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Goals of algorithm design
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**Goals of algorithm design**
1. Design efficient algorithms to solve problems
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Goals of algorithm design
1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems
Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming
Common Paradigms for Algorithm Design

- Greedy Algorithms
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Greedy algorithms are often for optimization problems.
Greedy Algorithms
Divide and Conquer
Dynamic Programming

Greedy algorithms are often for optimization problems.
They often run in polynomial time due to their simplicity.
**Greedy Algorithm**

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy

**Analysis of Greedy Algorithm**

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

**Def.** A strategy is **safe**: there is always an optimum solution that agrees with the decision made according to the strategy.
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1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
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5. Summary
Box Packing

**Input:** $n$ boxes of capacities $c_1, c_2, \cdots, c_n$
$m$ items of sizes $s_1, s_2, \cdots, s_m$

Can put **at most 1** item in a box

Item $j$ can be put into box $i$ if $s_j \leq c_i$

**Output:** A way to put as many items as possible in the boxes.
Box Packing

**Input:**  
$n$ boxes of capacities $c_1, c_2, \cdots, c_n$
$m$ items of sizes $s_1, s_2, \cdots, s_m$
Can put at most 1 item in a box
Item $j$ can be put into box $i$ if $s_j \leq c_i$

**Output:** A way to put as many items as possible in the boxes.

**Example:**
- Box capacities: 60, 40, 25, 15, 12
- Item sizes: 45, 42, 20, 19, 16
- Can put 3 items in boxes: 45 $\rightarrow$ 60, 20 $\rightarrow$ 40, 19 $\rightarrow$ 25
Greedy Algorithm

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Greedy Algorithm
- Build up the solutions in steps
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Designing a Reasonable Strategy for Box Packing
- Q: Take box 1. Which item should we put in box 1?
Greedy Algorithm

- Build up the solutions in steps
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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.
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**Lemma** The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.
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- Intuition: putting the item gives us the easiest residual problem.
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Lemma  The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
- formal proof via exchanging argument:
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.
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Proof.

- Let $j =$ largest item that box 1 can hold.
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Proof.

- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**
- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S$:

```
S: box 1
   □  □  □  ......  □
   ▼  ▼  ▼  ......  ▼
   item j
```

$s_j' \leq s_j$, and swapping gives another solution $S'$ is also an optimum solution. In $S'$, $j$ is put into Box 1.
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

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  $S'$:

<table>
<thead>
<tr>
<th>box 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>item $j'$</td>
</tr>
<tr>
<td>item $j$</td>
</tr>
<tr>
<td></td>
</tr>
<tr>
<td></td>
</tr>
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  - $s_{j'} \leq s_j$, and swapping gives another solution $S'$
  - $S'$ is also an optimum solution. In $S'$, $j$ is put into Box 1.

\[ \square \]
Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.
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**Analysis of Greedy Algorithm**

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**Analysis of Greedy Algorithm**

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
- Trivial: we decided to put Item $j$ into Box 1, and the remaining instance is obtained by removing Item $j$ and Box 1.
**Generic Greedy Algorithm**

1. while the instance is non-trivial do
2. make the choice using the greedy strategy
3. reduce the instance

**Greedy Algorithm for Box Packing**

1. \( T \leftarrow \{1, 2, 3, \ldots, m\} \)
2. for \( i \leftarrow 1 \) to \( n \) do
3. if some item in \( T \) can be put into box \( i \) then
4. \( j \leftarrow \) the largest item in \( T \) that can be put into box \( i \)
5. print("put item \( j \) in box \( i \)"")
6. \( T \leftarrow T \setminus \{j\} \)
Generic Greedy Algorithm

1: while the instance is non-trivial do
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Lemma  Generic algorithm is correct if and only if the greedy strategy is safe.
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- Greedy strategy is safe: we will not miss the optimum solution
Generic Greedy Algorithm

1: while the instance is non-trivial do
2: make the choice using the greedy strategy
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Lemma  Generic algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
- Greedy strategy is not safe: we will miss the optimum solution for some instance, since the choices we made are irrevocable.
Greedy Algorithm

- Build up the solutions in steps
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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
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**Def.**

A strategy is “safe” if there is always an optimum solution that is “consistent” with the decision made according to the strategy.
Greedy Algorithm

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Def. A strategy is “safe” if there is always an optimum solution that is “consistent” with the decision made according to the strategy.
Exchange argument: Proof of Safety of a Strategy

- let $S$ be an arbitrary optimum solution.
- if $S$ is consistent with the greedy choice, done.
- otherwise, show that it can be modified to another optimum solution $S'$ that is consistent with the choice.
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The procedure is not a part of the algorithm.
Outline

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**Interval Scheduling**

**Input:** $n$ jobs, job $i$ with start time $s_i$ and finish time $f_i$.

$i$ and $j$ are compatible if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint.

**Output:** A maximum-size subset of mutually compatible jobs.
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Greedy Algorithm for Interval Scheduling

Which of the following strategies are safe?

- Schedule the job with the smallest size?
  
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Greedy Algorithm for Interval Scheduling

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![Interval Scheduling Diagram]

- Jobs represented by intervals on a timeline.
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![Diagram of intervals and scheduling]

0 1 2 3 4 5 6 7 8 9
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![Diagram of Interval Scheduling]

The diagram illustrates the intervals of different jobs on a timeline, with varying start and end times. The safe strategy is to schedule the job with the earliest finish time.
Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
- Schedule the job with the smallest size? No!
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![Diagram showing the earliest finish time strategy for interval scheduling]
Lemma It is safe to schedule the job $j$ with the earliest finish time: There is an optimum solution where the job $j$ with the earliest finish time is scheduled.

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$S$: ______ ______ ______ ______ ______ ______ 

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Greedy Algorithm for Interval Scheduling

Schedule\((s, f, n)\)

1. \(A \leftarrow \{1, 2, \cdots, n\}, S \leftarrow \emptyset\)
2. \textbf{while } \(A \neq \emptyset\) \textbf{ do}
3. \hspace{1em} \(j \leftarrow \arg \min_{j' \in A} f_{j'}\)
4. \hspace{1em} \(S \leftarrow S \cup \{j\}; A \leftarrow \{j' \in A : s_{j'} \geq f_j\}\)
5. \textbf{return } S
Greedy Algorithm for Interval Scheduling

**Schedule**(*s, f, n*)

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Running time of algorithm?
Greedy Algorithm for Interval Scheduling

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Running time of algorithm?

- Naive implementation: \( O(n^2) \) time
Greedy Algorithm for Interval Scheduling

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Running time of algorithm?

- Naive implementation: \( O(n^2) \) time
- Clever implementation: \( O(n \lg n) \) time
Clever Implementation of Greedy Algorithm

Schedule\((s, f, n)\)

1: sort jobs according to \(f\) values
2: \(t \leftarrow 0, S \leftarrow \emptyset\)
3: for every \(j \in [n]\) according to non-decreasing order of \(f_j\) do
   4: if \(s_j \geq t\) then
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5: \hspace{2em} $S \leftarrow S \cup \{j\}$
6: \hspace{2em} $t \leftarrow f_j$
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4: \hspace{1em} if \(s_j \geq t\) then
5: \hspace{2em} \(S \leftarrow S \cup \{j\}\)
6: \hspace{2em} \(t \leftarrow f_j\)
7: return \(S\)
Clever Implementation of Greedy Algorithm

Schedule \((s, f, n)\)

1: sort jobs according to \(f\) values
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Clever Implementation of Greedy Algorithm

Schedule($s, f, n$)

1: sort jobs according to $f$ values
2: $t \leftarrow 0$, $S \leftarrow \emptyset$
3: for every $j \in [n]$ according to non-decreasing order of $f_j$ do
4:   if $s_j \geq t$ then
5:     $S \leftarrow S \cup \{j\}$
6:     $t \leftarrow f_j$
7: return $S$
Clever Implementation of Greedy Algorithm

\textbf{Schedule}(s, f, n)

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3: \textbf{for} every \( j \in [n] \) according to non-decreasing order of \( f_j \) \textbf{do}
4: \hspace{1em} \textbf{if} \( s_j \geq t \) \textbf{then}
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Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests

Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary. Cache hit happens if requested page already in cache. Goal: minimize the number of cache misses.
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests

```plaintext
page sequence
1 5 4 2 5 3 2 1
```
Offline Caching

- Cache that can store $k$ pages
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```
<table>
<thead>
<tr>
<th>page sequence</th>
<th>cache</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>×</td>
</tr>
<tr>
<td>5</td>
<td>×</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
```
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- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
Offline Caching

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Cache that can store $k$ pages

Sequence of page requests

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<thead>
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<th>Page Sequence</th>
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<tr>
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<tr>
<td>4</td>
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</tr>
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<td>2</td>
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</tr>
<tr>
<td>2</td>
<td>✗ 1 2 3</td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
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<table>
<thead>
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<tr>
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</tr>
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</table>

misses = 6
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
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- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.

<table>
<thead>
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<th>Page Sequence</th>
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<tbody>
<tr>
<td>1</td>
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<tr>
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<td>✓ 1 2 3</td>
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<tr>
<td>1</td>
<td>✓ 1 2 3</td>
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</table>

misses = 6
### A Better Solution for Example

<table>
<thead>
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<th>Cache</th>
<th>Cache</th>
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<td>✗ 1</td>
</tr>
<tr>
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<td>✗ 1</td>
</tr>
<tr>
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<td>✗ 1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>3</td>
<td>✗ 1</td>
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**Misses:** 6

<table>
<thead>
<tr>
<th>Page Sequence</th>
<th>Cache</th>
<th>Cache</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>✗ 1</td>
</tr>
<tr>
<td>5</td>
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<tr>
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</tr>
<tr>
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<td></td>
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<td></td>
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<tr>
<td>1</td>
<td></td>
<td>✅ 1</td>
</tr>
</tbody>
</table>

**Misses:** 5
## Offline Caching Problem

| **Input:** | $k$: the size of cache  
|           | $n$: number of pages  
|           | $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests  
| **Output:** | $i_1, i_2, i_3, \cdots, i_T \in \{\text{hit}, \text{empty}\} \cup [n]$: indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)  

We use $[n]$ for $\{1, 2, 3, \cdots, n\}$.  

Offline Caching: we know the whole sequence ahead of time.  

Online Caching: we have to make decisions on the fly, before seeing future requests.  

**Q:** Which one is more realistic?  

**A:** Online caching.
Offline Caching Problem

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- **Offline Caching**: we know the whole sequence ahead of time.
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### Offline Caching Problem

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#### Details
- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

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**A:** Online caching
- Offline Caching: we know the whole sequence ahead of time.
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Q: Which one is more realistic?

A: Online caching

Q: Why do we study the offline caching problem?
Offline Caching: we know the whole sequence ahead of time.

Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

Q: Why do we study the offline caching problem?

A: Use the offline solution as a benchmark to measure the “competitive ratio” of online algorithms
Offline Caching: Potential Greedy Algorithms

- FIFO (First-In-First-Out): always evict the first page in cache
Offline Caching: Potential Greedy Algorithms

- FIFO (First-In-First-Out): always evict the first page in cache
- LRU (Least-Recently-Used): Evict page whose most recent access was earliest

All the above algorithms are not optimum! Indeed all the algorithms are "online", i.e., the decisions can be made without knowing future requests. Online algorithms cannot be optimum.
Offline Caching: Potential Greedy Algorithms

- FIFO (First-In-First-Out): always evict the first page in cache
- LRU (Least-Recently-Used): Evict page whose most recent access was earliest
- LFU (Least-Frequently-Used): Evict page that was least frequently requested
Offline Caching: Potential Greedy Algorithms

- **FIFO (First-In-First-Out):** always evict the first page in cache
- **LRU (Least-Recently-Used):** Evict page whose most recent access was earliest
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- All the above algorithms are not optimum!
- Indeed all the algorithms are “online”, i.e, the decisions can be made without knowing future requests. Online algorithms can not be optimum.
FIFO is not optimum

requests
1
2
3
4

FIFO

1

2

3

4

1
FIFO is not optimum

requests

1
2
3
4

FIFO

X
FIFO is not optimum

requests

<table>
<thead>
<tr>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

FIFO

- 1
- 1

- 1
FIFO is not optimum

<table>
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<th>FIFO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<tr>
<td>2</td>
<td>x</td>
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<tr>
<td>3</td>
<td>x</td>
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</tr>
<tr>
<td>1</td>
<td>x</td>
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</tbody>
</table>
FIFO is not optimum

FIFO

requests

1
2
3
4
1
2
3
4
1

FIFO

1
2
3
4
1
FIFO is not optimum

requests

FIFO

1 1 1 1
2 1 2
3
4
1
FIFO is not optimum

requests

1
2
3
4
1
2
3

FIFO

1
2
3
FIFO is not optimum

requests
1
2
3
4
1
FIFO
1
1
2
1
2
3

 crosses indicate requests processed
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
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<tbody>
<tr>
<td>1</td>
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FIFO is not optimum
FIFO is not optimum
FIFO is not optimum

<table>
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<th>FIFO</th>
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<tbody>
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<td>🗑️ 1</td>
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<td>🗑️ 1</td>
</tr>
<tr>
<td>3</td>
<td>🗑️ 1</td>
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<tr>
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<td>🗑️ 4</td>
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</table>

misses = 5
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗ 1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>2</td>
<td>✗ 1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>3</td>
<td>✗ 1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>4</td>
<td>✗ 4</td>
<td>✗ 1</td>
</tr>
</tbody>
</table>

FIFO misses = 5

Furthest-in-Future misses = 4
Furthest-in-Future (FF)

- Algorithm: every time, evict the page that is not requested until furthest in the future, if we need to evict one.

- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.
### Furthest-in-Future (FF)

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
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<td>1</td>
<td>❌ 1</td>
<td>❌ 1</td>
</tr>
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<td>❌ 1  2</td>
<td>❌ 1  2</td>
</tr>
<tr>
<td>3</td>
<td>❌ 1  2  3</td>
<td>❌ 1  2  3</td>
</tr>
<tr>
<td>4</td>
<td>❌ 4  2  3</td>
<td>❌ 4  2  3</td>
</tr>
<tr>
<td>1</td>
<td>❌ 4  1  3</td>
<td>✔️ 1  4  3</td>
</tr>
</tbody>
</table>

- **FIFO**
  - misses = 5
- **Furthest-in-Future**
  - misses = 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

□  □  □  □  □  □  □  □  □  □  □  □
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X

1  1  1

5  5

4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

1  1  1  1  2

1  5  5  5

4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

x  x  x  x

1  1  1  2

5  5  5

4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

x  x  x  x  ✓

1  1  1  2  2

5  5  5  5

4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X  X  ✓

☐  1  1  1  2  2
☐  ☐  5  5  5  5
☐  ☐  ☐  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

1  1  1  2  2  2

5  5  5  5  5  3

4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X  X  X  X  ✓  X

☐  1  1  1  2  2  2

☐  ☐  5  5  5  5  3

☐  ☐  ☐  4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

\[\begin{array}{ccccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 2 & 5 & 5 & 5 & 5 & 3 & 3 & 4 & 4 & 4 & 4 & 4
\end{array}\]
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×  ✔  ×  ✔  ✔

1  1  1  2  2  2  2  2

5  5  5  5  3  3  3

4  4  4  4  4  4  4
## Example

### Requests

|   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| 1 | 5 | 4 | 2 | 5 | 3 | 2 | 4 | 3 | 1 | 5 | 3 |   |   |   |   |   |   |   |

![requests diagram]
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

X X X X ✓ ✓ ✓ ✓ ✓

☐  1  1  1  2  2  2  2  2  2

☐  ☐  5  5  5  5  3  3  3  3

☐  ☐  ☐  4  4  4  4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

❌ ❌ ❌ ❌ ❌ ✔️ ❌ ✔️ ✔️ ✔️ ❌

☐  1  1  1  2  2  2  2  2  2  1

☐  ☐  5  5  5  5  3  3  3  3  3

☐  ☐  ☐  4  4  4  4  4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3

requests

1  5  4  2  5  3  2  4  3  1  5  3
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

1  1  1  1  2  2  2  2  2  1  5  5

5  5  5  5  3  3  3  3  3  3  3  3

4  4  4  4  4  4  4  4  4  4  4  4
Recall: Designing and Analyzing Greedy Algorithms

**Greedy Algorithm**
- Build up the solutions in steps
- At each step, make an *irrevocable* decision using a “reasonable” strategy

**Analysis of Greedy Algorithm**
- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
# Recall: Designing and Analyzing Greedy Algorithms

## Greedy Algorithm
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- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
Offline Caching Problem

**Input:**  
$k$: the size of cache  
$n$: number of pages  
$\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

**Output:**  
$i_1, i_2, i_3, \cdots, i_t \in \{\text{hit, empty}\} \cup [n]$  
- empty stands for an empty page  
- “hit” means evicting no pages
Offline Caching Problem

**Input:**
- \( k \): the size of cache
- \( n \): number of pages
- \( \rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n] \): sequence of requests
- \( p_1, p_2, \cdots, p_k \in \{ \text{empty} \} \cup [n] \): initial set of pages in cache

**Output:**
- \( i_1, i_2, i_3, \cdots, i_t \in \{ \text{hit}, \text{empty} \} \cup [n] \)
  - empty stands for an empty page
  - “hit” means evicting no pages
Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
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Analysis of Greedy Algorithm

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**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.
Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. **There is an optimum solution in which $p^*$ is evicted at time 1.**
Proof.

1 \( S \): any optimum solution
2 \( p^* \): page in cache not requested until furthest in the future.
   - In the example, \( p^* = 3 \).
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$.
3. Assume $S$ evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$. 
**Proof.**

1. *S*: any optimum solution

2. *p*\(^*\): page in cache not requested until furthest in the future.
   - In the example, *p*\(^*\) = 3.

3. Assume *S* evicts some *p*\(^\prime\) \(\neq p^*\) at time 1; otherwise done.
   - In the example, *p*\(^\prime\) = 2.
Proof.

Create $S'$.

$S'$ evicts $p^*$ (=3) instead of $p^\prime$ (=2) at time 1.

After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p^\prime$ (=2) and $S$ contains $p^*$ (=3).

From now on, $S'$ will "copy" $S$.
Proof.

Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.
Proof.

4 Create $S'$. $S'$ evicts $p^* (=3)$ instead of $p' (=2)$ at time 1.
Proof.

4 Create $S'$. $S'$ evicts $p^*(=3)$ instead of $p'(=2)$ at time 1.

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![Diagram showing cache states](image-url)
**Proof.**


5. After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$.

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6 From now on, $S'$ will “copy” $S$. 
Proof.

If $S$ evicted the page $p^*$, then $S'$ will evict the page $p'^*$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

Assume $S$ did not evict $p^*$ (=3) before we see $p'^*$ (=2).
**Proof.**

If $S$ evicted the page $p^*$, $S'$ will evict the page $p'$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.
Proof.

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Proof.

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8 Assume $S$ did not evict $p^*(=3)$ before we see $p'(=2)$. 
Proof. If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won't be optimum. Assume otherwise. So far, $S'$ has 1 less page-miss than $S$ does. The status of $S'$ and that of $S$ only differ by 1 page.
Proof.

If $S$ evicts $p^\ast (=3)$ for $p^\prime (=2)$, then $S$ won't be optimum. Assume otherwise.

So far, $S'$ has 1 less page-miss than $S$ does.

The status of $S'$ and that of $S$ only differ by 1 page.
Proof. If \( S \) evicts \( p^* = 3 \) for \( p' = 2 \), then \( S \) won't be optimum. Assume otherwise.

So far, \( S' \) has 1 less page-miss than \( S \) does.

The status of \( S' \) and that of \( S \) only differ by 1 page.
Proof.

9 If \( S \) evicts \( p^*(=3) \) for \( p' (=2) \), then \( S \) won’t be optimum. Assume otherwise.
Proof.

If \( S \) evicts \( p^*(=3) \) for \( p'(=2) \), then \( S \) won’t be optimum. Assume otherwise.
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9 If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.

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Proof.

9 If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.

10 So far, $S'$ has 1 less page-miss than $S$ does.

11 The status of $S'$ and that of $S$ only differ by 1 page.
\[ S : \begin{array}{cccccccc} 1 & 1 & 5 & 5 & \cdots & 6 & 2 \\ 2 & 4 & 4 & 4 & \cdots & 8 & 8 \\ 3 & 3 & 3 & 3 & \cdots & 3 & 3 \end{array} \]

\[ S' : \begin{array}{cccccccc} 1 & 1 & 5 & 5 & \cdots & 6 & 6 \\ 2 & 4 & 4 & 4 & \cdots & 8 & 8 \\ 3 & 2 & 2 & 2 & \cdots & 2 & 2 \end{array} \]

**Proof.**
Proof.

We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.
\textbf{Proof.} 

12 We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.

- Idea: if $S$ has a page-hit and $S'$ has a page-miss, we use the opportunity to make the status of $S'$ the same as that of $S$. 

\[
\begin{array}{ccccccc}
4 & 5 & 4 & \cdots & 2 & 3 \\
\times & \times & \checkmark & \times \\
1 & 1 & 5 & 5 & \cdots & 6 & 2 \\
2 & 4 & 4 & 4 & 8 & 8 \\
3 & 3 & 3 & 3 & \cdots & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{ccccccc}
1 & 1 & 5 & 5 & \cdots & 6 & 6 \\
2 & 4 & 4 & 4 & 8 & 8 \\
3 & 2 & 2 & 2 & \cdots & 2 & 2 \\
\end{array}
\]
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
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**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.
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**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. **It is safe to evict $p^*$ at time 1.**

**Theorem** The furthest-in-future strategy is optimum.
1: for $t \leftarrow 1$ to $T$ do
2: if $\rho_t$ is in cache then do nothing
3: else if there is an empty page in cache then
4: evict the empty page and load $\rho_t$ in cache
5: else
6: $p^* \leftarrow$ page in cache that is not used furthest in the future
7: evict $p^*$ and load $\rho_t$ in cache
Q: How can we make the algorithm as fast as possible?

A: The running time can be made to be $O(n + T \log k)$. For each page $p$, use a linked list (or an array with dynamic size) to store the time steps in which $p$ is requested. We can find the next time a page is requested easily. Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
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A:

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  - We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
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<tr>
<td>pages</td>
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<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

| P1: | 1 | 10 |
| P2: | 4 | 7  |
| P3: | 6 | 9  | 12 |
| P4: | 3 | 8  |
| P5: | 2 | 5  | 11 |

priority queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<tr>
<td>pages</td>
<td>P1</td>
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**Priority Queue**

<table>
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<th>priority values</th>
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</tbody>
</table>

**Process Pages**

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
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</table>

**Pages**: P1: 1 10, P2: 4 7, P3: 6 9 12, P4: 3 8, P5: 2 5 11
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<tr>
<td>P4</td>
<td></td>
</tr>
<tr>
<td>P5</td>
<td></td>
</tr>
</tbody>
</table>
The diagram shows a priority queue with the following pages:

- **P1**: 1, 10
- **P2**: 4, 7
- **P3**: 6, 9, 12
- **P4**: 3, 8
- **P5**: 2, 5, 11

The priority queue is represented by a table with columns for **pages** and **priority values**. The priority values are:

- **P1**: 10

The time and pages are as follows:

<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
<tr>
<td>time</td>
<td>pages</td>
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</tr>
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<td>0</td>
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<tr>
<td>9</td>
<td>P1</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>priority queue</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
</tr>
<tr>
<td>-------</td>
</tr>
<tr>
<td>P1</td>
</tr>
<tr>
<td>P5</td>
</tr>
</tbody>
</table>

P1:  
- page 1: 1
- page 10

P2:  
- page 4
- page 7

P3:  
- page 6
- page 9
- page 12

P4:  
- page 3
- page 8

P5:  
- page 2
- page 5
- page 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>P1</th>
<th>P5</th>
</tr>
</thead>
<tbody>
<tr>
<td>priority values</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
</tr>
</tbody>
</table>

priority queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

P1:  
\[\begin{array}{c}
1 \\
10
\end{array}\]

P2:  
\[\begin{array}{c}
4 \\
7
\end{array}\]

P3:  
\[\begin{array}{c}
6 \\
9 \\
12
\end{array}\]

P4:  
\[\begin{array}{c}
3 \\
8
\end{array}\]

P5:  
\[\begin{array}{c}
2 \\
5 \\
11
\end{array}\]

priority queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>10</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>
The diagram depicts a time-line with pages assigned to processes (P1, P2, P3, P4, P5). The time-line is labeled with time in columns and pages in rows.

The priority queue is shown with the following assignments:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>

The processes are arranged as follows:

- **P1:** time 1, pages 10
- **P2:** time 4, pages 7
- **P3:** time 6, pages 12
- **P4:** time 2, pages 8
- **P5:** time 2, pages 11
<table>
<thead>
<tr>
<th>Time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>Pages</th>
<th>Priority Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>5</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>
The table shows the time progression and the corresponding pages for each process. The priority queue is also depicted, listing the pages and their corresponding priority values.

- **P1:** Time 1, Pages 10
- **P2:** Time 4, Pages 7
- **P3:** Time 6, Pages 9, 12
- **P4:** Time 3, Pages 8
- **P5:** Time 2, Pages 5, 11

The priority queue contains:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>11</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

Priority Queue:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P5</td>
<td>11</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>
The figure illustrates a priority queue and its corresponding pages over time. The priority queue contains the following values:

<table>
<thead>
<tr>
<th>Pages</th>
<th>Priority Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>

The pages over time are as follows:

<table>
<thead>
<tr>
<th>Time</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>P1</td>
</tr>
<tr>
<td>1</td>
<td>P5</td>
</tr>
<tr>
<td>2</td>
<td>P4</td>
</tr>
<tr>
<td>3</td>
<td>P2</td>
</tr>
<tr>
<td>4</td>
<td>P5</td>
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<td>5</td>
<td>P3</td>
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<td>6</td>
<td>P2</td>
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<td>7</td>
<td>P4</td>
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<tr>
<td>8</td>
<td>P3</td>
</tr>
<tr>
<td>9</td>
<td>P1</td>
</tr>
<tr>
<td>10</td>
<td>P5</td>
</tr>
<tr>
<td>11</td>
<td>P3</td>
</tr>
<tr>
<td>12</td>
<td>P1</td>
</tr>
</tbody>
</table>

The diagram shows the pages and their times, with red crosses indicating the pages that have been requested.
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>7</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>

**Pages:**
- P1: 1 10
- P2: 4 7
- P3: 6 9 12
- P4: 3 8
- P5: 2 5 11
<table>
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<tr>
<th>time</th>
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<th>3</th>
<th>4</th>
<th>5</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
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</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
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<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
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</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
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<tr>
<td>P3</td>
<td>9</td>
</tr>
<tr>
<td>P4</td>
<td>8</td>
</tr>
</tbody>
</table>

**Pages**

- P1: 1 10
- P2: 4 7
- P3: 6 9 12
- P4: 3 8
- P5: 2 5 11
<table>
<thead>
<tr>
<th>time</th>
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<th>12</th>
</tr>
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<tbody>
<tr>
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<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
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<td>P3</td>
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**Priority Queue**

<table>
<thead>
<tr>
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<tbody>
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</tr>
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<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
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**Priority Queue**

<table>
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<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
<tr>
<td>P3</td>
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</tr>
<tr>
<td>P4</td>
<td>$\infty$</td>
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</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
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</tbody>
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### Priority Queue

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

#### Pages

- **P1:**
  - 1
  - 10

- **P2:**
  - 4
  - 7

- **P3:**
  - 6
  - 9
  - 12

- **P4:**
  - 3
  - 8

- **P5:**
  - 2
  - 5
  - 11
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</tr>
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<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
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<td>P5</td>
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<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>
The diagram illustrates the page replacement process using a priority queue. The table shows the pages and their priority values over time.

### Priority Queue

<table>
<thead>
<tr>
<th>Pages</th>
<th>Priority Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

The priority queue contains the following entries:

- P1 with priority ∞
- P3 with priority 12
- P4 with priority ∞
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

**Pages Accessed**

- P1: 1, 10
- P2: 4, 7
- P3: 6, 9, 12
- P4: 3, 8
- P5: 2, 5, 11
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:**
  - time: 10

- **P2:**
  - time: 7

- **P3:**
  - time: 12

- **P4:**
  - time: 8

- **P5:**
  - time: 11

**Priority Queue:***

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>
The diagram illustrates a priority queue for managing pages over time.

- **Pages:**
  - P1: [1, 10]
  - P2: [4, 7]
  - P3: [6, 9, 12]
  - P4: [3, 8]
  - P5: [2, 5, 11]

- **Priority Queue Values:**
  - P5: ∞
  - P3: 12
  - P4: ∞
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:** time 1, pages 10
- **P2:** time 4, pages 7
- **P3:** time 6, pages 9, 12
- **P4:** time 3, pages 8
- **P5:** time 2, pages 5, 11

Priority Queue:

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P5</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>∞</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>
1: for every $p \leftarrow 1$ to $n$ do
2: \hspace{1em} \textit{times}[p] \leftarrow \text{array of times in which } p \text{ is requested, in increasing order} \quad \triangleright \text{ put } \infty \text{ at the end of array}
3: \hspace{1em} \textit{pointer}[p] \leftarrow 1
4: $Q \leftarrow \text{empty priority queue}$
5: for every $t \leftarrow 1$ to $T$ do
6: \hspace{1em} \textit{pointer}[$\rho_t$] \leftarrow \textit{pointer}[$\rho_t$] + 1
7: \hspace{1em} if $\rho_t \in Q$ then
8: \hspace{2em} $Q.$\text{increase-key}(\rho_t, \textit{times}[\rho_t, \textit{pointer}[$\rho_t$]]), print “hit”, continue
9: \hspace{1em} if $Q.$\text{size}() < $k$ then
10: \hspace{2em} print “load $\rho_t$ to an empty page”
11: \hspace{1em} else
12: \hspace{2em} $p \leftarrow Q.$\text{extract-max}(), print “evict $p$ and load $\rho_t$”
13: \hspace{2em} $Q.$\text{insert}(\rho_t, \textit{times}[\rho_t, \textit{pointer}[$\rho_t$]]) \quad \triangleright \text{ add } \rho_t \text{ to } Q \text{ with key value } \textit{times}[\rho_t, \textit{pointer}[$\rho_t$]]
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
Let $V$ be a ground set of size $n$.

**Def.** A *priority queue* is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- **insert**$(v, key\_value)$: insert an element $v \in V \setminus U$, with associated key value $key\_value$.
- **decrease\_key**$(v, new\_key\_value)$: decrease the key value of an element $v \in U$ to $new\_key\_value$
- **extract\_min**(): return and remove the element in $U$ with the smallest key value
- ...
Simple Implementations for Priority Queue

- $n = \text{size of ground set } V$

<table>
<thead>
<tr>
<th>data structures</th>
<th>insert</th>
<th>extract_min</th>
<th>decrease_key</th>
</tr>
</thead>
<tbody>
<tr>
<td>array</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>sorted array</td>
<td></td>
<td></td>
<td></td>
</tr>
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</table>
**Simple Implementations for Priority Queue**

- \( n = \) size of ground set \( V \)

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</thead>
<tbody>
<tr>
<td>array</td>
<td>( O(1) )</td>
<td>( O(n) )</td>
<td>( O(1) )</td>
</tr>
<tr>
<td>sorted array</td>
<td></td>
<td></td>
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**Simple Implementations for Priority Queue**

- \( n = \text{size of ground set } V \)

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Simple Implementations for Priority Queue

\[ n = \text{size of ground set } V \]

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<tr>
<td>sorted array</td>
<td>(O(n))</td>
<td>(O(1))</td>
<td>(O(n))</td>
</tr>
<tr>
<td>heap</td>
<td>(O(\lg n))</td>
<td>(O(\lg n))</td>
<td>(O(\lg n))</td>
</tr>
</tbody>
</table>
Heap

The elements in a heap is organized using a complete binary tree:

- Nodes are indexed as \{1, 2, 3, \ldots, s\}
- Parent of node $i$: $\lfloor i/2 \rfloor$
- Left child of node $i$: $2i$
- Right child of node $i$: $2i + 1$
A heap $H$ contains the following fields:

- $s$: size of $U$ (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node $i$ of the tree
- $p[v], v \in U$: the index of node containing $v$
- $key[v], v \in U$: the key value of element $v$

Diagram:

- $s = 5$
- $A = (\text{'f'}, \text{'g'}, \text{'c'}, \text{'e'}, \text{'b'})$
- $p[\text{'f'}] = 1, p[\text{'g'}] = 2, p[\text{'c'}] = 3, p[\text{'e'}] = 4, p[\text{'b'}] = 5$
The following heap property is satisfied:

- For any two nodes $i, j$ such that $i$ is the parent of $j$, we have $key[A[i]] \leq key[A[j]]$.

A heap. Numbers in the circles denote key values of elements.
$\text{insert}(v, \text{key}_v)$
$\text{insert}(v, key\_value)$
\text{insert}(v, \text{key\_value})
\textbf{insert}(v, \textit{key\_value})
$\text{insert}(v, key\_value)$
**insert**(*v, key_value*)

1: \( s \leftarrow s + 1 \)
2: \( A[s] \leftarrow v \)
3: \( p[v] \leftarrow s \)
4: \( key[v] \leftarrow key_value \)
5: **heapify_up**(*s*)

**heapify-up**(*i*)

1: \[ while \ i > 1 \ do \]
2: \[ j \leftarrow \lfloor i/2 \rfloor \]
3: \[ if \ key[A[i]] < key[A[j]] \ then \]
4: \[ swap \ A[i] \ and \ A[j] \]
5: \[ p[A[i]] \leftarrow i, p[A[j]] \leftarrow j \]
6: \[ i \leftarrow j \]
7: \[ else \ break \]
extract_min()
extract_min()
extract_min()
extract_min()
extract_min()
extract_min()
**extract_min()**

1. \( \text{ret} \leftarrow A[1] \)
2. \( A[1] \leftarrow A[s] \)
3. \( p[A[1]] \leftarrow 1 \)
4. \( s \leftarrow s - 1 \)
5. if \( s \geq 1 \) then
6. \( \text{heapify-down}(1) \)
7. return \( \text{ret} \)

**heapify-down(i)**

1. while \( 2i \leq s \) do
2. if \( 2i = s \) or
   \( \text{key}[A[2i]] \leq \text{key}[A[2i + 1]] \) then
3. \( j \leftarrow 2i \)
4. else
5. \( j \leftarrow 2i + 1 \)
6. if \( \text{key}[A[j]] < \text{key}[A[i]] \) then
7. swap \( A[i] \) and \( A[j] \)
8. \( p[A[i]] \leftarrow i, p[A[j]] \leftarrow j \)
9. \( i \leftarrow j \)
10. else break

**decrease_key(v, key_value)**

1. \( \text{key}[v] \leftarrow \text{key_value} \)
2. \( \text{heapify-up}(p[v]) \)
Running time of heapify\_up and heapify\_down: $O(\lg n)$
• Running time of heapify\_up and heapify\_down: $O(lg\ n)$
• Running time of insert, exact\_min and decrease\_key: $O(lg\ n)$
- Running time of heapify\_up and heapify\_down: $O(lg\ n)$
- Running time of insert, exact\_min and decrease\_key: $O(lg\ n)$

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<tr>
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<tr>
<td>heap</td>
<td>$O(lg\ n)$</td>
<td>$O(lg\ n)$</td>
<td>$O(lg\ n)$</td>
</tr>
</tbody>
</table>
Two Definitions Needed to Prove that the Procedures Maintain Heap Property

**Def.** We say that $H$ is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make $H$ a heap.

**Def.** We say that $H$ is almost a heap except that $key[A[i]]$ is too big if we can decrease $key[A[i]]$ to make $H$ a heap.
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - **Heap**: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
8 letters $a, b, c, d, e, f, g, h$ in a language

need to encode a message using bits

idea: use 3 bits per letter

$$
\begin{array}{cccccccc}
  a & b & c & d & e & f & g & h \\
  000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\end{array}
$$

$deacfg \rightarrow 011100000010101110$

**Q:** Can we have a better encoding scheme?

Seems unlikely: must use 3 bits per letter

**Q:** What if some letters appear more frequently than the others?
Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and
  more bits for letters that are less frequently used.
What is the issue with the following encoding scheme?

- $a$: 0
- $b$: 1
- $c$: 00

Solution: Use prefix codes to guarantee a unique decoding.
Q: What is the issue with the following encoding scheme?

- $a$: 0
- $b$: 1
- $c$: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to $aa$ or $c$. Use prefix codes to guarantee a unique decoding.
Q: What is the issue with the following encoding scheme?

- $a$: 0
- $b$: 1
- $c$: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to $aa$ or $c$.

Solution

Use prefix codes to guarantee a unique decoding.
Prefix Codes

**Def.** A prefix code for a set $S$ of letters is a function $\gamma : S \to \{0, 1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.
Def. A prefix code for a set $S$ of letters is a function $\gamma : S \rightarrow \{0, 1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$.
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>001</td>
<td>0000</td>
<td>0001</td>
<td>100</td>
</tr>
<tr>
<td>e</td>
<td>11</td>
<td>1010</td>
<td>1011</td>
<td>01</td>
</tr>
</tbody>
</table>

Diagram:

- Node b connects to 0.
- Node c connects to 1.
- Node d connects to 0.
- Node e connects to 1.
- Node f connects to 0.
- Node g connects to 1.
Prefix Codes Guarantee Unique Decoding

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<table>
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<th>a</th>
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<table>
<thead>
<tr>
<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
</tr>
</thead>
<tbody>
<tr>
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</table>

00010011000000001011110100001001
Prefix Codes Guarantee Unique Decoding

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<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>$a$</td>
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</tr>
<tr>
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<p>| | | | |</p>
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<td>$e$</td>
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- $0001/001100000001011110100001001$
- $c$
Prefix Codes Guarantee Unique Decoding

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- $0001/001/100000001011110100001001$
- ca
Prefix Codes Guarantee Unique Decoding

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- 0001/001/100/000001011110100001001
- cad
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- 0001/001/100/0000/01011110100001001
- cadb
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0001/001/100/0000/01/011110100001001

cadbh
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0001/001/100/0000/01/01/1110100001001

cadbh
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<th>b</th>
<th>c</th>
<th>d</th>
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<td>0001</td>
<td>100</td>
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<th>e</th>
<th>f</th>
<th>g</th>
<th>h</th>
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<td>1010</td>
<td>1011</td>
<td>01</td>
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- 0001/001/100/0000/01/01/11/10100001001
- cadbhhe
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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<td>0000</td>
<td>0001</td>
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<tr>
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Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some letter
- If coding scheme is not wasteful: a non-leaf has exactly two children

Best Prefix Codes
- Input: frequencies of letters in a message
- Output: prefix coding scheme with the shortest encoding for the message
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Best Prefix Codes
- **Input:** frequencies of letters in a message
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**example**

<table>
<thead>
<tr>
<th>letters</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequencies</td>
<td>18</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>scheme 1 length</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>total = 89</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>scheme 2 length</td>
<td>1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>total = 87</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>scheme 3 length</td>
<td>1</td>
<td>4</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>total = 84</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

scheme 1

```
a  d  e
b  c
```

scheme 2

```
a
b  c  d  e
```

scheme 3

```
a
 e
 d
 b  c
```
### Example

<table>
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<tr>
<th>letters</th>
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| scheme 1 length | 2 | 3 | 3 | 2 | 2 |
| total = 89 |

| scheme 2 length | 1 | 3 | 3 | 3 | 3 |
| total = 87 |

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| total = 84 |

**Scheme Diagrams**

- **Scheme 1**: a → d → e, b → c
- **Scheme 2**: a → b → c → d → e
- **Scheme 3**: a → c → d, b → e
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)
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Q: What types of decisions should we make?
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- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.
Which Two Letters Can Be Safely Put Together As Brothers?

Focus on the “structure” of the optimum encoding tree
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers
Which Two Letters Can Be Safely Put Together As Brothers?

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best to put the two least frequent symbols here!
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers

**Lemma** It is safe to make the two least frequent letters brothers.
**Lemma** There is an optimum encoding tree, where the two least frequent letters are brothers.

So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.
Lemma There is an optimum encoding tree, where the two least frequent letters are brothers.

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Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.
- \( f_x \): the frequency of the letter \( x \) in the support.
- \( x_1 \) and \( x_2 \): the two letters we decided to put together.
- \( d_x \) the depth of letter \( x \) in our output encoding tree.

\[
\sum_{x \in S} f_x d_x
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2}
\]

\[
= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1}
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= \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x'} (d_{x'} + 1)
\]

Def: $f_{x'} = f_{x_1} + f_{x_2}$
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\[
\text{Def: } f'_x = f_{x_1} + f_{x_2}
\]

\[
\begin{align*}
\sum_{x \in S} f_x d_x \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2}) d_{x_1} \\
= & \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f'_x (d'_x + 1) \\
= & \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x + f'_x
\end{align*}
\]
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\end{align*}
\]

Def: \( f_{x'} = f_{x_1} + f_{x_2} \)
In order to minimize

$$\sum_{x \in S} f_x d_x,$$

we need to minimize

$$\sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} f_x d_x,$$

subject to that $d$ is the depth function for an encoding tree of $S \setminus \{x_1, x_2\}$.

This is exactly the best prefix codes problem, with letters $S \setminus \{x_1, x_2\} \cup \{x'\}$ and frequency vector $f$. 
Example

A 27  B 15  C 11  D 9  E 8  F 5
Example
Example

A    B    C    D    E    F  5  8  9  11  15  27
13  20
28

\[ \begin{array}{cccc}
A & B & C & D \\
27 & 15 & 11 & 9 \\
 & 20 & 8 & 5 \\
 & 28 & 13 & \\
\end{array} \]
Example
Example
Example

A : 00
B : 10
C : 010
D : 011
E : 110
F : 111
Def. The codes given the greedy algorithm is called the **Huffman codes**.
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**Huffman**($S$, $f$)

1. **while** $|S| > 1$ **do**
2. let $x_1, x_2$ be the two letters with the smallest $f$ values
3. introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
4. let $x_1$ and $x_2$ be the two children of $x'$
5. $S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}$
6. **return** the tree constructed
Huffman($S, f$)

1: $Q \leftarrow \text{build-priority-queue}(S)$
2: while $Q$.size $> 1$ do
3:     $x_1 \leftarrow Q$.extract-min()
4:     $x_2 \leftarrow Q$.extract-min()
5:     introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
6:     let $x_1$ and $x_2$ be the two children of $x'$
7:     $Q$.insert($x', f_{x'}$)
8: return the tree constructed
Outline

1. Toy Example: Box Packing
2. Interval Scheduling
3. Offline Caching
   - Heap: Concrete Data Structure for Priority Queue
4. Data Compression and Huffman Code
5. Summary
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy
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### Summary for Greedy Algorithms

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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
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Def. A strategy is “safe” if there is always an optimum solution that “agrees with” the decision made according to the strategy.
Take an arbitrary optimum solution $S$
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
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  - Offline caching: a complicated “copying” algorithm
  - Huffman codes: move the two least frequent letters to the deepest leaves.
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- Prove that the reasonable strategy is “safe” (key)
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- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one