CSE 431/531: Algorithm Analysis and Design (Fall 2021)

Introduction and Syllabus

Lecturer: Shi Li
Department of Computer Science and Engineering
University at Buffalo
Outline

Syllabus

Introduction
   What is an Algorithm?
   Example: Insertion Sort
   Analysis of Insertion Sort

Asymptotic Notations

Common Running times
Course Webpage (contains schedule, policies, homeworks and slides):
http://www.cse.buffalo.edu/~shil/courses/CSE531/

Please sign up course on Piazza via link on course webpage
- announcements, polls, asking/answering questions
CSE 431/531: Algorithm Analysis and Design

- **Time & Location**: 5:30pm-6:45pm, Davis 101
- **Hybrid mode**
  - Session 1: In person on Tuesdays, Remote on Thursdays
  - Session 2: Remote on Tuesdays, In person on Thursdays
- **Instructor**:
  - Shi Li, shil@buffalo.edu
- **TAs**:
  - Xiaoyu Zhang, Charles Wiechec, Yunus Esencayi
COVID-Related Information

- Get vaccinated
- Wear a mask

<table>
<thead>
<tr>
<th>What do I do if I don’t feel well?</th>
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<tbody>
<tr>
<td>Your safety and the safety of your class-mates comes first</td>
</tr>
<tr>
<td>Follow UB procedure</td>
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<tr>
<td>Do not come to class– just send me an email, and we can meet on Zoom temporarily while you sort things out– even if it is false alarm!</td>
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<td>Your privacy will be protected to the extent that is reasonably possible</td>
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You **should** already have/know:

- Mathematical Background
  - basic reasoning skills, inductive proofs
- Basic data Structures
  - linked lists, arrays
  - stacks, queues
- Some Programming Experience
  - Python, C, C++ or Java
You Will Learn

▶ Classic algorithms for classic problems
  ▶ Sorting, shortest paths, minimum spanning tree, · · ·
▶ How to analyze algorithms
  ▶ Correctness
  ▶ Running time (efficiency)
▶ Meta techniques to design algorithms
  ▶ Greedy algorithms
  ▶ Divide and conquer
  ▶ Dynamic programming
  ▶ · · ·
▶ NP-completeness
# Tentative Schedule

- 75 Minutes/Lecture \( \times \) 29 Lectures

<table>
<thead>
<tr>
<th>Topic</th>
<th>Lectures</th>
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<tr>
<td>Introduction</td>
<td>3 lectures</td>
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<tr>
<td>Graph Basics</td>
<td>2 lectures</td>
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<tr>
<td>Greedy Algorithms</td>
<td>5 lectures</td>
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<tr>
<td>Divide and Conquer</td>
<td>5 lectures</td>
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<tr>
<td>Dynamic Programming</td>
<td>5 lectures</td>
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<td>Graph Algorithms</td>
<td>5 lectures</td>
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<tr>
<td>NP-Completeness</td>
<td>3 lectures</td>
</tr>
<tr>
<td>Final Review</td>
<td>1 lecture</td>
</tr>
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</table>
Textbook

Textbook (Highly Recommended):

▶ Algorithm Design, 1st Edition, by Jon Kleinberg and Eva Tardos

Other Reference Books

▶ Introduction to Algorithms, Third Edition, Thomas Cormen, Charles Leiserson, Rondald Rivest, Clifford Stein
Highly recommended: read the correspondent sections from the textbook (or reference book) before classes.

Sections for each lecture can be found on the course webpage.

Slides are posted on course webpage. They may get updated before the classes start.

In last lecture of a major topic (Greedy Algorithms, Divide and Conquer, Dynamic Programming, Graph Algorithms), I will discuss exercise problems, which will be posted on the course webpage before class.
Grading

- 40% for theory homeworks
  - 8 points × 5 theory homeworks
- 20% for programming problems
  - 10 points × 2 programming assignments
- 40% for final exam
For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussions
  - **Must write down solutions on your own, in your own words**
  - Write down names of students you collaborated with
For Homeworks, You Are Not Allowed to

- Use external resources
  - Can’t Google or ask questions online for solutions
  - Can’t read posted solutions from other algorithm course webpages
- Copy solutions from other students
For Programming Problems

- Need to implement the algorithms by yourself
- Can not copy codes from others or the Internet
- We use Moss (https://theory.stanford.edu/~aiken/moss/) to detect similarity of programs
Late Policy

- You have 1 “late credit”, using it allows you to submit an assignment solution for three days.
- With no special reasons, no other late submissions will be accepted.
Final Exam will be closed-book

Academic Integrity (AI) Policy for the Course

- minor violation:
  - 0 score for the involved homework/prog. assignment, and
  - 1-letter grade down
- 2 minor violations = 1 major violation
  - failure for the course
  - case will be reported to the department and university
  - further sanctions may include a dishonesty mark on transcript or expulsion from university

Questions?
Outline

Syllabus

Introduction
  What is an Algorithm?
  Example: Insertion Sort
  Analysis of Insertion Sort

Asymptotic Notations

Common Running times
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Asymptotic Notations

Common Running times
What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.

- Computational problem: specifies the input/output relationship.

- An algorithm solves a computational problem if it produces the correct output for any given input.
**Examples**

**Greatest Common Divisor**

**Input:** two integers \(a, b > 0\)

**Output:** the greatest common divisor of \(a\) and \(b\)

---

**Example:**

- **Input:** 210, 270
- **Output:** 30

**Algorithm:** Euclidean algorithm

\[
gcd(270, 210) = gcd(210, 270 \mod 210) = gcd(210, 60)
\]

\[
(270, 210) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)
\]
Examples

Sorting

**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

**Output:** a permutation \((a'_1, a'_2, \cdots, a'_n)\) of the input sequence such that \(a'_1 \leq a'_2 \leq \cdots \leq a'_n\)

Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

- **Algorithms:** insertion sort, merge sort, quicksort, \ldots
Examples

Shortest Path

Input: directed graph $G = (V, E)$, $s, t \in V$
Output: a shortest path from $s$ to $t$ in $G$

▶ Algorithm: Dijkstra’s algorithm
Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, using a particular programming language
Pseudo-Code:

Euclidean\((a, b)\)

1: **while** \(b > 0\) **do**
2: \((a, b) \leftarrow (b, a \ mod \ b)\)
3: **return** \(a\)

C++ program:

```cpp
int Euclidean(int a, int b) {
    int c;
    while (b > 0) {
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```
Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g., GUI)
  - ...
- Why is it important to study the running time (efficiency) of an algorithm?
  1. feasible vs. infeasible
  2. efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g., python)
  3. fundamental
  4. it is fun!
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Sorting Problem

**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

**Output:** a permutation \((a'_1, a'_2, \cdots, a'_n)\) of the input sequence such that \(a'_1 \leq a'_2 \leq \cdots \leq a'_n\)

**Example:**

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59
At the end of $j$-th iteration, the first $j$ numbers are sorted.

iteration 1: 53, 12, 35, 21, 59, 15
iteration 2: 12, 53, 35, 21, 59, 15
iteration 3: 12, 35, 53, 21, 59, 15
iteration 4: 12, 21, 35, 53, 59, 15
iteration 5: 12, 21, 35, 53, 59, 15
iteration 6: 12, 15, 21, 35, 53, 59
Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

**insertion-sort**($A, n$)

1: for $j \leftarrow 2$ to $n$ do
2:     $key \leftarrow A[j]$
3:     $i \leftarrow j - 1$
4:     while $i > 0$ and $A[i] > key$ do
5:         $A[i + 1] \leftarrow A[i]$
6:         $i \leftarrow i - 1$
7:     $A[i + 1] \leftarrow key$

- $j = 6$
- $key = 15$

12 15 21 35 53 59

↑
i
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Asymptotic Notations

Common Running times
Analysis of Insertion Sort

- Correctness
- Running time
Correctness of Insertion Sort


  after $j = 1 : 53, 12, 35, 21, 59, 15$
  after $j = 2 : 12, 53, 35, 21, 59, 15$
  after $j = 3 : 12, 35, 53, 21, 59, 15$
  after $j = 4 : 12, 21, 35, 53, 59, 15$
  after $j = 5 : 12, 21, 35, 53, 59, 15$
  after $j = 6 : 12, 15, 21, 35, 53, 59$
Analyzing Running Time of Insertion Sort

Q1: what is the size of input?

A1: Running time as the function of size

possible definition of size:

- Sorting problem: # integers,
- Greatest common divisor: total length of two integers
- Shortest path in a graph: # edges in graph

Q2: Which input?

- For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.

A2: Worst-case analysis:

- Running time for size $n = \text{worst running time over all possible arrays of length } n$
Q3: How fast is the computer?

Q4: Programming language?

A: They do not matter!

**Important idea: asymptotic analysis**

Focus on growth of running-time as a function, not any particular value.
Asymptotic Analysis: $O$-notation

Informal way to define $O$-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$
- $n^2/100 - 3n + 10 = O(n^2)$
Asymptotic Analysis: $O$-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n^2 + 10 = O(n^2)$

$O$-notation allows us to ignore
- architecture of computer
- programming language
- how we measure the running time: seconds or \# instructions?

to execute $a \leftarrow b + c$:
- program 1 requires 10 instructions, or $10^{-8}$ seconds
- program 2 requires 2 instructions, or $10^{-9}$ seconds
- they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation
Asymptotic Analysis: $O$-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$
- Does not tell which algorithm is faster for a specific $n$!
- Algorithm 2 will eventually beat algorithm 1 as $n$ increases.
- For Algorithm 1: if we increase $n$ by a factor of 2, running time increases by a factor of 4
- For Algorithm 2: if we increase $n$ by a factor of 2, running time increases by a factor of 2
Asymptotic Analysis of Insertion Sort

insertion-sort(\(A, n\))

1: for \(j \leftarrow 2\) to \(n\) do
2: key \(\leftarrow A[j]\)
3: \(i \leftarrow j - 1\)
4: while \(i > 0\) and \(A[i] > key\) do
5: \(A[i + 1] \leftarrow A[i]\)
6: \(i \leftarrow i - 1\)
7: \(A[i + 1] \leftarrow key\)

Worst-case running time for iteration \(j\) of the outer loop?
Answer: \(O(j)\)

Total running time = \(\sum_{j=2}^{n} O(j) = O(\sum_{j=2}^{n} j)\)
= \(O\left(\frac{n(n+1)}{2} - 1\right) = O(n^2)\)
Computation Model

- Random-Access Machine (RAM) model
  - reading and writing $A[j]$ takes $O(1)$ time
- Basic operations such as addition, subtraction and multiplication take $O(1)$ time
- Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough
  - Reason: often we need to read the integer $n$ and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.
- What is the precision of real numbers?
  - Most of the time, we only consider integers.
- Can we do better than insertion sort asymptotically?
  - Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time
Remember to sign up for Piazza.

Questions?
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Asymptotic Notations

Common Running times
Asymptotically Positive Functions

**Def.** $f : \mathbb{N} \to \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

- In other words, $f(n)$ is positive for large enough $n$.

- $n^2 - n - 30$ Yes
- $2^n - n^{20}$ Yes
- $100n - n^2/10 + 50?$ No

- We only consider asymptotically positive functions.
**O-Notation**  For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$  

In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c > 0$ and every large enough $n$. 

---

![Diagram of functions](image)
**O-Notation: Asymptotic Upper Bound**

**O-Notation**  For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c > 0$ and every large enough $n$.
- $3n^2 + 2n \in O(n^2 - 10n)$

**Proof.**

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

$$3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n) = -n^2 + 40n \leq 0.$$

$$3n^2 + 2n \leq c(n^2 - 10n)$$
**O-Notation**  For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$

- In other words, $f(n) \in O(g(n))$ if $f(n) \leq cg(n)$ for some $c$ and large enough $n$.

- $3n^2 + 2n \in O(n^2 - 10n)$
- $3n^2 + 2n \in O(n^3 - 5n^2)$
- $n^{100} \in O(2^n)$
- $n^3 \notin O(10n^2)$

<table>
<thead>
<tr>
<th>Asymptotic Notations</th>
<th>$O$</th>
<th>$\Omega$</th>
<th>$\Theta$</th>
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</thead>
<tbody>
<tr>
<td>Comparison Relations</td>
<td>$\leq$</td>
<td></td>
<td></td>
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</tbody>
</table>
Conventions

- We use \( f(n) = O(g(n)) \) to denote \( f(n) \in O(g(n)) \)

\[
3n^2 + 2n = O(n^3 - 10n)
\]

\[
3n^2 + 2n = O(n^2 + 5n)
\]

\[
3n^2 + 2n = O(n^2)
\]

"=" is asymmetric! Following equalities are wrong:

\[
O(n^3 - 10n) = 3n^2 + 2n
\]

\[
O(n^2 + 5n) = 3n^2 + 2n
\]

\[
O(n^2) = 3n^2 + 2n
\]

- Analogy: Mike is a student. A student is Mike.
**Ω-Notation: Asymptotic Lower Bound**

### O-Notation

For a function \( g(n) \),

\[
O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.
\]

### Ω-Notation

For a function \( g(n) \),

\[
Ω(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \}.
\]

▶ In other words, \( f(n) \in Ω(g(n)) \) if \( f(n) \geq cg(n) \) for some \( c \) and large enough \( n \).
Ω-Notation: Asymptotic Lower Bound

Ω-Notation For a function $g(n)$,

$$\Omega(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \}.$$
Again, we use “=” instead of $\in$.

$4n^2 = \Omega(n - 10)$

$3n^2 - n + 10 = \Omega(n^2 - 20)$

<table>
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</tbody>
</table>

**Theorem**  
$f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$. 
**Θ-Notation: Asymptotic Tight Bound**

**Θ-Notation**  For a function $g(n)$,

$$\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}. $$

- $f(n) = \Theta(g(n))$, then for large enough $n$, we have “$f(n) \approx g(n)$”. 

![Graph showing the relationship between $f(n)$ and $g(n)$ with a notation of $\Theta$ indicating asymptotic tight bound.](image-url)
\(\Theta\)-Notation: Asymptotic Tight Bound

**\(\Theta\)-Notation**  For a function \(g(n)\),

\[
\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}.
\]

- \(3n^2 + 2n = \Theta(n^2 - 20n)\)
- \(2^{n/3} + 100 = \Theta(2^{n/3})\)

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</table>

**Theorem**  \(f(n) = \Theta(g(n))\) if and only if \(f(n) = O(g(n))\) and \(f(n) = \Omega(g(n))\).
### Trivial Facts on Comparison Relations

- $a \leq b \iff b \geq a$
- $a = b \iff a \leq b \text{ and } a \geq b$
- $a \leq b \text{ or } a \geq b$

### Correct Analogies

- $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
- $f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \text{ and } f(n) = \Omega(g(n))$

### Incorrect Analogy

- $f(n) = O(g(n)) \text{ or } f(n) = \Omega(g(n))$
Incorrect Analogy

\[ f(n) = O(g(n)) \text{ or } f(n) = \Omega(f(n)) \]

\[ f(n) = n^2 \]

\[ g(n) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
n^3 & \text{if } n \text{ is even}
\end{cases} \]
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2)$, $3n^2 - 10n - 5 = \Theta(n^2)$

- In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.
- $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, though weird
- $3n^2 - 10n - 5 = O(n^2)$ is the most natural since $n^2$ is the simplest term we can have inside $O(\cdot)$. 
Notice that $O$ denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.
- We say: the running time of the insertion sort algorithm is $O(n^2)$ and the bound is tight.
- We do not use $\Omega$ and $\Theta$ very often when we upper bound running times.
Exercise

For each pair of functions $f, g$ in the following table, indicate whether $f$ is $O, \Omega$ or $\Theta$ of $g$.

<table>
<thead>
<tr>
<th>$f$</th>
<th>$g$</th>
<th>$O$</th>
<th>$\Omega$</th>
<th>$\Theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n^3 - 100n$</td>
<td>$5n^2 + 3n$</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$3n - 50$</td>
<td>$n^2 - 7n$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$n^2 - 100n$</td>
<td>$5n^2 + 30n$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$\log_2 n$</td>
<td>$\log_{10} n$</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>$\log_{10} n$</td>
<td>$n^{0.1}$</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>$2^n$</td>
<td>$2^{n/2}$</td>
<td>No</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>$\sqrt{n}$</td>
<td>$n^{\sin n}$</td>
<td>No</td>
<td>No</td>
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We often use $\log n$ for $\log_2 n$. But for $O(\log n)$, the base is not important.
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Asymptotic Notations

Common Running times
Computing the sum of $n$ numbers

\[
\text{sum}(A, n)
\]

1: $S \leftarrow 0$
2: for $i \leftarrow 1$ to $n$
3: \hspace{1em} $S \leftarrow S + A[i]$
4: return $S$
Merge two sorted arrays

\[
\begin{array}{cccccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5 & 7 & 8 & 9 & 12 & 20 & 25 & 29 & 32 & 48
\end{array}
\]
**O(n) (Linear) Running Time**

merge($B, C, n_1, n_2$) \ \ \ \ $B$ and $C$ are sorted, with length $n_1$ and $n_2$

1: $A \leftarrow []; i \leftarrow 1; j \leftarrow 1$
2: while $i \leq n_1$ and $j \leq n_2$ do
3: \hspace{1em} if $B[i] \leq C[j]$ then
4: \hspace{2em} append $B[i]$ to $A$; $i \leftarrow i + 1$
5: \hspace{1em} else
6: \hspace{2em} append $C[j]$ to $A$; $j \leftarrow j + 1$
7: if $i \leq n_1$ then append $B[i..n_1]$ to $A$
8: if $j \leq n_2$ then append $C[j..n_2]$ to $A$
9: return $A$

Running time $= O(n)$ where $n = n_1 + n_2$. 
merge-sort\( (A, n) \)

1: \textbf{if} \( n = 1 \) \textbf{then}
2: \hspace{1em} \textbf{return} \( A \)
3: \textbf{else}
4: \hspace{1em} \( B \leftarrow \text{merge-sort} \left( A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor \right) \)
5: \hspace{1em} \( C \leftarrow \text{merge-sort} \left( A[\lceil n/2 \rceil + 1..n], n - \lfloor n/2 \rfloor \right) \)
6: \hspace{1em} \textbf{return} \( \text{merge}(B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor) \)
$O(n \log n)$ Running Time

- Merge-Sort

Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$
Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

**Output:** the pair of points that are closest
**Closest Pair**

**Input:** \( n \) points in plane: \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

**Output:** the pair of points that are closest

\[
\text{closest-pair}(x, y, n)
\]

1: \( \text{bestd} \leftarrow \infty \)
2: \( \textbf{for} \ i \leftarrow 1 \ \text{to} \ n - 1 \ \textbf{do} \)
3: \( \quad \textbf{for} \ j \leftarrow i + 1 \ \text{to} \ n \ \textbf{do} \)
4: \( \quad \quad d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2} \)
5: \( \quad \quad \textbf{if} \ d < \text{bestd} \ \textbf{then} \)
6: \( \quad \quad \quad \text{besti} \leftarrow i, \text{bestj} \leftarrow j, \text{bestd} \leftarrow d \)
7: \( \textbf{return} \ (\text{besti}, \text{bestj}) \)

Closest pair can be solved in \( O(n \log n) \) time!
Multiply two matrices of size $n \times n$

matrix-multiplication($A$, $B$, $n$)

1: $C \leftarrow$ matrix of size $n \times n$, with all entries being 0
2: for $i \leftarrow 1$ to $n$ do
3: for $j \leftarrow 1$ to $n$ do
4: for $k \leftarrow 1$ to $n$ do
5: \[ C[i, k] \leftarrow C[i, k] + A[i, j] \times B[j, k] \]
6: return $C$
Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

Independent set of size $k$

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$
$O(n^k)$ Running Time for Integer $k \geq 4$

**Independent Set of Size $k$**

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$

```python
def independent_set(G = (V, E)):
    for every set $S \subseteq V$ of size $k$ do
        $b \leftarrow$ true
    for every $u, v \in S$ do
        if $(u, v) \in E$ then $b \leftarrow$ false
    if $b$ return true
    return false
```

Running time $= O(\frac{n^k}{k!} \times k^2) = O(n^k)$ (assume $k$ is a constant)
### Maximum Independent Set Problem

**Input:** graph $G = (V, E)$

**Output:** the maximum independent set of $G$

#### max-independent-set($G = (V, E)$)

1. $R \leftarrow \emptyset$
2. for every set $S \subseteq V$ do
3.   $b \leftarrow \text{true}$
4.   for every $u, v \in S$ do
5.     if $(u, v) \in E$ then $b \leftarrow \text{false}$
6.   if $b$ and $|S| > |R|$ then $R \leftarrow S$
7. return $R$

Running time = $O(2^n n^2)$. 
Beyond Polynomial Time: $n!$

**Hamiltonian Cycle Problem**

**Input:** a graph with $n$ vertices

**Output:** a cycle that visits each node exactly once, or say no such cycle exists
Beyond Polynomial Time: $n!$

**Hamiltonian**\((G = (V, E))\)

1. for every permutation \((p_1, p_2, \cdots, p_n)\) of \(V\) do
2. \(b \leftarrow\) true
3. for \(i \leftarrow 1\) to \(n - 1\) do
4. \(\) if \((p_i, p_{i+1}) \notin E\) then \(b \leftarrow\) false
5. if \((p_n, p_1) \notin E\) then \(b \leftarrow\) false
6. if \(b\) then return \((p_1, p_2, \cdots, p_n)\)
7. return “No Hamiltonian Cycle”

Running time \(= O(n! \times n)\)
Binary search
- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

E.g, search 35 in the following array:
**$O(\log n)$ (Logarithmic) Running Time**

Binary search

- **Input:** sorted array $A$ of size $n$, an integer $t$;
- **Output:** whether $t$ appears in $A$.

**binary-search($A, n, t$)**

1: $i \leftarrow 1$, $j \leftarrow n$
2: **while** $i \leq j$ **do**
3: $k \leftarrow \lfloor (i + j)/2 \rfloor$
4: if $A[k] = t$ return true
5: if $t < A[k]$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6: **return** false

Running time $= O(\log n)$
Comparing the Orders

- Sort the functions from smallest to largest asymptotically: \( \log n, \ n, \ n^2, \ n \log n, \ n!, \ 2^n, \ e^n, \ n^n \)
- \( \log n = O(n) \)
- \( n = O(n^2) n = O(n \log n) \)
- \( n \log n = O(n^2) \)
- \( n^2 = O(n!) n^2 = O(2^n) \)
- \( 2^n = O(n!) 2^n = O(e^n) \)
- \( e^n = O(n!) \)
- \( n! = O(n^n) \)
When we talk about upper bound on running time:

- **Logarithmic time:** \( O(\log n) \)
- **Linear time:** \( O(n) \)
- **Quadratic time:** \( O(n^2) \)
- **Cubic time:** \( O(n^3) \)
- **Polynomial time:** \( O(n^k) \) for some constant \( k \)
- **Exponential time:** \( O(c^n) \) for some \( c > 1 \)
- **Sub-linear time:** \( o(n) \)
- **Sub-quadratic time:** \( o(n^2) \)
Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.
- Using asymptotic analysis allows us to ignore the leading constants and lower order terms.
- Makes our life much easier! (E.g., the leading constant depends on the implementation, compiler and computer architecture of computer.)
Q: Does ignoring the leading constant cause any issues?

- e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
- For “natural” algorithms, constants are not so big!
- So, for reasonably large $n$, algorithm with lower order running time beats algorithm with higher order running time.