<tr>
<td>c</td>
<td>∞</td>
<td>∞</td>
<td>-3</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>d</td>
<td>∞</td>
<td>-2</td>
<td>7</td>
<td>7</td>
<td>4</td>
</tr>
</tbody>
</table>

**Notes:**
- Length-0 edge
- **Path:** s → a → c → d
dynamic-programming($G, w, s$)

1. $f^0[s] \leftarrow 0$ and $f^0[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2. for $\ell \leftarrow 1$ to $n - 1$ do
   3. copy $f^{\ell-1} \rightarrow f^\ell$
   4. for each $(u, v) \in E$ do
   5. if $f^{\ell-1}[u] + w(u, v) < f^\ell[v]$ then
   6. $f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)$
7. return $(f^{n-1}[v])_{v \in V}$

Obs. Assuming there are no negative cycles, then a shortest path contains at most $n - 1$ edges.

Proof. If there is a path containing at least $n$ edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length.
dynamic-programming\((G, w, s)\)

\begin{enumerate}
\item \(f^0[s] \leftarrow 0\) and \(f^0[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
\item \textbf{for} \(\ell \leftarrow 1\) to \(n - 1\) \textbf{do}
\item \quad \text{copy} \(f^{\ell-1} \rightarrow f^\ell\)
\item \textbf{for} each \((u, v) \in E\) \textbf{do}
\item \quad \textbf{if} \(f^{\ell-1}[u] + w(u, v) < f^\ell[v]\) \textbf{then}
\item \quad \(f^\ell[v] \leftarrow f^{\ell-1}[u] + w(u, v)\)
\item \textbf{return} \((f^{n-1}[v])_{v \in V}\)
\end{enumerate}

\textbf{Obs.} Assuming there are no negative cycles, then a shortest path contains at most \(n - 1\) edges
dynamic-programming\((G, w, s)\)

1: \( f^0[s] \leftarrow 0 \) and \( f^0[v] \leftarrow \infty \) for any \( v \in V \setminus \{s\} \)
2: for \( \ell \leftarrow 1 \) to \( n - 1 \) do
3: copy \( f^{\ell-1} \rightarrow f^{\ell} \)
4: for each \((u, v) \in E\) do
5: if \( f^{\ell-1}[u] + w(u, v) < f^{\ell}[v] \) then
6: \( f^{\ell}[v] \leftarrow f^{\ell-1}[u] + w(u, v) \)
7: return \((f^{n-1}[v])_{v \in V}\)

**Obs.** Assuming there are no negative cycles, then a shortest path contains at most \( n - 1 \) edges

**Proof.**

If there is a path containing at least \( n \) edges, then it contains a cycle. Removing the cycle gives a path with the same or smaller length. \(\Box\)
Dynamic Programming with Better Space Usage

dynamic-programming\((G, w, s)\)

1. \(f^{\text{old}}[s] \leftarrow 0\) and \(f^{\text{old}}[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2. for \(\ell \leftarrow 1\) to \(n - 1\) do
3. copy \(f^{\text{old}} \rightarrow f^{\text{new}}\)
4. for each \((u, v) \in E\) do
5. if \(f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v]\) then
6. \(f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)\)
7. copy \(f^{\text{new}} \rightarrow f^{\text{old}}\)
8. return \(f^{\text{old}}\)

\(f^\ell\) only depends on \(f^{\ell-1}\): only need 2 vectors
Dynamic Programming with Better Space Usage

dynamic-programming\((G, w, s)\)

1: \(f^{\text{old}}[s] \leftarrow 0\) and \(f^{\text{old}}[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: \textbf{for } \ell \leftarrow 1 \text{ to } n - 1 \text{ do}
3: \quad \text{copy } f^{\text{old}} \rightarrow f^{\text{new}}
4: \quad \textbf{for each } (u, v) \in E \text{ do}
5: \quad \quad \textbf{if } f^{\text{old}}[u] + w(u, v) < f^{\text{new}}[v] \text{ then}
6: \quad \quad \quad f^{\text{new}}[v] \leftarrow f^{\text{old}}[u] + w(u, v)
7: \quad \text{copy } f^{\text{new}} \rightarrow f^{\text{old}}
8: \quad \textbf{return } f^{\text{old}}

- \(f^\ell\) only depends on \(f^{\ell-1}\): only need 2 vectors
- only need 1 vector!
Dynamic Programming with Better Space Usage

dynamic-programming\((G, w, s)\)

1: \(f[s] \leftarrow 0 \) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)

2: \textbf{for} \(\ell \leftarrow 1\) to \(n - 1\) \textbf{do}

3: \hspace{1em} copy \(f \rightarrow f\)

4: \hspace{1em} \textbf{for} each \((u, v) \in E\) \textbf{do}

5: \hspace{2em} \textbf{if} \(f[u] + w(u, v) < f[v]\) \textbf{then}

6: \hspace{3em} \(f[v] \leftarrow f[u] + w(u, v)\)

7: \hspace{1em} copy \(f \rightarrow f\)

8: \textbf{return} \(f\)

- \(f^\ell\) only depends on \(f^{\ell-1}\): only need 2 vectors
- only need 1 vector!
Dynamic Programming with Better Space Usage

dynamic-programming($G, w, s$)

1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n - 1$ do
3:   for each $(u, v) \in E$ do
4:     if $f[u] + w(u, v) < f[v]$ then
5:       $f[v] \leftarrow f[u] + w(u, v)$
6: return $f$

- $f^\ell$ only depends on $f^{\ell-1}$: only need 2 vectors
- only need 1 vector!
Bellman-Ford Algorithm

Bellman-Ford\( (G, w, s) \)

1: \( f[s] \leftarrow 0 \) and \( f[v] \leftarrow \infty \) for any \( v \in V \setminus \{s\} \)
2: \textbf{for } \ell \leftarrow 1 \textbf{ to } n - 1 \textbf{ do}
3: \textbf{for each } (u, v) \in E \textbf{ do}
4: \textbf{if } f[u] + w(u, v) < f[v] \textbf{ then}
5: \hspace{1cm} f[v] \leftarrow f[u] + w(u, v)
6: \textbf{return } f

- \( f^\ell \) only depends on \( f^{\ell-1} \): only need 2 vectors
- only need 1 vector!
Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: for \(\ell \leftarrow 1\) to \(n - 1\) do
3: for each \((u, v) \in E\) do
4: if \(f[u] + w(u, v) < f[v]\) then
5: \(f[v] \leftarrow f[u] + w(u, v)\)
6: return \(f\)

- Issue: when we compute \(f[u] + w(u, v)\), \(f[u]\) may be changed since the end of last iteration
Bellman-Ford Algorithm

Bellman-Ford \((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
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- Issue: when we compute \(f[u] + w(u, v)\), \(f[u]\) may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
Bellman-Ford Algorithm

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4: if \( f[u] + w(u, v) < f[v] \) then
5: \( f[v] \leftarrow f[u] + w(u, v) \)
6: return \( f \)

- Issue: when we compute \( f[u] + w(u, v) \), \( f[u] \) may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
- After iteration \( \ell \), \( f[v] \) is at most the length of the shortest path from \( s \) to \( v \) that uses at most \( \ell \) edges
Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: \textbf{for} \(\ell \leftarrow 1\) to \(n - 1\) \textbf{do}
3: \hspace{1em} \textbf{for each} \((u, v) \in E\) \textbf{do}
4: \hspace{2em} \textbf{if} \(f[u] + w(u, v) < f[v]\) \textbf{then}
5: \hspace{3em} \(f[v] \leftarrow f[u] + w(u, v)\)
6: \textbf{return} \(f\)

- Issue: when we compute \(f[u] + w(u, v)\), \(f[u]\) may be changed since the end of last iteration
- This is OK: it can only “accelerate” the process!
- After iteration \(\ell\), \(f[v]\) is at most the length of the shortest path from \(s\) to \(v\) that uses at most \(\ell\) edges
- \(f[v]\) is always the length of some path from \(s\) to \(v\)
Bellman-Ford Algorithm

- After iteration \( \ell \):
  
  \[
  \text{length of shortest } s-v \text{ path} \\
  \leq f[v] \\
  \leq \text{length of shortest } s-v \text{ path using at most } \ell \text{ edges}
  \]

- Assuming there are no negative cycles:
  
  \[
  \text{length of shortest } s-v \text{ path} \\
  = \text{length of shortest } s-v \text{ path using at most } n-1 \text{ edges}
  \]

- So, assuming there are no negative cycles, after iteration \( n-1 \): 
  
  \[
  f[v] = \text{length of shortest } s-v \text{ path}
  \]
order in which we consider edges:

\[(s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)\]

<table>
<thead>
<tr>
<th>vertices</th>
<th>(s)</th>
<th>(a)</th>
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end of iteration 1: 0, 2, 7, 2, 4
end of iteration 2: 0, 2, 7, -2, 4
end of iteration 3: 0, 2, 7, -2, 4
Algorithm terminates in 3 iterations,
instead of 4.

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(s, a), (s, b), (a, b), (a, c), (b, d), (c, d), (d, a)

vertices

\[
\begin{array}{c|c|c|c|c|c}
\text{vertices} & s & a & b & c & d \\
\hline
f & 0 & 6 & 7 & 2 & \infty \\
\end{array}
\]
order in which we consider edges:

- $(s, a)$
- $(s, b)$
- $(a, b)$
- $(a, c)$
- $(b, d)$
- $(c, d)$
- $(d, a)$

vertices

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vertices

\[
\begin{array}{ccccc}
\text{vertices} & s & a & b & c & d \\
 f & 0 & 6 & 7 & 2 & 4 \\
\end{array}
\]
order in which we consider edges:

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vertices

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Algorithm terminates in 3 iterations, instead of 4.
order in which we consider edges:
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(c, d), (d, a)

vertices
\begin{array}{|c|c|c|c|c|}
\hline
\text{f} & s & a & b & c & d \\
\hline
0 & 2 & 7 & 2 & 4 \\
\hline
\end{array}

end of iteration 1: 0, 2, 7, 2, 4

Algorithm terminates in 3 iterations, instead of 4.
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Algorithm terminates in 3 iterations,
order in which we consider edges: 
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draws
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(s, a), (s, b), (a, b), (a, c), (b, d), 
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Bellman-Ford Algorithm

Bellman-Ford\((G, w, s)\)

1: \(f[s] \leftarrow 0\) and \(f[v] \leftarrow \infty\) for any \(v \in V \setminus \{s\}\)
2: \textbf{for} \(\ell \leftarrow 1\) to \(n\) \textbf{do}
3: \hspace{1em} \textit{updated} \leftarrow \text{false}
4: \hspace{1em} \textbf{for each} \((u, v) \in E\) \textbf{do}
5: \hspace{2em} \textbf{if} \(f[u] + w(u, v) < f[v]\) \textbf{then}
6: \hspace{3em} \(f[v] \leftarrow f[u] + w(u, v)\)
7: \hspace{2em} \textit{updated} \leftarrow \text{true}
8: \hspace{1em} \textbf{if not} \textit{updated}, then \textbf{return} \(f\)
9: \hspace{1em} \textbf{output} “negative cycle exists”
Bellman-Ford Algorithm

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1: $f[s] \leftarrow 0$ and $f[v] \leftarrow \infty$ for any $v \in V \setminus \{s\}$
2: for $\ell \leftarrow 1$ to $n$ do
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5: \hspace{2em} if $f[u] + w(u, v) < f[v]$ then
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- $\pi[v]$: the parent of $v$ in the shortest path tree
Bellman-Ford Algorithm

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- \(\pi[v]\): the parent of \(v\) in the shortest path tree
- Running time = \(O(nm)\)
Outline

1. Minimum Spanning Tree
   - Kruskal’s Algorithm
   - Reverse-Kruskal’s Algorithm
   - Prim’s Algorithm

2. Single Source Shortest Paths
   - Dijkstra’s Algorithm

3. Shortest Paths in Graphs with Negative Weights

4. All-Pair Shortest Paths and Floyd-Warshall
All-Pair Shortest Paths

**Input:** directed graph $G = (V, E)$,
$w : E \rightarrow \mathbb{R}$ (can be negative)

**Output:** shortest path from $u$ to $v$ for every $u, v \in V$

1: for every starting point $s \in V$
2: run Bellman-Ford ($G, w, s$)

Running time $= O(n^2 m)$
All-Pair Shortest Paths

**Input:** directed graph $G = (V, E)$,

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## Summary of Shortest Path Algorithms we learned

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Graph</th>
<th>Weights</th>
<th>SS?</th>
<th>Running Time</th>
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<tbody>
<tr>
<td>Simple DP</td>
<td>DAG</td>
<td>$\mathbb{R}$</td>
<td>SS</td>
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<td>Floyd-Warshall</td>
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</tbody>
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- **DAG** = directed acyclic graph
- **U** = undirected
- **D** = directed
- **SS** = single source
- **AP** = all pairs
Design a Dynamic Programming Algorithm

- It is convenient to assume \( V = \{1, 2, 3, \cdots, n\} \)
Design a Dynamic Programming Algorithm

- It is convenient to assume $V = \{1, 2, 3, \cdots, n\}$
- For simplicity, extend the $w$ values to non-edges:

$$w(i, j) = \begin{cases} 
0 & i = j \\
\text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\
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For now assume there are no negative cycles
Design a Dynamic Programming Algorithm

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Cells for Floyd-Warshall Algorithm

- First try: \( f[i, j] \) is length of shortest path from \( i \) to \( j \)
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- First try: $f[i, j]$ is length of shortest path from $i$ to $j$
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$f^k[i, j]$: length of shortest path from $i$ to $j$ that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices
Example for Definition of $f^k[i, j]$'s

\[ f^0[1, 4] = \infty \]
\[ f^1[1, 4] = \infty \]
\[ f^2[1, 4] = 140 \quad (1 \rightarrow 2 \rightarrow 4) \]
\[ f^3[1, 4] = 90 \quad (1 \rightarrow 3 \rightarrow 2 \rightarrow 4) \]
\[ f^4[1, 4] = 90 \quad (1 \rightarrow 3 \rightarrow 2 \rightarrow 4) \]
\[ f^5[1, 4] = 60 \quad (1 \rightarrow 3 \rightarrow 5 \rightarrow 4) \]
\[
w(i, j) = \begin{cases} 
0 & i = j \\
\text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\
\infty & i \neq j, (i, j) \notin E 
\end{cases}
\]

- \(f_k[i, j]\): length of shortest path from \(i\) to \(j\) that only uses vertices \(\{1, 2, 3, \ldots, k\}\) as intermediate vertices
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- \( f^k[i, j] \): length of shortest path from \( i \) to \( j \) that only uses vertices \( \{1, 2, 3, \ldots, k\} \) as intermediate vertices

\[ f^k[i, j] = \begin{cases} 
k = 0 \\
k = 1, 2, \ldots, n 
\end{cases} \]
\[
\begin{align*}
    w(i, j) &= \begin{cases} 
        0 & \text{if } i = j \\
        \text{weight of edge } (i, j) & \text{if } i \neq j, (i, j) \in E \\
        \infty & \text{if } i \neq j, (i, j) \notin E
    \end{cases}
\end{align*}
\]

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        w(i, j) & \text{if } k = 0 \\
        k = 1, 2, \ldots, n
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- $f^k[i, j]$: length of shortest path from $i$ to $j$ that only uses vertices $\{1, 2, 3, \cdots, k\}$ as intermediate vertices

$$f^k[i, j] = \begin{cases} 
w(i, j) & k = 0 \\
\min \{ & k = 1, 2, \cdots, n 
\} \end{cases}$$
\[ w(i, j) = \begin{cases} 
0 & i = j \\
\text{weight of edge } (i, j) & i \neq j, (i, j) \in E \\
\infty & i \neq j, (i, j) \not\in E
\end{cases} \]

\[ f^k[i, j] \]: length of shortest path from \( i \) to \( j \) that only uses vertices \( \{1, 2, 3, \cdots, k\} \) as intermediate vertices

\[ f^k[i, j] = \begin{cases} 
w(i, j) & k = 0 \\
\min \left\{ f^{k-1}[i, j] \right\} & k = 1, 2, \cdots, n
\end{cases} \]
\[ w(i, j) = \begin{cases} 
0 & i = j \\
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- \( f^k[i, j] \): length of shortest path from \( i \) to \( j \) that only uses vertices \( \{1, 2, 3, \cdots, k\} \) as intermediate vertices

\[ f^k[i, j] = \begin{cases} 
w(i, j) & k = 0 \\
\min \left\{ f^{k-1}[i, j], f^{k-1}[i, k] + f^{k-1}[k, j] \right\} & k = 1, 2, \cdots, n 
\end{cases} \]
Floyd-Warshall($G, w$)

1: $f^0 \leftarrow w$
2: for $k \leftarrow 1$ to $n$ do
3: copy $f^{k-1} \rightarrow f^k$
4: for $i \leftarrow 1$ to $n$ do
5: for $j \leftarrow 1$ to $n$ do
6: \textbf{if} $f^{k-1}[i, k] + f^{k-1}[k, j] < f^k[i, j]$ \textbf{then}
7: $f^k[i, j] \leftarrow f^{k-1}[i, k] + f^{k-1}[k, j]$
Floyd-Warshall($G, w$)

1: $f^{old} \leftarrow w$
2: for $k \leftarrow 1$ to $n$ do
3:     copy $f^{old} \rightarrow f^{new}$
4: for $i \leftarrow 1$ to $n$ do
5:     for $j \leftarrow 1$ to $n$ do
6:         if $f^{old}[i, k] + f^{old}[k, j] < f^{new}[i, j]$ then
7:             $f^{new}[i, j] \leftarrow f^{old}[i, k] + f^{old}[k, j]$
Floyd-Warshall($G, w$)

1: $f^{\text{old}} \leftarrow w$
2: for $k \leftarrow 1$ to $n$ do
3:     copy $f^{\text{old}} \rightarrow f^{\text{new}}$
4: for $i \leftarrow 1$ to $n$ do
5:     for $j \leftarrow 1$ to $n$ do
6:         if $f^{\text{old}}[i, k] + f^{\text{old}}[k, j] < f^{\text{new}}[i, j]$ then
7:             $f^{\text{new}}[i, j] \leftarrow f^{\text{old}}[i, k] + f^{\text{old}}[k, j]$

Lemma
Assume there are no negative cycles in $G$. After iteration $k$, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from $i$ to $j$ that only uses vertices in \{1, 2, 3, ..., $k$\} as intermediate vertices.

Running time = $O(n^3)$. 
Floyd-Warshall($G, w$)

1: $f \leftarrow w$
2: **for** $k \leftarrow 1$ **to** $n$ **do**
3: \hspace{1em} **copy** $f \rightarrow f$
4: **for** $i \leftarrow 1$ **to** $n$ **do**
5: \hspace{1em} **for** $j \leftarrow 1$ **to** $n$ **do**
6: \hspace{2em} **if** $f[i, k] + f[k, j] < f[i, j]$ **then**
7: \hspace{2em} $f[i, j] \leftarrow f[i, k] + f[k, j]$

**Lemma** Assume there are no negative cycles in $G$. After iteration $k$, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from $i$ to $j$ that only uses vertices in \{1, 2, 3, \ldots, k\} as intermediate vertices.

**Running time** = $O(n^3)$. 


Floyd-Warshall\((G, w)\)

1: \( f \leftarrow w \)
2: \textbf{for} \( k \leftarrow 1 \) \textbf{to} \( n \) \textbf{do}
3: \hskip 1em \textbf{for} \( i \leftarrow 1 \) \textbf{to} \( n \) \textbf{do}
4: \hskip 2em \textbf{for} \( j \leftarrow 1 \) \textbf{to} \( n \) \textbf{do}
5: \quad \textbf{if} \( f[i, k] + f[k, j] < f[i, j] \) \textbf{then}
6: \hskip 2.5em \hspace{1em} f[i, j] \leftarrow f[i, k] + f[k, j]
Floyd-Warshall($G, w$)

1: $f \leftarrow w$
2: for $k \leftarrow 1$ to $n$ do
3: for $i \leftarrow 1$ to $n$ do
4: for $j \leftarrow 1$ to $n$ do
5: if $f[i, k] + f[k, j] < f[i, j]$ then
6: $f[i, j] \leftarrow f[i, k] + f[k, j]$

Lemma Assume there are no negative cycles in $G$. After iteration $k$, for $i, j \in V$, $f[i, j]$ is exactly the length of shortest path from $i$ to $j$ that only uses vertices in $\{1, 2, 3, \cdots, k\}$ as intermediate vertices.
Floyd-Warshall \((G, w)\)

1: \(f \leftarrow w\)
2: \(\text{for } k \leftarrow 1 \text{ to } n \text{ do}\)
3: \(\text{for } i \leftarrow 1 \text{ to } n \text{ do}\)
4: \(\text{for } j \leftarrow 1 \text{ to } n \text{ do}\)
5: \(\text{if } f[i, k] + f[k, j] < f[i, j] \text{ then}\)
6: \(f[i, j] \leftarrow f[i, k] + f[k, j]\)

**Lemma** Assume there are no negative cycles in \(G\). After iteration \(k\), for \(i, j \in V\), \(f[i, j]\) is **exactly** the length of shortest path from \(i\) to \(j\) that only uses vertices in \(\{1, 2, 3, \cdots, k\}\) as intermediate vertices.

- Running time = \(O(n^3)\).
\[
\begin{array}{c|ccccc}
& 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 90 & 30 & \infty & \infty \\
2 & 10 & 0 & \infty & 50 & \infty \\
3 & 60 & 10 & 0 & 70 & 20 \\
4 & \infty & \infty & \infty & 0 & 20 \\
5 & \infty & \infty & \infty & 10 & 0 \\
\end{array}
\]
\[ i = 2, \; k = 1, \; j = 3 \]
\[ i = 2, \ k = 1, \ j = 3 \]
$i = 1, \ k = 2, \ j = 4$
\[
i = 1, \ k = 2, \ j = 4
\]
\[ i = 3, \ k = 2, \ j = 1, \]

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 0 & 90 & 30 & 140 & \infty \\
2 & 10 & 0 & 40 & 50 & \infty \\
3 & 60 & 10 & 0 & 70 & 20 \\
4 & \infty & \infty & \infty & 0 & 20 \\
5 & \infty & \infty & \infty & 10 & 0 \\
\end{array}
\]
\begin{itemize}
\item \( i = 3, \ k = 2, \ j = 1 \),
\end{itemize}

\begin{table}
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\begin{tabular}{c|ccccc}
 & 1 & 2 & 3 & 4 & 5 \\
\hline
1 & 0 & 90 & 30 & 140 & \infty \\
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\end{tabular}
\end{table}
\[ i = 3, \quad k = 2, \quad j = 4 \]
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<table>
<thead>
<tr>
<th></th>
<th>1</th>
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3 & 20 & 10 & 0 & 60 & 20 \\
4 & \infty & \infty & \infty & 0 & 20 \\
5 & \infty & \infty & \infty & 10 & 0 \\
\end{array} \]
\( i = 1, \ k = 3, \ j = 2 \)
Recovering Shortest Paths

Floyd-Warshall($G, w$)

1: $f \leftarrow w$, $\pi[i, j] \leftarrow \perp$ for every $i, j \in V$
2: for $k \leftarrow 1$ to $n$ do
3:  for $i \leftarrow 1$ to $n$ do
4:   for $j \leftarrow 1$ to $n$ do
5:      if $f[i, k] + f[k, j] < f[i, j]$ then
6:         $f[i, j] \leftarrow f[i, k] + f[k, j]$, $\pi[i, j] \leftarrow k$
Floyd-Warshall($G, w$)

1: $f \leftarrow w$, $\pi[i, j] \leftarrow \bot$ for every $i, j \in V$
2: for $k \leftarrow 1$ to $n$ do
3: for $i \leftarrow 1$ to $n$ do
4: for $j \leftarrow 1$ to $n$ do
5: if $f[i, k] + f[k, j] < f[i, j]$ then
6: $f[i, j] \leftarrow f[i, k] + f[k, j]$, $\pi[i, j] \leftarrow k$

print-path($i, j$)

1: if $\pi[i, j] = \bot$ then
2: if $i \neq j$ then print($i, \"","\")
3: else
4: print-path($i, \pi[i, j]$), print-path($\pi[i, j], j$)
Detecting Negative Cycles

Floyd-Warshall\((G, w)\)

1: \(f \leftarrow w, \pi[i, j] \leftarrow \bot\) for every \(i, j \in V\)
2: \textbf{for } \(k \leftarrow 1\) to \(n\) \textbf{do}
3: \hspace{1em} \textbf{for } \(i \leftarrow 1\) to \(n\) \textbf{do}
4: \hspace{2em} \textbf{for } \(j \leftarrow 1\) to \(n\) \textbf{do}
5: \hspace{3em} \textbf{if } \(f[i, k] + f[k, j] < f[i, j]\) \textbf{ then}
6: \hspace{4em} \(f[i, j] \leftarrow f[i, k] + f[k, j], \pi[i, j] \leftarrow k\)
Detecting Negative Cycles

**Floyd-Warshall**($G, w$)

1: $f \leftarrow w, \pi[i, j] \leftarrow \perp$ for every $i, j \in V$
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6:                 $f[i, j] \leftarrow f[i, k] + f[k, j], \pi[i, j] \leftarrow k$
7: **for** $k \leftarrow 1$ to $n$ **do**
8:     **for** $i \leftarrow 1$ to $n$ **do**
9:         **for** $j \leftarrow 1$ to $n$ **do**
10:     if $f[i, k] + f[k, j] < f[i, j]$ then
11:         report “negative cycle exists” and exit
## Summary of Shortest Path Algorithms

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- **DAG** = directed acyclic graph
- **U** = undirected
- **D** = directed
- **SS** = single source
- **AP** = all pairs