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**Trivial Algorithm for an Optimization Problem**

Enumerate all valid solutions, compare them and output the best one.
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Goals of algorithm design
**Def.** In an optimization problem, our goal is to find a valid solution with the minimum cost (or maximum value).

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**Goals of algorithm design**

1. Design efficient algorithms to solve problems
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**Goals of algorithm design**

1. Design efficient algorithms to solve problems
2. Design more efficient algorithms to solve problems
Common Paradigms for Algorithm Design

- Greedy Algorithms
- Divide and Conquer
- Dynamic Programming

Greedy algorithms are often for optimization problems. They often run in polynomial time due to their simplicity.
Common Paradigms for Algorithm Design

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- Greedy algorithms are often for optimization problems.
Greedy Algorithms

Divide and Conquer

Dynamic Programming

Greedy algorithms are often for optimization problems.
They often run in polynomial time due to their simplicity.
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an **irrevocable** decision using a “reasonable” strategy
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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem
Greedy Algorithm

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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe" (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
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Def. A strategy is safe: there is always an optimum solution that agrees with the decision made according to the strategy.
Outline

Toy Example: Box Packing

Interval Scheduling

Offline Caching
   Heap: Concrete Data Structure for Priority Queue

Data Compression and Huffman Code

Summary
Box Packing

**Input:** $n$ boxes of capacities $c_1, c_2, \cdots, c_n$
$m$ items of sizes $s_1, s_2, \cdots, s_m$

Can put at most 1 item in a box

Item $j$ can be put into box $i$ if $s_j \leq c_i$

**Output:** A way to put as many items as possible in the boxes.
**Box Packing**

**Input:** \( n \) boxes of capacities \( c_1, c_2, \cdots, c_n \)  
\( m \) items of sizes \( s_1, s_2, \cdots, s_m \)  
Can put at most 1 item in a box  
Item \( j \) can be put into box \( i \) if \( s_j \leq c_i \)

**Output:** A way to put as many items as possible in the boxes.

**Example:**

- Box capacities: 60, 40, 25, 15, 12  
- Item sizes: 45, 42, 20, 19, 16  
- Can put 3 items in boxes: 45 \( \rightarrow \) 60, 20 \( \rightarrow \) 40, 19 \( \rightarrow \) 25
Greedy Algorithm

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Greedy Algorithm

- Build up the solutions in steps
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Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a “reasonable” strategy

Designing a Reasonable Strategy for Box Packing

- Q: Take box 1. Which item should we put in box 1?
- A: The item of the largest size that can be put into the box.
Analysis of Greedy Algorithm

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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
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Lemma The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.
Analysis of Greedy Algorithm

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Lemma  The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
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**Lemma** The strategy that put into box 1 the largest item it can hold is “safe”: There is an optimum solution in which box 1 contains the largest item it can hold.

- Intuition: putting the item gives us the easiest residual problem.
- formal proof via exchanging argument:
Lemma  There is an optimum solution in which box 1 contains the largest item it can hold.
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**Proof.**

- Let $j$ = largest item that box 1 can hold.
Lemma  There is an optimum solution in which box 1 contains the largest item it can hold.

Proof.

- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
Lemma  There is an optimum solution in which box 1 contains the largest item it can hold.

Proof.

- Let $j = \text{largest item that box 1 can hold}$.  
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.  
- Otherwise, assume this is what happens in $S$: 

\[
S: \quad \begin{array}{c}
\text{box 1} \\
\text{item } j \\
\text{......}
\end{array}
\]
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

**Proof.**

- Let $j =$ largest item that box 1 can hold.
- Take any optimum solution $S$. If $j$ is put into Box 1 in $S$, done.
- Otherwise, assume this is what happens in $S$:

  $S'$: box 1

  \[ \text{item } j' \hspace{1cm} \text{item } j \hspace{1cm} \Diamond \text{S} \Diamond \hspace{1cm} \Diamond \]

  $s_{j'} \leq s_j$, and swapping gives another solution $S'$
**Lemma** There is an optimum solution in which box 1 contains the largest item it can hold.

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  \begin{array}{cccc}
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  \]

  - $s_{j'} \leq s_j$, and swapping gives another solution $S'$
  - $S'$ is also an optimum solution. In $S'$, $j$ is put into Box 1. □
Notice that the exchanging operation is only for the sake of analysis; it is not a part of the algorithm.
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**Analysis of Greedy Algorithm**

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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe”
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- Trivial: we decided to put Item $j$ into Box 1, and the remaining instance is obtained by removing Item $j$ and Box 1.
### Generic Greedy Algorithm

1. **while** the instance is non-trivial **do**
2. make the choice using the greedy strategy
3. reduce the instance

### Greedy Algorithm for Box Packing

1. $T \leftarrow \{1, 2, 3, \cdots, m\}$
2. **for** $i \leftarrow 1$ to $n$ **do**
3. **if** some item in $T$ can be put into box $i$ **then**
4. $j \leftarrow$ the largest item in $T$ that can be put into box $i$
5. print(“put item $j$ in box $i$”)
6. $T \leftarrow T \setminus \{j\}$
Generic Greedy Algorithm

1: while the instance is non-trivial do
2: make the choice using the greedy strategy
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**Lemma** Generic algorithm is correct if and only if the greedy strategy is safe.
**Generic Greedy Algorithm**

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- Greedy strategy is safe: we will not miss the optimum solution
Generic Greedy Algorithm

1: **while** the instance is non-trivial **do**
2: make the choice using the greedy strategy
3: reduce the instance

**Lemma**  Generic algorithm is correct if and only if the greedy strategy is safe.

- Greedy strategy is safe: we will not miss the optimum solution
- Greedy strategy is not safe: we will miss the optimum solution for some instance, since the choices we made are irrevocable.
Greedy Algorithm

- Build up the solutions in steps
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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is "safe"
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Def. A strategy is "safe" if there is always an optimum solution that is "consistent" with the decision made according to the strategy.
Exchange argument: Proof of Safety of a Strategy

- let $S$ be an arbitrary optimum solution.
- if $S$ is consistent with the greedy choice, done.
- otherwise, show that it can be modified to another optimum solution $S'$ that is consistent with the choice.
Exchange argument: Proof of Safety of a Strategy

- let $S$ be an arbitrary optimum solution.
- if $S$ is consistent with the greedy choice, done.
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- The procedure is not a part of the algorithm.
Outline

Toy Example: Box Packing

Interval Scheduling

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Summary
Interval Scheduling

**Input:** $n$ jobs, job $i$ with start time $s_i$ and finish time $f_i$

$i$ and $j$ are **compatible** if $[s_i, f_i)$ and $[s_j, f_j)$ are disjoint

**Output:** A maximum-size subset of mutually compatible jobs
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**Output:** A maximum-size subset of mutually compatible jobs
Greedy Algorithm for Interval Scheduling

- Which of the following strategies are safe?
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- Which of the following strategies are safe?
- Schedule the job with the smallest size?
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![Diagram showing intervals on a timeline]
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![Diagram showing intervals and conflicts between jobs.](image-url)
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**Lemma** It is safe to schedule the job $j$ with the earliest finish time: There is an optimum solution where the job $j$ with the earliest finish time is scheduled.

**Proof.**
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- Take an arbitrary optimum solution $S$
**Greedy Algorithm for Interval Scheduling**

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\[ S: \]

\[ \text{[Orange boxes]} \]
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- What is the remaining task after we decided to schedule $j$?
- Is it another instance of interval scheduling problem?
Greedy Algorithm for Interval Scheduling

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Greedy Algorithm for Interval Scheduling

Schedule\((s, f, n)\)

1. \(A \leftarrow \{1, 2, \ldots, n\}, S \leftarrow \emptyset\)
2. while \(A \neq \emptyset\) do
3. \(j \leftarrow \arg \min_{j' \in A} f_{j'}\)
4. \(S \leftarrow S \cup \{j\}; A \leftarrow \{j' \in A : s_{j'} \geq f_j\}\)
5. return \(S\)
Greedy Algorithm for Interval Scheduling

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Running time of algorithm?
Greedy Algorithm for Interval Scheduling

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Running time of algorithm?  
- **Naive implementation:** \(O(n^2)\) time
Greedy Algorithm for Interval Scheduling

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Running time of algorithm?

- Naive implementation: \(O(n^2)\) time
- Clever implementation: \(O(n \log n)\) time
Clever Implementation of Greedy Algorithm

Schedule\((s, f, n)\)

1: sort jobs according to \(f\) values
2: \(t \leftarrow 0, S \leftarrow \emptyset\)
3: for every \(j \in \left[ n \right]\) according to non-decreasing order of \(f_j\) do
4: \quad if \(s_j \geq t\) then
5: \quad \quad \(S \leftarrow S \cup \{j\}\)
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Clever Implementation of Greedy Algorithm

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Schedule\( (s, f, n) \)

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2: \( t \leftarrow 0, \ S \leftarrow \emptyset \)
3: for every \( j \in [n] \) according to non-decreasing order of \( f_j \) do
4: \hspace{1cm} if \( s_j \geq t \) then
5: \hspace{2cm} \( S \leftarrow S \cup \{j\} \)
6: \hspace{1cm} \( t \leftarrow f_j \)
7: return \( S \)
Clever Implementation of Greedy Algorithm

Schedule(s, f, n)

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2: $t \leftarrow 0$, $S \leftarrow \emptyset$
3: for every $j \in [n]$ according to non-decreasing order of $f_j$ do
4:   if $s_j \geq t$ then
5:     $S \leftarrow S \cup \{j\}$
6:     $t \leftarrow f_j$
7: return $S$
Clever Implementation of Greedy Algorithm

Schedule\((s, f, n)\)

1: sort jobs according to \(f\) values
2: \(t \leftarrow 0, S \leftarrow \emptyset\)
3: for every \(j \in [n]\) according to non-decreasing order of \(f_j\) do
4: \hspace{1em} if \(s_j \geq t\) then
5: \hspace{2em} \(S \leftarrow S \cup \{j\}\)
6: \hspace{2em} \(t \leftarrow f_j\)
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4: \hspace{1em} if \(s_j \geq t\) then
5: \hspace{2em} \(S \leftarrow S \cup \{j\}\)
6: \hspace{1em} \(t \leftarrow f_j\)
7: return \(S\)
Clever Implementation of Greedy Algorithm

Schedule($s, f, n$)

1: sort jobs according to $f$ values
2: $t \leftarrow 0$, $S \leftarrow \emptyset$
3: for every $j \in [n]$ according to non-decreasing order of $f_j$ do
4: \hspace{1em} if $s_j \geq t$ then
5: \hspace{2em} $S \leftarrow S \cup \{j\}$
6: \hspace{2em} $t \leftarrow f_j$
7: return $S$
Clever Implementation of Greedy Algorithm

Schedule\((s, f, n)\)

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   5: \(S \leftarrow S \cup \{j\}\)
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7: return \(S\)
Clever Implementation of Greedy Algorithm

**Schedule**(*s*, *f*, *n*)

1: sort jobs according to *f* values
2: *t* ← 0, *S* ← ∅
3: for every *j* ∈ [n] according to non-decreasing order of *f*<sub>*j*</sub> do
4:   if *s*<sub>*j*</sub> ≥ *t* then
5:     *S* ← *S* ∪ {*j*}
6:     *t* ← *f*<sub>*j*</sub>
7: return *S*
Clever Implementation of Greedy Algorithm

**Schedule** \( (s, f, n) \)

1. sort jobs according to \( f \) values
2. \( t \leftarrow 0, \ S \leftarrow \emptyset \)
3. **for every** \( j \in [n] \) **according to non-decreasing order of** \( f_j \) **do**
4. \[ \text{if } s_j \geq t \text{ then} \]
5. \( S \leftarrow S \cup \{j\} \)
6. \( t \leftarrow f_j \)
7. **return** \( S \)
Outline

Toy Example: Box Packing

Interval Scheduling

Offline Caching
   Heap: Concrete Data Structure for Priority Queue

Data Compression and Huffman Code

Summary
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests

Cache miss happens if requested page not in cache. We need to bring the page into cache, and evict some existing page if necessary. Cache hit happens if requested page already in cache. Goal: minimize the number of cache misses.
Offline Caching

- Cache that can store \( k \) pages
- Sequence of page requests

```
page sequence
1
5
4
2
5
3
2
1
```

```
cache
[]
[]
[]
```
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
Offline Caching

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<table>
<thead>
<tr>
<th>page sequence</th>
<th>cache</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>5</td>
<td>✗ 5</td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.

```
page sequence: 1 5 4 2 5 3 2 1
cache: [●] [●] [●]
```
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
Offline Caching

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Offline Caching

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## Offline Caching

- **Cache** that can store $k$ pages
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- **Cache miss** happens if requested page not in cache. We need bring the page into cache, and evict some existing page if necessary.
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<table>
<thead>
<tr>
<th>Page Request Sequence</th>
<th>Cache</th>
</tr>
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<tbody>
<tr>
<td>1</td>
<td>✔️</td>
</tr>
<tr>
<td>5</td>
<td>✗</td>
</tr>
<tr>
<td>4</td>
<td>✗</td>
</tr>
<tr>
<td>2</td>
<td>✗</td>
</tr>
<tr>
<td>5</td>
<td>✗</td>
</tr>
<tr>
<td>3</td>
<td>✗</td>
</tr>
<tr>
<td>4</td>
<td>✗</td>
</tr>
<tr>
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<tr>
<td>2</td>
<td>✗</td>
</tr>
<tr>
<td>1</td>
<td>✔️</td>
</tr>
</tbody>
</table>
Offline Caching

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<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>

- Goal: minimize the number of cache misses.
Offline Caching

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- Sequence of page requests
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<th>cache</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗</td>
</tr>
<tr>
<td>5</td>
<td>✗</td>
</tr>
<tr>
<td>4</td>
<td>✗</td>
</tr>
<tr>
<td>2</td>
<td>✗</td>
</tr>
<tr>
<td>5</td>
<td>✗</td>
</tr>
<tr>
<td>3</td>
<td>✗</td>
</tr>
<tr>
<td>2</td>
<td>✔</td>
</tr>
</tbody>
</table>

misses = 7
Offline Caching

- Cache that can store $k$ pages
- Sequence of page requests
- Cache miss happens if requested page not in cache. We need to bring the page into cache, and evict some existing page if necessary.
- Cache hit happens if requested page already in cache.
- Goal: minimize the number of cache misses.
A Better Solution for Example

<table>
<thead>
<tr>
<th>page sequence</th>
<th>cache</th>
<th>cache</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗ 1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>5</td>
<td>✗ 5</td>
<td>✗ 5</td>
</tr>
<tr>
<td>4</td>
<td>✗ 4</td>
<td>✗ 4</td>
</tr>
<tr>
<td>2</td>
<td>✗ 4</td>
<td>✗ 4</td>
</tr>
<tr>
<td>5</td>
<td>✗ 425</td>
<td>✗ 542</td>
</tr>
<tr>
<td>3</td>
<td>✗ 423</td>
<td>✗ 532</td>
</tr>
<tr>
<td>2</td>
<td>✓ 423</td>
<td>✓ 532</td>
</tr>
<tr>
<td>1</td>
<td>✗ 123</td>
<td>✗ 132</td>
</tr>
</tbody>
</table>

misses = 7

misses = 6
Offline Caching Problem

**Input:**
- \( k \): the size of cache
- \( n \): number of pages
- \( \rho_1, \rho_2, \rho_3, \ldots, \rho_T \in [n] \): sequence of requests

**Output:**
- \( i_1, i_2, i_3, \ldots, i_T \in \{\text{hit, empty}\} \cup [n] \): indices of pages to evict ("hit" means evicting no page, "empty" means evicting empty page)

We use \([n]\) for \(\{1, 2, 3, \ldots, n\}\).
Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages

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**Offline Caching:** we know the whole sequence ahead of time.

**Online Caching:** we have to make decisions on the fly, before seeing future requests.
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**Q:** Which one is more realistic?
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Q: Which one is more realistic?

A: Online caching
- Offline Caching: we know the whole sequence ahead of time.
- Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?

A: Online caching

Q: Why do we study the offline caching problem?
Offline Caching: we know the whole sequence ahead of time.
Online Caching: we have to make decisions on the fly, before seeing future requests.

Q: Which one is more realistic?
A: Online caching

Q: Why do we study the offline caching problem?
A: Use the offline solution as a benchmark to measure the “competitive ratio” of online algorithms
Offline Caching: Potential Greedy Algorithms

- FIFO (First-In-First-Out): always evict the first page in cache
- LRU (Least-Recently-Used): evict page whose most recent access was earliest
- LFU (Least-Frequently-Used): evict page that was least frequently requested

All the above algorithms are not optimum! Indeed all the algorithms are "online", i.e., the decisions can be made without knowing future requests. Online algorithms cannot be optimum.
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- All the above algorithms are not optimum!
- Indeed all the algorithms are “online”, i.e, the decisions can be made without knowing future requests. Online algorithms can not be optimum.
FIFO is not optimum

requests

1
2
3
4
1
FIFO is not optimum

requests

1
2
3
4
1

FIFO

×
FIFO is not optimum

requests
1 2 3 4

FIFO

1 1 1 1
FIFO is not optimum

requests

FIFO

1

2

3

4

1
FIFO is not optimum

requests

1  2  3  4  1  2

FIFO

\[\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
1 \\
\end{array}\]
FIFO is not optimum

requests

1
2
3
4
1

FIFO

1

1

2

1

2

1

2

1

2
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>X</td>
</tr>
</tbody>
</table>
FIFO is not optimum
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>☒</td>
</tr>
<tr>
<td>2</td>
<td>☒</td>
</tr>
<tr>
<td>3</td>
<td>☒</td>
</tr>
<tr>
<td>4</td>
<td>☒</td>
</tr>
</tbody>
</table>

The figure shows a sequence of requests (1, 2, 3, 4) and the corresponding FIFO sequence. The FIFO sequence is not optimum as it does not reflect the order of the requests.
FIFO is not optimum
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>✗ 1</td>
</tr>
<tr>
<td>2</td>
<td>✗ 1</td>
</tr>
<tr>
<td>3</td>
<td>✗ 1</td>
</tr>
<tr>
<td>4</td>
<td>✗ 1</td>
</tr>
<tr>
<td>1</td>
<td>✗ 1</td>
</tr>
</tbody>
</table>
FIFO is not optimum

requests

1
2
3
4

FIFO

misses = 5
FIFO is not optimum

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>2</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>3</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>4</td>
<td>×</td>
<td>×</td>
</tr>
<tr>
<td>1</td>
<td>×</td>
<td>✓</td>
</tr>
</tbody>
</table>

misses = 5

misses = 4
Optimum Offline Caching

**Furthest-in-Future (FF)**

- Algorithm: every time, evict the item that is not requested until furthest in the future, if we need to evict one.
- The algorithm is not an online algorithm, since the decision at a step depends on the request sequence in the future.
## Furthest-in-Future (FF)

<table>
<thead>
<tr>
<th>requests</th>
<th>FIFO</th>
<th>Furthest-in-Future</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1 2</td>
<td>1 2</td>
</tr>
<tr>
<td>3</td>
<td>1 2 3</td>
<td>1 2 3</td>
</tr>
<tr>
<td>4</td>
<td>4 2 3</td>
<td>1 4 3</td>
</tr>
<tr>
<td>1</td>
<td>4 1 3</td>
<td>1 4 3</td>
</tr>
</tbody>
</table>

- Misses for FIFO: 5
- Misses for FF: 4
Example

requests

1 5 4 2 5 3 2 4 3 1 5 3
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×

☐  1  1  1

☐  ☐  5  5

☐  ☐  ☐  4
Example
Example

requests

1  5  4  2  5
3  2  4  3  1
5  3

requests

1  1  1  2
0  0  5  5  5
0  0  0  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×

1  1  1  2

5  5  5

4  4
Example

requests
1 5 4 2 5 3 2 4 3 1 5 3

requests
1 1 1 2 2
5 5 5 5
4 4 4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×  ✓

☐  1  1  1  2  2

☐  ☐  5  5  5  5

☐  ☐  ☐  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

x  x  x  x  ✓  x

1  1  1  2  2  2

x  x  5  5  5  5  3

x  x  4  4  4  4
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

×  ×  ×  ×  ✓  ×  ✓

1  1  1  2  2  2  2

5  5  5  5  3  3

4  4  4  4  4
**Example**

requests

```
1  5  4  2  5  3  2  4  3  1  5  3
```

```
×  ×  ×  ×  ✓  ×  ✓  ✓
```

```
☐  1  1  1  2  2  2  2  2
☐  ☐  5  5  5  5  3  3  3
☐  ☐  ☐  4  4  4  4  4  4
```
Example

requests

<table>
<thead>
<tr>
<th>1</th>
<th>5</th>
<th>4</th>
<th>2</th>
<th>5</th>
<th>3</th>
<th>2</th>
<th>4</th>
<th>3</th>
<th>1</th>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
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<td>✗</td>
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Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

- [x]  [x]  [x]  [x]  [✓]  [x]  [✓]  [✓]  [✓]
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

1  1  1  2  2  2  2  2  2  1

5  5  5  5  3  3  3  3  3  3

4  4  4  4  4  4  4  4  4  4


Example
Example

requests

1  5  4  2  5  3  2  4  3  1  5  3

x  x  x  x  ✓  x  ✓  ✓  ✓  x  x  ✓

1  1  1  2  2  2  2  2  2  1  5  5

5  5  5  5  3  3  3  3  3  3  3  3

4  4  4  4  4  4  4  4  4  4  4  4
Recall: Designing and Analyzing Greedy Algorithms

**Greedy Algorithm**
- Build up the solutions in steps
- At each step, make an *irrevocable* decision using a “reasonable” strategy

**Analysis of Greedy Algorithm**
- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
Recall: Designing and Analyzing Greedy Algorithms

<table>
<thead>
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<tr>
<td>▶ Build up the solutions in steps</td>
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<tr>
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Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests

**Output:**
- $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit, empty}\} \cup [n]$
  - empty stands for an empty page
  - “hit” means evicting no pages
### Offline Caching Problem

**Input:**
- $k$: the size of cache
- $n$: number of pages
- $\rho_1, \rho_2, \rho_3, \cdots, \rho_T \in [n]$: sequence of requests
- $p_1, p_2, \cdots, p_k \in \{\text{empty}\} \cup [n]$: initial set of pages in cache

**Output:**
- $i_1, i_2, i_3, \cdots, i_t \in \{\text{hit, empty}\} \cup [n]$
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  - “hit” means evicting no pages
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Analysis of Greedy Algorithm

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- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. **It is safe to evict** $p^*$ **at time 1.**
Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

Lemma  Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
Proof.

1. $S$: any optimum solution
2. $p^*$: page in cache not requested until furthest in the future.
   - In the example, $p^* = 3$. 
Proof.

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   - In the example, $p^* = 3$.
3. Assume $S$ evicts some $p' \neq p^*$ at time 1; otherwise done.
   - In the example, $p' = 2$. 
Proof.

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   ▶ In the example, $p' = 2$. 

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\[ \text{\textbf{X}} \]

\[ \text{\textbf{S}} : \]
\[
\begin{array}{c|c|c}
\hline
1 & 4 & 2 \\
\hline
2 & 1 & 3 \\
\hline
3 & 4 & 3 \\
\hline
\end{array}
\]
Proof.

Create $S'$.

$S'$ evicts $p^*$ (=3) instead of $p'$ (=2) at time 1.

After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S$ contains $p'$ (=2) and $S$ contains $p^*$ (=3).

From now on, $S'$ will "copy" $S$.
Proof.

**Proof.**

Proof.


5. After time 1, cache status of $S$ and that of $S'$ differ by only 1 page. $S'$ contains $p'(=2)$ and $S$ contains $p^*(=3)$. 

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Proof.
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6. From now on, $S'$ will “copy” $S$. 
Proof.

If $S$ evicts the page $p'$, then $S'$ will evict the page $p^*$, so the cache status of $S$ and $S'$ will be the same and they will be exactly the same from now on.

Assume $S$ did not evict $p'$ before we see $p'$.

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Proof.

7. If $S$ evicted the page $p'$, $S'$ will evict the page $p^*$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.
Proof.

7. If $S$ evicted the page $p'$, $S'$ will evict the page $p^*$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

8. Assume $S$ did not evict $p'(=2)$ before we see $p'(=2)$. 
Proof.

7. If $S$ evicted the page $p'$, $S'$ will evict the page $p^*$. Then, the cache status of $S$ and that of $S'$ will be the same. $S$ and $S'$ will be exactly the same from now on.

8. Assume $S$ did not evict $p'(=2)$ before we see $p'(=2)$. 
Proof.

If $S$ evicts $p^*$ ($=3$) for $p'$ ($=2$), then $S$ won't be optimum. Assume otherwise.

So far, $S'$ has 1 less page-miss than $S$ does.

The status of $S'$ and that of $S$ only differ by 1 page.
Proof.

If $S$ evicts $p^* (=3)$ for $p' (=2)$, then $S$ won’t be optimum. Assume otherwise.

So far, $S'$ has 1 less page-miss than $S$ does.

The status of $S'$ and that of $S$ only differ by 1 page.
**Proof.**

If $S$ evicts $p^*(=3)$ for $p' (=2)$, then $S$ won't be optimum. Assume otherwise. So far, $S'$ has 1 less page-miss than $S$ does. The status of $S'$ and that of $S$ only differ by 1 page.

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Proof.

9. If $S$ evicts $p^* (=3)$ for $p' (=2)$, then $S$ won’t be optimum. Assume otherwise.
Proof.

9. If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.
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### Proof.

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### Proof.

9. If $S$ evicts $p^*(=3)$ for $p'(=2)$, then $S$ won’t be optimum. Assume otherwise.

10. So far, $S'$ has 1 less page-miss than $S$ does.
**Proof.**

9. If $S$ evicts $p^* (=3)$ for $p' (=2)$, then $S$ won’t be optimum. Assume otherwise.

10. So far, $S'$ has 1 less page-miss than $S$ does.

11. The status of $S'$ and that of $S$ only differ by 1 page.
Proof.
Proof.

12. We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.
Proof.

12. We can then guarantee that $S'$ make at most the same number of page-misses as $S$ does.

   Idea: if $S$ has a page-hit and $S'$ has a page-miss, we use the opportunity to make the status of $S'$ the same as that of $S$. □
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. There is an optimum solution in which $p^*$ is evicted at time 1.
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma** Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. **It is safe to evict $p^*$ at time 1.**
Thus, we have shown how to create another solution $S'$ with the same number of page-misses as that of the optimum solution $S$. Thus, we proved

**Lemma**  Assume at time 1 a page fault happens and there are no empty pages in the cache. Let $p^*$ be the page in cache that is not requested until furthest in the future. It is safe to evict $p^*$ at time 1.

**Theorem**  The furthest-in-future strategy is optimum.
1: for $t \leftarrow 1$ to $T$ do
2: if $\rho_t$ is in cache then do nothing
3: else if there is an empty page in cache then
4: evict the empty page and load $\rho_t$ in cache
5: else
6: $p^* \leftarrow$ page in cache that is not used furthest in the future
7: evict $p^*$ and load $\rho_t$ in cache
Q: How can we make the algorithm as fast as possible?

A: The running time can be made to be $O(n + T \log k)$. For each page $p$, use a linked list (or an array with dynamic size) to store the time steps in which $p$ is requested. We can find the next time a page is requested easily. Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
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  - We can find the next time a page is requested easily.
- Use a priority queue data structure to hold all the pages in cache, so that we can easily find the page that is requested furthest in the future.
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Priority queue:

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Tasks:

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P2: 4 7
P3: 6 9 12
P4: 3 8
P5: 2 5 11
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**Priority Queue**

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**Priority Queue Entries**

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
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- **P5:** 2 5 11
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P5:  

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priority queue

pages

values

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<td>9</td>
<td></td>
<td></td>
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<tr>
<td>10</td>
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<tr>
<td>11</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P2</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

- **P1:** 1 | 10

- **P2:** 4 | 7

- **P3:** 6 | 9 | 12

- **P4:** 3 | 8

- **P5:** 2 | 5 | 11

**Priority Queue**

<table>
<thead>
<tr>
<th>pages</th>
<th>priority values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>$\infty$</td>
</tr>
</tbody>
</table>
### Priority Queue

<table>
<thead>
<tr>
<th>Pages</th>
<th>Priority Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>∞</td>
</tr>
<tr>
<td>P3</td>
<td>12</td>
</tr>
<tr>
<td>P4</td>
<td>∞</td>
</tr>
</tbody>
</table>

### Time and Pages

<table>
<thead>
<tr>
<th>Time</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>P1</td>
</tr>
<tr>
<td>1</td>
<td>P5</td>
</tr>
<tr>
<td>2</td>
<td>P4</td>
</tr>
<tr>
<td>3</td>
<td>P2</td>
</tr>
<tr>
<td>4</td>
<td>P5</td>
</tr>
<tr>
<td>5</td>
<td>P3</td>
</tr>
<tr>
<td>6</td>
<td>P2</td>
</tr>
<tr>
<td>7</td>
<td>P4</td>
</tr>
<tr>
<td>8</td>
<td>P3</td>
</tr>
<tr>
<td>9</td>
<td>P1</td>
</tr>
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</tr>
<tr>
<td>12</td>
<td></td>
</tr>
<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
</tr>
<tr>
<td>pages</td>
<td>P1</td>
</tr>
</tbody>
</table>

P1:  

- 1
- 10

P2:  

- 4
- 7

P3:  

- 6
- 9
- 12

P4:  

- 3
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P5:  

- 2
- 5
- 11

**Priority Queue**

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</tbody>
</table>

Red X marks indicate pages that were swapped,
Green checkmarks indicate pages that were not swapped.
The diagram illustrates the priority queue algorithm for handling page faults in a computer system. The time axis is marked from 0 to 12, and the pages are listed for each process (P1 to P5) at specific time points.

The priority queue is structured as a table with two columns: "pages" and "priority values." The pages column lists the pages of each process, and the priority values indicate the priority level for each page. The green check marks represent pages that have been successfully retrieved from memory, while the red X marks indicate pages that are still waiting to be fetched.

For example, at time 4, the priority queue shows:
- P3: 12
- P4: ∞

This indicates that P3 has the highest priority and P4 has an infinite priority, indicating a different priority scheme for each page.
<table>
<thead>
<tr>
<th>time</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>pages</td>
<td>P1</td>
<td>P5</td>
<td>P4</td>
<td>P2</td>
<td>P5</td>
<td>P3</td>
<td>P2</td>
<td>P4</td>
<td>P3</td>
<td>P1</td>
<td>P5</td>
<td>P3</td>
<td></td>
</tr>
</tbody>
</table>

- **P1:** 1 10
- **P2:** 4 7
- **P3:** 6 9 12
- **P4:** 3 8
- **P5:** 2 5 11

---

**Priority Queue**

<table>
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<td>∞</td>
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<tr>
<td>time</td>
<td>0</td>
</tr>
<tr>
<td>------</td>
<td>---</td>
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<td>pages</td>
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<td>∞</td>
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</table>

**Pages Accessed**
- P1: 1 10
- P2: 4 7
- P3: 6 9 12
- P4: 3 8
- P5: 2 5 11
1: for every \( p \leftarrow 1 \) to \( n \) do
2: \( \text{times}[p] \leftarrow \) array of times in which \( p \) is requested, in increasing order \( \triangleright \) put \( \infty \) at the end of array
3: \( \text{pointer}[p] \leftarrow 1 \)
4: \( Q \leftarrow \) empty priority queue
5: for every \( t \leftarrow 1 \) to \( T \) do
6: \( \text{pointer}[\rho_t] \leftarrow \text{pointer}[\rho_t] + 1 \)
7: \( \text{nexttime}[\rho_t] \leftarrow \text{times}[\rho_t, \text{pointer}[\rho_t]] \)
8: if \( \rho_t \in Q \) then
9: \( Q.\text{increase-key}(\rho_t, \text{nexttime}[\rho_t]), \) print “hit”, continue
10: if \( Q.\text{size}() \leq k \) then
11: print “load \( \rho_t \) to an empty page”
12: else
13: \( p \leftarrow Q.\text{extract-max}(), \) print “evict \( p \) and load \( \rho_t \)”
14: \( Q.\text{insert}(\rho_t, \text{nexttime}[\rho_t]) \) \( \triangleright \) add \( \rho_t \) to \( Q \) with key value
Outline

Toy Example: Box Packing

Interval Scheduling

Offline Caching
  Heap: Concrete Data Structure for Priority Queue

Data Compression and Huffman Code

Summary
Let $V$ be a ground set of size $n$.

**Def.** A priority queue is an abstract data structure that maintains a set $U \subseteq V$ of elements, each with an associated key value, and supports the following operations:

- **insert($v$, key_value):** insert an element $v \in V \setminus U$, with associated key value key_value.
- **decrease_key($v$, new_key_value):** decrease the key value of an element $v \in U$ to new_key_value.
- **extract_min():** return and remove the element in $U$ with the smallest key value.
- ...
Simple Implementations for Priority Queue

- $n = \text{size of ground set } V$

<table>
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<tr>
<th>data structures</th>
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<tbody>
<tr>
<td>array</td>
<td></td>
<td></td>
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Simple Implementations for Priority Queue

$n =$ size of ground set $V$

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Simple Implementations for Priority Queue

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**Simple Implementations for Priority Queue**

$\n = \text{size of ground set } V$

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</tr>
<tr>
<td>heap</td>
<td>$O(\log n)$</td>
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</table>
Heap

The elements in a heap is organized using a complete binary tree:

- Nodes are indexed as \{1, 2, 3, \ldots , s}\}
- Parent of node \(i\): \(\lfloor i/2 \rfloor\)
- Left child of node \(i\): \(2i\)
- Right child of node \(i\): \(2i + 1\)
A heap $H$ contains the following fields:

- $s$: size of $U$ (number of elements in the heap)
- $A[i], 1 \leq i \leq s$: the element at node $i$ of the tree
- $p[v], v \in U$: the index of node containing $v$
- $key[v], v \in U$: the key value of element $v$

1. $s = 5$
2. $A = (\textsc{f}, \textsc{g}, \textsc{c}, \textsc{e}, \textsc{b})$
3. $p[\textsc{f}] = 1, p[\textsc{g}] = 2, p[\textsc{c}] = 3, p[\textsc{e}] = 4, p[\textsc{b}] = 5$
The following **heap property** is satisfied:

- for any two nodes \( i, j \) such that \( i \) is the parent of \( j \), we have \( key[A[i]] \leq key[A[j]] \).

A heap. Numbers in the circles denote key values of elements.
\( \text{insert}(v, \text{key_value}) \)
\textbf{insert}(v, \textit{key\_value})
$insert(v, key_{\text{value}})$
\textbf{insert}(v, \textit{key\_value})
\text{insert}(v, key_value)
**insert**\((v, key_value)\)

1: \( s \leftarrow s + 1 \)
2: \( A[s] \leftarrow v \)
3: \( p[v] \leftarrow s \)
4: \( key[v] \leftarrow key_value \)
5: **heapify-up**\((s)\)

**heapify-up**\((i)\)

1: **while** \( i > 1 \) **do**
2: \( j \leftarrow \lfloor i/2 \rfloor \)
3: **if** \( key[A[i]] < key[A[j]] \) **then**
4: \( \text{swap } A[i] \text{ and } A[j] \)
5: \( p[A[i]] \leftarrow i, p[A[j]] \leftarrow j \)
6: \( i \leftarrow j \)
7: **else** break
extract_min()
extract_min()
extract_min()
extract_min()
extract_min()
extract_min()
**extract_min()**

1: \( \text{ret} \leftarrow A[1] \)
3: \( p[A[1]] \leftarrow 1 \)
4: \( s \leftarrow s - 1 \)
5: if \( s \geq 1 \) then
6: \( \text{heapify\_down}(1) \)
7: return \( \text{ret} \)

**heapify-down(i)**

1: while \( 2i \leq s \) do
2: if \( 2i = s \) or \( \text{key}[A[2i]] \leq \text{key}[A[2i + 1]] \) then
3: \( j \leftarrow 2i \)
4: else
5: \( j \leftarrow 2i + 1 \)
6: if \( \text{key}[A[j]] < \text{key}[A[i]] \) then
7: swap \( A[i] \) and \( A[j] \)
8: \( p[A[i]] \leftarrow i, p[A[j]] \leftarrow j \)
9: \( i \leftarrow j \)
10: else break

**decrease_key(v, key_val)**

1: \( \text{key}[v] \leftarrow \text{key\_value} \)
2: \( \text{heapify\_up}(p[v]) \)
Running time of heapify\_up and heapify\_down: $O(\lg n)$
- Running time of heapify\_up and heapify\_down: $O(lg\ n)$
- Running time of insert, exact\_min and decrease\_key: $O(lg\ n)$
Running time of heapify\_up and heapify\_down: $O(\lg n)$

Running time of insert, exact\_min and decrease\_key: $O(\lg n)$

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Two Definitions Needed to Prove that the Procedures Maintain Heap Property

Def. We say that $H$ is almost a heap except that $key[A[i]]$ is too small if we can increase $key[A[i]]$ to make $H$ a heap.

Def. We say that $H$ is almost a heap except that $key[A[i]]$ is too big if we can decrease $key[A[i]]$ to make $H$ a heap.
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Data Compression and Huffman Code

Summary
Encoding Letters Using Bits

- 8 letters $a, b, c, d, e, f, g, h$ in a language
- need to encode a message using bits
- idea: use 3 bits per letter

\[
\begin{array}{cccccccc}
  a & b & c & d & e & f & g & h \\
  000 & 001 & 010 & 011 & 100 & 101 & 110 & 111 \\
\end{array}
\]

\[
deacfg \rightarrow 011100000010101110
\]

Q: Can we have a better encoding scheme?
- Seems unlikely: must use 3 bits per letter

Q: What if some letters appear more frequently than the others?
Q: If some letters appear more frequently than the others, can we have a better encoding scheme?

A: Using variable-length encoding scheme might be more efficient.

Idea

- using fewer bits for letters that are more frequently used, and more bits for letters that are less frequently used.
Q: What is the issue with the following encoding scheme?

$\begin{align*}
a & : 0 \\
b & : 1 \\
c & : 00
\end{align*}$

A: Can not guarantee a unique decoding. For example, 00 can be decoded to $a$ or $c$.

Solution: Use prefix codes to guarantee a unique decoding.
Q: What is the issue with the following encoding scheme?
▶ $a: 0 \quad b: 1 \quad c: 00$

A: Can not guarantee a unique decoding. For example, 00 can be decoded to $aa$ or $c$.  

Solution: Use prefix codes to guarantee a unique decoding.
Q: What is the issue with the following encoding scheme?

▶

a: 0  
b: 1  
c: 00

A: Can not guarantee a unique decoding. For example, 00 can be decoded to $aa$ or $c$.

Solution

Use prefix codes to guarantee a unique decoding.
Prefix Codes

**Def.** A prefix code for a set $S$ of letters is a function $\gamma : S \rightarrow \{0, 1\}^*$ such that for two distinct $x, y \in S$, $\gamma(x)$ is not a prefix of $\gamma(y)$. 
Prefix Codes

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<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>001</td>
<td>0000</td>
<td>0001</td>
<td>100</td>
</tr>
<tr>
<td>$e$</td>
<td>11</td>
<td>1010</td>
<td>1011</td>
<td>01</td>
</tr>
</tbody>
</table>
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.
Prefix Codes Guarantee Unique Decoding

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Diagram:
```
  0 1
  v v
 a  h
  0 1
  v v
 b c
```
```
  0 1
  v v
 d e
  0 1
  v v
 f g
```
Prefix Codes Guarantee Unique Decoding

▶ Reason: there is only one way to cut the first code.

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▶ 0001001100000001011110100001001
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- 0001/001100000001011110100001001
- c
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<tr>
<td>e</td>
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<td>1010</td>
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- 0001/001/100/0000/01/011110100001001
- cadbh
Prefix Codes Guarantee Unique Decoding

- Reason: there is only one way to cut the first code.

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▶ 0001/001/100/0000/01/01/11/1010/0001/001/

▶ cadbhhefca
Properties of Encoding Tree

- Rooted binary tree
- Left edges labelled 0 and right edges labelled 1
- A leaf corresponds to a code for some letter
- If coding scheme is not wasteful: a non-leaf has exactly two children

Best Prefix Codes

Input: frequencies of letters in a message
Output: prefix coding scheme with the shortest encoding for the message
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### Example

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<tr>
<th>letters</th>
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<th>c</th>
<th>d</th>
<th>e</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequencies</td>
<td>18</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>10</td>
</tr>
</tbody>
</table>

- **Scheme 1**: a d e
- **Scheme 2**: b c b c d e
- **Scheme 3**: a

```plaintext
scheme 1
b c

scheme 2
b c d e

scheme 3
d
b c
e
```
## Example

<table>
<thead>
<tr>
<th>letters</th>
<th>a</th>
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<td>scheme 1 length</td>
<td>2</td>
<td>3</td>
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<tr>
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![Scheme 1](image1)

![Scheme 2](image2)

![Scheme 3](image3)
Example Input: \((a: 18, b: 3, c: 4, d: 6, e: 10)\)
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**Q:** What types of decisions should we make?
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- Not clear how to design the greedy algorithm

A: We can choose two letters and make them brothers in the tree.
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree

![Diagram of an encoding tree with two deepest leaves highlighted as brothers.]

Lemma
It is safe to make the two least frequent letters brothers.
Which Two Letters Can Be Safely Put Together As Brothers?

- Focus on the “structure” of the optimum encoding tree
- There are two deepest leaves that are brothers
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**Lemma** It is safe to make the two least frequent letters brothers.
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**Q:** Is the residual problem another instance of the best prefix codes problem?
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- So we can irrevocably decide to make the two least frequent letters brothers.

Q: Is the residual problem another instance of the best prefix codes problem?

A: Yes, though it is not immediate to see why.
- \( f_x \): the frequency of the letter \( x \) in the support.
- \( x_1 \) and \( x_2 \): the two letters we decided to put together.
- \( d_x \) the depth of letter \( x \) in our output encoding tree.

\[
\sum_{x \in S} f_x d_x = \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + f_{x_1} d_{x_1} + f_{x_2} d_{x_2} = \sum_{x \in S \setminus \{x_1, x_2\}} f_x d_x + (f_{x_1} + f_{x_2})d_{x_1}
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Def: \( f_{x'} = f_{x_1} + f_{x_2} \)
In order to minimize
\[ \sum_{x \in S} fx dx, \]
we need to minimize
\[ \sum_{x \in S \setminus \{x_1, x_2\} \cup \{x'\}} fx dx, \]
subject to that \( d \) is the depth function for an encoding tree of \( S \setminus \{x_1, x_2\} \).

▶ This is exactly the best prefix codes problem, with letters \( S \setminus \{x_1, x_2\} \cup \{x'\} \) and frequency vector \( f \)!
Example

A  27  B  15  C  11  D  9  E  8  F  5
Example
Example
Example
Example
Example
Example
Example

\[
\begin{align*}
A & : 00 \\
B & : 10 \\
C & : 010 \\
D & : 011 \\
E & : 110 \\
F & : 111
\end{align*}
\]
Def. The codes given the greedy algorithm is called the Huffman codes.
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**Huffman**\((S, f)\)

1: while \(|S| > 1\) do  
2: let \(x_1, x_2\) be the two letters with the smallest \(f\) values  
3: introduce a new letter \(x'\) and let \(f_{x'} = f_{x_1} + f_{x_2}\)  
4: let \(x_1\) and \(x_2\) be the two children of \(x'\)  
5: \(S \leftarrow S \setminus \{x_1, x_2\} \cup \{x'\}\)  
6: return the tree constructed
Huffman($S, f$)

1: $Q \leftarrow \text{build-priority-queue}(S)$
2: while $Q$.size > 1 do
3:     $x_1 \leftarrow Q$.extract-min()
4:     $x_2 \leftarrow Q$.extract-min()
5:     introduce a new letter $x'$ and let $f_{x'} = f_{x_1} + f_{x_2}$
6:     let $x_1$ and $x_2$ be the two children of $x'$
7:     $Q$.insert($x'$, $f_{x'}$)
8: return the tree constructed
Outline

Toy Example: Box Packing

Interval Scheduling

Offline Caching
  Heap: Concrete Data Structure for Priority Queue

Data Compression and Huffman Code

Summary
Summary for Greedy Algorithms

Greedy Algorithm

- Build up the solutions in steps
- At each step, make an irrevocable decision using a “reasonable” strategy

Interval scheduling problem: schedule the job \( j^* \) with the earliest deadline

Offline Caching: evict the page that is used furthest in the future

Huffman codes: make the two least frequent letters brothers
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Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)
## Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
- Show that the remaining task after applying the strategy is to solve a (many) smaller instance(s) of the same problem (usually easy)

## Def.
A strategy is “safe” if there is always an optimum solution that “agrees with” the decision made according to the strategy.
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
- If $S$ agrees with the decision made according to the strategy, done
Proving a Strategy is Safe

- Take an arbitrary optimum solution $S$
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- Change $S$ slightly to another optimum solution $S'$ that agrees with the decision
  - Interval scheduling problem: exchange $j^*$ with the first job in an optimal solution
  - Offline caching: a complicated “copying” algorithm
  - Huffman codes: move the two least frequent letters to the deepest leaves.
### Analysis of Greedy Algorithm

- Prove that the reasonable strategy is “safe” (key)
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- Interval scheduling problem: remove $j^*$ and the jobs it conflicts with
- Offline caching: trivial
- Huffman codes: merge two letters into one