CSE 431/531: Algorithm Analysis and Design (Fall 2022)
Graph Basics

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University at Buffalo

Outline

Examples of Graphs

Figure: Road Networks

Figure: Internet

Figure: Social Networks

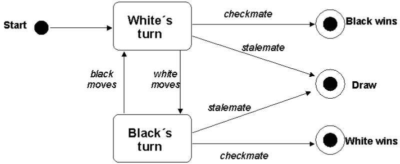
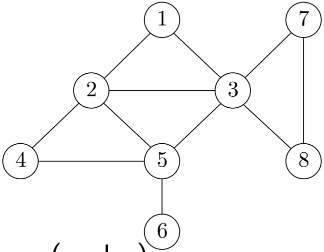


Figure: Transition Graphs

(Undirected) Graph $G = (V, E)$



- V: set of vertices (nodes);
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2

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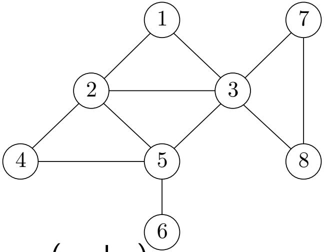
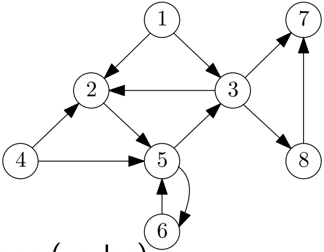


- V: set of vertices (nodes);
  - $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$
- E: pairwise relationships among V;
  - (undirected) graphs: relationship is symmetric, E contains subsets of size 2
  - $E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 5\}, \{3, 7\}, \{3, 8\}, \{4, 5\}, \{5, 6\}, \{7, 8\}\}$

Directed Graph $G = (V, E)$



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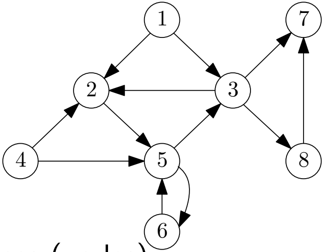

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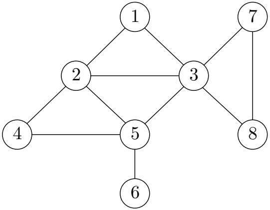
Abuse of Notations

- For (undirected) graphs, we often use (i, j) to denote the set $\{i, j\}$.
- We call (i, j) an unordered pair; in this case $(i, j) = (j, i)$.



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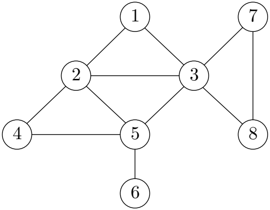
- Social Network : Undirected
- Transition Graph : Directed
- Road Network : Directed or Undirected
- Internet : Directed or Undirected

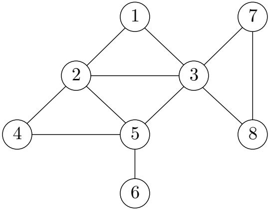
Representation of Graphs



$$\begin{array}{c|cccccccc}
 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
2 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
3 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
4 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
5 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
7 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
8 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0
\end{array}$$

- Adjacency matrix
  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected

Representation of Graphs



1: 2 → 3
2: 1 → 3 → 4 → 5
3: 1 → 2 → 5 → 7 → 8
4: 2 → 5
5: 2 → 3 → 4 → 6
6: 5
7: 3 → 8
8: 3 → 7

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- Linked lists
  - For every vertex v, there is a linked list containing all neighbours of v.

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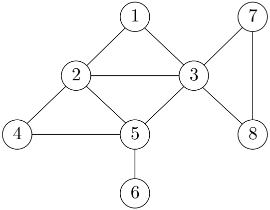

1: $\boxed{2}\boxed{3}$

2: $\boxed{1}\boxed{3}\boxed{4}\boxed{5}$

3: $\boxed{1}\boxed{2}\boxed{5}\boxed{7}\boxed{8}$

4: $\boxed{2}\boxed{5}$

5: $\boxed{2}\boxed{3}\boxed{4}\boxed{6}$

6: $\boxed{5}$

7: $\boxed{3}\boxed{8}$

8: $\boxed{3}\boxed{7}$

$d : (2, 4, 5, 2, 4, 1, 2, 2)$

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  - $n \times n$ matrix, $A[u, v] = 1$ if $(u, v) \in E$ and $A[u, v] = 0$ otherwise
  - A is symmetric if graph is undirected
- Linked lists
  - For every vertex v, there is a linked list containing all neighbours of v.
  - When graph is static, can use array of variant-length arrays.

Comparison of Two Representations

- Assuming we are dealing with undirected graphs
- n: number of vertices
- m: number of edges, assuming $n-1 \leq m \leq n(n-1)/2$
- d_v: number of neighbors of v

	Matrix	Linked Lists
memory usage		
time to check $(u, v) \in E$		
time to list all neighbours of v		

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	Matrix	Linked Lists
memory usage	$O(n^2)$	$O(m)$
time to check $(u, v) \in E$	$O(1)$	
time to list all neighbours of v		

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time to check $(u, v) \in E$	$O(1)$	$O(d_u)$
time to list all neighbours of v	$O(n)$	$O(d_v)$

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Connectivity Problem

Input: graph $G = (V, E)$, (using linked lists)
two vertices $s, t \in V$

Output: whether there is a path connecting s to t in G

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  - Breadth-First Search (BFS)
  - Depth-First Search (DFS)

Breadth-First Search (BFS)

- Build layers $L_0, L_1, L_2, L_3, \cdots$
- $L_0 = \{s\}$
- L_{j+1} contains all nodes that are not in $L_0 \cup L_1 \cup \cdots \cup L_j$ and have an edge to a vertex in L_j

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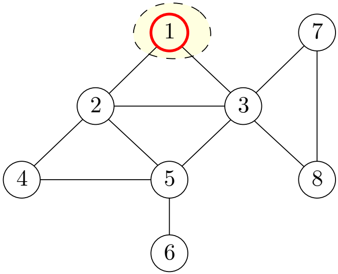

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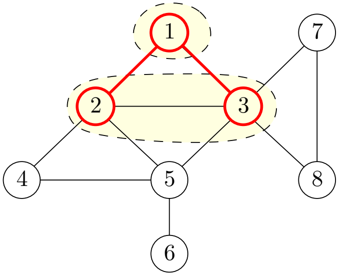

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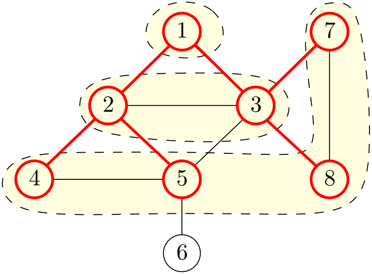

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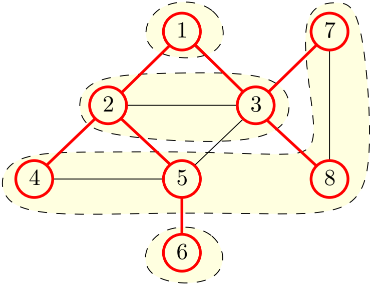


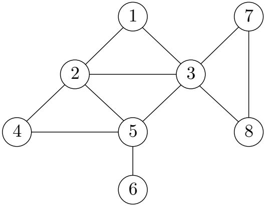
Implementing BFS using a Queue

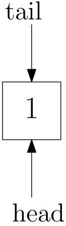
BFS(s)

```
$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark $s$ as "visited" and all other vertices as "unvisited"
while $head \le tail$ do
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbours $u$ of $v$ do
        if $u$ is "unvisited" then
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark $u$ as "visited"
```

- Running time: $O(n + m)$.

Example of BFS via Queue





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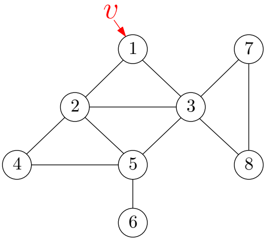
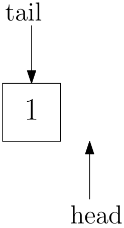


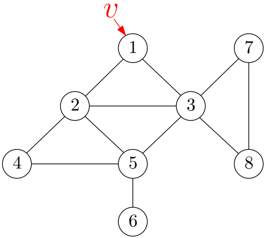


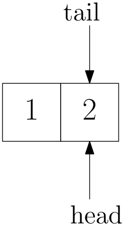
tail

1

head

Example of BFS via Queue

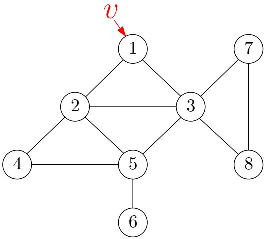


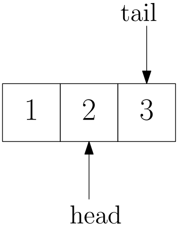

tail

1	2

head

Example of BFS via Queue



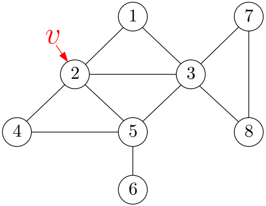


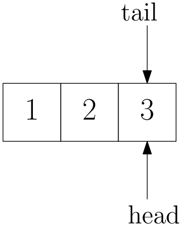
tail

1	2	3

head

Example of BFS via Queue





tail

1	2	3

head

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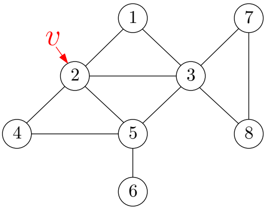

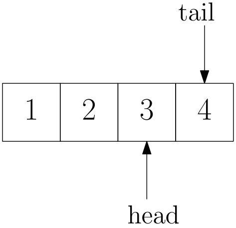

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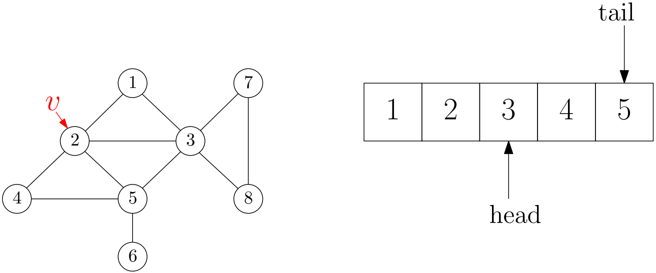

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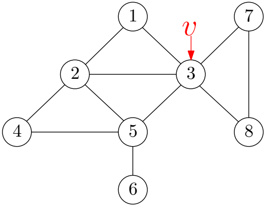

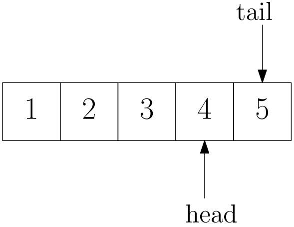

tail

1	2	3	4	5

head

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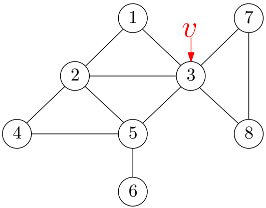
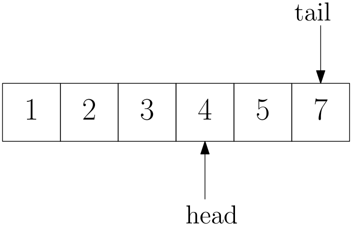
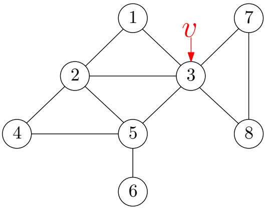


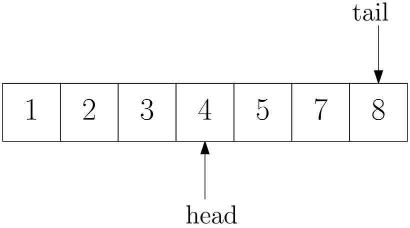

tail

1	2	3	4	5	7

head

Example of BFS via Queue





1	2	3	4	5	7	8

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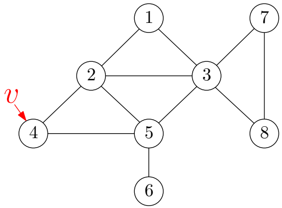

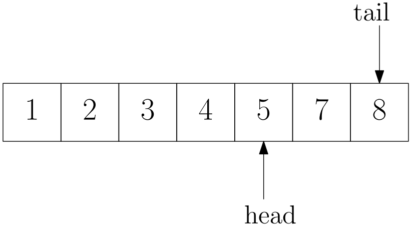

1	2	3	4	5	7	8

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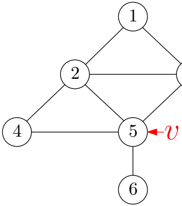

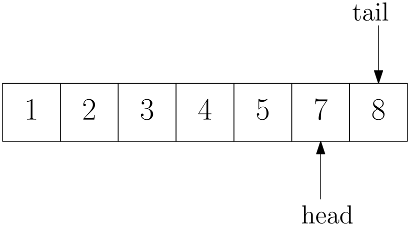

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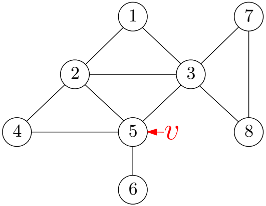

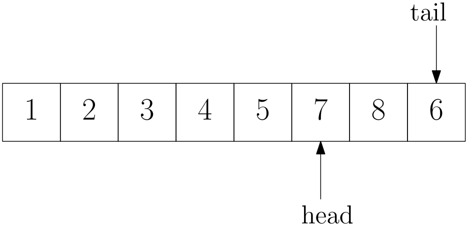

1	2	3	4	5	7	8	6

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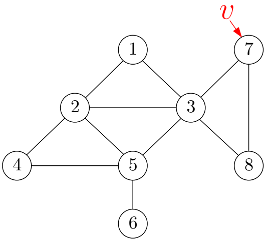

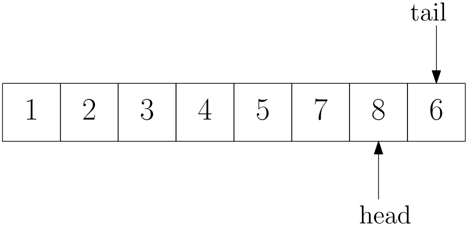

1	2	3	4	5	7	8	6

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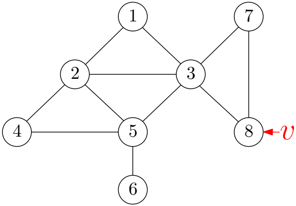

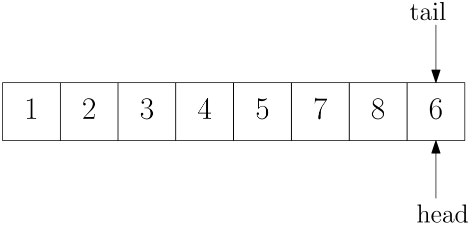

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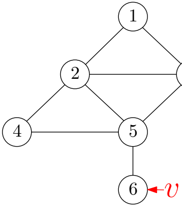

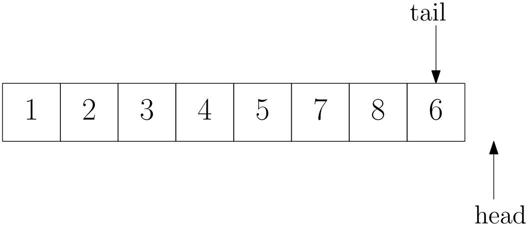

1	2	3	4	5	7	8	6

Depth-First Search (DFS)

- Starting from s
- Travel through the first edge leading out of the current vertex
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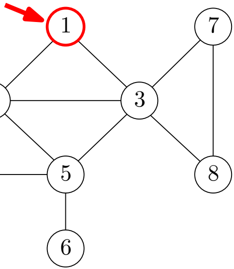

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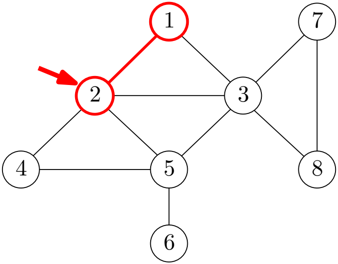

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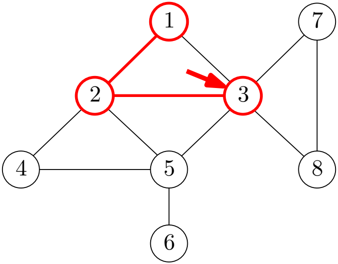

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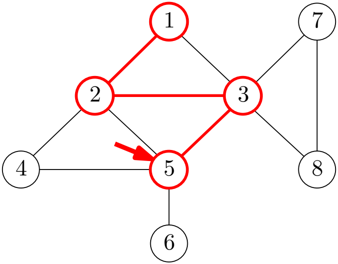

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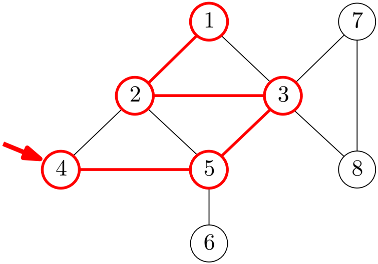

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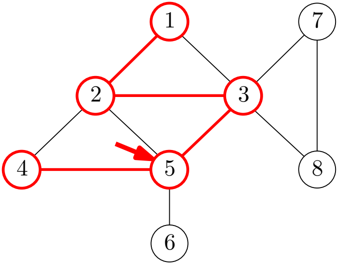

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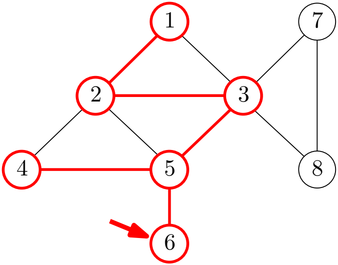

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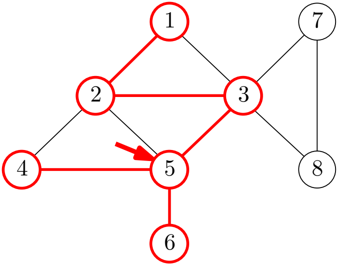

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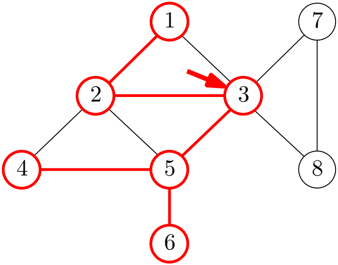

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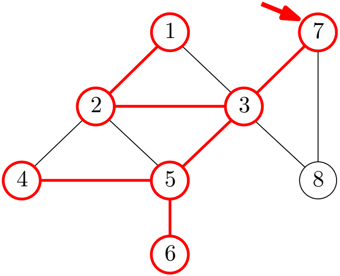

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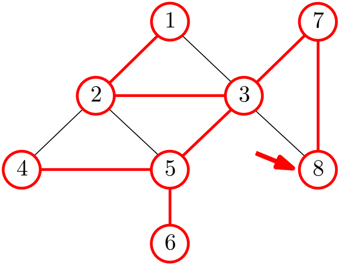


Implementing DFS using Recurrsion

DFS(s)

```
mark all vertices as "unvisited"
recursive-DFS($s$)
```

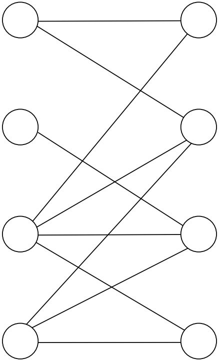
recursive-DFS(v)

```
mark $v$ as "visited"
for all neighbours $u$ of $v$ do
    if $u$ is unvisited then recursive-DFS($u$)
```

Outline

Testing Bipartiteness: Applications of BFS

Def. A graph $G = (V, E)$ is a bipartite graph if there is a partition of V into two sets L and R such that for every edge $(u, v) \in E$, we have either $u \in L, v \in R$ or $v \in L, u \in R$.



Testing Bipartiteness

- Taking an arbitrary vertex $s \in V$

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- $\cdots$
- Report “not a bipartite graph” if contradiction was found

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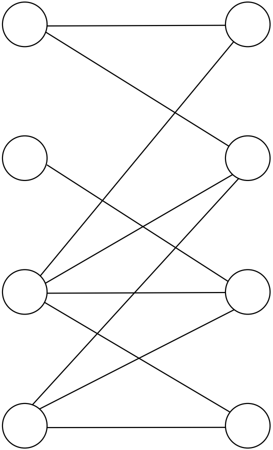
- Assuming $s \in L$ w.l.o.g
- Neighbors of s must be in R
- Neighbors of neighbors of s must be in L
- $\cdots$
- Report "not a bipartite graph" if contradiction was found
- If G contains multiple connected components, repeat above algorithm for each component

Test Bipartiteness



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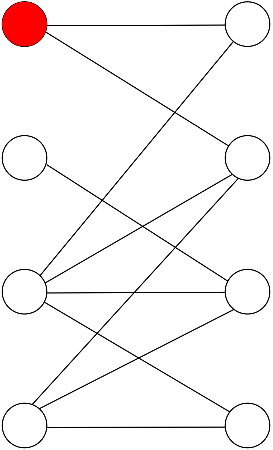

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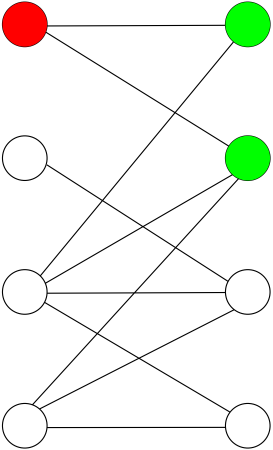

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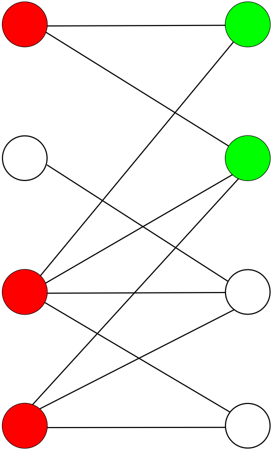

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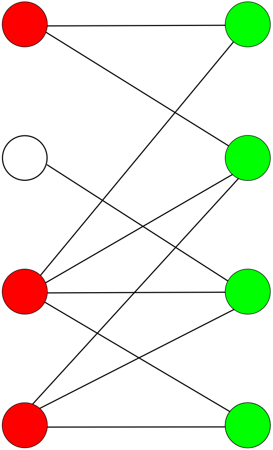

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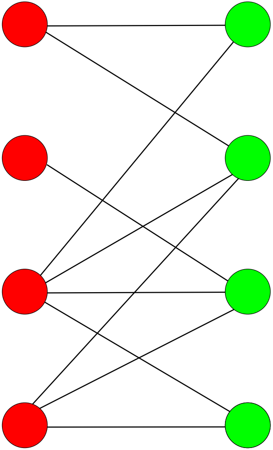

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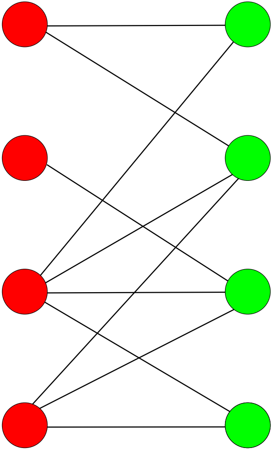

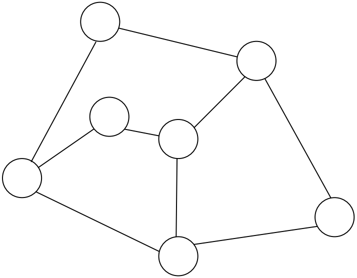

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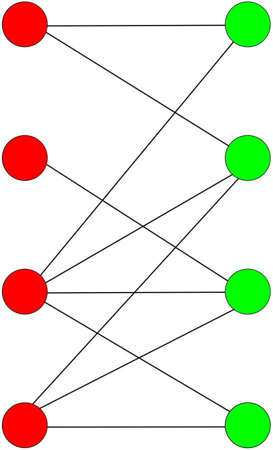

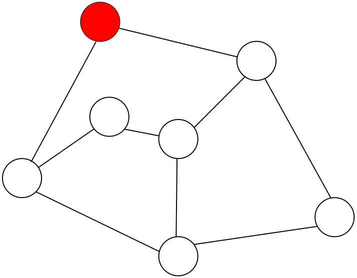

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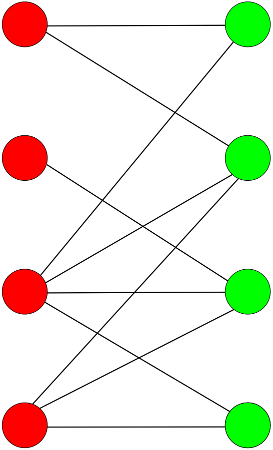

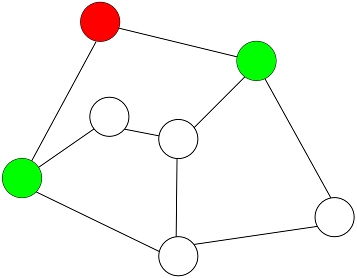

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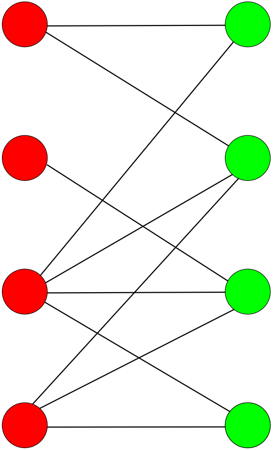

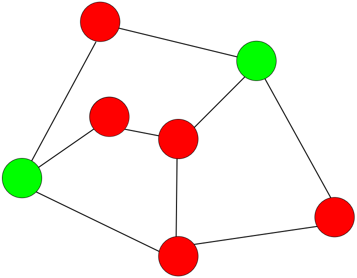

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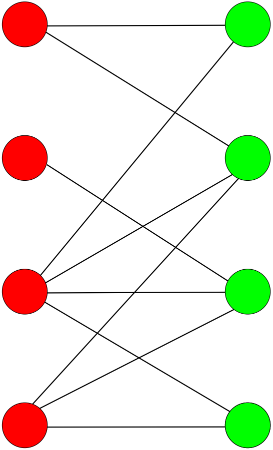

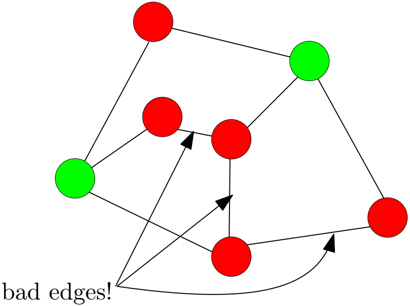

Testing Bipartiteness using BFS

BFS(s)

```
$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark $s$ as "visited" and all other vertices as "unvisited"
while $head \leq tail$ do
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbours $u$ of $v$ do
        if $u$ is "unvisited" then
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark $u$ as "visited"
```

Testing Bipartiteness using BFS

test-bipartiteness(s)

```
$head \leftarrow 1, tail \leftarrow 1, queue[1] \leftarrow s$
mark $s$ as "visited" and all other vertices as "unvisited"
$color[s] \leftarrow 0$
while $head \leq tail$ do
    $v \leftarrow queue[head], head \leftarrow head + 1$
    for all neighbours $u$ of $v$ do
        if $u$ is "unvisited" then
            $tail \leftarrow tail + 1, queue[tail] = u$
            mark $u$ as "visited"
            $color[u] \leftarrow 1 - color[v]$
        else if $color[u] = color[v]$ then
            print("$G$ is not bipartite") and exit
```

Testing Bipartiteness using BFS

```
mark all vertices as "unvisited"
for each vertex $v \in V$ do
    if $v$ is "unvisited" then
        test-bipartiteness($v$)
print("$G$ is bipartite")
```

Testing Bipartiteness using BFS

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mark all vertices as "unvisited"
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```

Obs. Running time of algorithm $= O(n + m)$

Outline

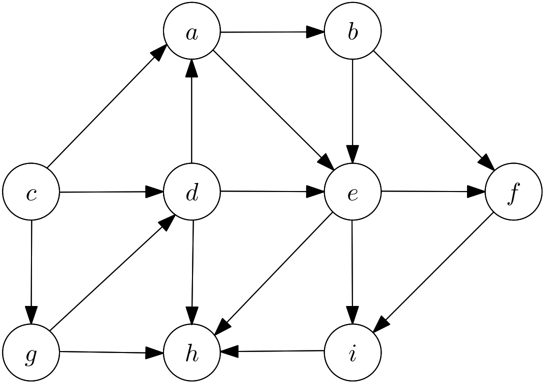
1. Graphs
2. Connectivity and Graph Traversal
   - Testing Bipartiteness
3. Topological Ordering

Topological Ordering Problem

Input: a directed acyclic graph (DAG) $G = (V, E)$

Output: 1-to-1 function $\pi : V \to \{1, 2, 3 \cdots, n\}$, so that

- if $(u, v) \in E$ then $\pi(u) < \pi(v)$



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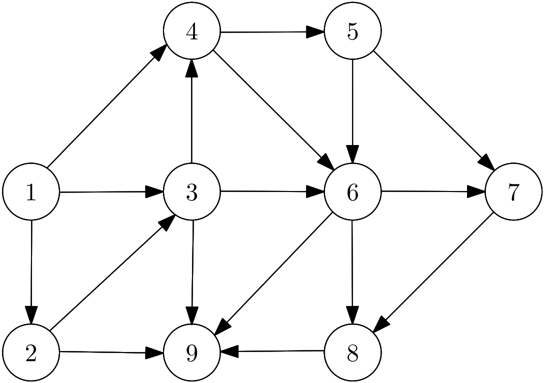
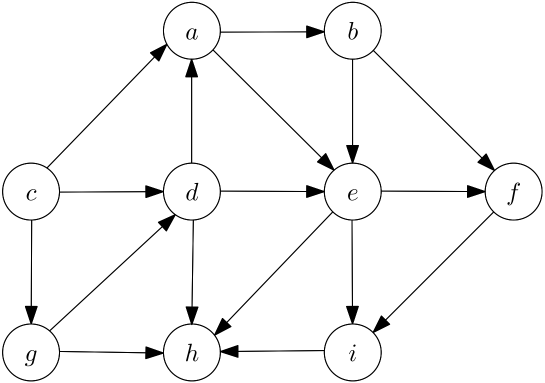


Topological Ordering

- Algorithm: each time take a vertex without incoming edges, then remove the vertex and all its outgoing edges.



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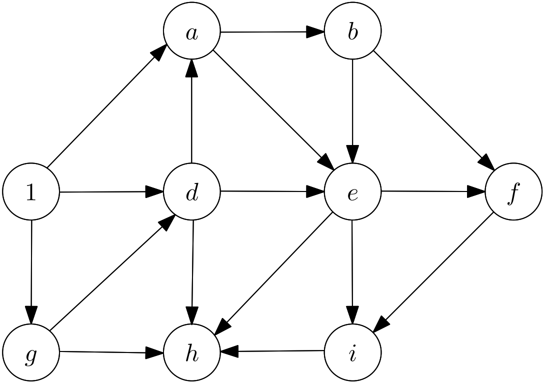

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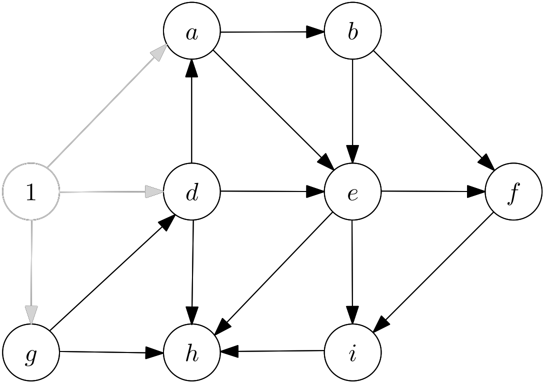

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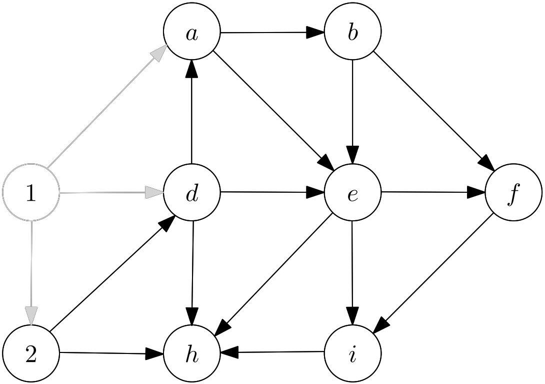

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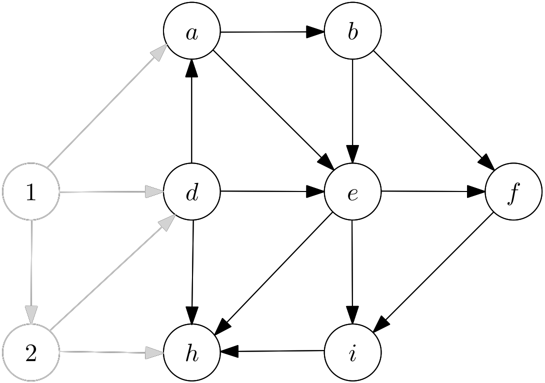

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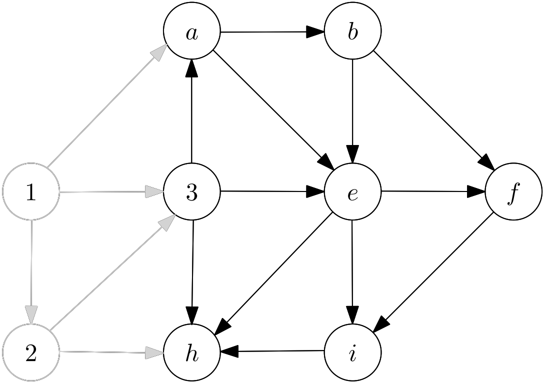

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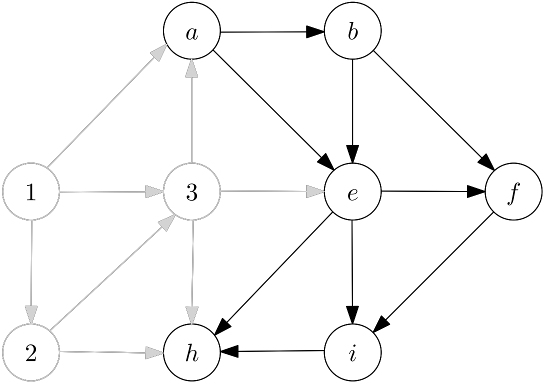

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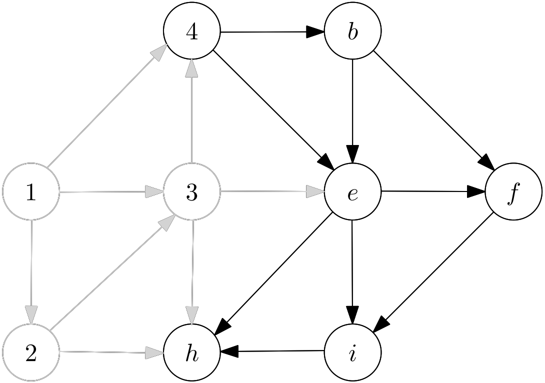

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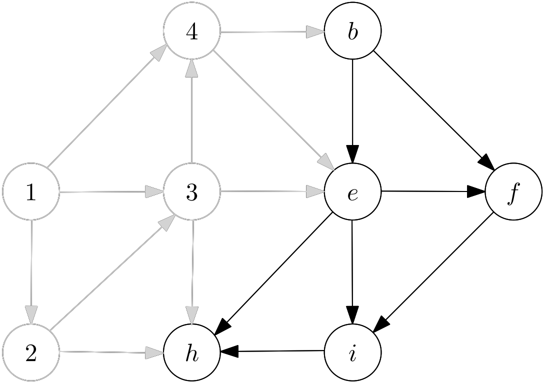

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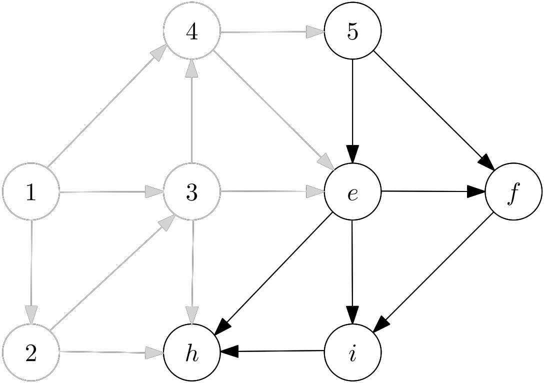

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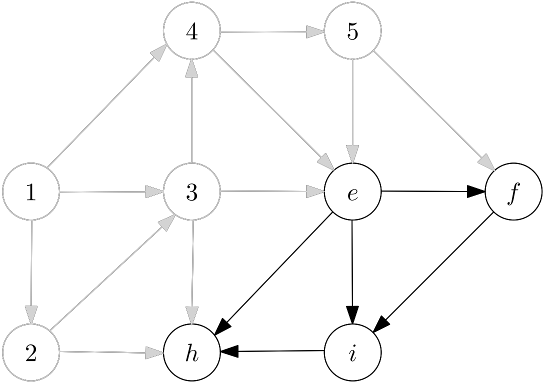

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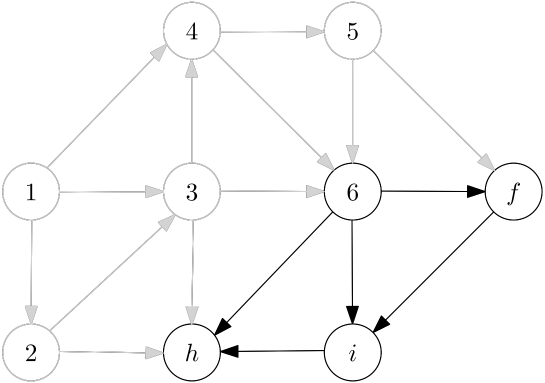

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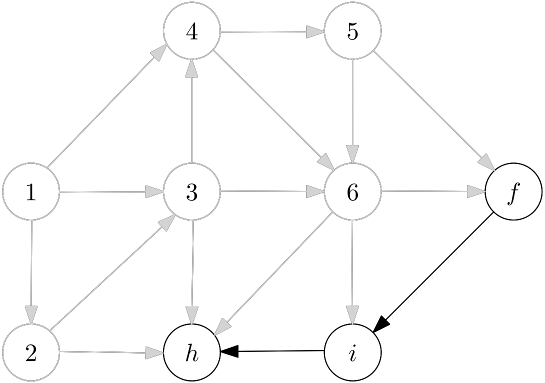

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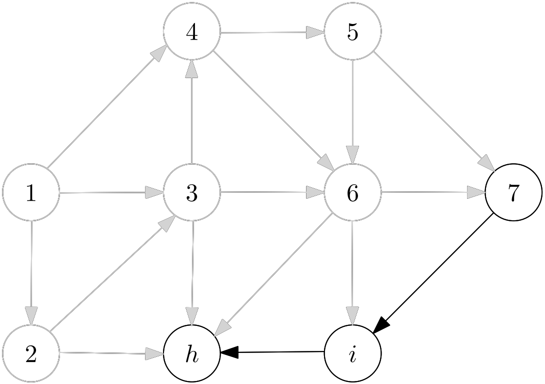

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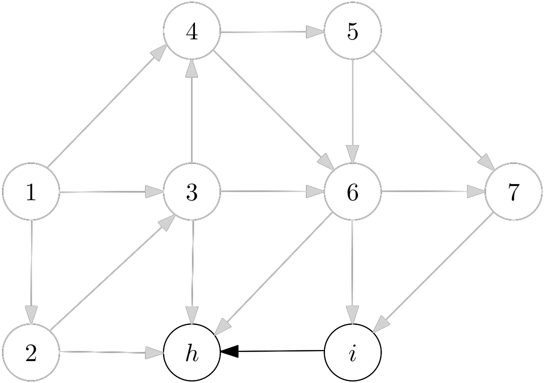

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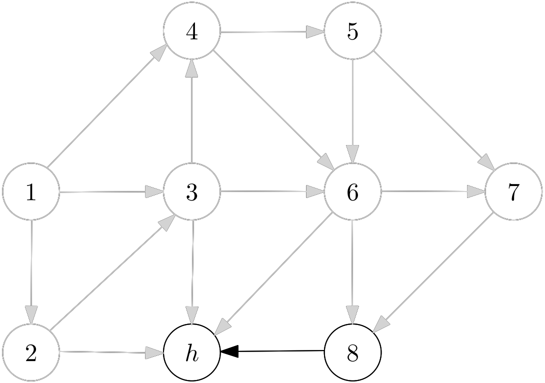

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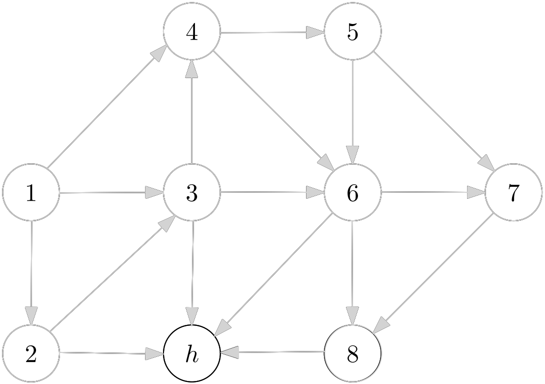

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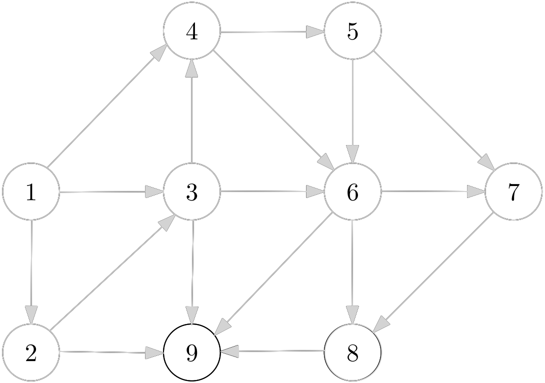

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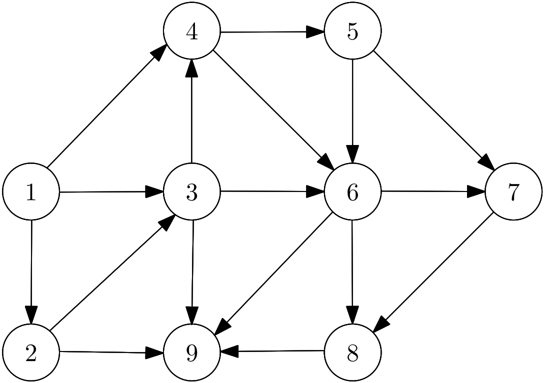

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Q: How to make the algorithm as efficient as possible?

Topological Ordering

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Q: How to make the algorithm as efficient as possible?

A:

- Use linked-lists of outgoing edges
- Maintain the in-degree d_v of vertices
- Maintain a queue (or stack) of vertices v with $d_v = 0$

topological-sort(G)

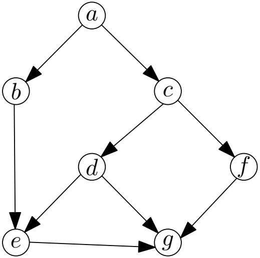
```
let $d_v \leftarrow 0$ for every $v \in V$
for every $v \in V$ do
    for every $u$ such that $(v, u) \in E$ do
        $d_u \leftarrow d_u + 1$
$S \leftarrow \{v : d_v = 0\}, i \leftarrow 0$
while $S \neq \emptyset$ do
    $v \leftarrow$ arbitrary vertex in $S$, $S \leftarrow S \setminus \{v\}$
    $i \leftarrow i + 1$, $\pi(v) \leftarrow i$
    for every $u$ such that $(v, u) \in E$ do
        $d_u \leftarrow d_u - 1$
        if $d_u = 0$ then add $u$ to $S$
if $i < n$ then output “not a DAG”
```

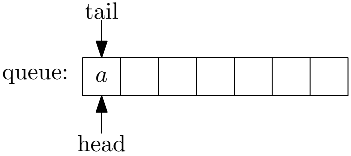
- S can be represented using a queue or a stack
- Running time $= O(n + m)$

S as a Queue or a Stack

DS	Queue	Stack
Initialization	$head \leftarrow 1$, $tail \leftarrow 0$	$top \leftarrow 0$
Non-Empty?	$head \le tail$	$top > 0$
Add(v)	$tail \leftarrow tail + 1$ $S[tail] \leftarrow v$	$top \leftarrow top + 1$ $S[top] \leftarrow v$
Retrieve v	$v \leftarrow S[head]$ $head \leftarrow head + 1$	$v \leftarrow S[top]$ $top \leftarrow top - 1$

Example



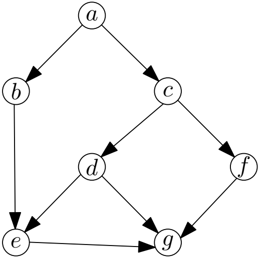

tail

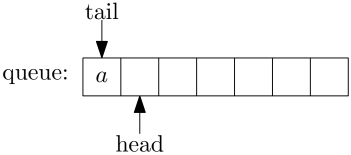
queue: a

head

	a	b	c	d	e	f	g
degree	0	1	1	1	2	1	3

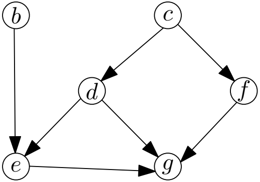
Example

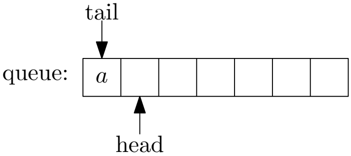




	a	b	c	d	e	f	g
degree	0	1	1	1	2	1	3

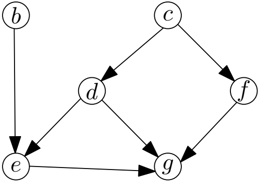
Example

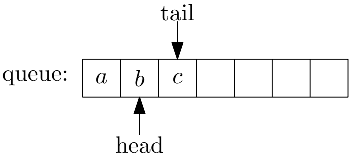




	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

Example

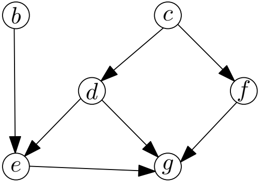


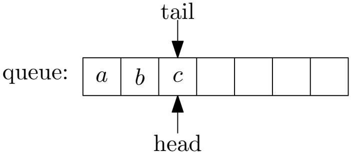


queue:	a	b	c				

	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

Example



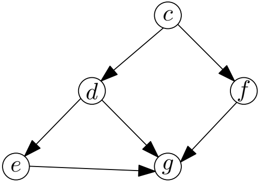


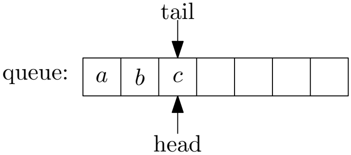
queue:

a	b	c				

	a	b	c	d	e	f	g
degree	0	0	0	1	2	1	3

Example



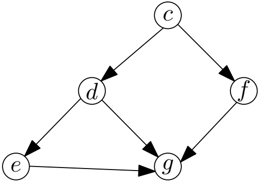

tail

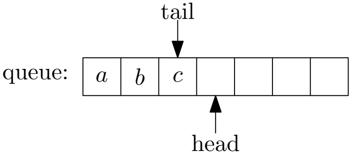
queue:	a	b	c				

head

	a	b	c	d	e	f	g
degree	0	0	0	1	1	1	3

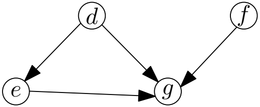
Example

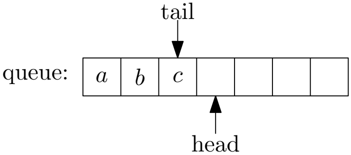




	a	b	c	d	e	f	g
degree	0	0	0	1	1	1	3

Example

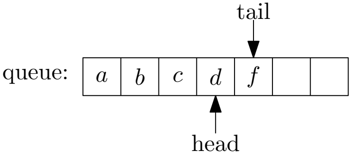




queue:	a	b	c				

	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

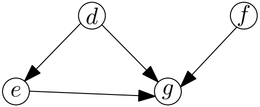
Example



tail

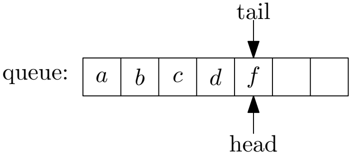
queue: a | b | c | d | f | |

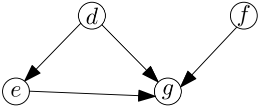
head



	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

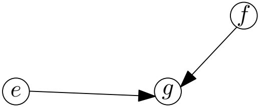
Example

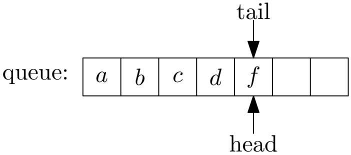




	a	b	c	d	e	f	g
degree	0	0	0	0	1	0	3

Example





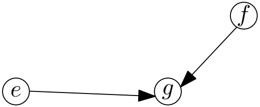
tail

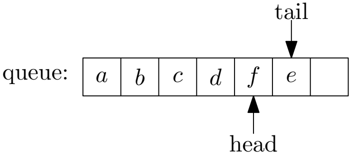
queue:	a	b	c	d	f		

head

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

Example



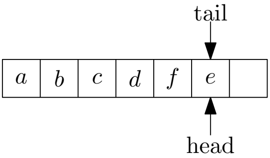


queue:

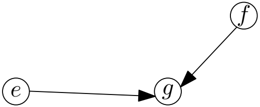
a	b	c	d	f	e	

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

Example

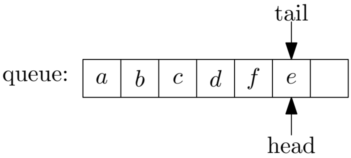


queue: a | b | c | d | f | e |



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	2

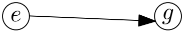
Example



tail

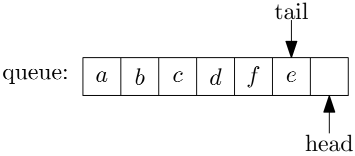
queue:	a	b	c	d	f	e	

head



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	1

Example



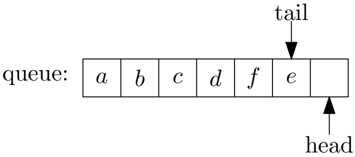
tail

queue:	a	b	c	d	f	e	

head

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	1

Example



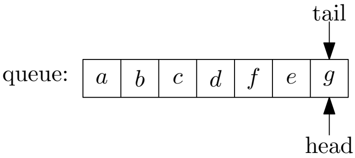
tail

queue: a b c d f e

head

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Example



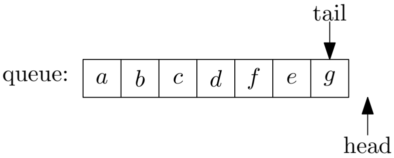
tail

queue:	a	b	c	d	f	e	g

head

	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0

Example



	a	b	c	d	e	f	g
degree	0	0	0	0	0	0	0