CSE 431/531: Algorithm Analysis and Design (Spring 2022)

Greedy Algorithms

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**Trivial Algorithm for an Optimization Problem**
Enumerate all valid solutions, compare them and output the best one.

Convention: polynomial time = efficient
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- However, trivial algorithm often runs in exponential time, as the number of potential solutions is often exponentially large.
Def. In an **optimization problem**, our goal is to find a valid solution with the minimum cost (or maximum value).

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- \( f(n) \) is a polynomial if \( f(n) = O(n^k) \) for some constant \( k > 0 \).
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- convention: polynomial time = **efficient**

Goals of algorithm design
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**Goals of algorithm design**
- Design efficient algorithms to solve problems
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Goals of algorithm design
1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems
Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
Greedy Algorithms
Divide and Conquer
Dynamic Programming

Greedy algorithms are often for optimization problems.
Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming

- Greedy algorithms are often for optimization problems.
- They often run in polynomial time due to their simplicity.
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy
**Greedy Algorithm**
- Build up the solutions in steps
- At each step, make an *irrevocable* decision using a "reasonable" strategy

**Analysis of Greedy Algorithm**
- Prove that the reasonable strategy is "safe"
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
Greedy Algorithm

- Build up the solutions in steps
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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” *(key)*
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem *(usually easy)*
Greedy Algorithm

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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Def. A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
Box Packing

**Input:** $n$ boxes of capacities $c_1, c_2, \cdots, c_n$
$m$ items of sizes $s_1, s_2, \cdots, s_m$
Can put at most 1 item in a box
Item $j$ can be put into box $i$ if $s_j \leq c_i$

**Output:** A way to put as many items as possible in the boxes.
**Box Packing**

**Input:** \( n \) boxes of capacities \( c_1, c_2, \ldots, c_n \)

\( m \) items of sizes \( s_1, s_2, \ldots, s_m \)

Can put at most 1 item in a box

Item \( j \) can be put into box \( i \) if \( s_j \leq c_i \)

**Output:** A way to put as many items as possible in the boxes.

**Example:**

- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: 45 \( \rightarrow \) 60, 20 \( \rightarrow \) 40, 19 \( \rightarrow \) 25
Greedy Algorithm

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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?


Greedy Algorithm

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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.
Analysis of Greedy Algorithm

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Analysis of Greedy Algorithm

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- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem

**Lemma** The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.
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**Lemma** The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
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Lemma  The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
- formal proof via exchanging argument:
Lemma  There is an optimum solution in which box 1 contains the largest item it can hold.

Proof. Let $j =$ largest item that box 1 can hold. Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done. Otherwise, assume this is what happens in $S$:

```
  S:
  box 1
  item j
  .....
```

$s_j' \leq s_j$, and swapping gives another solution $S'$ is also an optimum solution. In $S'$, $j$ is put into Box 1.
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**

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- Let $j$ = largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S$:

  $S$:  
  \[
  \begin{array}{cccc}
  & \text{box 1} & \text{item } j & \cdots \n  \end{array}
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  $s_j' \leq s_j$, and swapping gives another solution $S'$. In $S'$, $j$ is put into Box 1.
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**

- Let $j = \text{largest item that box 1 can hold}$.  
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.  
- Otherwise, assume this is what happens in $S'$:
  - $S'$:
    - box 1
    - item $j'$
    - item $j$
    - $s_{j'} \leq s_j$, and swapping gives another solution $S'$
Lemma There is an optimum solution in which box 1 contains the largest item it can hold.

Proof.

- Let \( j \) = largest item that box 1 can hold.
- Take any optimum solution \( S \). If \( j \) is put into Box 1 in \( S \), done.
- Otherwise, assume this is what happens in \( S \):

  \[
  S': \quad \begin{array}{cccc}
  \text{box 1} & \text{item } j' & \text{item } j & \hdots & \text{item } j' \\
  \wnode{C1}{C1} & \wnode{C2}{C2} & \wnode{C3}{C3} & \hdots & \wnode{C4}{C4}
  \end{array}
  \]

- \( s'_{j'} \leq s_j \), and swapping gives another solution \( S' \)
- \( S' \) is also an optimum solution. In \( S' \), \( j \) is put into Box 1.
Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.
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**Analysis of Greedy Algorithm**

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- Trivial: we decided to put Item $j$ into Box 1, and the remaining instance is obtained by removing Item $j$ and Box 1.
Generic Greedy Algorithm

1. while the instance is non-trivial do
2. make the choice using the greedy strategy
3. reduce the instance

Greedy Algorithm for Box Packing

1. $T \leftarrow \{1, 2, 3, \cdots, m\}$
2. for $i \leftarrow 1$ to $n$ do
3. if some item in $T$ can be put into box $i$ then
4. $j \leftarrow$ the largest item in $T$ that can be put into box $i$
5. print(“put item $j$ in box $i$”)
6. $T \leftarrow T \setminus \{j\}$
Generic Greedy Algorithm

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Lemma Generic algorithm is correct if and only if the greedy strategy is safe.
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Lemma  Generic algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
Generic Greedy Algorithm

1: while the instance is non-trivial do
2: make the choice using the greedy strategy
3: reduce the instance

Lemma  Generic algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
- Greedy strategy is not safe: we will miss the optimum solution for some instance, since the choices we made are irrevocable.
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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
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Def. A strategy is “safe” if there is always an optimum solution that is “consistent” with the decision made according to the strategy.
Exchange argument: Proof of Safety of a Strategy

- let $S$ be an arbitrary optimum solution.
- if $S$ is consistent with the greedy choice, done.
- otherwise, show that it can be modified to another optimum solution $S'$ that is consistent with the choice.
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otherwise, show that it can be modified to another optimum solution $S'$ that is consistent with the choice.

The procedure is not a part of the algorithm.
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**Interval Scheduling**

**Input:** 
$n$ jobs, job $i$ with start time $s_i$ and finish time $f_i$

$i$ and $j$ are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

**Output:** A maximum-size subset of mutually compatible jobs
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Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
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- Which of the following strategies are safe?
- Schedule the job with the smallest size?
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Schedule the job with the smallest size? No!
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- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!
- Schedule the job conflicting with smallest number of other jobs? No!
- Schedule the job with the earliest finish time?
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- Which of the following strategies are safe?
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Lemma. It is safe to schedule the job \( j \) with the earliest finish time: There is an optimum solution where the job \( j \) with the earliest finish time is scheduled.

Proof.
**Lemma** It is safe to schedule the job $j$ with the earliest finish time: There is an optimum solution where the job $j$ with the earliest finish time is scheduled.

**Proof.**
- Take an arbitrary optimum solution $S'$. 
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- Take an arbitrary optimum solution $S$
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- Otherwise, replace the first job in $S$ with $j$ to obtain another optimum schedule $S'$.
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20/80
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- What is the remaining task after we decided to schedule $j$?
- Is it another instance of interval scheduling problem?
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Greedy Algorithm for Interval Scheduling

Schedule\((s, f, n)\)

1. \(A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset\)
2. while \(A \neq \emptyset\) do
3. \(j \leftarrow \arg\min_{j^\prime \in A} f_{j^\prime}\)
4. \(S \leftarrow S \cup \{j\}; A \leftarrow \{j^\prime \in A : s_{j^\prime} \geq f_j\}\)
5. return \(S\)
Greedy Algorithm for Interval Scheduling

Schedule$(s, f, n)$

1: $A \leftarrow \{1, 2, \ldots, n\}, S \leftarrow \emptyset$
2: while $A \neq \emptyset$ do
3: \hspace{1em} $j \leftarrow \text{arg min}_{j' \in A} f_{j'}$
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Running time of algorithm?
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Running time of algorithm?

- Naive implementation: \( O(n^2) \) time
Greedy Algorithm for Interval Scheduling

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Running time of algorithm?

- Naive implementation: $O(n^2)$ time
- Clever implementation: $O(n \lg n)$ time
Clever Implementation of Greedy Algorithm

Schedule\((s, f, n)\)

1: sort jobs according to \(f\) values
2: \(t \leftarrow 0, S \leftarrow \emptyset\)
3: for every \(j \in [n]\) according to non-decreasing order of \(f_j\) do
4: \hspace{1em} if \(s_j \geq t\) then
5: \hspace{2em} \(S \leftarrow S \cup \{j\}\)
6: \hspace{2em} \(t \leftarrow f_j\)
7: return \(S\)
Clever Implementation of Greedy Algorithm

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4: \hspace{1em} if \(s_j \geq t\) then
5: \hspace{2em} \(S \leftarrow S \cup \{j\}\)
6: \hspace{2em} \(t \leftarrow f_j\)
7: return \(S\)
Clever Implementation of Greedy Algorithm

\textbf{Schedule}(s, f, n)

1: sort jobs according to $f$ values
2: $t \leftarrow 0$, $S \leftarrow \emptyset$
3: \textbf{for} every $j \in [n]$ according to non-decreasing order of $f_j$ \textbf{do}
4: \hspace{1em} \textbf{if} $s_j \geq t$ \textbf{then}
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**Schedule**(*s, f, n*)

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Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
Offline Caching

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<table>
<thead>
<tr>
<th>page sequence</th>
<th>cache</th>
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<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
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</tr>
</tbody>
</table>
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<tr>
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<table>
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<tr>
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</tr>
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</tr>
<tr>
<td>3</td>
<td>X</td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
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</table>
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```
page sequence    cache
1               □  □  □
5               □  □  □
4               □  □  □
2               □  □  □
5               □  □  □
4               □  □  □
2               □  □  □
3               □  □  □
5               □  □  □
4               □  □  □
2               □  □  □
3               □  □  □
2               □  □  □
1               □  □  □
misses = 7
```
Offline Caching

- Cache that can store $k$ pages
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- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.
A Better Solution for Example

<table>
<thead>
<tr>
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<tr>
<td>1</td>
<td>✗ 1</td>
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<td>✗ 4 2</td>
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<td>✗ 4 2 3</td>
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<tr>
<td>2</td>
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<td>✔ 5 4 2</td>
</tr>
<tr>
<td>1</td>
<td>✗ 1 2 3</td>
<td>✗ 1 3 2</td>
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</table>

misses = 7

misses = 6
Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \ldots, \rho_T \in [n]$: sequence of requests

**Output:**
- $i_1, i_2, i_3, \ldots, i_T \in \{\text{hit}, \text{empty}\} \cup [n]$: indices of pages to evict (“hit” means evicting no page, “empty” means evicting empty page)

We use $[n]$ for \{1, 2, 3, \ldots, n\}.

---

**Question:** Which one is more realistic?

**Answer:** Online caching
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- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.
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**Q:** Why do we study the offline caching problem?
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**Q:** Which one is more realistic?

**A:** Online caching

**Q:** Why do we study the offline caching problem?

**A:** Use the offline solution as a benchmark to measure the “competitive ratio” of online algorithms.
Offline Caching: Potential Greedy Algorithms

- FIFO (First-In-First-Out): always evict the first page in cache
Offline Caching: Potential Greedy Algorithms

- **FIFO (First-In-First-Out):** always evict the first page in cache
- **LRU (Least-Recently-Used):** Evict page whose most recent access was earliest

All the above algorithms are not optimum! Indeed all the algorithms are “online”, i.e., the decisions can be made without knowing future requests. Online algorithms cannot be optimum.
Offline Caching: Potential Greedy Algorithms

- FIFO (First-In-First-Out): always evict the first page in cache
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- LFU (Least-Frequently-Used): Evict page that was least frequently requested

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FIFO is not optimum

requests

1
2
3
4
1
FIFO

1 2 3
FIFO is not optimum
FIFO is not optimum

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</tr>
<tr>
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<td></td>
</tr>
<tr>
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<td>✗</td>
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FIFO is not optimum
FIFO is not optimum

requests

1 1
2 1 1
3 1 1 2
4

FIFO
FIFO is not optimum

requests
1
2
3
4
1
FIFO

1
1
2

1
1
2

1
1
2
FIFO is not optimum

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<td>3</td>
</tr>
<tr>
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FIFO

<p>| | | |</p>
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</table>

1  1  2  3
FIFO is not optimum

1
2
3
4
1
2
3

requests

FIFO

1
2
3
4
1
2
3
FIFO is not optimum

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FIFO is not optimum

requests
1
2
3
4
1

FIFO

1
2
3
FIFO is not optimum

requests

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</table>
FIFO is not optimum

misses = 5
FIFO is not optimum

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<td>✗ 1</td>
</tr>
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<tr>
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<td>✓ 1</td>
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</table>

misses = 5

misses = 4
Furthest-in-Future (FF)

- Algorithm: every time, evict the item that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.
### Furthest-in-Future (FF)

<table>
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<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
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<td>[X] 1</td>
<td>[X] 1</td>
</tr>
<tr>
<td>2</td>
<td>[X] 1 2</td>
<td>[X] 1 2</td>
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<tr>
<td>3</td>
<td>[X] 1 2 3</td>
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<td>[X] 4 2 3</td>
<td>[X] 1 4 3</td>
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<tr>
<td>1</td>
<td>[X] 4 1 3</td>
<td>✔ 1 4 3</td>
</tr>
</tbody>
</table>

**misses = 5**

**misses = 4**

---

[33/80]
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

\[\begin{array}{cccccc}
\times & \times & \times \\
1 & 1 & 1 \\
\times & \times & 5 & 5 \\
\times & \times & \times & 4 \\
\end{array}\]
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3  

x  x  x

1  1  1

5  5

1  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×  ×

1  1  1  2

5  5  5

4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×

1  1  1  1  2

5  5  5

4  4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

-  -  -  -  ✔️

-  -  -  -  

-  -  -  -  

-  -  -  -  

-  -  -  -  

1 1 1 2 2 5 5 5 5 4 4 4
Example

requests

\[ \begin{array}{cccccccc}
1 & 5 & 4 & 2 & 5 & 3 & 2 & 4 \\
\end{array} \]
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

x x x x ✓ x

1 1 1 2 2 2

5 5 5 5 3

4 4 4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X  X  ✔  X

☐  1  1  1  2  2  2

☐  ☐  5  5  5  3

☐  ☐  ☐  4  4  4  4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

❌ ❌ ❌ ❌ ✔️ ❌ ✔️

[ ] 1 1 1 2 2 2 2

[ ] 5 5 5 5 3 3

[ ] 4 4 4 4 4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

✓  ✓  ✓  ✓  ✓  ✔  ✓  ✔  ✔

1  1  1  2  2  2  2  2

5  5  5  5  5  3  3  3

4  4  4  4  4  4  4
Example

requests

\[\begin{array}{cccccccccccc}
1 & 5 & 4 & 2 & 5 & 3 & 2 & 4 & 3 & 1 & 5 & 3 \\
\end{array}\]

\[\begin{array}{cccccccccccc}
\times & \times & \times & \times & \checkmark & \times & \checkmark & \checkmark & \checkmark \\
\end{array}\]

\[\begin{array}{cccccccccccc}
\boxed{} & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 & 2 \\
\boxed{} & \boxed{} & 5 & 5 & 5 & 5 & 3 & 3 & 3 & 3 \\
\boxed{} & \boxed{} & \boxed{} & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}\]
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3

x x x x □  □  □  1 1 1 2 2 2 2 2 2 1

□ □ 5 5 5 5 3 3 3 3 3

□ □ □ 4 4 4 4 4 4 4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

✓  ✓  ✓  ✓  ✓  ✔  ✓  ✔  ✔  ☒  ☒  ☒

☐  1  1  1  2  2  2  2  2  2  1  5

☐  ☐  5  5  5  5  3  3  3  3  3  3

☐  ☐  ☐  4  4  4  4  4  4  4  4  4
Example

<p>| | | | | | | | | | | | | | | | | |</p>
<table>
<thead>
<tr>
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<td>3</td>
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</tr>
</tbody>
</table>

requests

```plaintext
1 1 1 1 2 2 2 2 2 2 2 1 5 5
5 5 5 5 5 3 3 3 3 3 3 3 3 3
4 4 4 4 4 4 4 4 4 4 4 4 4 4
```

![Diagram with checkmarks and crosses]

- Red crosses represent requests that are not matched.
- Green checkmarks represent requests that are matched.

Matched request: 3

34/80
Recall: Designing and Analyzing Greedy Algorithms

**Greedy Algorithm**
- Build up the solutions in steps
- At each step, make an *irrevocable* decision using a “reasonable” strategy

**Analysis of Greedy Algorithm**
- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
Recall: Designing and Analyzing Greedy Algorithms

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Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

**Output:**
- $i_1, i_2, i_3, \cdots, i_T \in \{\text{hit, empty}\} \cup [n]$
  - empty stands for an empty page
  - “hit” means evicting no pages
Offline Caching Problem

Input: \( k \) : the size of cache
\( n \) : number of pages
\( \rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n] \): sequence of requests
\( p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n] \): initial set of pages in cache

Output: \( i_1, i_2, i_3, \cdots, i_t \in \{\text{hit}, \text{empty}\} \cup [n] \)
- empty stands for an empty page
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Analysis of Greedy Algorithm

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Analysis of Greedy Algorithm

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**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.
Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$. 
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume $S$ evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$. 

$$
S : \begin{array}{cc}
1 & 1 \\
2 & 4 \\
3 & 3 \\
\end{array}
$$

$\ldots \begin{array}{ccccccccc}
\text{X} \\
\end{array} \ldots$
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume $S$ evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$. 
Proof.

Create $S'$. $S'$ evicts $p^*$ (i.e., 3) instead of $p'$ (i.e., 2) at time 1.

After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S$ contains $p'$ (i.e., 2) and $S$ contains $p^*$ (i.e., 3).

From now on, $S'$ will "copy" $S$. 
Proof.

Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
Proof.

Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
Proof.

4 Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.

5 After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S$ contains $p'(=2)$ and $S$ contains $p^*(=3)$. 
Proof.

4 Create $S'$. $S'$ evicts $p^* (=3)$ instead of $p'(=2)$ at time 1.

5 After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

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**Table**

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**Table**

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Proof.

Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.

After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

From now on, $S'$ will “copy” $S$. 

Proof.
Proof.

1. Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
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Proof.

4 Create \( S' \). \( S' \) evicts \( p^*(=3) \) instead of \( p'(=2) \) at time 1.

5 After time 1, cache status of \( S \) and that of \( S' \) differ by only 1 page. \( S \) contains \( p'(=2) \) and \( S \) contains \( p^*(=3) \).

6 From now on, \( S' \) will “copy” \( S \).
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Proof.

If \( S \) evicted the page \( p' \), then \( S' \) will evict the page \( p^* \). Then, the cache status of \( S \) and that of \( S' \) will be the same. \( S \) and \( S' \) will be exactly the same from now on.

Assume \( S \) did not evict \( p' (=2) \) before we see \( p' (=2) \).
Proof.

If $S$ evicted the page $p'$, $S'$ will evict the page $p^*$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.
Proof.

7 If $S$ evicted the page $p'$, $S'$ will evict the page $p^*$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

8 Assume $S$ did not evict $p'(=2)$ before we see $p'(=2)$. 

---

Proof.
Proof.

7. If $S$ evicted the page $p'$, $S'$ will evict the page $p^*$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

8. Assume $S$ did not evict $p'(=2)$ before we see $p'(=2)$. 
Proof.

If \( S \) evicts \( p^* (=3) \) for \( p' (=2) \), then \( S \) won't be optimum. Assume otherwise. So far, \( S' \) has 1 less page-miss than \( S \) does. The status of \( S' \) and that of \( S \) only differ by 1 page.
Proof.

If $S$ evicts $p^\ast (=3)$ for $p\,' (=2)$, then $S$ won't be optimum. Assume otherwise. So far, $S\,'$ has 1 less page-miss than $S$ does. The status of $S\,'$ and that of $S$ only differ by 1 page.
Proof.

If \( S \) evicts \( p^* (=3) \) for \( p^' (=2) \), then \( S \) won't be optimum. Assume otherwise.

So far, \( S' \) has 1 less page-miss than \( S \) does.

The status of \( S' \) and that of \( S \) only differ by 1 page.
If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.
Proof.

If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.
### Proof.

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Proof.

If $S$ evicts $p^* (= 3)$ for $p' (= 2)$, then $S$ won’t be optimum. Assume otherwise.
Proof.

9 If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.

10 So far, $S'$ has 1 less page-miss than $S$ does.
**Proof.**

9. If \( S \) evicts \( p^*(=3) \) for \( p'(=2) \), then \( S \) won’t be optimum. Assume otherwise.

10. So far, \( S' \) has 1 less page-miss than \( S \) does.

11. The status of \( S' \) and that of \( S \) only differ by 1 page.
**Proof.**

We can then guarantee that \( S' \) makes at most the same number of page-misses as \( S \) does.

Idea: if \( S \) has a page-hit and \( S' \) has a page-miss, we use the opportunity to make the status of \( S' \) the same as that of \( S \).
**Proof.**

We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.
**Proof.**

We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.

- Idea: if $S$ has a page-hit and $S'$ has a page-miss, we use the opportunity to make the status of $S'$ the same as that of $S$. 

\[ S : \begin{array}{cccccccc}
1 & 1 & 5 & 5 & \cdots & 6 & 2 \\
2 & 4 & 4 & 4 & \cdots & 8 & 8 \\
3 & 3 & 3 & 3 & \cdots & 3 & 3 \\
\end{array} \]

\[ S' : \begin{array}{cccccccc}
1 & 1 & 5 & 5 & \cdots & 6 & 6 \\
2 & 4 & 4 & 4 & \cdots & 8 & 8 \\
3 & 2 & 2 & 2 & \cdots & 2 & 2 \\
\end{array} \]
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.

**Theorem** The furthest-in-future strategy is optimum.
1: **for** $t \leftarrow 1$ to $T$ **do**
2: \hspace{1em} **if** $\rho_t$ is in cache **then** do nothing
3: \hspace{1em} **else if** there is an empty page in cache **then**
4: \hspace{2em} evict the empty page and load $\rho_t$ in cache
5: \hspace{1em} **else**
6: \hspace{2em} $p^*$ $\leftarrow$ page in cache that is not used furthest in the future
7: \hspace{2em} evict $p^*$ and load $\rho_t$ in cache
Q: How can we make the algorithm as fast as possible?

A: The running time can be made to be $O(n + T \log k)$. For each page $p$, use a linked list (or an array with dynamic size) to store the time steps in which $p$ is requested. We can find the next time a page is requested easily. Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
Q: How can we make the algorithm as fast as possible?

A: The running time can be made to be $O(n + T \log k)$. 

For each page $p$, use a linked list (or an array with dynamic size) to store the time steps in which $p$ is requested. We can find the next time a page is requested easily. Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
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Q: How can we make the algorithm as fast as possible?

A:
- The running time can be made to be $O(n + T \log k)$.
- For each page $p$, use a linked list (or an array with dynamic size) to store the time steps in which $p$ is requested.
- We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
A priority queue is used to manage the pages. Each page has a value associated with it, which determines its priority. The priority queue maintains the pages in order of their values, with the page having the lowest value at the front. The table shows the time, pages, and values for each page.

- **P1**: Pages 1 and 10
- **P2**: Pages 4 and 7
- **P3**: Pages 6, 9, and 12
- **P4**: Pages 3 and 8
- **P5**: Pages 2, 5, and 11
<table>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

P1:  

| 1 | 10 |

P2:  

| 4 | 7 |

P3:  

| 6 | 9 | 12 |

P4:  

| 3 | 8 |

P5:  

| 2 | 5 | 11 |

Priority queue:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
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<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Diagram:
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<th>2</th>
<th>3</th>
<th>4</th>
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<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

The priority queue contains the following pages and values:

- **P1:** 1, 10
- **P2:** 4, 7
- **P3:** 6, 9, 12
- **P4:** 3, 8
- **P5:** 2, 5, 11

The priority queue is structured as follows:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<tr>
<td></td>
<td></td>
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<td></td>
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</tbody>
</table>

This structure helps in managing the pages and their associated values efficiently.
<table>
<thead>
<tr>
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<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<td>P3</td>
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</table>

**Priority Queue**

<table>
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</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
</tbody>
</table>

**Pages:**
- **P1:** 1, 10
- **P2:** 4, 7
- **P3:** 6, 9, 12
- **P4:** 3, 8
- **P5:** 2, 5, 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P2</td>
<td></td>
</tr>
<tr>
<td>P3</td>
<td></td>
</tr>
<tr>
<td>P4</td>
<td></td>
</tr>
<tr>
<td>P5</td>
<td></td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

| P1: | 1 | 10 |
| P2: | 4 | 7  |
| P3: | 6 | 9 | 12 |
| P4: | 3 | 8  |
| P5: | 2 | 5 | 11 |

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
</tbody>
</table>

Diagram shows a timeline with pages scheduled for time slots and a priority queue with priorities assigned to each page.
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

Priority queue:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
</tbody>
</table>
The document contains a table showing the time and pages with priority queue values. The priority queue table includes pages and their corresponding priority values.

**Time and Pages Table**

<table>
<thead>
<tr>
<th>Time</th>
<th>P1</th>
<th>P5</th>
<th>P4</th>
<th>P2</th>
<th>P5</th>
<th>P3</th>
<th>P2</th>
<th>P4</th>
<th>P3</th>
<th>P1</th>
<th>P5</th>
<th>P3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>1</td>
<td>1</td>
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<td>2</td>
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<td>11</td>
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<td></td>
<td></td>
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<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

**Priority Queue Table**

<table>
<thead>
<tr>
<th>Pages</th>
<th>Priority Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>
A priority queue is shown with pages and priority values:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>11</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>11</td>
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<tr>
<td>P4</td>
<td>8</td>
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</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

P1:  

P2:  4 7

P3:  6 9 12

P4:  3 8

P5:  2 5 11

priority queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

P1: 1 10
P2: 4 7
P3: 6 9 12
P4: 3 8
P5: 2 5 11

### Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>

### Pages

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11
A priority queue is shown with the following values:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

The pages and priority queue at each time step are shown in the diagram.
The image shows a priority queue and a table managing pages and their priority values over time. Here is the information extracted from the image:

**Priority Queue:**

- **P1:** Pages 1 and 10
- **P2:** Pages 4 and 7
- **P3:** Pages 6, 9, and 12
- **P4:** Pages 3 and 8
- **P5:** Pages 2, 5, and 11

**Pages and Time Table:**

<table>
<thead>
<tr>
<th>time</th>
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<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
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</tr>
</tbody>
</table>

**Priority Values Table:**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

P1: 1 10

P2: 4 7

P3: 6 9 12

P4: 3 8

P5: 2 5 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:**
  - Pages: 1, 10
- **P2:**
  - Pages: 4, 7
- **P3:**
  - Pages: 6, 9, 12
- **P4:**
  - Pages: 3, 8
- **P5:**
  - Pages: 2, 5, 11

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>
A priority queue is used to manage pages in a system. The table shows the priority values of each page, with P1, P3, and P4 having infinite values. The diagram illustrates the pages being accessed over time, with checks indicating which pages are currently being used. The priority queue helps in deciding which page to access next based on its priority value.
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
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<td>time</td>
<td>0</td>
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<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
<tr>
<td>check</td>
<td>×</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>
### Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>∞</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

### Pages and Time Table

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:** 1, 10
- **P2:** 4, 7
- **P3:** 6, 9, 12
- **P4:** 3, 8
- **P5:** 2, 5, 11

 prioritize queue

- **P5:** ∞
- **P3:** ∞
- **P4:** ∞
1: for every $p \leftarrow 1$ to $n$ do
2: \hspace{1em} times[$p$] $\leftarrow$ array of times in which $p$ is requested, in increasing order $\triangleright$ put $\infty$ at the end of array
3: \hspace{1em} pointer[$p$] $\leftarrow$ 1
4: $Q \leftarrow$ empty priority queue
5: for every $t \leftarrow 1$ to $T$ do
6: \hspace{1em} pointer[$\rho_t$] $\leftarrow$ pointer[$\rho_t$] + 1
7: \hspace{1em} nexttime[$\rho_t$] $\leftarrow$ times[$\rho_t$, pointer[$\rho_t$]]
8: \hspace{1em} if $\rho_t \in Q$ then
9: \hspace{2em} $Q$.increase-key($\rho_t$, nexttime[$\rho_t$]), print “hit”, continue
10: \hspace{1em} if $Q$.size() < $k$ then
11: \hspace{2em} print “load $\rho_t$ to an empty page”
12: \hspace{1em} else
13: \hspace{2em} $p \leftarrow Q$.extract-max(), print “evict $p$ and load $\rho_t$”
14: \hspace{2em} $Q$.insert($\rho_t$, nexttime[$\rho_t$]) $\triangleright$ add $\rho_t$ to $Q$ with key value nexttime[$\rho_t$]
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
Let $V$ be a ground set of size $n$.

**Def.** A *priority queue* is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- **insert**($v, key_value$): insert an element $v \in V \setminus U$, with associated key value $key_value$.
- **decrease_key**($v, new_key_value$): decrease the key value of an element $v \in U$ to $new_key_value$
- **extract_min**(): return and remove the element in $U$ with the smallest key value
- ...
Simple Implementations for Priority Queue

- $n = \text{size of ground set } V$

<table>
<thead>
<tr>
<th>data structures</th>
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</tr>
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<td>array</td>
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<tr>
<td>sorted array</td>
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**Simple Implementations for Priority Queue**

- \( n = \text{size of ground set } V \)

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Simple Implementations for Priority Queue

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Simple Implementations for Priority Queue

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<tr>
<td>heap</td>
<td>( O(\lg n) )</td>
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</table>
The elements in a heap is organized using a complete binary tree:

- Nodes are indexed as \( \{1, 2, 3, \cdots, s\} \)
- Parent of node \( i \): \( \lfloor i/2 \rfloor \)
- Left child of node \( i \): \( 2i \)
- Right child of node \( i \): \( 2i + 1 \)
Heap

A heap $H$ contains the following fields:

- $s$: size of $U$ (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node $i$ of the tree
- $p[v], v \in U$: the index of node containing $v$
- $key[v], v \in U$: the key value of element $v$

- $s = 5$
- $A = (’f’, ’g’, ’c’, ’e’, ’b’) $
- $p[’f’] = 1, p[’g’] = 2, p[’c’] = 3, p[’e’] = 4, p[’b’] = 5$
The following **heap property** is satisfied:

- for any two nodes $i, j$ such that $i$ is the parent of $j$, we have $key[A[i]] \leq key[A[j]]$.

A heap. Numbers in the circles denote key values of elements.
\text{insert}(v, \text{key\_value})
\( \text{insert}(v, \text{key_value}) \)
insert\((v, key_value)\)
\textbf{insert}(v, \textit{key\_value})
insert($v, key_value$)
insert($v$, key_value)

1: $s \leftarrow s + 1$
2: $A[s] \leftarrow v$
3: $p[v] \leftarrow s$
4: $key[v] \leftarrow key_value$
5: heapify_up($s$)

heapify-up($i$)

1: while $i > 1$ do
2:  $j \leftarrow \lfloor i/2 \rfloor$
3:  if $key[A[i]] < key[A[j]]$ then
4:     swap $A[i]$ and $A[j]$
5:     $p[A[i]] \leftarrow i$, $p[A[j]] \leftarrow j$
6:     $i \leftarrow j$
7: else break
extract_min()
extract_min()
extract_min()
extract_min()
extract_min()
extract_min()
**extract_min()**

1: $ret \leftarrow A[1]$
3: $p[A[1]] \leftarrow 1$
4: $s \leftarrow s - 1$
5: if $s \geq 1$ then
6: \hspace{1em} heapify_down(1)
7: return $ret$

**heapify-down(i)**

1: while $2i \leq s$ do
2: \hspace{1em} if $2i = s$ or $key[A[2i]] \leq key[A[2i + 1]]$ then
3: \hspace{2em} $j \leftarrow 2i$
4: \hspace{1em} else
5: \hspace{3em} $j \leftarrow 2i + 1$
6: \hspace{1em} if $key[A[j]] < key[A[i]]$ then
7: \hspace{2em} swap $A[i]$ and $A[j]$
8: \hspace{2em} $p[A[i]] \leftarrow i, p[A[j]] \leftarrow j$
9: \hspace{2em} $i \leftarrow j$
10: else break

**decrease_key(v, key_value)**

1: $key[v] \leftarrow key_value$
2: heapify-up($p[v]$)
Running time of heapify\_up and heapify\_down: $O(\lg n)$
- Running time of heapify\_up and heapify\_down: $O(\lg n)$
- Running time of insert, exact\_min and decrease\_key: $O(\lg n)$
- Running time of `heapify_up` and `heapify_down`: $O(\lg n)$
- Running time of `insert`, `exact_min` and `decrease_key`: $O(\lg n)$

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Two Definitions Needed to Prove that the Procedures Maintain Heap Property

**Def.** We say that $H$ is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make $H$ a heap.

**Def.** We say that $H$ is almost a heap except that $key[A[i]]$ is too big if we can decrease $key[A[i]]$ to make $H$ a heap.
Outline

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8 letters $a, b, c, d, e, f, g, h$ in a language

need to encode a message using bits

idea: use 3 bits per letter

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<tr>
<th></th>
<th>a</th>
<th>b</th>
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<tr>
<td></td>
<td>000</td>
<td>001</td>
<td>010</td>
<td>011</td>
<td>100</td>
<td>101</td>
<td>110</td>
<td>111</td>
</tr>
</tbody>
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$\text{deacfg} \rightarrow 011100000010101110$

Q: Can we have a better encoding scheme?

Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?
Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea
- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.
Q: What is the issue with the following encoding scheme?

- $a$: 0
- $b$: 1
- $c$: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to $aa$ or $c$. Solution: Use prefix codes to guarantee a unique decoding.
Q: What is the issue with the following encoding scheme?

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Q: What is the issue with the following encoding scheme?

a: 0  
b: 1  
c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to aa or c.

Solution

Use prefix codes to guarantee a unique decoding.
Def. A prefix code for a set $S$ of letters is a function $\gamma: S \rightarrow \{0, 1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$. 
**Prefix Codes**

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</tr>
<tr>
<td>$f$</td>
<td>11</td>
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![Tree diagram showing the prefix code]
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.
Prefix Codes Guarantee Unique Decoding

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- 00010011000000001011110100001001
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- 0001/001100000000101111101100001001
- c
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0001/001/100000001011110100001001

c, a
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0001/001/100/000001011110100001001

cad
Prefix Codes Guarantee Unique Decoding

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- $0001/001/100/0000/01011110100001001$
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- 0001/001/100/0000/01/01/11/1010/0001001
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- cadbhhefca
Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some letter
- If coding scheme is not wasteful: a non-leaf has exactly two children

Best Prefix Codes
- Input: frequencies of letters in a message
- Output: prefix coding scheme with the shortest encoding for the message
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Best Prefix Codes

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example

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<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequencies</td>
<td>18</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

scheme 1 length
2
3
3
2
2
total = 89

scheme 2 length
1
3
3
3
3
total = 87

scheme 3 length
1
4
4
3
2
total = 84

scheme 1
scheme 2
scheme 3
example

<table>
<thead>
<tr>
<th>letters</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequencies</td>
<td>18</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>scheme 1 length</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>scheme 2 length</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>scheme 3 length</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>

scheme 1

```
a
 o----------------------o
   |                     |
   v                     v
b   c
```

scheme 2

```
a
 o----------------------o
   |                     |
   v                     v
b   c   d   e
```

scheme 3

```
a
 o----------------------o
   |                     |
   v                     v
e   d
   o----------------------o
   |                     |
   v                     v
b   c
```
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)

Q: What types of decisions should we make?
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)

Q: What types of decisions should we make?

Can we directly give a code for some letter?
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)

Q: What types of decisions should we make?

- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.
Example Input: \((a: 18, \ b: 3, \ c: 4, \ d: 6, \ e: 10)\)

**Q:** What types of decisions should we make?

- Can we directly give a code for some letter?
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Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)

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- Can we directly give a code for some letter?
- Hard to design a strategy; residual problem is complicated.
- Can we partition the letters into left and right sub-trees?
- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
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best to put the two least frequent symbols here!
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers

**Lemma**  It is safe to make the two least frequent letters brothers.
Lemma  There is an optimum encoding tree, where the two least frequent letters are brothers.
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So we can irrevocably decide to make the two least frequent letters brothers.
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Q: Is the residual problem another instance of the best prefix codes problem?
Lemma  There is an optimum encoding tree, where the two least frequent letters are brothers.

- So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.
• \( f_x \): the frequency of the letter \( x \) in the support.
• \( x_1 \) and \( x_2 \): the two letters we decided to put together.
• \( d_x \) the depth of letter \( x \) in our output encoding tree.

\[
\sum_{x \in S} f_x d_x = \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2}
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
\]
- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
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\]

**Def:** $f_{x'} = f_{x_1} + f_{x_2}$
- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
- $d_x$ the depth of letter $x$ in our output encoding tree.

\[
\sum_{x \in S} f_x d_x
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_1 d_1 + f_2 d_2
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_1 + f_2) d_1
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f' (d' + 1)
\]

**Def:** $f' = f_1 + f_2$
- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
- $d_x$ the depth of letter $x$ in our output encoding tree.

\[
def x' = f_{x_1} + f_{x_2}
\]
- $f_x$: the frequency of the letter $x$ in the support.
- $x_1$ and $x_2$: the two letters we decided to put together.
- $d_x$ the depth of letter $x$ in our output encoding tree.

\[
\sum_{x \in S} f_x d_x \\
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \\
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x'} (d_{x'} + 1) \\
= \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x + f_{x'}
\]

Def: $f_{x'} = f_{x_1} + f_{x_2}$
In order to minimize \[ \sum_{x \in S} f_x d_x, \]
we need to minimize \[ \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x, \]
subject to that \(d\) is the depth function for an encoding tree of \(S \setminus \{x_1, x_2\}\).

This is exactly the best prefix codes problem, with letters \(S \setminus \{x_1, x_2\} \cup \{x'\}\) and frequency vector \(f\)!
Example

A  27  B  15  C  11  D  9  E  8  F  5
Example

A 27  B 15  C 11  D 9  E 8  F 5
Example

\[ A \quad 27 \quad B \quad 15 \quad C \quad 11 \quad D \quad 9 \quad E \quad 8 \quad F \quad 5 \]
Example
Example
Example

```
A  B  C  D  E  F  5  8
11 15 27 13 20 28 47 75
```
Example
Example

A: 00
B: 10
C: 010
D: 011
E: 110
F: 111
Def. The codes given the greedy algorithm is called the **Huffman** codes.
**Def.** The codes given the greedy algorithm is called the **Huffman codes**.

**Huffman**($S$, $f$)

1: **while** $|S| > 1$ **do**
2: let $x_1$, $x_2$ be the two letters with the smallest $f$ values
3: introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
4: let $x_1$ and $x_2$ be the two children of $x'$
5: $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
6: **return** the tree constructed
Algorithm using Priority Queue

Huffman($S, f$)

1: $Q \leftarrow \text{build-priority-queue}(S)$
2: while $Q$.size $> 1$ do
3:     $x_1 \leftarrow Q$.extract-min()
4:     $x_2 \leftarrow Q$.extract-min()
5:     introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
6:     let $x_1$ and $x_2$ be the two children of $x'$
7:     $Q$.insert($x', f_{x'}$)
8: return the tree constructed
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
Summary for Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy
Greedy Algorithm

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- Interval scheduling problem: schedule the job $j^*$ with the earliest deadline
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### Summary for Greedy Algorithms

#### Greedy Algorithm
- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy

- Interval scheduling problem: schedule the job $j^*$ with the earliest deadline
- Offline Caching: evict the page that is used furthest in the future
- Huffman codes: make the two least frequent letters brothers
## Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
## Summary for Greedy Algorithms

### Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

### Def.
A strategy is “safe” if there is always an optimum solution that “agrees with” the decision made according to the strategy.
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S'$
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
- So assume $S$ does not agree with decision
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
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- Change $S$ slightly to another optimum solution $S'$ that agrees with the decision
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
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- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
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  - Offline caching: a complicated “copying” algorithm
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$.
- If $S$ agrees with the decision made according to the strategy, done.
- So assume $S$ does not agree with the decision.
- Change $S$ slightly to another optimum solution $S'$ that agrees with the decision.
  - Interval scheduling problem: exchange $j^*$ with the first job in an optimal solution.
  - Offline caching: a complicated “copying” algorithm.
  - Huffman codes: move the two least frequent letters to the deepest leaves.
## Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
Summary for Greedy Algorithms

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- Prove that the reasonable strategy is “safe” (key)
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- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
Summary for Greedy Algorithms

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- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
Summary for Greedy Algorithms

Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one