CSE 431/531: Algorithm Analysis and Design (Fall 2022)

Greedy Algorithms

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Def. In an optimization problem, our goal of is to find a valid solution with the minimum cost (or maximum value).

Trivial Algorithm for an Optimization Problem

Enumerate all valid solutions, compare them and output the best one.

- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.
- $f(n)$ is a polynomial if $f(n) = O(n^k)$ for some constant $k > 0$.
- convention: polynomial time $=$ efficient

Goals of algorithm design

1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems

Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
- Greedy algorithms are often for optimization problems.
- They often run in polynomial time due to their simplicity.

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Def. A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.

Outline

Box Packing

Input: n boxes of capacities $c_1, c_2, \cdots, c_n$

m items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item j can be put into box i if $s_j \leq c_i$

Output: A way to put as many items as possible in the boxes.

Example:

- Box capacities: $60, 40, 25, 15, 12$
- Item sizes: $45, 42, 20, 19, 16$
- Can put 3 items in boxes: $45 \rightarrow 60, 20 \rightarrow 40, 19 \rightarrow 25$

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

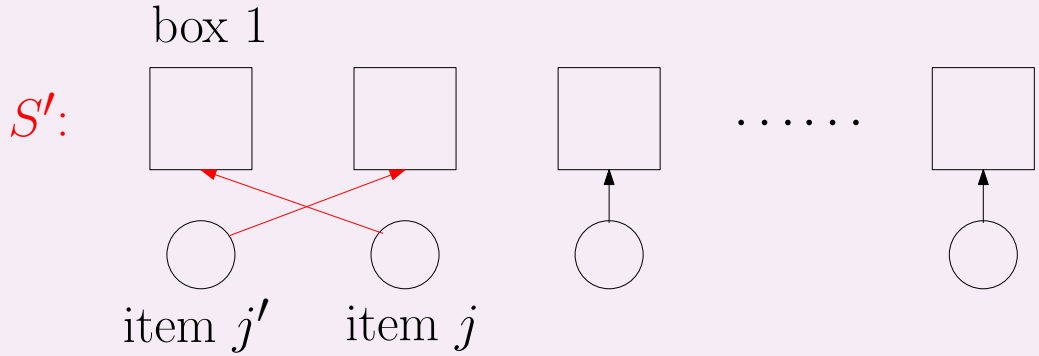
Lemma The strategy that put into box 1 the largest item it can hold is "safe": There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
- formal proof via exchanging argument:

Lemma There is an optimum solution in which box 1 contains the largest item it can hold.

Proof.

- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution S. If j is put into Box 1 in S, done.
- Otherwise, assume this is what happens in S:



- $s_{j'} \leq s_j$, and swapping gives another solution S'
- S' is also an optimum solution. In S', j is put into Box 1. □

- Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

- Trivial: we decided to put Item j into Box 1, and the remaining instance is obtained by removing Item j and Box 1.

Generic Greedy Algorithm

1: **while** the instance is non-trivial **do**
2: make the choice using the greedy strategy
3: reduce the instance

Greedy Algorithm for Box Packing

1: $T \leftarrow \{1, 2, 3, \cdots, m\}$
2: **for** $i \leftarrow 1$ to n **do**
3: **if** some item in T can be put into box i **then**
4: $j \leftarrow$ the largest item in T that can be put into box i
5: print("put item j in box i")
6: $T \leftarrow T \setminus \{j\}$

Generic Greedy Algorithm

1: **while** the instance is non-trivial **do**
2: make the choice using the greedy strategy
3: reduce the instance

Lemma Generic algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
- Greedy stretegy is not safe: we will miss the optimum solution for some instance, since the choices we made are irrevocable.

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

Def. A strategy is "safe" if there is always an optimum solution that is "consistent" with the decision made according to the strategy.

Exchange argument: Proof of Safety of a Strategy

- let S be an arbitrary optimum solution.
- if S is consistent with the greedy choice, done.
- otherwise, show that it can be modified to another optimum solution S' that is consistent with the choice.

- The procedure is not a part of the algorithm.

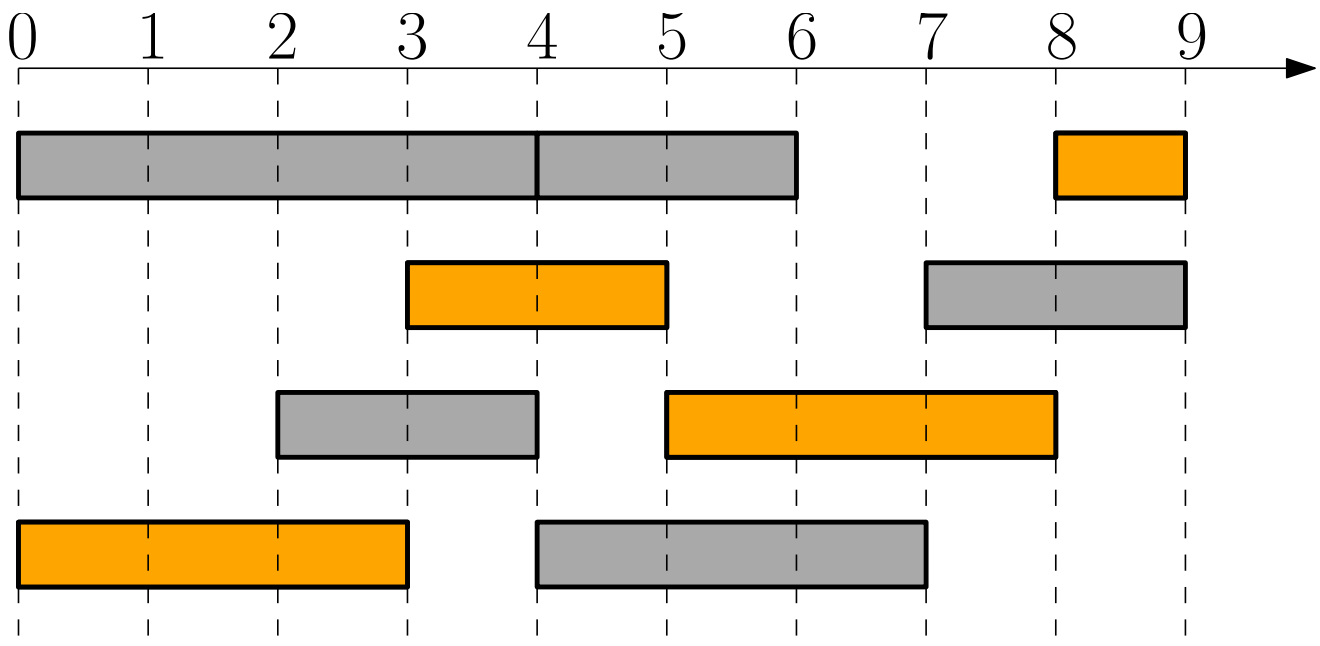
Outline

Interval Scheduling

Input: n jobs, job i with start time s_i and finish time f_i

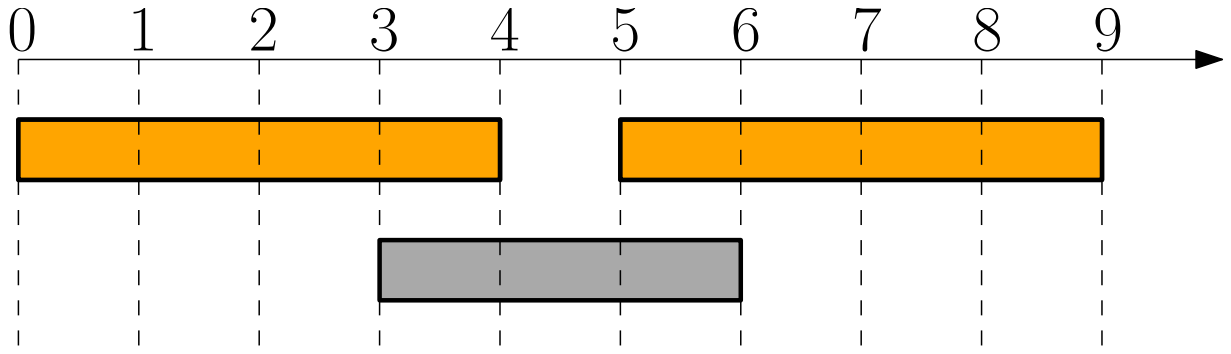
i and j are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

Output: A maximum-size subset of mutually compatible jobs



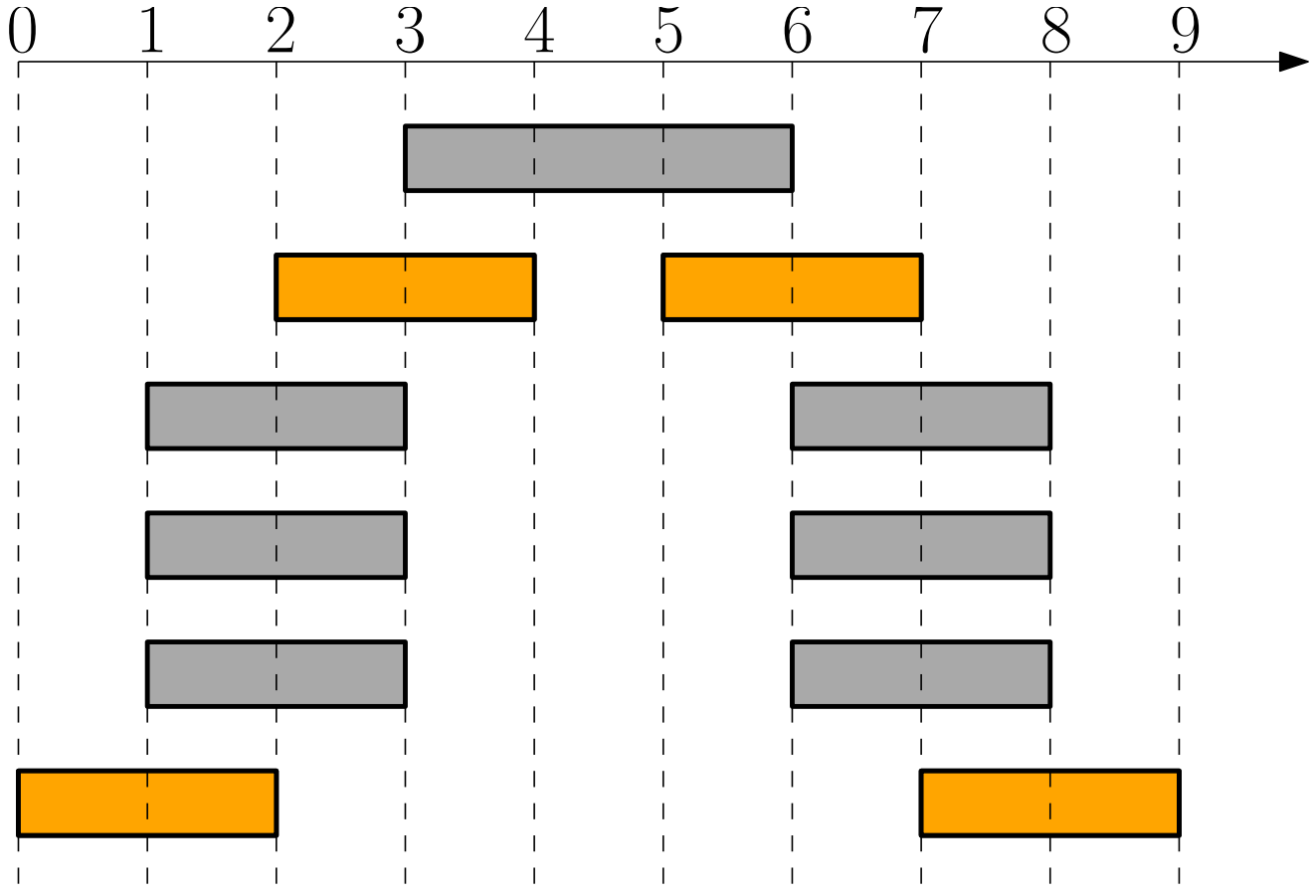
Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!



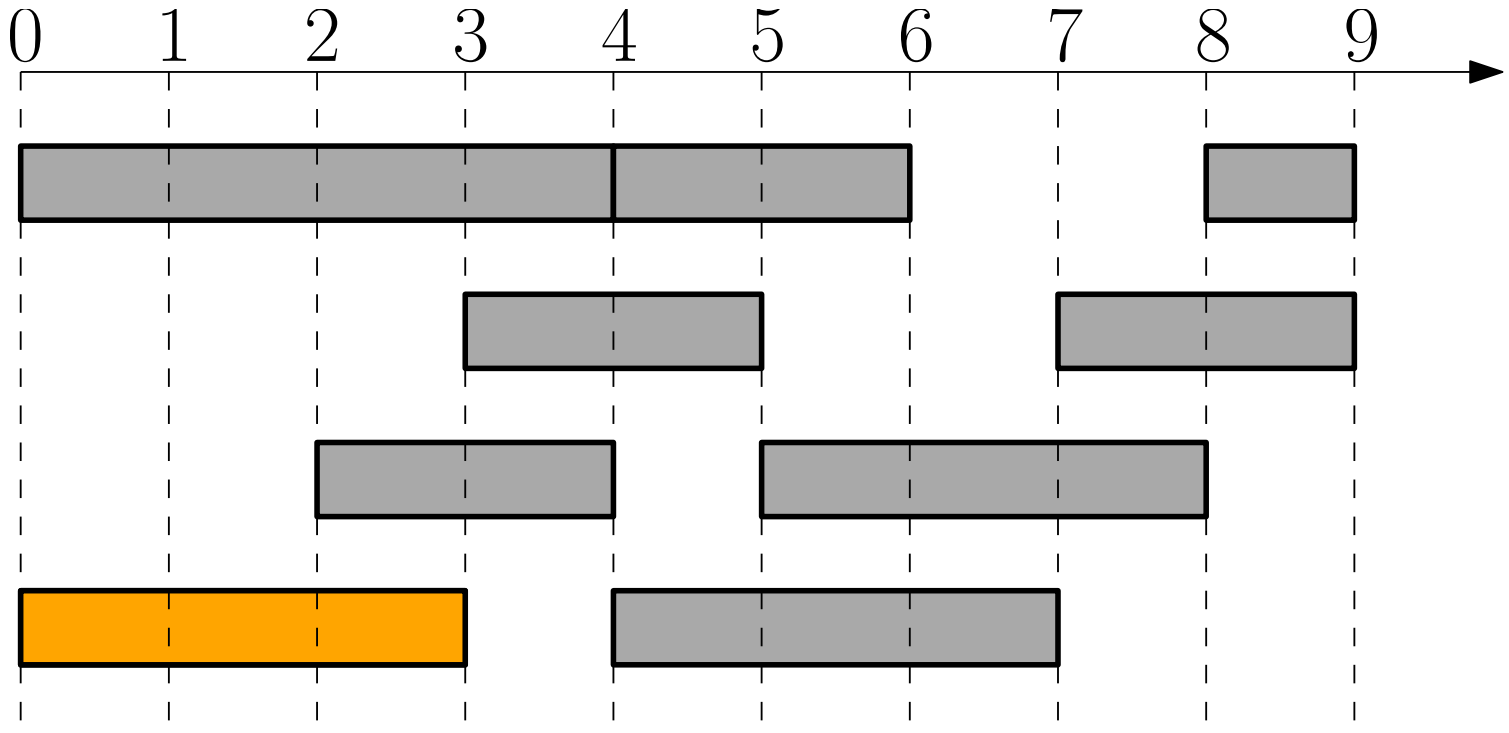
Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!



Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time? Yes!

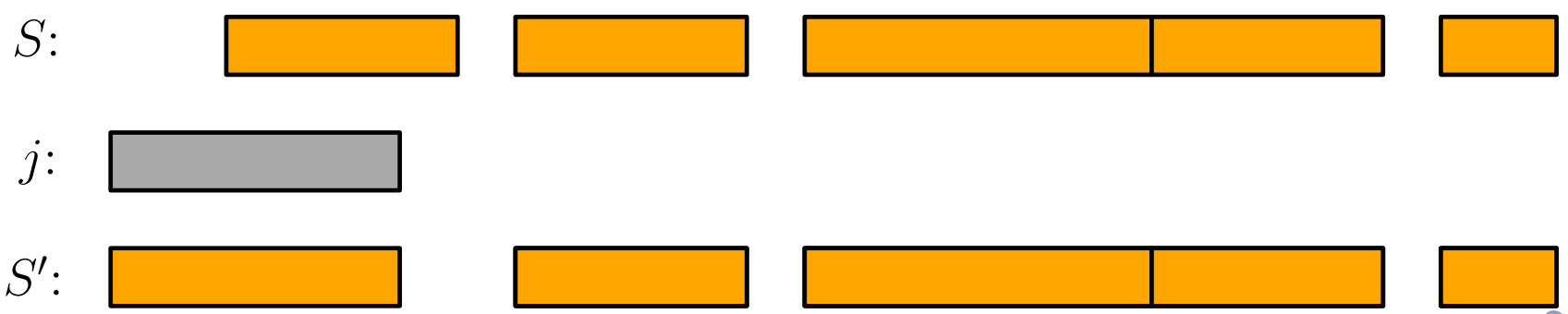


Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: There is an optimum solution where the job j with the earliest finish time is scheduled.

Proof.

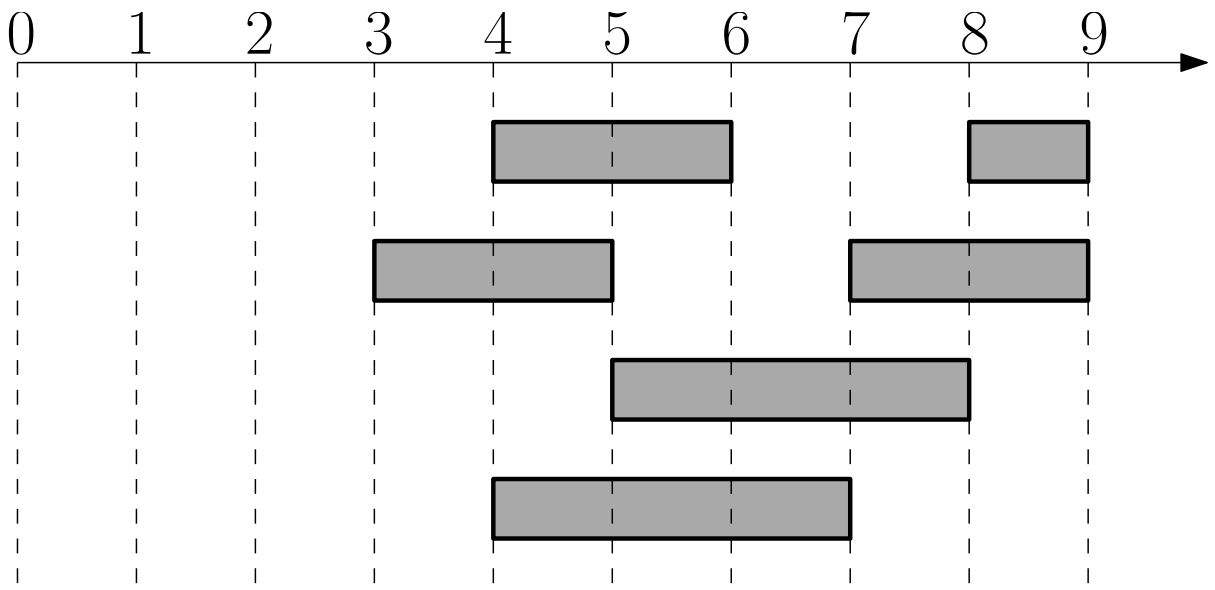
- Take an arbitrary optimum solution S
- If it contains j, done
- Otherwise, replace the first job in S with j to obtain another optimum schedule S'. □



Greedy Algorithm for Interval Scheduling

Lemma It is safe to schedule the job j with the earliest finish time: There is an optimum solution where the job j with the earliest finish time is scheduled.

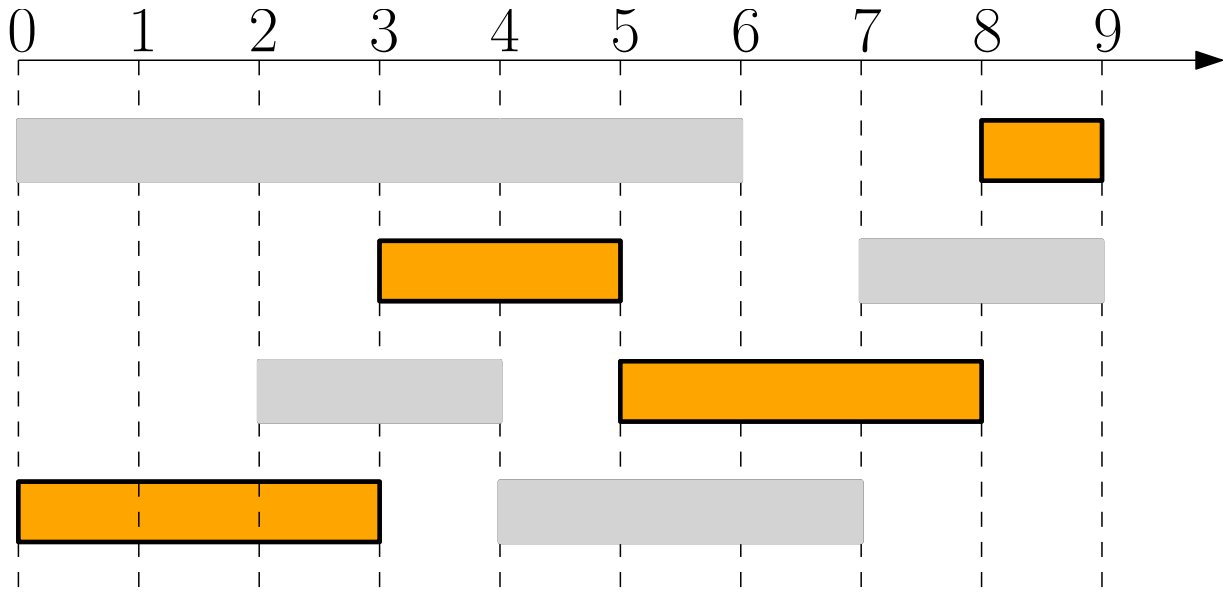
- What is the remaining task after we decided to schedule j?
- Is it another instance of interval scheduling problem? Yes!



Greedy Algorithm for Interval Scheduling

Schedule(s, f, n)

1: $A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset$
2: **while** $A \neq \emptyset$ **do**
3: $\quad j \leftarrow \arg\min_{j' \in A} f_{j'}$
4: $\quad S \leftarrow S \cup \{j\}$; $A \leftarrow \{j' \in A : s_{j'} \geq f_j\}$
5: **return** S



Greedy Algorithm for Interval Scheduling

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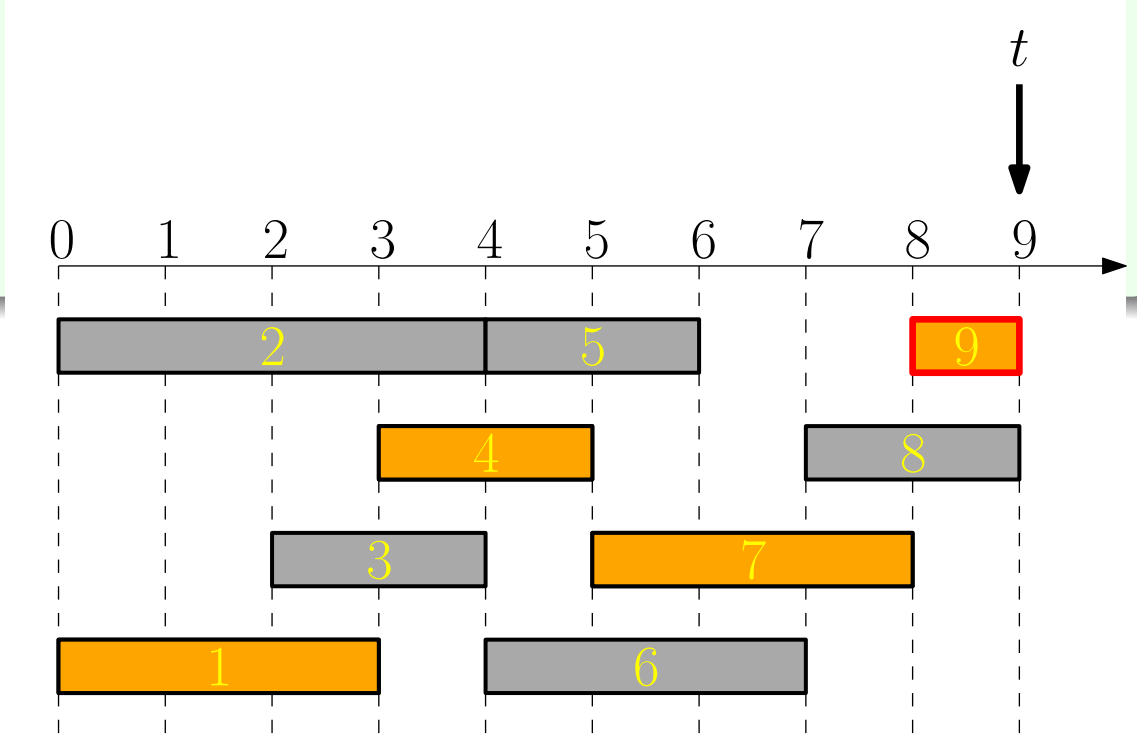
Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time

Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

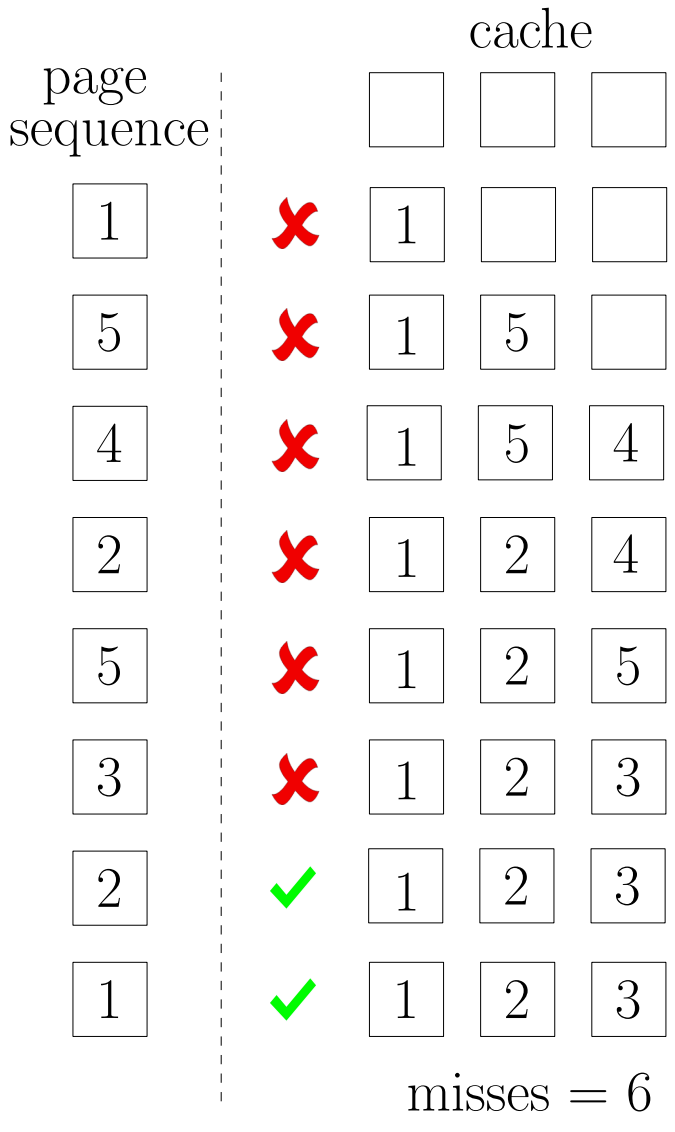
1: sort jobs according to f values
2: $t \leftarrow 0$, $S \leftarrow \emptyset$
3: **for** every $j \in [n]$ according to non-decreasing order of f_j **do**
4: **if** $s_j \geq t$ **then**
5: $S \leftarrow S \cup \{j\}$
6: $t \leftarrow f_j$
7: **return** S



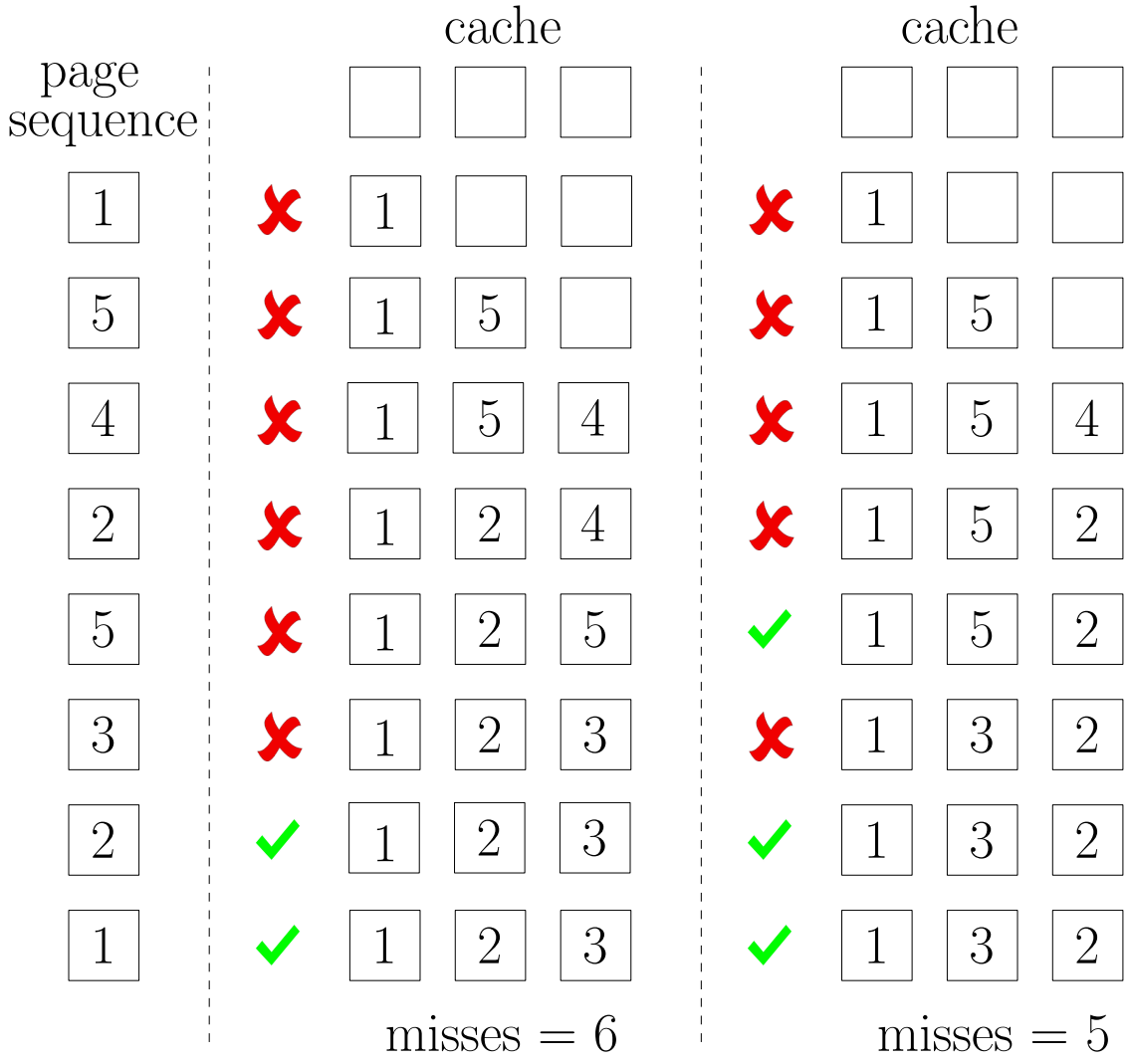
Outline

Offline Caching

- Cache that can store k pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.



A Better Solution for Example



Offline Caching Problem

Input: k : the size of cache

n : number of pages

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

Output: $i_1, i_2, i_3, \cdots, i_T \in \{\text{hit}, \text{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

Q: Why do we study the offline caching problem?

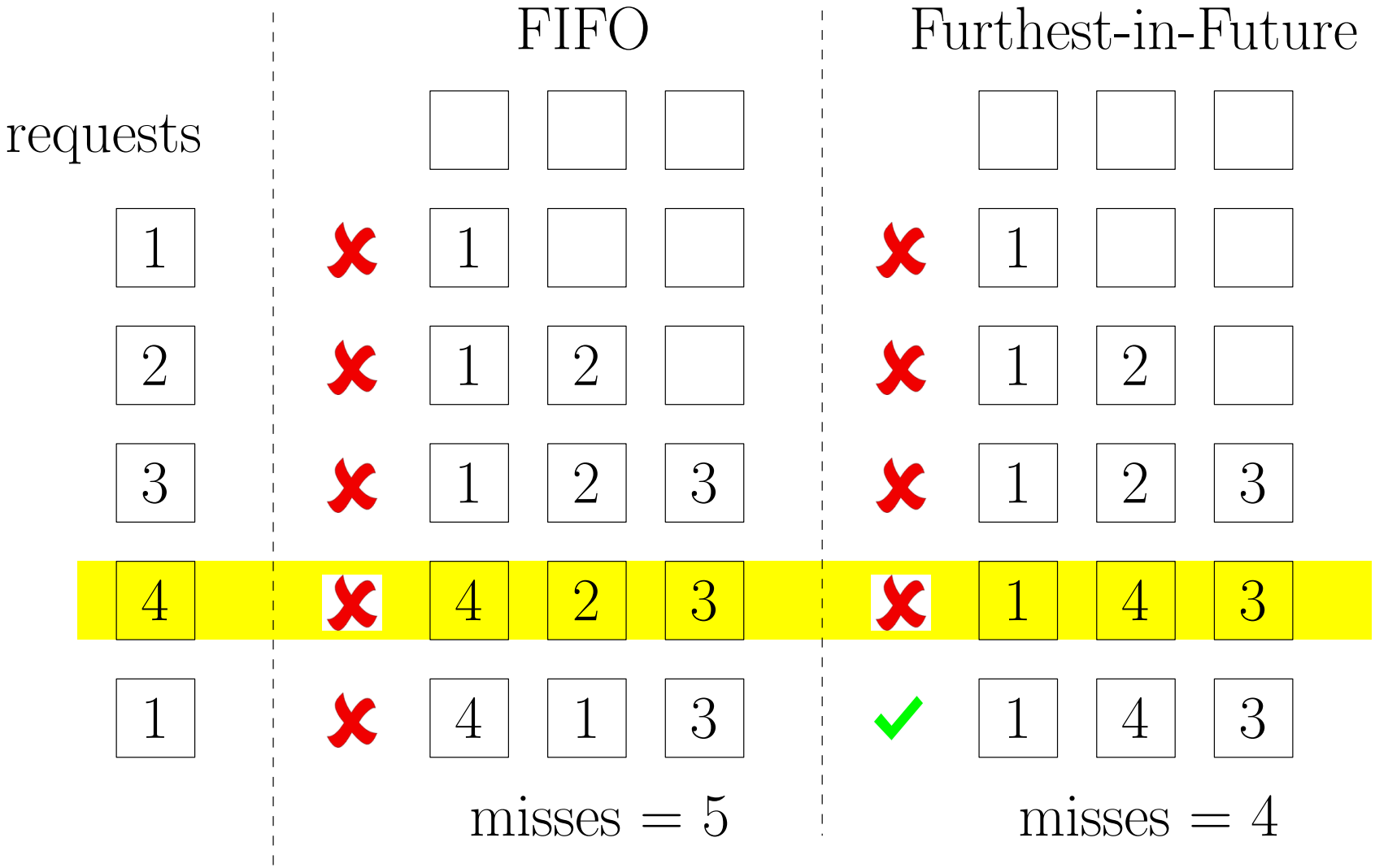
A: Use the offline solution as a benchmark to measure the "competitive ratio" of online algorithms

Offline Caching: Potential Greedy Algorithms

- FIFO(First-In-First-Out): always evict the first page in cache
- LRU(Least-Recently-Used): Evict page whose most recent access was earliest
- LFU(Least-Frequently-Used): Evict page that was least frequently requested

- All the above algorithms are not optimum!
- Indeed all the algorithms are "online", i.e, the decisions can be made without knowing future requests. Online algorithms can not be optimum.

FIFO is not optimum

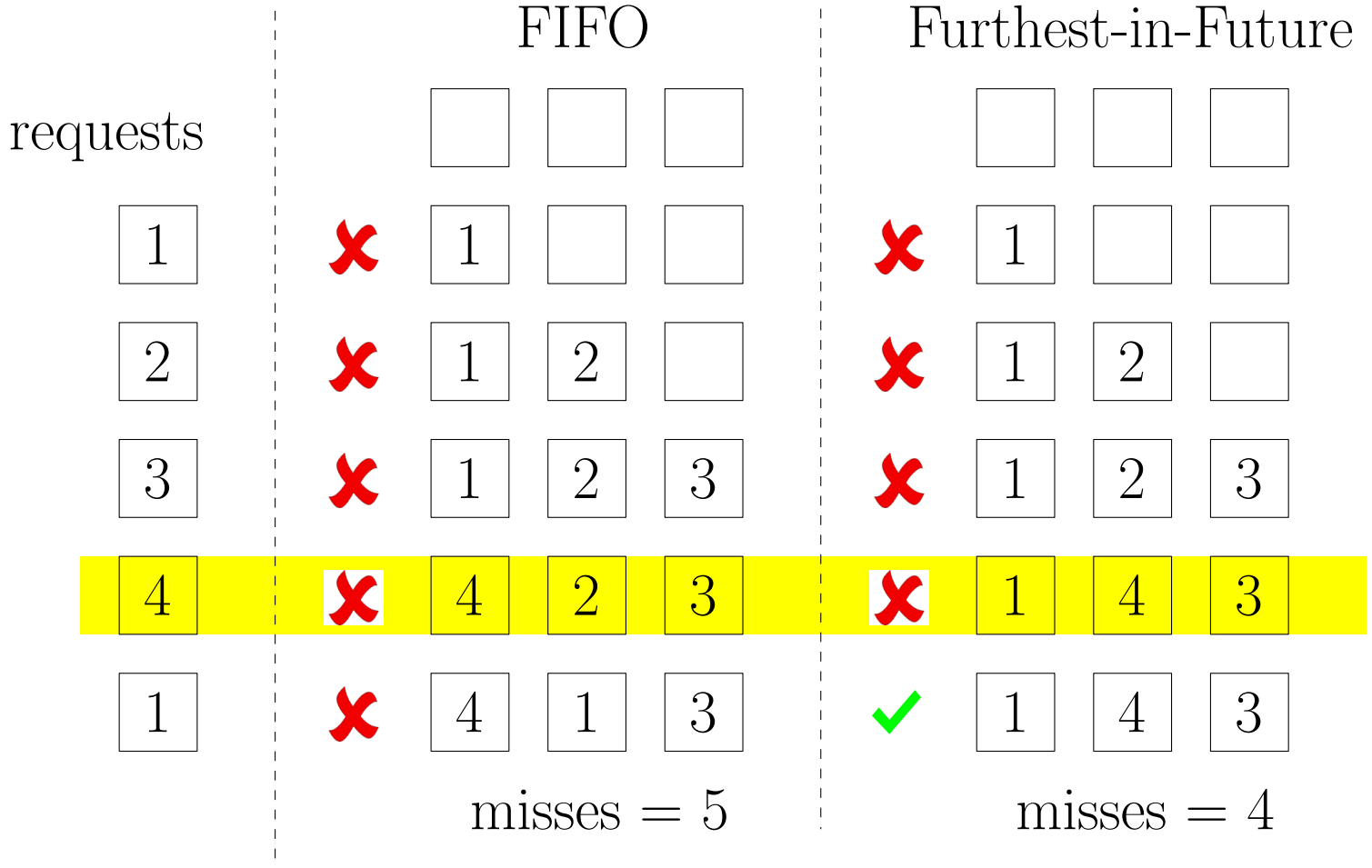


Optimum Offline Caching

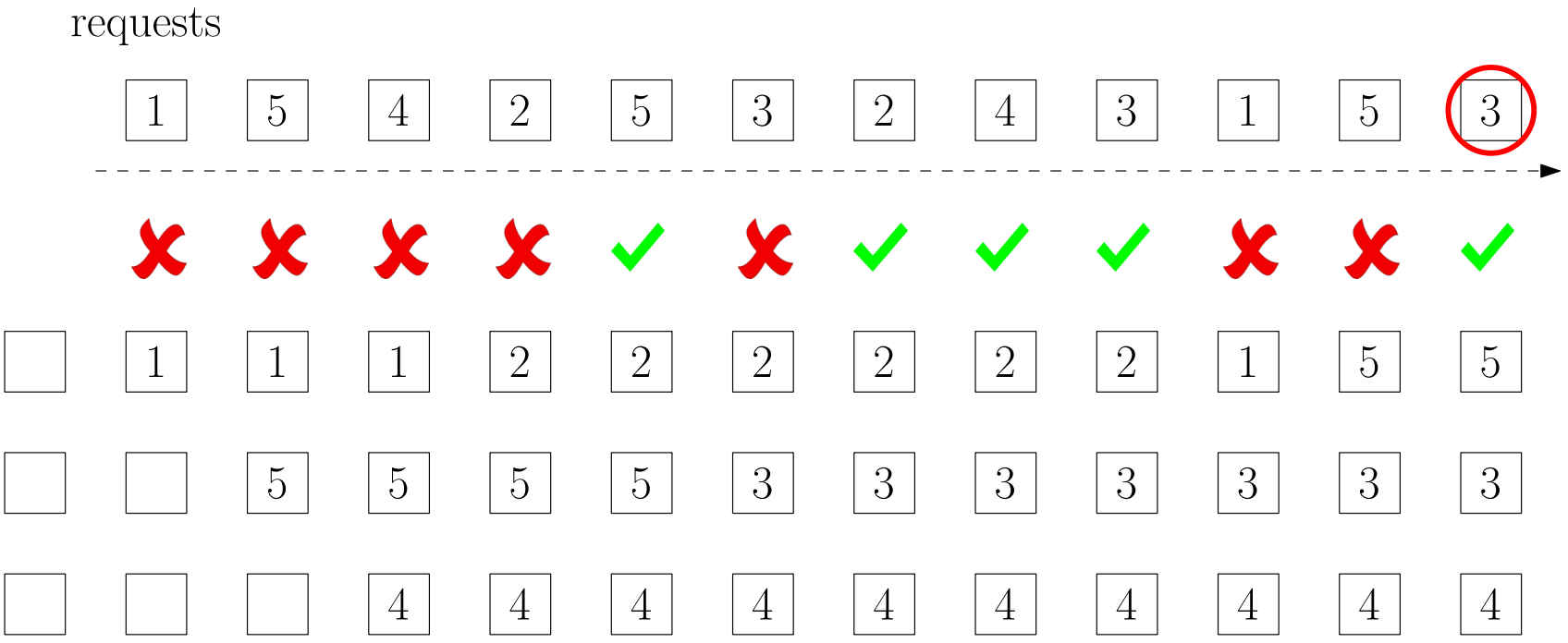
Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.

Furthest-in-Future (FF)



Example


requests
1 5 4 2 5 3 2 4 3 1 5 3
1 1 1 2 2 2 2 2 2 1 5 5
5 5 5 5 3 3 3 3 3 3 3
4 4 4 4 4 4 4 4 4 4 4

Recall: Designing and Analyzing Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Offline Caching Problem

Input: k : the size of cache

n : number of pages

$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

$p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

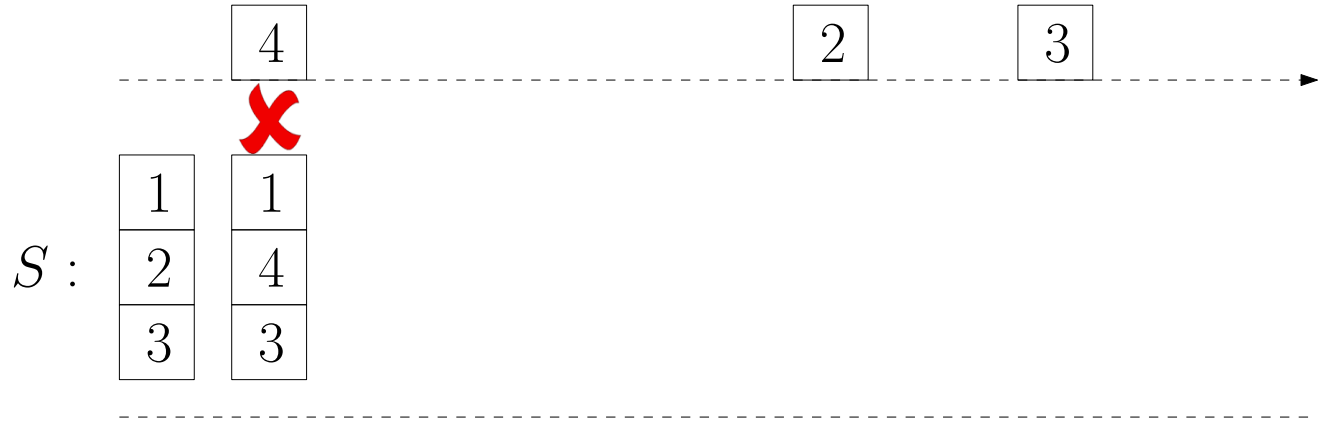
Output: $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit}, \text{empty}\} \cup [n]$

- empty stands for an empty page
- "hit" means evicting no pages

Analysis of Greedy Algorithm

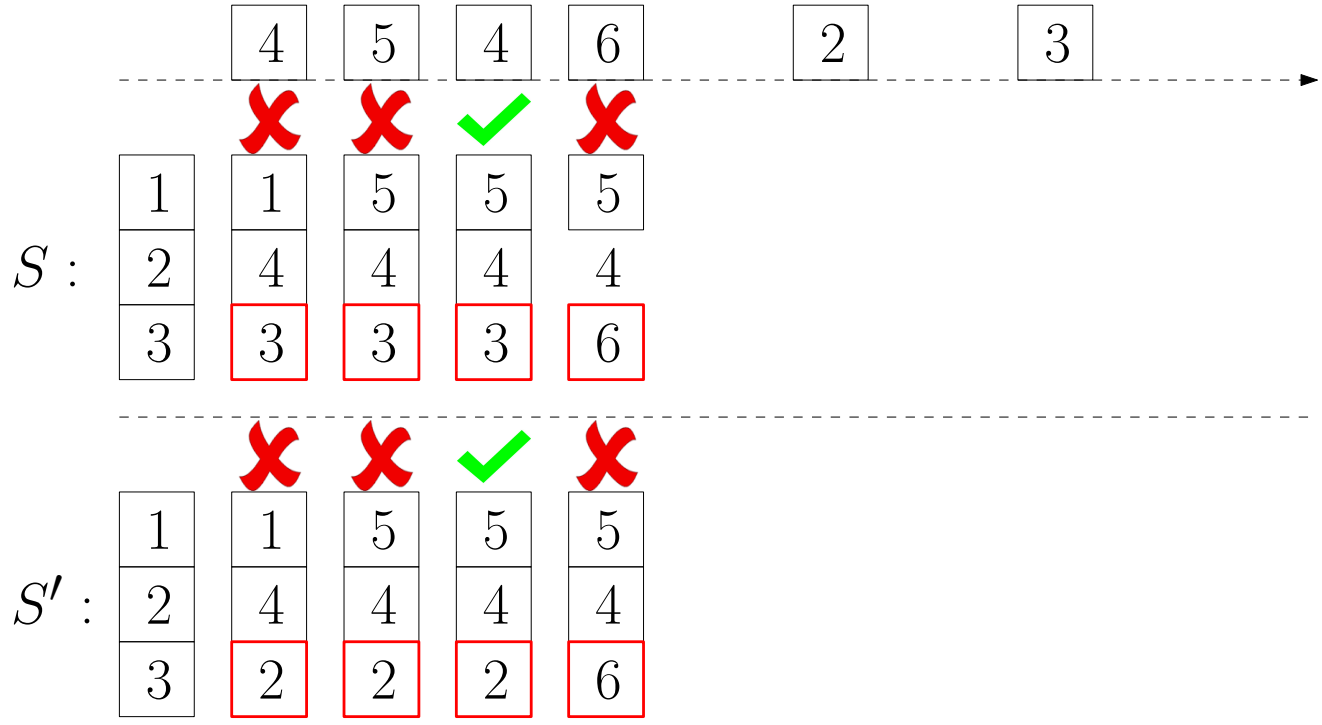
- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. There is an optimum solution in which p^* is evicted at time 1.



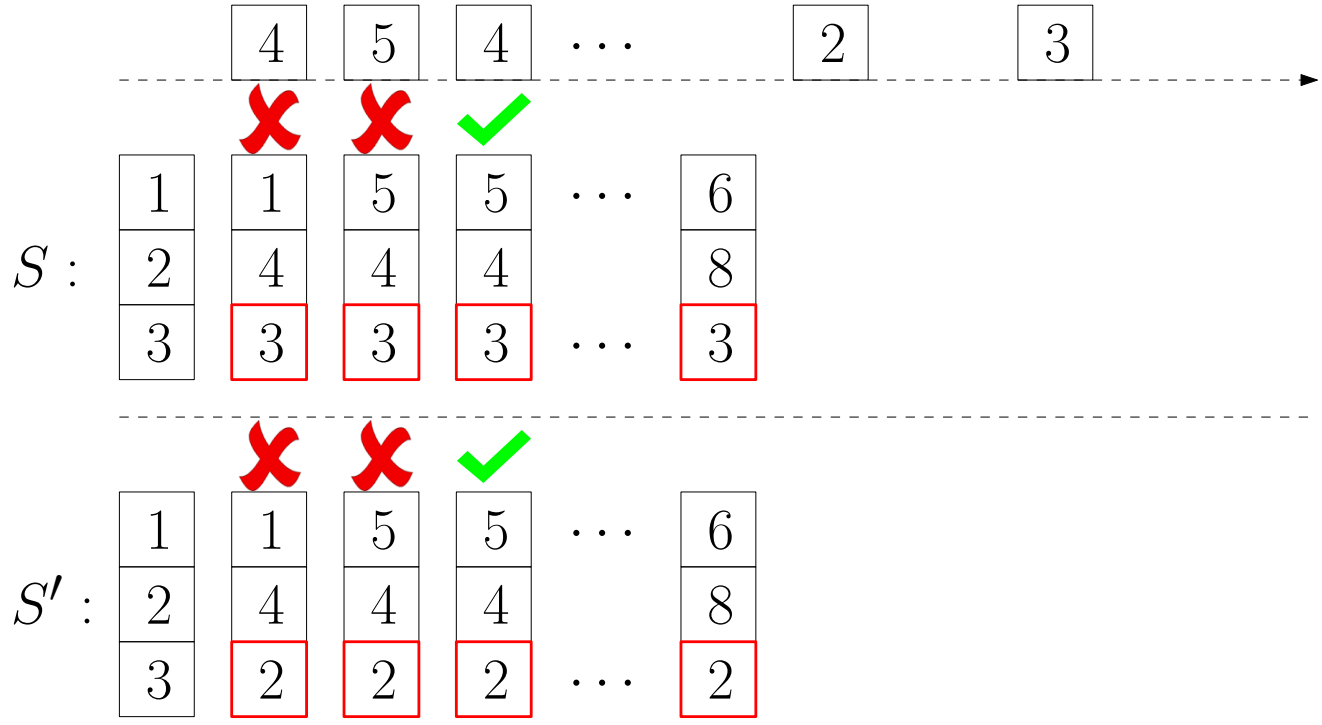
Proof.

1. S: any optimum solution
2. p^*: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume S evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$.



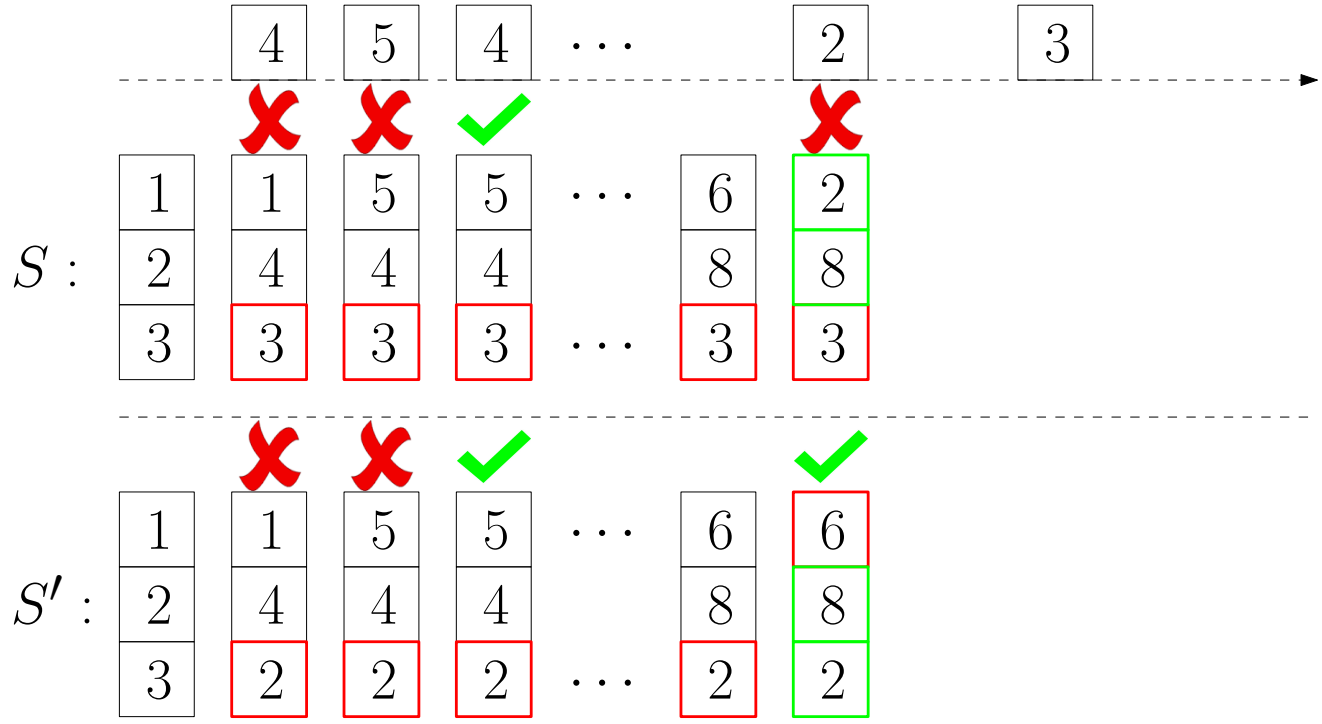
Proof.

4. Create S'. S' evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
5. After time 1, cache status of S and that of S' differ by only 1 page. S' contains $p'(=2)$ and S contains $p^*(=3)$.
6. From now on, S' will "copy" S.



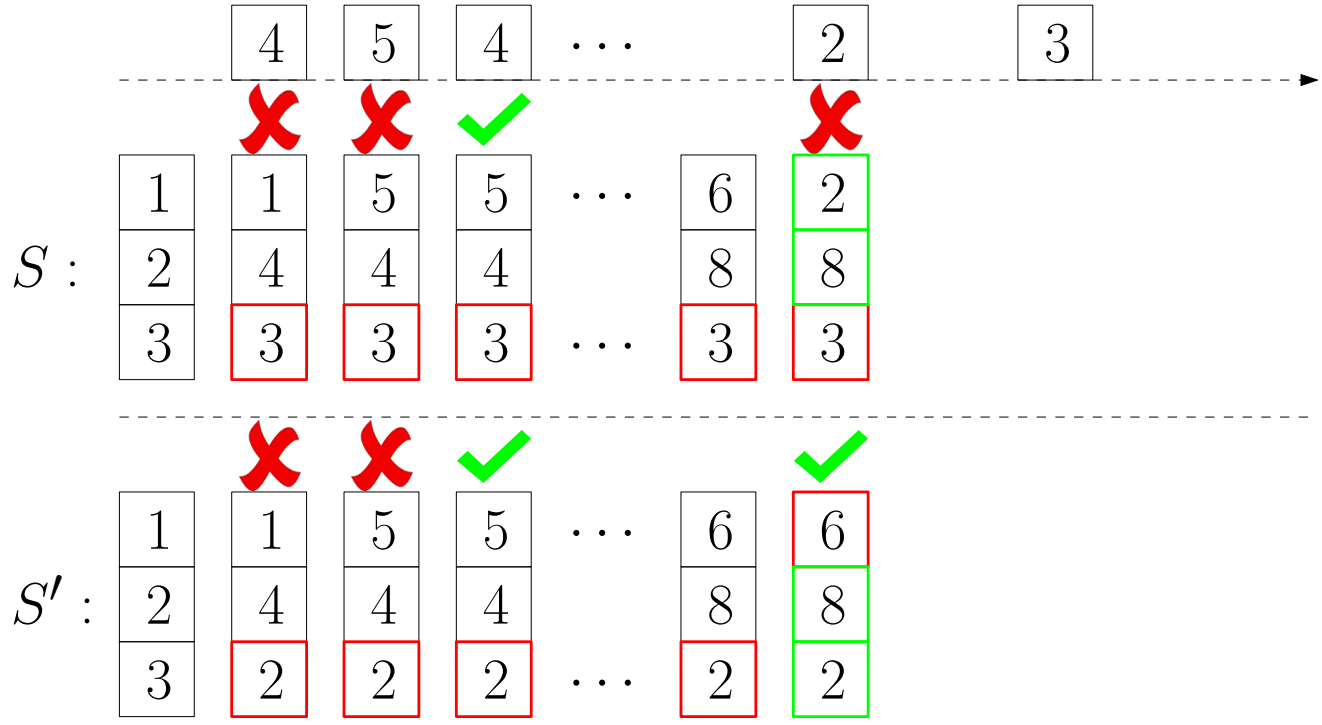
Proof.

7. If S evicted the page p^*, S' will evict the page p'. Then, the cache status of S and that of S' will be the same. S and S' will be exactly the same from now on.
8. Assume S did not evict $p^*(=3)$ before we see $p'(=2)$.



Proof.

9. If S evicts $p^*(=3)$ for $p'(=2)$, then S won't be optimum. Assume otherwise.
10. So far, S' has 1 less page-miss than S does.
11. The status of S' and that of S only differ by 1 page.



Proof.

12. We can then guarantee that S' make at most the same number of page-misses as S does.
    - Idea: if S has a page-hit and S' has a page-miss, we use the opportunity to make the status of S' the same as that of S. □

- Thus, we have shown how to create another solution S' with the same number of page-misses as that of the optimum solution S. Thus, we proved

Lemma Assume at time 1 a page fault happens and there are no empty pages in the cache. Let p^* be the page in cache that is not requested until furthest in the future. It is safe to evict p^* at time 1.

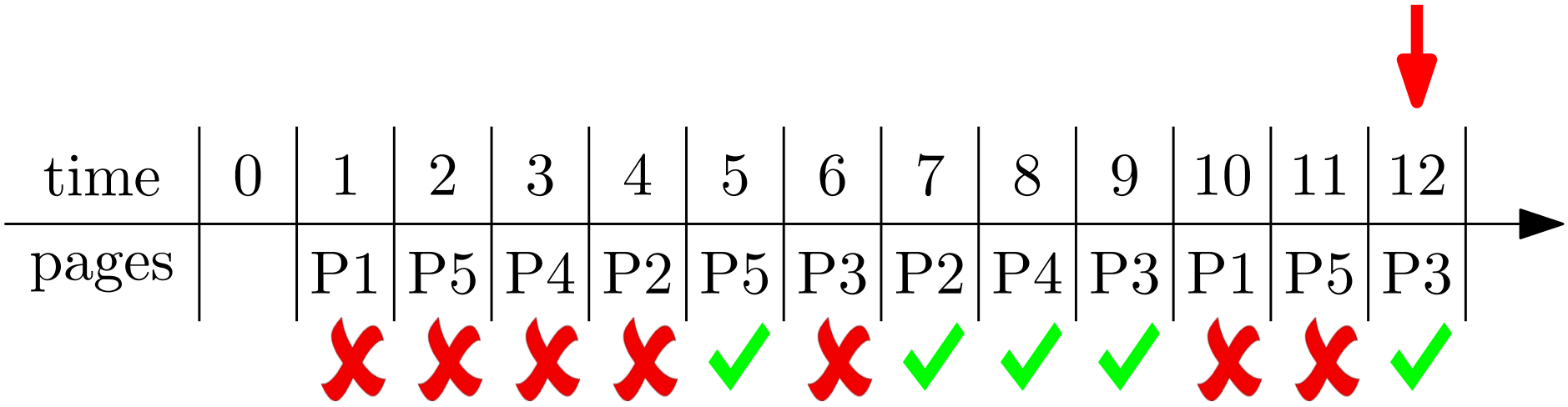
Theorem The furthest-in-future strategy is optimum.

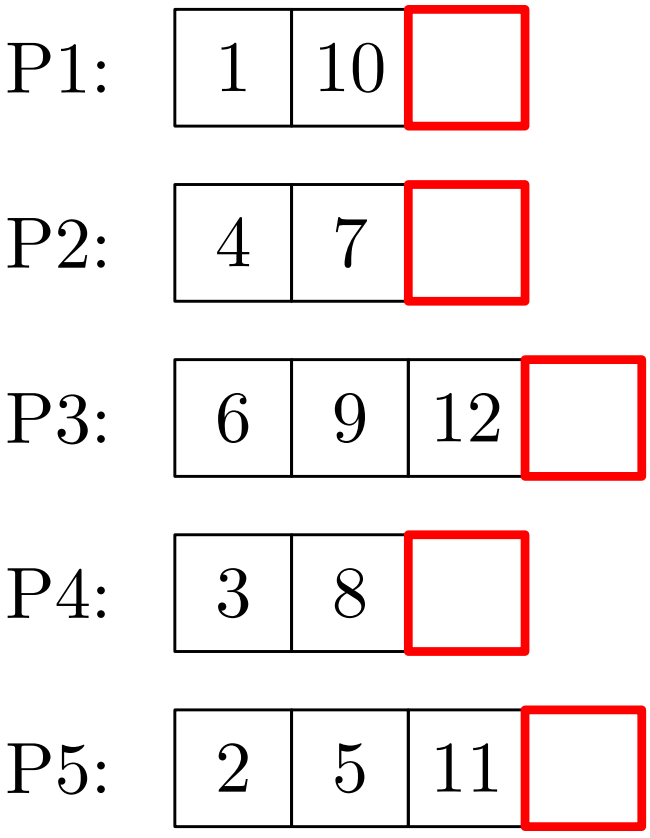
1: **for** $t \leftarrow 1$ to T **do**
2: **if** ρ_t is in cache **then** do nothing
3: **else if** there is an empty page in cache **then**
4: evict the empty page and load ρ_t in cache
5: **else**
6: $p^* \leftarrow$ page in cache that is not used furthest in the future
7: evict p^* and load ρ_t in cache

Q: How can we make the algorithm as fast as possible?

A:

- The running time can be made to be $O(n + T \log k)$.
- For each page p, use a linked list (or an array with dynamic size) to store the time steps in which p is requested.
  - We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.





priority queue

pages	priority values
P5	∞
P3	∞
P4	∞

1: **for** every $p \leftarrow 1$ to n **do**
2: $times[p] \leftarrow$ array of times in which p is requested, in increasing order ▷ put ∞ at the end of array
3: $pointer[p] \leftarrow 1$
4: $Q \leftarrow$ empty priority queue
5: **for** every $t \leftarrow 1$ to T **do**
6: $pointer[\rho_t] \leftarrow pointer[\rho_t] + 1$
7: **if** $\rho_t \in Q$ **then**
8: Q.increase-key$(\rho_t, times[\rho_t, pointer[\rho_t]])$, **print** "hit", **continue**
9: **if** $Q.size() < k$ **then**
10: **print** "load ρ_t to an empty page "
11: **else**
12: $p \leftarrow Q$.extract-max(), **print** "evict p and load ρ_t"
13: Q.insert$(\rho_t, times[\rho_t, pointer[\rho_t]])$ ▷ add ρ_t to Q with key value $times[\rho_t, pointer[\rho_t]]$

Outline

- Let V be a ground set of size n.

Def. A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- insert(v, key_value): insert an element $v \in V \setminus U$, with associated key value key_value.
- decrease_key(v, new_key_value): decrease the key value of an element $v \in U$ to new_key_value
- extract_min$()$: return and remove the element in U with the smallest key value
- $\ldots$

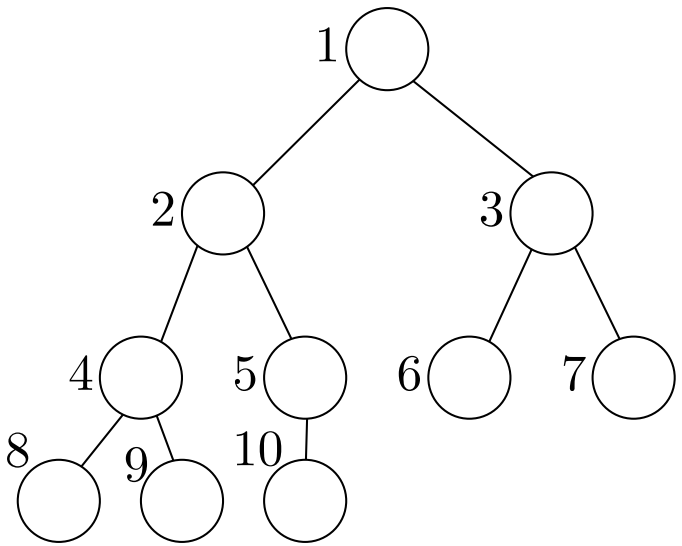
Simple Implementations for Priority Queue

- $n =$ size of ground set V

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Heap

The elements in a heap is organized using a complete binary tree:

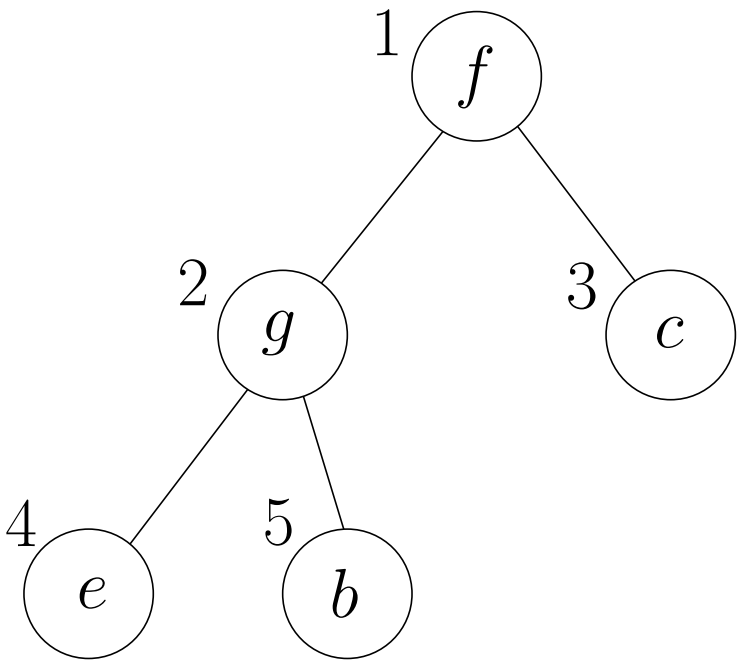


- Nodes are indexed as $\{1, 2, 3, \cdots, s\}$
- Parent of node i: $\lfloor i/2 \rfloor$
- Left child of node i: $2i$
- Right child of node i: $2i + 1$

Heap

A heap H contains the following fields

- s: size of U (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node i of the tree
- $p[v], v \in U$: the index of node containing v
- $key[v], v \in U$: the key value of element v

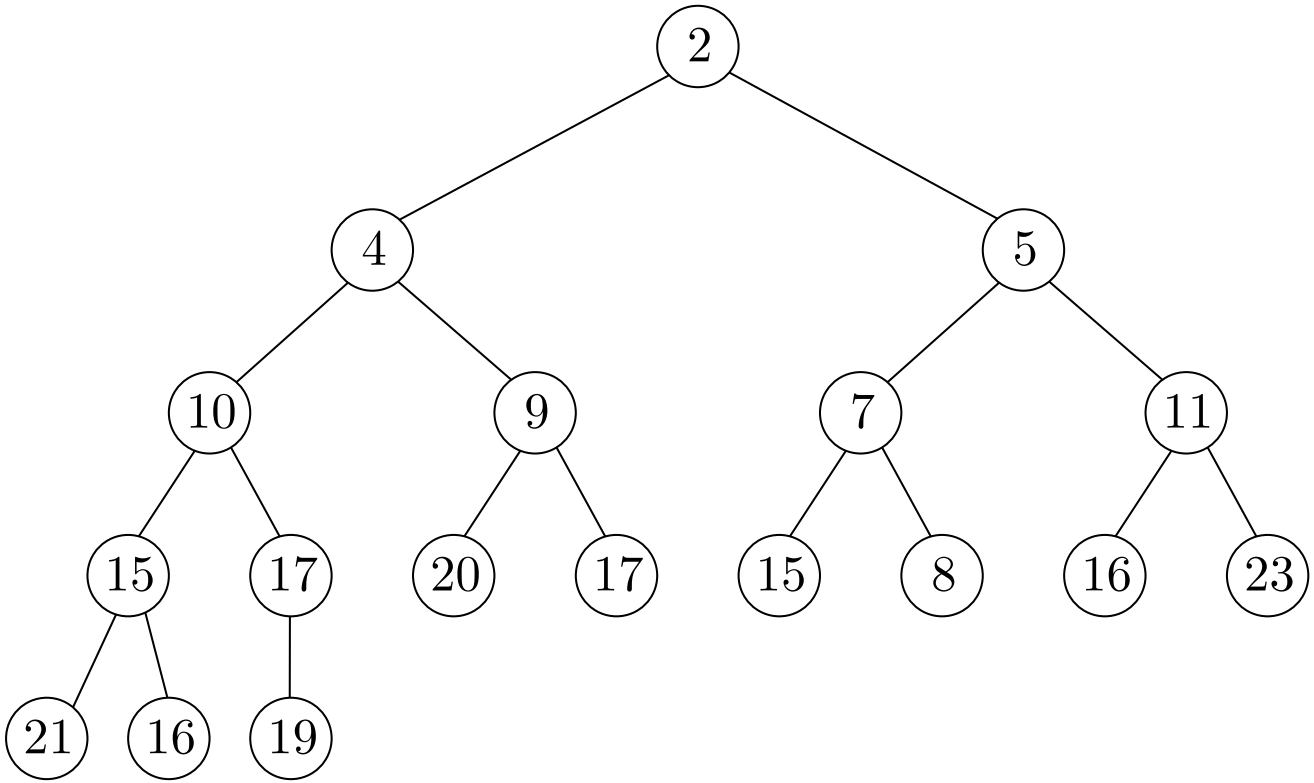


- $s = 5$
- $A = (\text{`}f\text{'}, \text{`}g\text{'}, \text{`}c\text{'}, \text{`}e\text{'}, \text{`}b\text{'})$
- $p[\text{`}f\text{'}] = 1, p[\text{`}g\text{'}] = 2, p[\text{`}c\text{'}] = 3,$
  $p[\text{`}e\text{'}] = 4, p[\text{`}b\text{'}] = 5$

Heap

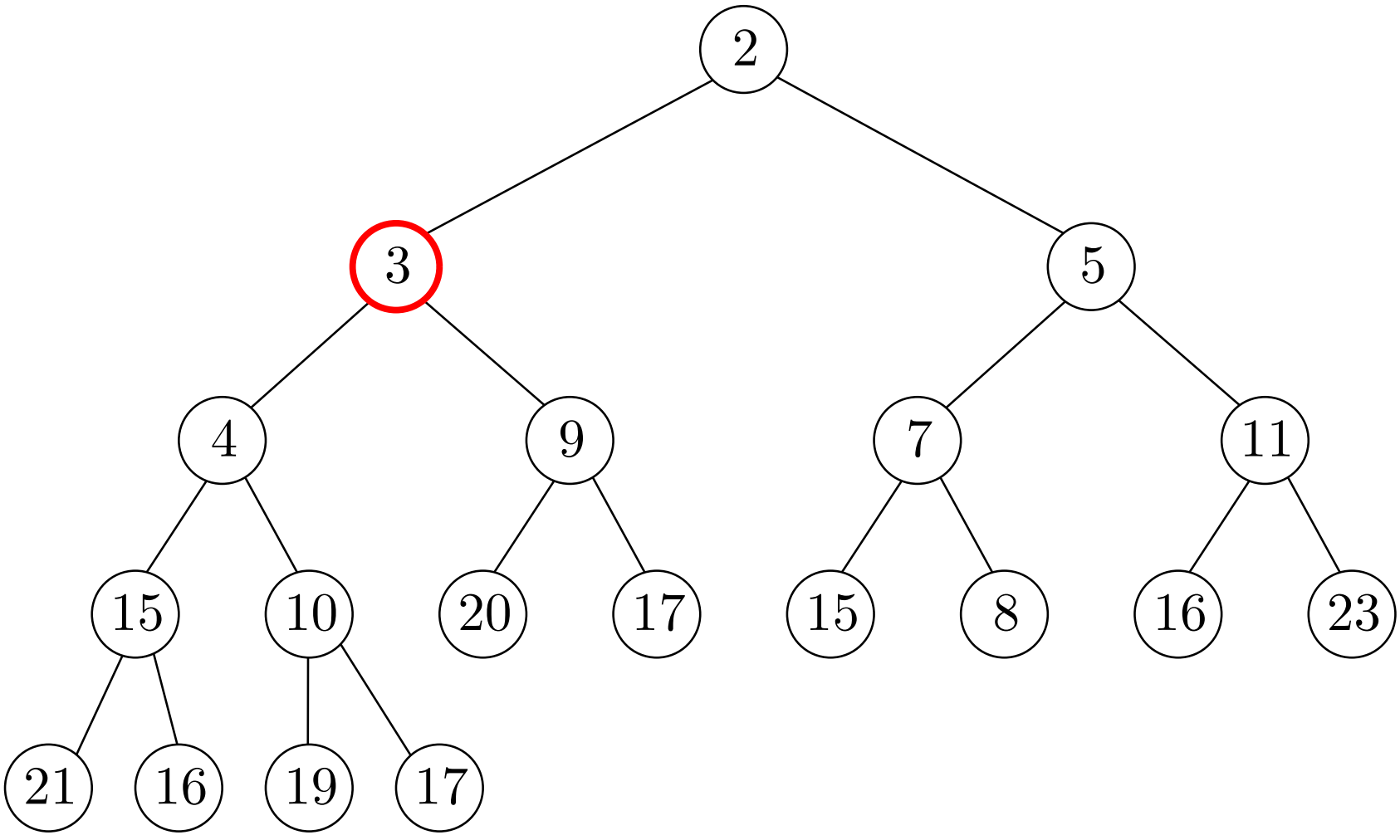
The following heap property is satisfied:

- for any two nodes i, j such that i is the parent of j, we have $key[A[i]] \leq key[A[j]]$.



A heap. Numbers in the circles denote key values of elements.

insert(v, key_value)



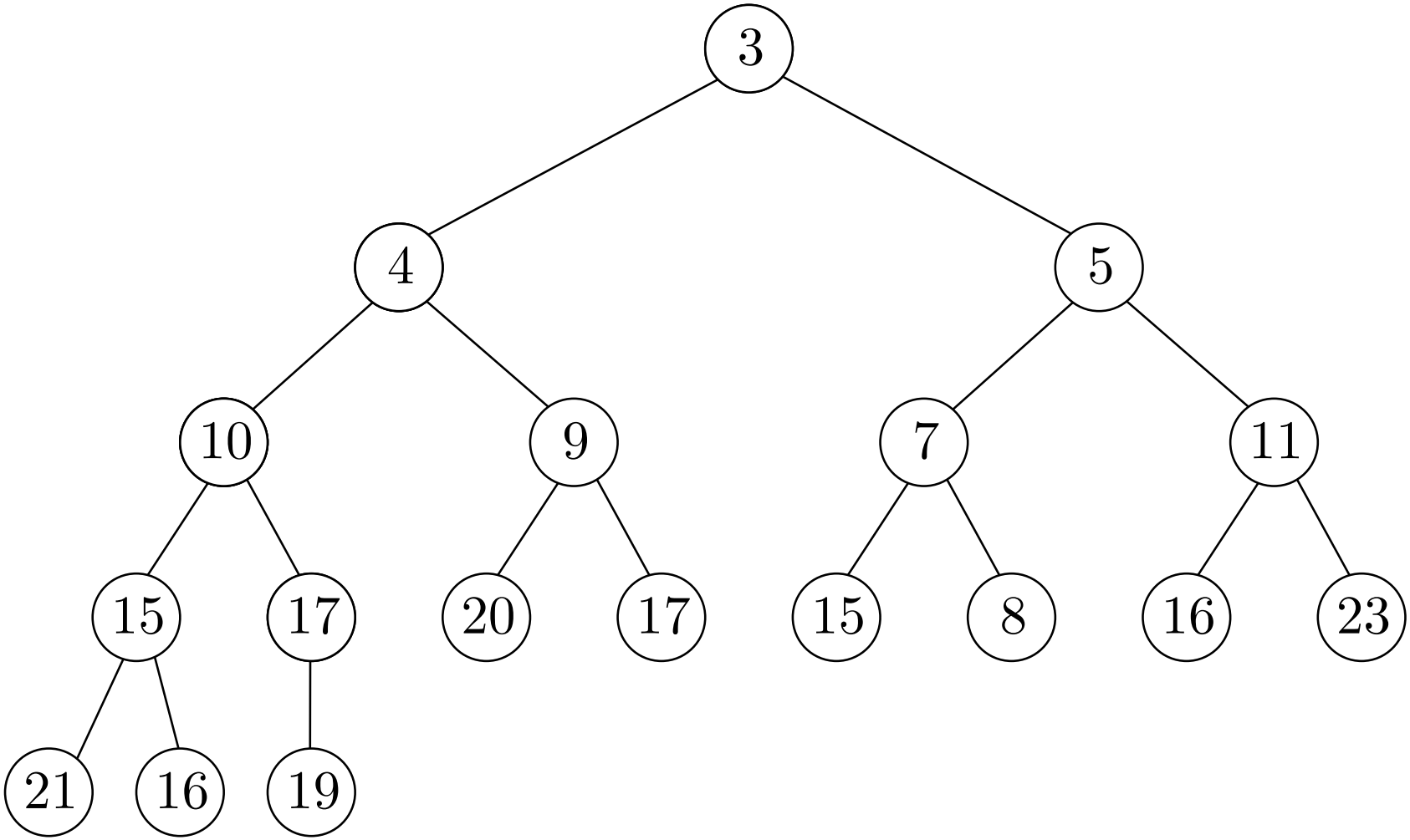
insert(v, key_value)

1: $s \leftarrow s + 1$
2: $A[s] \leftarrow v$
3: $p[v] \leftarrow s$
4: $key[v] \leftarrow key\_value$
5: heapify_up(s)

heapify-up(i)

1: **while** $i > 1$ **do**
2: $\quad j \leftarrow \lfloor i/2 \rfloor$
3: $\quad$ **if** $key[A[i]] < key[A[j]]$ **then**
4: $\quad\quad$ swap $A[i]$ and $A[j]$
5: $\quad\quad p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
6: $\quad\quad i \leftarrow j$
7: $\quad$ **else** break

extract_min()



extract_min()

1: $ret \leftarrow A[1]$
2: $A[1] \leftarrow A[s]$
3: $p[A[1]] \leftarrow 1$
4: $s \leftarrow s - 1$
5: **if** $s \geq 1$ **then**
6: heapify_down(1)
7: **return** ret

decrease_key$(v, key_val$

1: $key[v] \leftarrow key\_value$
2: heapify-up$(p[v])$

heapify-down(i)

1: **while** $2i \leq s$ **do**
2: **if** $2i = s$ or $key[A[2i]] \leq key[A[2i+1]]$ **then**
3: $j \leftarrow 2i$
4: **else**
5: $j \leftarrow 2i + 1$
6: **if** $key[A[j]] < key[A[i]]$ **then**
7: swap $A[i]$ and $A[j]$
8: $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
9: $i \leftarrow j$
10: **else** break

- Running time of heapify_up and heapify_down: $O(\lg n)$
- Running time of insert, exact_min and decrease_key: $O(\lg n)$

data structures	**insert**	**extract_min**	**decrease_key**
array	$O(1)$	$O(n)$	$O(1)$
sorted array	$O(n)$	$O(1)$	$O(n)$
heap	$O(\lg n)$	$O(\lg n)$	$O(\lg n)$

Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that H is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make H a heap.

Def. We say that H is almost a heap except that $key[A[i]]$ is too big if we can decrease $key[A[i]]$ to make H a heap.

Outline

Encoding Letters Using Bits

- 8 letters a, b, c, d, e, f, g, h in a language
- need to encode a message using bits
- idea: use 3 bits per letter

a	b	c	d	e	f	g	h
000	001	010	011	100	101	110	111

$$deacfg \to 011100000010101110$$

Q: Can we have a better encoding scheme?

- Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?

Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.

Q: What is the issue with the following encoding scheme?

- a: 0 b: 1 c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

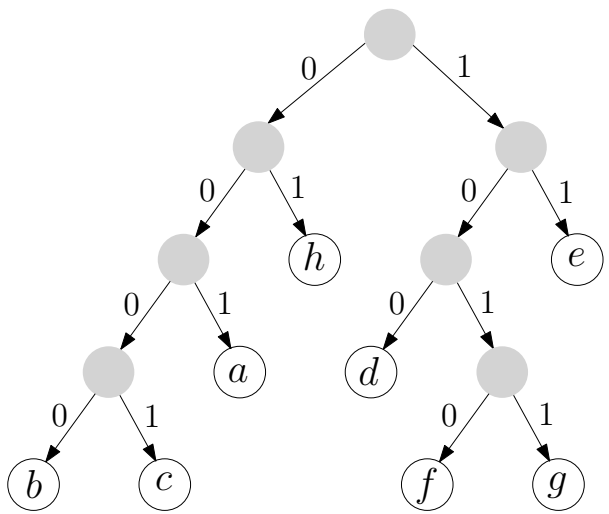
Solution

Use prefix codes to guarantee a unique decoding.

Prefix Codes

Def. A prefix code for a set S of letters is a function $\gamma : S \to \{0,1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.

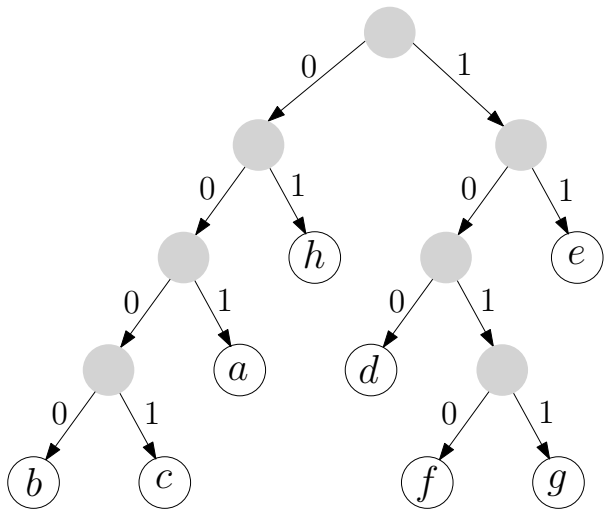
a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



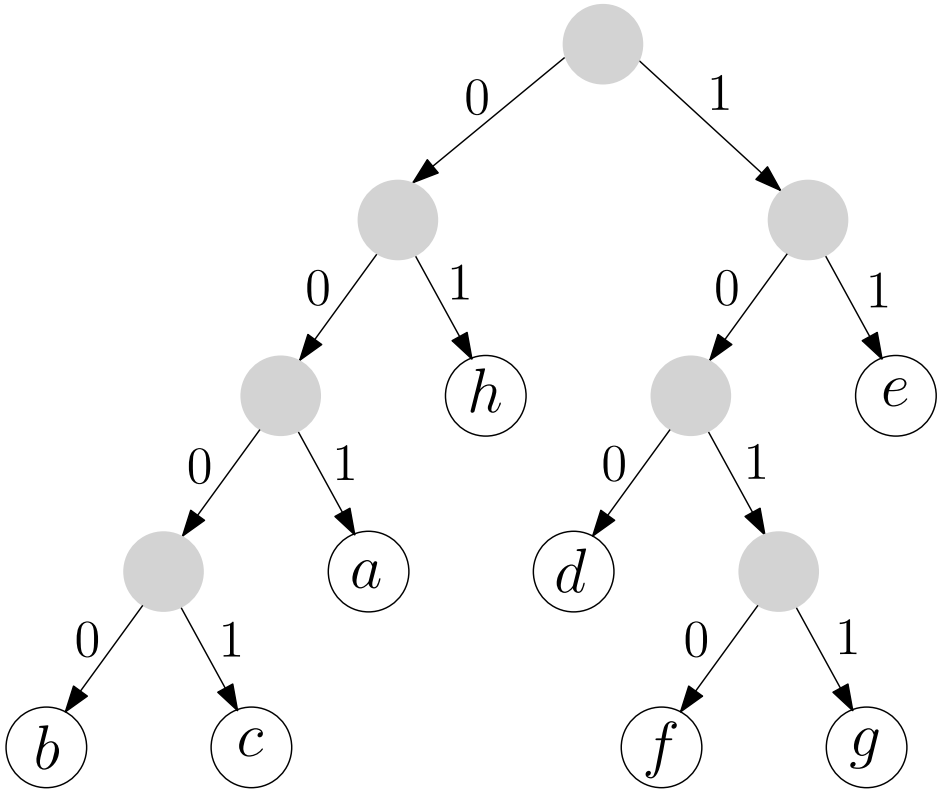
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

a	b	c	d
001	0000	0001	100
e	f	g	h
11	1010	1011	01



- 0001/001/100/0000/01/01/11/1010/0001/001/
- cadbhhefca



Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some letter
- If coding scheme is not wasteful: a non-leaf has exactly two children

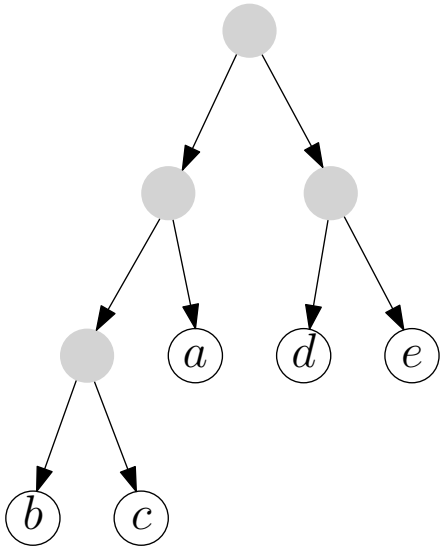
Best Prefix Codes

Input: frequencies of letters in a message

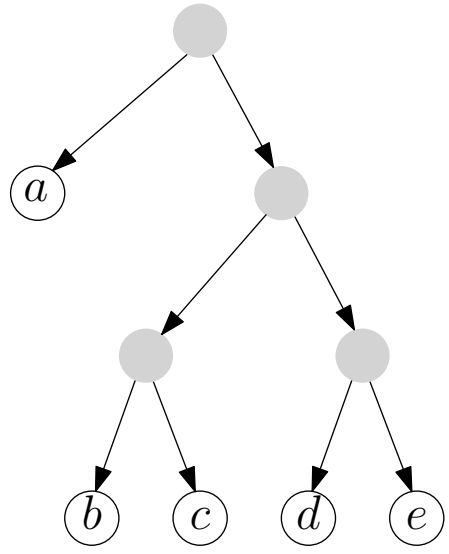
Output: prefix coding scheme with the shortest encoding for the message

example

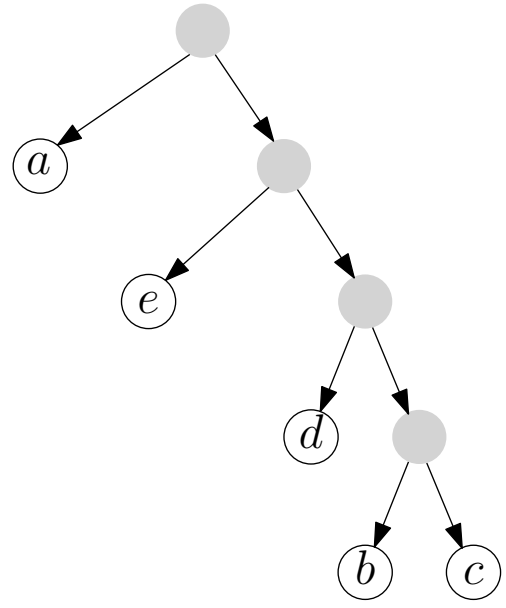
letters	a	b	c	d	e	
frequencies	18	3	4	6	10	
scheme 1 length	2	3	3	2	2	total = 89
scheme 2 length	1	3	3	3	3	total = 87
scheme 3 length	1	4	4	3	2	total = 84



scheme 1



scheme 2



scheme 3

- Example Input: (a: 18, b: 3, c: 4, d: 6, e: 10)

Q: What types of decisions should we make?

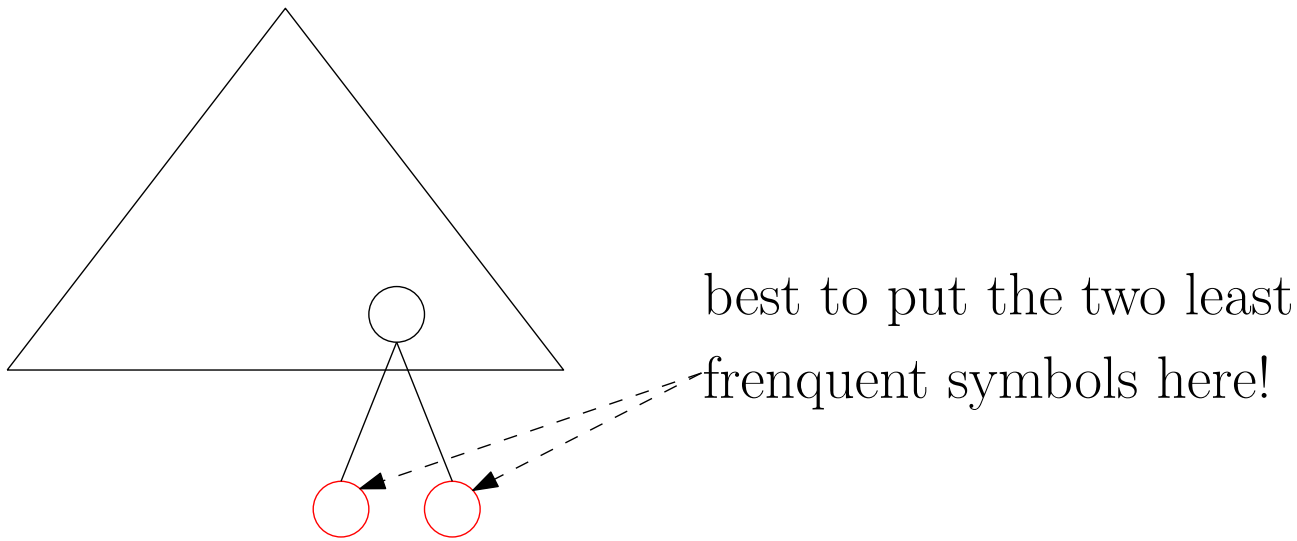
- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.

- Can we partition the letters into left and right sub-trees?
- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.

Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the "structure" of the optimum encoding tree
- There are two deepest leaves that are brothers



Lemma It is safe to make the two least frequent letters brothers.

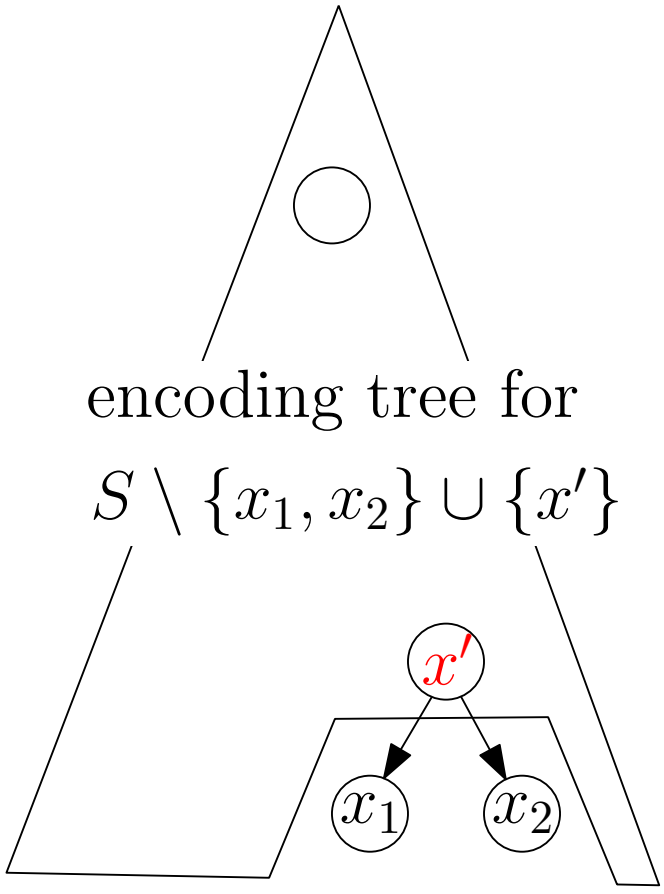
Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

- So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.

- f_x: the frequency of the letter x in the support.
- x_1 and x_2: the two letters we decided to put together.
- d_x the depth of letter x in our output encoding tree.



$$\sum_{x \in S} f_x d_x$$

$$= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2}$$

$$= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}$$

$$= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x'}(d_{x'} + 1)$$

$$= \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x + f_{x'}$$

Def: $f_{x'} = f_{x_1} + f_{x_2}$

In order to minimize

$$\sum_{x \in S} f_x d_x,$$

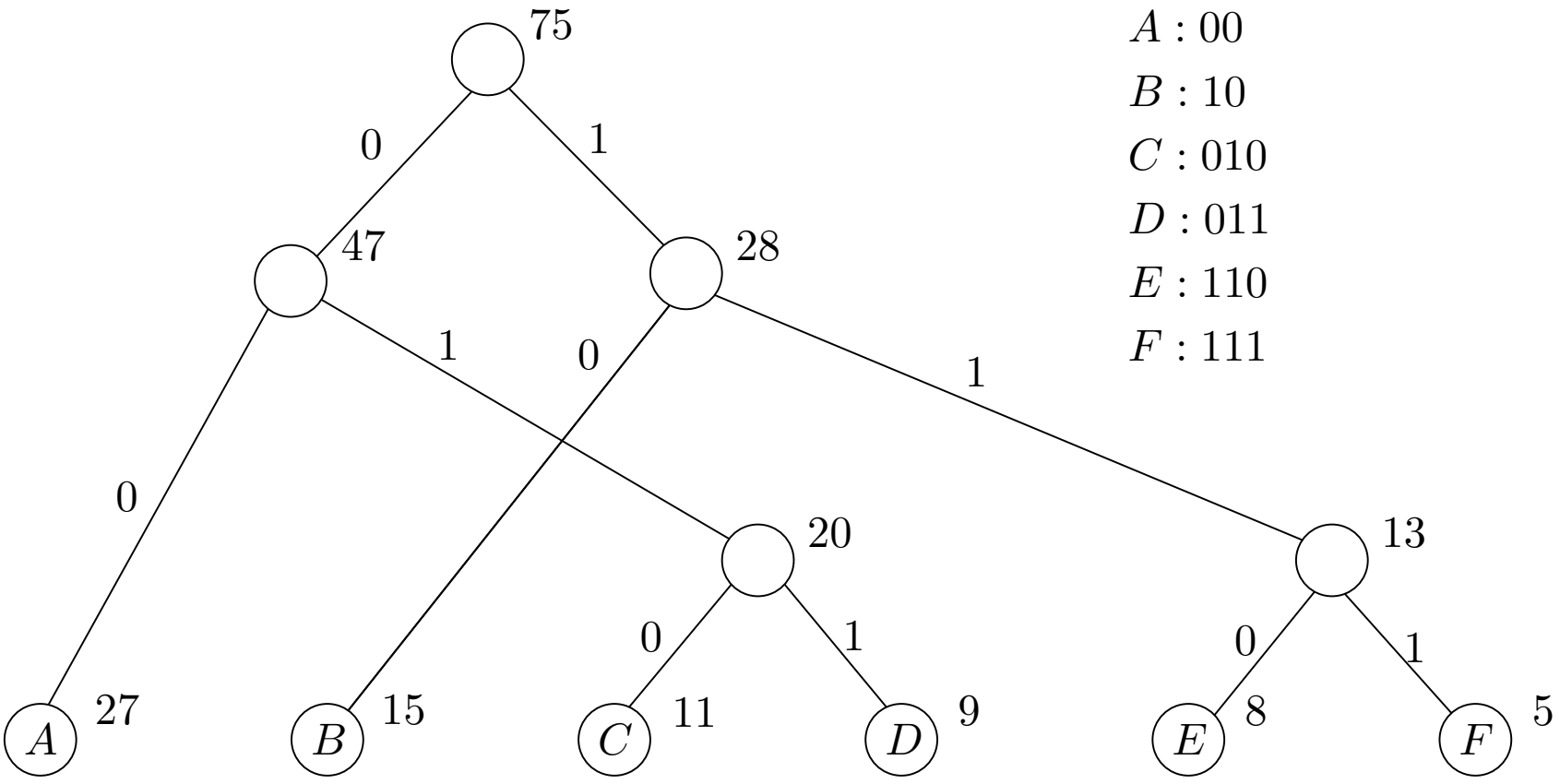
we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that d is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

- This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector f!

Example


75
0
1
47
28
1
0
1
0
20
13
0
1
0
1
27
15
11
9
8
5
A
B
C
D
E
F
A : 00
B : 10
C : 010
D : 011
E : 110
F : 111

Def. The codes given the greedy algorithm is called the Huffman codes.

Huffman(S, f)

1: **while** $|S| > 1$ **do**
2: let x_1, x_2 be the two letters with the smallest f values
3: introduce a new letter x' and let $f_{x'} = f_{x_1} + f_{x_2}$
4: let x_1 and x_2 be the two children of x'
5: $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
6: **return** the tree constructed

Algorithm using Priority Queue

Huffman(S, f)

1: $Q \leftarrow$ build-priority-queue(S)
2: **while** Q.size > 1 **do**
3: $\quad x_1 \leftarrow Q$.extract-min()
4: $\quad x_2 \leftarrow Q$.extract-min()
5: $\quad$ introduce a new letter x' and let $f_{x'} = f_{x_1} + f_{x_2}$
6: $\quad$ let x_1 and x_2 be the two children of x'
7: $\quad Q$.insert($x', f_{x'}$)
8: **return** the tree constructed

Outline

Summary for Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a "reasonable" strategy

- Interval scheduling problem: schedule the job j^* with the earliest deadline
- Offline Caching: evict the page that is used furthest in the future
- Huffman codes: make the two least frequent letters brothers

Summary for Greedy Algorithms

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Def. A strategy is "safe" if there is always an optimum solution that "agrees with" the decision made according to the strategy.

Proving a Strategy is Safe

- Take an arbitrary optimum solution S
- If S agrees with the decision made according to the strategy, done
- So assume S does not agree with decision
- Change S slightly to another optimum solution S' that agrees with the decision
  - Interval scheduling problem: exchange j^* with the first job in an optimal solution
  - Offline caching: a complicated "copying" algorithm
  - Huffman codes: move the two least frequent letters to the deepest leaves.

Summary for Greedy Algorithms

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove j^* and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one