CSE 431/531: Algorithm Analysis and Design (Spring 2022)

Graph Basics

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Outline

1. Graphs

2. Connectivity and Graph Traversal
   - Testing Bipartiteness

3. Topological Ordering
Examples of Graphs

Figure: Road Networks

Figure: Social Networks

Figure: Internet

Figure: Transition Graphs
(Undirected) Graph $G = (V, E)$

- $V$: set of vertices (nodes);
- $E$: pairwise relationships among $V$;
  - (undirected) graphs: relationship is symmetric, $E$ contains subsets of size 2
\((\text{Undirected}) \text{ Graph } G = (V, E)\)

- $V$: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- $E$: pairwise relationships among $V$;
  - (undirected) graphs: relationship is symmetric, $E$ contains subsets of size 2
  - $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$
Directed Graph $G = (V, E)$

- $V$: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

- $E$: pairwise relationships among $V$;
  - directed graphs: relationship is asymmetric, $E$ contains ordered pairs
Directed Graph $G = (V, E)$

- $V$: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$

- $E$: pairwise relationships among $V$;
  - directed graphs: relationship is asymmetric, $E$ contains ordered pairs
  - $E = \{(1, 2), (1, 3), (3, 2), (4, 2), (2, 5), (5, 3), (3, 7), (3, 8), (4, 5), (5, 6), (6, 5), (8, 7)\}$
Abuse of Notations

- For (undirected) graphs, we often use $(i, j)$ to denote the set $\{i, j\}$.
- We call $(i, j)$ an unordered pair; in this case $(i, j) = (j, i)$.

$$E = \{(1, 2), (1, 3), (2, 3), (2, 4), (2, 5), (3, 5), (3, 7), (3, 8), (4, 5), (5, 6), (7, 8)\}$$
- Social Network: Undirected
- Transition Graph: Directed
- Road Network: Directed or Undirected
- Internet: Directed or Undirected
**Adjacency matrix**

- $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
- $A$ is symmetric if graph is undirected
Representation of Graphs

- **Adjacency matrix**
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - $A$ is symmetric if graph is undirected

- **Linked lists**
  - For every vertex $v$, there is a linked list containing all neighbours of $v$. 
### Adjacency matrix
- $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
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### Linked lists
- For every vertex $v$, there is a linked list containing all neighbours of $v$.
- When graph is static, can use array of variant-length arrays.
Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- \( n \): number of vertices
- \( m \): number of edges, assuming \( n - 1 \leq m \leq n(n - 1)/2 \)
- \( d_v \): number of neighbors of \( v \)

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Connectivity Problem

**Input:** graph $G = (V, E)$, (using linked lists)

two vertices $s, t \in V$

**Output:** whether there is a path connecting $s$ to $t$ in $G$
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- Breadth-First Search (BFS)
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  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)
Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- $L_{j+1}$ contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in $L_j$
Breadth-First Search (BFS)

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Implementing BFS using a Queue

**BFS**(s)

1: \( \text{head} \leftarrow 1, \text{tail} \leftarrow 1, \text{queue}[1] \leftarrow s \)
2: mark \( s \) as “visited” and all other vertices as “unvisited”
3: \textbf{while} \( \text{head} \leq \text{tail} \) \textbf{do}
4: \( v \leftarrow \text{queue}[\text{head}], \text{head} \leftarrow \text{head} + 1 \)
5: \textbf{for} all neighbours \( u \) of \( v \) \textbf{do}
6: \quad \textbf{if} \( u \) is “unvisited” \textbf{then}
7: \quad \quad \text{tail} \leftarrow \text{tail} + 1, \text{queue}[\text{tail}] = u
8: \quad \text{mark} \( u \) as “visited”

- Running time: \( O(n + m) \).
Example of BFS via Queue
Example of BFS via Queue

![Graph example](image.png)
Example of BFS via Queue
Example of BFS via Queue
Example of BFS via Queue

\[ \begin{array}{c}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 &
\end{array} \]

\[ \begin{array}{c}
1 \rightarrow 2 \rightarrow 3 \\
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\end{array} \]
Example of BFS via Queue
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\[
\begin{array}{c}
\text{1} & \text{2} & \text{3} & \text{4} & \text{5} \\
\text{7} & \text{8} & \text{6} & \text{5} & \text{4} & \text{3} & \text{2} & \text{1}
\end{array}
\]
Example of BFS via Queue
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```
1 2 3 4 5 7 8
```

```
head
tail

```
Example of BFS via Queue
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Depth-First Search (DFS)

- Starting from $s$
- Travel through the first edge leading out of the current vertex
- When reach an already-visited vertex ("dead-end"), go back
- Travel through the next edge
- If tried all edges leading out of the current vertex, go back
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![Graph Diagram](image-url)
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Implementing DFS using Recursion

**DFS(s)**
1. mark all vertices as “unvisited”
2. recursive-DFS(s)

**recursive-DFS(v)**
1. mark v as “visited”
2. for all neighbours u of v do
3. if u is unvisited then recursive-DFS(u)
Outline

1. Graphs

2. Connectivity and Graph Traversal
   - Testing Bipartiteness

3. Topological Ordering
Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of $V$ into two sets $L$ and $R$ such that for every edge $(u, v) \in E$, we have either $u \in L, v \in R$ or $v \in L, u \in R$. 
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
Testing Bipartiteness

- Taking an arbitrary vertex \( s \in V \)
- Assuming \( s \in L \) w.l.o.g
- Neighbors of \( s \) must be in \( R \)

If \( G \) contains multiple connected components, repeat above algorithm for each component.
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of $s$ must be in $R$
- Neighbors of neighbors of $s$ must be in $L$
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
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Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L$ w.l.o.g
- Neighbors of $s$ must be in $R$
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- ...
- Report “not a bipartite graph” if contradiction was found
Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$
- Assuming $s \in L \text{ w.l.o.g.}$
- Neighbors of $s$ must be in $R$
- Neighbors of neighbors of $s$ must be in $L$
- \ldots
- Report “not a bipartite graph” if contradiction was found
- If $G$ contains multiple connected components, repeat above algorithm for each component
Test Bipartiteness
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bad edges!
Testing Bipartiteness using BFS

**BFS(s)**

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4: \( v \leftarrow \text{queue}[\text{head}], \text{head} \leftarrow \text{head} + 1 \)
5: \( \textbf{for} \ \text{all} \ \text{neighbours} \ u \ \text{of} \ v \ \textbf{do} \)
6: \( \quad \textbf{if} \ u \ \text{is “unvisited” then} \)
7: \( \quad \text{tail} \leftarrow \text{tail} + 1, \text{queue}[\text{tail}] = u \)
8: \( \quad \text{mark} \ u \ \text{as “visited”} \)
function test-bipartiteness(s) {
    head ← 1, tail ← 1, queue[1] ← s
    mark s as “visited” and all other vertices as “unvisited”
    color[s] ← 0
    while head ≤ tail do
        v ← queue[head], head ← head + 1
        for all neighbours u of v do
            if u is “unvisited” then
                tail ← tail + 1, queue[tail] = u
                mark u as “visited”
                color[u] ← 1 − color[v]
            else if color[u] = color[v] then
                print(“G is not bipartite”) and exit
    }
}
Testing Bipartiteness using BFS

1: mark all vertices as “unvisited”
2: for each vertex \( v \in V \) do
3: \[ \text{if } v \text{ is “unvisited” then} \]
4: test-bipartiteness\( (v) \)
5: print(“\( G \) is bipartite”)
Testing Bipartiteness using BFS

1: mark all vertices as “unvisited”
2: for each vertex $v \in V$ do
3:   if $v$ is “unvisited” then
4:     test-bipartiteness($v$)
5:   print(“$G$ is bipartite”)

Obs. Running time of algorithm = $O(n + m)$
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**Topological Ordering Problem**

**Input:** a directed acyclic graph (DAG) $G = (V, E)$

**Output:** 1-to-1 function $\pi : V \rightarrow \{1, 2, 3 \cdots , n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$
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Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.
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Q: How to make the algorithm as efficient as possible?

A: Use linked-lists of outgoing edges, maintain the in-degree \(d_v\) of vertices, and maintain a queue (or stack) of vertices \(v\) with \(d_v = 0\).
Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.

**Q:** How to make the algorithm as efficient as possible?

**A:**
- Use linked-lists of outgoing edges
- Maintain the in-degree $d_v$ of vertices
- Maintain a queue (or stack) of vertices $v$ with $d_v = 0$
topological-sort($G$)

1: let $d_v \leftarrow 0$ for every $v \in V$
2: for every $v \in V$ do
3:     for every $u$ such that $(v, u) \in E$ do
4:         $d_u \leftarrow d_u + 1$
5:     $S \leftarrow \{v : d_v = 0\}$, $i \leftarrow 0$
6:     while $S \neq \emptyset$ do
7:         $v \leftarrow$ arbitrary vertex in $S$, $S \leftarrow S \setminus \{v\}$
8:         $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
9:     for every $u$ such that $(v, u) \in E$ do
10:        $d_u \leftarrow d_u - 1$
11:    if $d_u = 0$ then add $u$ to $S$
12: if $i < n$ then output “not a DAG”

- $S$ can be represented using a queue or a stack
- Running time $= O(n + m)$
### $S$ as a Queue or a Stack

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<th>Stack</th>
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<td>$head \leftarrow 1, tail \leftarrow 0$</td>
<td>$top \leftarrow 0$</td>
</tr>
<tr>
<td>Non-Empty?</td>
<td>$head \leq tail$</td>
<td>$top &gt; 0$</td>
</tr>
<tr>
<td>Add($v$)</td>
<td>$tail \leftarrow tail + 1$</td>
<td>$top \leftarrow top + 1$</td>
</tr>
<tr>
<td></td>
<td>$S[tail] \leftarrow v$</td>
<td>$S[top] \leftarrow v$</td>
</tr>
<tr>
<td>Retrieve $v$</td>
<td>$v \leftarrow S[head]$</td>
<td>$v \leftarrow S[top]$</td>
</tr>
<tr>
<td></td>
<td>$head \leftarrow head + 1$</td>
<td>$top \leftarrow top - 1$</td>
</tr>
</tbody>
</table>
Example

The diagram shows a graph with nodes labeled as follows:
- Node $a$
- Node $b$
- Node $c$
- Node $d$
- Node $e$
- Node $f$
- Node $g$

The graph contains directed edges:
- $a$ to $b$
- $a$ to $c$
- $a$ to $d$
- $b$ to $c$
- $b$ to $d$
- $b$ to $e$
- $c$ to $d$
- $c$ to $e$
- $d$ to $e$
- $d$ to $f$
- $e$ to $d$
- $e$ to $f$
- $f$ to $g$
- $g$ to $d$

The queue is represented as:
- Head at $a$
- Tail at $g$

The degree of each node is given in the following table:

<table>
<thead>
<tr>
<th>Degree</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$f$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
**Example**

![Graph Diagram]

**Queue:**
```
head
```

```
queue:
```

```

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
```

**Degree:**
```
```
Example

![Graph with nodes labeled a, b, c, d, e, f, g. The graph shows directed edges between the nodes with arrows indicating the direction of the edges.]

<table>
<thead>
<tr>
<th>queue:</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>head</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>tail</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>degree</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>
Example

![Graph Diagram]

**Example**

- **Queue:**
  - **Head:** 1
  - **Tail:** 3

- **Degree:**
  - 0
  - 1
  - 2
  - 3

- **Nodes:**
  - *a*
  - *b*
  - *c*
  - *d*
  - *e*
  - *f*
  - *g*
Example

![Graph Diagram]

<table>
<thead>
<tr>
<th>head</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>tail</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
<td>e</td>
<td>f</td>
<td>g</td>
</tr>
</tbody>
</table>

| degree | 0 | 0 | 0 | 1 | 2 | 1 | 3 |
Example

![Graph Diagram]

**queue:**

```plaintext
queue: a b c
degree
e
queue:
c
d
e
f
g
0
0
0
1
1
1
3
head
tail
```
Example

Queue:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Head

Tail

Graph:

- Vertices: a, b, c, d, e, f, g
- Edges: a → b, b → c, c → d, d → e, e → f, f → g, g → c
Example

```
queue:

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
```
Example

![Diagram showing a graph with nodes e, d, f, and g.](image)

queue: [a, b, c, d, f, ]

head

tail

<table>
<thead>
<tr>
<th>degree</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>
Example

![Graph with nodes e, d, f, and g with edges between them]

**Queue:**

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>degree</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

**Head**

**Tail**
Example
Example

```
queue: a b c d f e

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>
```

degree

e
f
g
e
head
tail
Example

\begin{itemize}
\item[	extbf{queue:}] \begin{array}{cccccccc}
    a & b & c & d & f & e \\
\end{array}
\end{itemize}

\begin{itemize}
\item[	extbf{degree}] \begin{array}{cccccccc}
    0 & 0 & 0 & 0 & 0 & 0 & 2 \\
\end{array}
\end{itemize}
Example

- **Queue:**
  - **Elements:** a, b, c, d, f, e
  - **Head:** e
  - **Tail:** g

- **Degree:**
  - **Indices:**
    - a: 0
    - b: 0
    - c: 0
    - d: 0
    - e: 0
    - f: 0
    - g: 1
Example

queue: | a | b | c | d | f | e |

degree | 0 | 0 | 0 | 0 | 0 | 0 | 1

e

head

tail

e

g

0

a

b

c

d

e

f

g

degree

queue:

head

tail

e

g

0

a

b

c

d

e

f

g

degree

queue:

head

tail

e

g

0

a

b

c

d

e

f

g

degree
Example

queue: \[ a \ b \ c \ d \ f \ e \]

\[
\begin{array}{c|c|c|c|c|c|c|c|c}
   & a & b & c & d & e & f & g \\
\hline
\text{degree} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]
Example

queue: \[ a \ b \ c \ d \ f \ e \ g \]

<table>
<thead>
<tr>
<th>degree</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

g
Example

queue: a b c d f e g

<table>
<thead>
<tr>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
<th>f</th>
<th>g</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

degree