CSE 431/531: Algorithm Analysis and Design (Fall 2021)

Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo
Outline

Syllabus

Introduction
   What is an Algorithm?
   Example: Insertion Sort
   Analysis of Insertion Sort

Asymptotic Notations

Common Running times
Course Webpage (contains schedule, policies, homeworks and slides):
http://www.cse.buffalo.edu/~shil/courses/CSE531/

Please sign up course on Piazza via link on course webpage
- announcements, polls, asking/answering questions
CSE 431/531: Algorithm Analysis and Design

- **Time & Location**: 5:30pm-6:45pm, Davis 101
- **Hybrid mode**
  - **Session 1**: In person on Tuesdays, Remote on Thursdays
  - **Session 2**: Remote on Tuesdays, In person on Thursdays
- **Instructor**:
  - Shi Li, shil@buffalo.edu
- **TAs**:
  - Xiaoyu Zhang, Charles Wiechec, Yunus Esencayi
COVID-Related Information

- Get vaccinated
- Wear a mask

What do I do if I don’t feel well?

- Your safety and the safety of your classmates comes first
- Follow UB procedure
- Do not come to class—just send me an email, and we can meet on Zoom temporarily while you sort things out—even if is false alarm!
- Your privacy will be protected to the extent that is reasonably possible
You should already have/know:
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- **Mathematical Background**
  - basic reasoning skills, inductive proofs
You should already have/know:

- Mathematical Background
  - basic reasoning skills, inductive proofs

- Basic data Structures
  - linked lists, arrays
  - stacks, queues
You should already have/know:

- **Mathematical Background**
  - basic reasoning skills, inductive proofs

- **Basic data Structures**
  - linked lists, arrays
  - stacks, queues

- **Some Programming Experience**
  - Python, C, C++ or Java
You Will Learn

▶ Classic algorithms for classic problems
▶ Sorting, shortest paths, minimum spanning tree, · · ·
You Will Learn

▶ Classic algorithms for classic problems
  ▶ Sorting, shortest paths, minimum spanning tree, · · ·
▶ How to analyze algorithms
  ▶ Correctness
  ▶ Running time (efficiency)
You Will Learn

- Classic algorithms for classic problems
  - Sorting, shortest paths, minimum spanning tree, ⋯
- How to analyze algorithms
  - Correctness
  - Running time (efficiency)
- Meta techniques to design algorithms
  - Greedy algorithms
  - Divide and conquer
  - Dynamic programming
  - ⋯
You Will Learn

- Classic algorithms for classic problems
  - Sorting, shortest paths, minimum spanning tree, · · ·

- How to analyze algorithms
  - Correctness
  - Running time (efficiency)

- Meta techniques to design algorithms
  - Greedy algorithms
  - Divide and conquer
  - Dynamic programming
  - · · ·

- NP-completeness
## Tentative Schedule

- **75 Minutes/Lecture × 29 Lectures**

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<tr>
<td>NP-Completeness</td>
<td>3 lectures</td>
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<tr>
<td>Final Review</td>
<td>1 lecture</td>
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Textbook (Highly Recommended):

- **Algorithm Design**, 1st Edition, by Jon Kleinberg and Eva Tardos

Other Reference Books

Highly recommended: read the correspondent sections from the textbook (or reference book) before classes

Sections for each lecture can be found on the course webpage.

Slides are posted on course webpage. They may get updated before the classes start.

In last lecture of a major topic (Greedy Algorithms, Divide and Conquer, Dynamic Programming, Graph Algorithms), I will discuss exercise problems, which will be posted on the course webpage before class.
Grading

- 40% for theory homeworks
  - 8 points × 5 theory homeworks
- 20% for programming problems
  - 10 points × 2 programming assignments
- 40% for final exam
For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussions
  - **Must write down solutions on your own, in your own words**
  - Write down names of students you collaborated with
For Homeworks, You Are Not Allowed to

- Use external resources
  - Can’t Google or ask questions online for solutions
  - Can’t read posted solutions from other algorithm course webpages
- Copy solutions from other students
For Programming Problems

- Need to implement the algorithms by yourself
- Can not copy codes from others or the Internet
- We use Moss (https://theory.stanford.edu/~aiken/moss/) to detect similarity of programs
Late Policy

- You have 1 “late credit”, using it allows you to submit an assignment solution for three days
- With no special reasons, no other late submissions will be accepted
Final Exam will be closed-book

### Academic Integrity (AI) Policy for the Course

- **minor violation:**
  - 0 score for the involved homework/prog. assignment, and
  - 1-letter grade down

- **2 minor violations = 1 major violation**
  - failure for the course
  - case will be reported to the department and university
  - further sanctions may include a dishonesty mark on transcript or expulsion from university
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Questions?
Outline

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  What is an Algorithm?
  Example: Insertion Sort
  Analysis of Insertion Sort

Asymptotic Notations

Common Running times
Outline

Syllabus

Introduction
  What is an Algorithm?
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Asymptotic Notations

Common Running times
What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.

- Computational problem: specifies the input/output relationship.

- An algorithm solves a computational problem if it produces the correct output for any given input.
# Greatest Common Divisor

**Input:** two integers $a, b > 0$

**Output:** the greatest common divisor of $a$ and $b$

Example:

- **Input:** 210, 270
- **Output:** 30

Algorithm: Euclidean algorithm

$$\text{gcd}(270, 210) = \text{gcd}(210, 270 \mod 210) = \text{gcd}(210, 60)$$

$$\begin{align*}
(270, 210) & \rightarrow (210, 60) \\
(210, 60) & \rightarrow (60, 30) \\
(60, 30) & \rightarrow (30, 0)
\end{align*}$$
**Greatest Common Divisor**

**Input:** two integers $a, b > 0$

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**Example:**

- **Input:** 210, 270
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Examples

Greatest Common Divisor

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Greatest Common Divisor

**Input:** two integers \(a, b > 0\)

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Example:

- **Input:** 210, 270
- **Output:** 30

**Algorithm:** Euclidean algorithm

\[
gcd(270, 210) = gcd(210, 270 \mod 210) = gcd(210, 60)
\]
**Examples**

### Greatest Common Divisor

**Input:** two integers $a, b > 0$

**Output:** the greatest common divisor of $a$ and $b$

#### Example:

- **Input:** 210, 270
- **Output:** 30

#### Algorithm: Euclidean algorithm

- $\text{gcd}(270, 210) = \text{gcd}(210, 270 \mod 210) = \text{gcd}(210, 60)$
- $(270, 210) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)$
Examples

### Sorting

**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

**Output:** a permutation \((a'_1, a'_2, \cdots, a'_n)\) of the input sequence such that \(a'_1 \leq a'_2 \leq \cdots \leq a'_n\)
Examples

Sorting

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

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Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59
Examples

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- Algorithms: insertion sort, merge sort, quicksort, ...
Examples

Shortest Path

**Input:** directed graph $G = (V, E)$, $s, t \in V$

**Output:** a shortest path from $s$ to $t$ in $G$
**Examples**

### Shortest Path

**Input:** directed graph $G = (V, E)$, $s, t \in V$

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Shortest Path

**Input:** directed graph $G = (V, E)$, $s, t \in V$

**Output:** a shortest path from $s$ to $t$ in $G$

Algorithm: Dijkstra’s algorithm
Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, using a particular programming language
Pseudo-Code

Euclidean\((a, b)\)

1. \textbf{while} \(b > 0\) do
2. \((a, b) \leftarrow (b, a \mod b)\)
3. \textbf{return} \(a\)

C++ program:

```cpp
int Euclidean(int a, int b) {
    int c;
    while (b > 0) {
        c = b;
        b = a % b;
        a = c;
    }
    return a;
}
```
Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
- Extensibility
- Modularity
- Object-oriented model
- User-friendliness (e.g., GUI)

Why is it important to study the running time (efficiency) of an algorithm?

1. Feasible vs. infeasible
2. Efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g., Python)
3. Fundamental
4. It is fun!
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  Example: Insertion Sort
  Analysis of Insertion Sort

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Sorting Problem

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

**Output:** a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

**Example:**

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59
Insertion-Sort

At the end of $j$-th iteration, the first $j$ numbers are sorted.

iteration 1: 53, 12, 35, 21, 59, 15
iteration 2: 12, 53, 35, 21, 59, 15
iteration 3: 12, 35, 53, 21, 59, 15
iteration 4: 12, 21, 35, 53, 59, 15
iteration 5: 12, 21, 35, 53, 59, 15
iteration 6: 12, 15, 21, 35, 53, 59
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort**($A, n$)

1: for $j \leftarrow 2$ to $n$ do
2: \hspace{1em} key $\leftarrow A[j]$
3: \hspace{1em} $i \leftarrow j - 1$
4: \hspace{1em} while $i > 0$ and $A[i] > key$ do
   5: \hspace{2em} $A[i + 1] \leftarrow A[i]$
   6: \hspace{2em} $i \leftarrow i - 1$
7: \hspace{1em} $A[i + 1] \leftarrow key$
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort** \((A, n)\)

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2. \(key \leftarrow A[j]\)
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4. **while** \(i > 0\) and \(A[i] > key\) **do**
5. \(A[i + 1] \leftarrow A[i]\)
6. \(i \leftarrow i - 1\)
7. \(A[i + 1] \leftarrow key\)

\(j = 6\)

\(key = 15\)

12 21 35 53 59 15
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort**($A, n$)

1: **for** $j \leftarrow 2$ to $n$ **do**
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3: \hspace{1em} $i \leftarrow j - 1$
4: **while** $i > 0$ and $A[i] > key$ **do**
5: \hspace{1.5em} $A[i + 1] \leftarrow A[i]$
6: \hspace{1.5em} $i \leftarrow i - 1$
7: \hspace{1.5em} $A[i + 1] \leftarrow key$

- $j = 6$
- $key = 15$

12 21 35 53 59 59
Example:

- Input: 53, 12, 35, 21, 59, 15
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**insertion-sort**(\(A, n\))

1: for \(j \leftarrow 2\) to \(n\) do
2: \(key \leftarrow A[j]\)
3: \(i \leftarrow j - 1\)
4: while \(i > 0\) and \(A[i] > key\) do
5: \(A[i + 1] \leftarrow A[i]\)
6: \(i \leftarrow i - 1\)
7: \(A[i + 1] \leftarrow key\)

- \(j = 6\)
- \(key = 15\)
- \(12\ 21\ 35\ 53\ 59\ 59\)

\(\uparrow\)

\(i\)
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort(A, n)**

1. for \( j \leftarrow 2 \) to \( n \) do
2.  \( key \leftarrow A[j] \)
3.  \( i \leftarrow j - 1 \)
4.  while \( i > 0 \) and \( A[i] > key \) do
5.      \( A[i + 1] \leftarrow A[i] \)
6.  \( i \leftarrow i - 1 \)
7.  \( A[i + 1] \leftarrow key \)

- \( j = 6 \)
- \( key = 15 \)

\[ 12 \quad 21 \quad 35 \quad 53 \quad 53 \quad 59 \]

\[ \uparrow \]

\[ i \]

29/77
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort(\(A, n\))**

1. for \(j \leftarrow 2\) to \(n\) do
2.   \(key \leftarrow A[j]\)
3.   \(i \leftarrow j - 1\)
4.   while \(i > 0\) and \(A[i] > key\) do
   5.     \(A[i + 1] \leftarrow A[i]\)
   6.     \(i \leftarrow i - 1\)
7.   \(A[i + 1] \leftarrow key\)
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort**($A, n$)

1: **for** $j \leftarrow 2$ **to** $n$ **do**
2: \hspace{1em} \textit{key} \leftarrow A[j]
3: \hspace{1em} $i \leftarrow j - 1$
4: \hspace{2em} **while** $i > 0$ **and** $A[i] > \textit{key}$ **do**
5: \hspace{3em} $A[i + 1] \leftarrow A[i]$
6: \hspace{3em} $i \leftarrow i - 1$
7: \hspace{2em} $A[i + 1] \leftarrow \textit{key}$

- $j = 6$
- $\textit{key} = 15$

12 21 35 35 53 59

\[\uparrow\]

i
Example:

- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

**insertion-sort**(*A, n*)

1. **for** *j* ← 2 to *n* **do**
2.  \(key \leftarrow A[j]\)
3.  \(i \leftarrow j - 1\)
4.  **while** *i* > 0 and *A*[*i*] > *key* **do**
5.  \(A[i + 1] \leftarrow A[i]\)
6.  \(i \leftarrow i - 1\)
7.  \(A[i + 1] \leftarrow key\)
**Example:**

- **Input:** 53, 12, 35, 21, 59, 15
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### insertion-sort($A, n$)

1. **for** $j \leftarrow 2$ to $n$ **do**
2. 
   $\text{key} \leftarrow A[j]$
3. 
   $i \leftarrow j - 1$
4. **while** $i > 0$ **and** $A[i] > \text{key}$ **do**
   5. 
      $A[i + 1] \leftarrow A[i]$
6. 
   $i \leftarrow i - 1$
7. 
   $A[i + 1] \leftarrow \text{key}$

- **$j = 6$**
- **$key = 15$**

<table>
<thead>
<tr>
<th>12</th>
<th>21</th>
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$i$
Example:

- Input: 53, 12, 35, 21, 59, 15
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insertion-sort\( (A, n) \)

1: for \( j \leftarrow 2 \) to \( n \) do
2: \hspace{1em} \( key \leftarrow A[j] \)
3: \hspace{1em} \( i \leftarrow j - 1 \)
4: \hspace{1em} while \( i > 0 \) and \( A[i] > key \) do
5: \hspace{2.5em} \( A[i + 1] \leftarrow A[i] \)
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7: \hspace{1em} \hspace{1em} \hspace{1em} \( A[i + 1] \leftarrow key \)

\( \uparrow \)

- \( j = 6 \)
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Example:

- Input: 53, 12, 35, 21, 59, 15
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**insertion-sort**\((A, n)\)

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Asymptotic Notations

Common Running times
Analysis of Insertion Sort

- Correctness
- Running time
Correctness of Insertion Sort

Invariant: after iteration $j$ of outer loop, $A[1..j]$ is the sorted array for the original $A[1..j]$.

- After $j = 1$: 53, 12, 35, 21, 59, 15
- After $j = 2$: 12, 53, 35, 21, 59, 15
- After $j = 3$: 12, 35, 53, 21, 59, 15
- After $j = 4$: 12, 21, 35, 53, 59, 15
- After $j = 5$: 12, 21, 35, 53, 59, 15
- After $j = 6$: 12, 15, 21, 35, 53, 59
Q1: what is the size of input?
Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size
Analyzing Running Time of Insertion Sort

➡️ Q1: what is the size of input?
➡️ A1: Running time as the function of size
➡️ possible definition of size:
  ➡️ Sorting problem: # integers,
  ➡️ Greatest common divisor: total length of two integers
  ➡️ Shortest path in a graph: # edges in graph
Analyzing Running Time of Insertion Sort

Q1: what is the size of input?
A1: Running time as the function of size
possible definition of size:
  Sorting problem: \# integers,
  Greatest common divisor: total length of two integers
  Shortest path in a graph: \# edges in graph

Q2: Which input?
  For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
Analyzing Running Time of Insertion Sort

Q1: what is the size of input?
A1: Running time as the function of size
possible definition of size:
- Sorting problem: # integers,
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- Shortest path in a graph: # edges in graph

Q2: Which input?
- For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.

A2: Worst-case analysis:
- Running time for size $n = \text{worst running time over all possible arrays of length } n$
Q3: How fast is the computer?
Q4: Programming language?
Analyzing Running Time of Insertion Sort

Q3: How fast is the computer?
Q4: Programming language?
A: They do not matter!
Analyzing Running Time of Insertion Sort

- Q3: How fast is the computer?
- Q4: Programming language?
- A: They do not matter!

**Important idea: asymptotic analysis**

- Focus on growth of running-time as a function, not any particular value.
Asymptotic Analysis: $O$-notation

Informal way to define $O$-notation:

- Ignoring lower order terms
- Ignoring leading constant
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$3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
Asymptotic Analysis: $O$-notation

Informal way to define $O$-notation:
- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
Informal way to define $O$-notation:

- Ignoring lower order terms
- Ignoring leading constant

- $3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n + 10 \Rightarrow n^2/100 \Rightarrow n^2$
Asymptotic Analysis: $O$-notation

Informal way to define $O$-notation:

- Ignoring lower order terms
- Ignoring leading constant

\[
3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3
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Asymptotic Analysis: \( O \)-notation

- \( 3n^3 + 2n^2 - 18n + 1028 = O(n^3) \)
- \( n^2/100 - 3n^2 + 10 = O(n^2) \)
Asymptotic Analysis: $O$-notation

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$O$-notation allows us to ignore
- architecture of computer
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- how we measure the running time: seconds or $\#$ instructions?
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To execute $a \leftarrow b + c$:

- program 1 requires 10 instructions, or $10^{-8}$ seconds
- program 2 requires 2 instructions, or $10^{-9}$ seconds
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- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or $10^{-8}$ seconds
  - program 2 requires 2 instructions, or $10^{-9}$ seconds
  - they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation
 Algorithm 1 runs in time $O(n^2)$
 Algorithm 2 runs in time $O(n)$
Asymptotic Analysis: $O$-notation

- Algorithm 1 runs in time $O(n^2)$
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- Does not tell which algorithm is faster for a specific $n$!
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### Asymptotic Analysis of Insertion Sort

**insertion-sort**\((A, n)\)

1. **for** \(j \leftarrow 2 \) to \(n\) **do**
2. \(key \leftarrow A[j]\)
3. \(i \leftarrow j - 1\)
4. **while** \(i > 0\) and \(A[i] > key\) **do**
5. \(A[i + 1] \leftarrow A[i]\)
6. \(i \leftarrow i - 1\)
7. \(A[i + 1] \leftarrow key\)

**Worst-case running time for iteration** \(j\) of the outer loop?

**Answer:** \(O(j)\)

**Total running time:**

\[
\sum_{j=2}^{n} O(j) = O\left(\sum_{j=2}^{n} j\right) = O\left(n^2 - 1\right) = O\left(n^2\right)
\]
Asymptotic Analysis of Insertion Sort

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Asymptotic Analysis of Insertion Sort

```plaintext
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\sum_{j=2}^{n} O(j) = O\left(\sum_{j=2}^{n} j\right) = O\left(\frac{n(n+1)}{2} - 1\right) = O\left(n^2\right)
\]
Computation Model

Random-Access Machine (RAM) model

- Reading and writing $A[j]$ takes $O(1)$ time.

Basic operations such as addition, subtraction and multiplication take $O(1)$ time.

Each integer (word) has $c \log n$ bits, $c \geq 1$ large enough.

Reason: often we need to read the integer $n$ and handle integers within range $[-n^c, n^c]$, it is convenient to assume this takes $O(1)$ time.

What is the precision of real numbers?

Most of the time, we only consider integers.

Can we do better than insertion sort asymptotically?

Yes: merge sort, quicksort and heap sort take $O(n \log n)$ time.
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Remember to sign up for Piazza.

Questions?
Outline

Syllabus

Introduction
  What is an Algorithm?
  Example: Insertion Sort
  Analysis of Insertion Sort

Asymptotic Notations

Common Running times
Asymptotically Positive Functions

**Def.** $f : \mathbb{N} \rightarrow \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$

In other words, $f(n)$ is positive for large enough $n$. We only consider asymptotically positive functions.
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$\Rightarrow$ $2^n - n^{20}$ Yes

$\Rightarrow$ $100n - n^2/10 + 50?$ No

$\Rightarrow$ We only consider asymptotically positive functions.
\textbf{O-Notation: Asymptotic Upper Bound}

\textbf{O-Notation} For a function \( g(n) \),
\[
O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.
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▶ $3n^2 + 2n \in O(n^2 - 10n)$

**Proof.**

Let $c = 4$ and $n_0 = 50$, for every $n > n_0 = 50$, we have,

\[
3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)
\]
\[
= -n^2 + 40n \leq 0.
\]

\[
3n^2 + 2n \leq c(n^2 - 10n)
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<tr>
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Conventions

We use “\( f(n) = O(g(n)) \)” to denote “\( f(n) \in O(g(n)) \)”
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“=” is asymmetric! Following equalities are wrong:

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$O(n^2) = 3n^2 + 2n$

Analogy: Mike is a student. A student is Mike.
$\Omega$-Notation: Asymptotic Lower Bound

**$O$-Notation** For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$  

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- **Notation: Asymptotic Lower Bound**

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**Ω-Notation** For a function $g(n)$,  
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- In other words, $f(n) \in \Omega(g(n))$ if $f(n) \geq cg(n)$ for some $c$ and large enough $n$. 
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\(\Omega\)-Notation  For a function \(g(n)\),
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\Omega(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \geq cg(n), \forall n \geq n_0 \}.
\]
Again, we use “=” instead of $\in$.

- $4n^2 = \Omega(n - 10)$
- $3n^2 - n + 10 = \Omega(n^2 - 20)$
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4n^2 = \Omega(n - 10)

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Theorem \( f(n) = O(g(n)) \iff g(n) = \Omega(f(n)) \).
**Θ-Notation: Asymptotic Tight Bound**

**Θ-Notation**  For a function \( g(n) \),

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\Theta(g(n)) = \{ \text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}. 
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▶ \( f(n) = \Theta(g(n)) \), then for large enough \( n \), we have \( “f(n) \approx g(n)” \).
**Θ-Notation**: Asymptotic Tight Bound

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$\triangleright f(n) = \Theta(g(n))$, then for large enough $n$, we have “$f(n) \approx g(n)$”.

![Diagram showing the relationship between $f(n)$, $c_1 g(n)$, and $c_2 g(n)$ for large enough $n$.](image)
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$\blacktriangleright$ $3n^2 + 2n = \Theta(n^2 - 20n)$
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- $3n^2 + 2n = \Theta(n^2 - 20n)$
- $2^{n/3} + 100 = \Theta(2^{n/3})$
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- $3n^2 + 2n = Θ(n^2 - 20n)$
- $2^{n/3+100} = Θ(2^{n/3})$

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\(\Theta\)-Notation: Asymptotic Tight Bound

\[ \Theta(g(n)) = \{ \text{function } f : \exists \, c_2 \geq c_1 > 0, n_0 > 0 \text{ such that } c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \} \]

\[3n^2 + 2n = \Theta(n^2 - 20n)\]

\[2^{n/3} + 100 = \Theta(2^{n/3})\]

<table>
<thead>
<tr>
<th>Asymptotic Notations</th>
<th>(O)</th>
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**Theorem** \(f(n) = \Theta(g(n))\) if and only if \(f(n) = O(g(n))\) and \(f(n) = \Omega(g(n))\).
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**Trivial Facts on Comparison Relations**

- $a \leq b \iff b \geq a$
- $a = b \iff a \leq b \text{ and } a \geq b$
- $a \leq b \text{ or } a \geq b$
### Asymptotic Notations

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#### Correct Analogies

- $f(n) = O(g(n)) \iff g(n) = \Omega(f(n))$
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**Correct Analogies**

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**Incorrect Analogy**

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Incorrect Analogy

\[ f(n) = O(g(n)) \text{ or } f(n) = \Omega(f(n)) \]

\[ f(n) = n^2 \]

\[ g(n) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
 n^3 & \text{if } n \text{ is even} 
\end{cases} \]
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2), 3n^2 - 10n - 5 = \Theta(n^2)$
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- In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.
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- $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, though weird
- $3n^2 - 10n - 5 = O(n^2)$ is the most natural since $n^2$ is the simplest term we can have inside $O(\cdot)$. 
Notice that \( O \) denotes asymptotic upper bound

- \( n^2 + 2n = O(n^3) \) is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is \( O(n^4) \).
- We say: the running time of the insertion sort algorithm is \( O(n^2) \) and the bound is tight.
Notice that $O$ denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.
- We say: the running time of the insertion sort algorithm is $O(n^2)$ and the bound is tight.
- We do not use $\Omega$ and $\Theta$ very often when we upper bound running times.
Exercise

For each pair of functions $f, g$ in the following table, indicate whether $f$ is $O, \Omega$ or $\Theta$ of $g$.

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Questions?
Outline

Syllabus

Introduction
What is an Algorithm?
Example: Insertion Sort
Analysis of Insertion Sort

Asymptotic Notations

Common Running times
Computing the sum of $n$ numbers

\[
\text{sum}(A, n)
\]

1: $S \leftarrow 0$
2: for $i \leftarrow 1$ to $n$
3: \hspace{1em} $S \leftarrow S + A[i]$
4: return $S$
\( O(n) \) (Linear) Running Time

- Merge two sorted arrays

<p>| | | | | | |</p>
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Merge two sorted arrays

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\begin{array}{cccccc}
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3 & \\
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<td>3</td>
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</tbody>
</table>
Merge two sorted arrays

3 8 12 20 32 48

5 7 9 25 29

3 5
\( O(n) \) (Linear) Running Time

- Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5 & 7
\end{array}
\]
\( O(n) \) (Linear) Running Time

- Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
\end{array}
\]

\[
\begin{array}{cccccc}
5 & 7 & 9 & 25 & 29 \\
\end{array}
\]

\[
\begin{array}{cccc}
3 & 5 & 7 \\
\end{array}
\]
Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5 & 7 & 8
\end{array}
\]
Merge two sorted arrays

3 8 12 20 32 48
5 7 9 25 29
3 5 7 8
Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
\end{array}
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\begin{array}{cccccc}
5 & 7 & 9 & 25 & 29 \\
\end{array}
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\[
\begin{array}{cccccccc}
3 & 5 & 7 & 8 & 9 & 12 & 20 & 25 & 29 \\
\end{array}
\]
Merge two sorted arrays

3 8 12 20 32 48
5 7 9 25 29
3 5 7 8 9 12 20 25 29 32 48

$O(n)$ (Linear) Running Time
\textit{O(n)} (Linear) Running Time

merge(\(B, C, n_1, n_2\)) \hspace{5mm} \text{B and C are sorted, with length } n_1 \text{ and } n_2

1: \(A \leftarrow []; i \leftarrow 1; j \leftarrow 1\)
2: \textbf{while} \(i \leq n_1 \text{ and } j \leq n_2\) \textbf{do}
3: \hspace{5mm} \textbf{if} \(B[i] \leq C[j]\) \textbf{then}
4: \hspace{10mm} \text{append } B[i] \text{ to } A; \ i \leftarrow i + 1
5: \hspace{5mm} \textbf{else}
6: \hspace{10mm} \text{append } C[j] \text{ to } A; \ j \leftarrow j + 1
7: \hspace{5mm} \textbf{if} \ i \leq n_1 \text{ then append } B[i..n_1] \text{ to } A
8: \hspace{5mm} \textbf{if} \ j \leq n_2 \text{ then append } C[j..n_2] \text{ to } A
9: \textbf{return } A

Running time = \(O(n)\) where \(n = n_1 + n_2\).
merge$(B, C, n_1, n_2)$  \\ $B$ and $C$ are sorted, with length $n_1$ and $n_2$

1: $A ← []; i ← 1; j ← 1$
2: while $i ≤ n_1$ and $j ≤ n_2$ do
3:    if $B[i] ≤ C[j]$ then
4:       append $B[i]$ to $A; i ← i + 1$
5:    else
6:       append $C[j]$ to $A; j ← j + 1$
7: if $i ≤ n_1$ then append $B[i..n_1]$ to $A$
8: if $j ≤ n_2$ then append $C[j..n_2]$ to $A$
9: return $A$

Running time $= O(n)$ where $n = n_1 + n_2$. 
**Running Time**

\[ O(n \log n) \]

**merge-sort** \((A, n)\)

1. **if** \( n = 1 \) **then**
2.      **return** \( A \)
3. **else**
4.      \( B \leftarrow \text{merge-sort} \left( A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor \right) \)
5.      \( C \leftarrow \text{merge-sort} \left( A[\lfloor n/2 \rfloor + 1..n], n - \lfloor n/2 \rfloor \right) \)
6.      **return** \( \text{merge}(B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor) \)
$O(n \log n)$ Running Time

- Merge-Sort

The diagram illustrates the divide-and-conquer strategy of the Merge-Sort algorithm. Each level represents a division into two subproblems of half the size, and the tree structure shows $\log n$ levels, leading to a total running time of $O(n \log n)$. The rectangles indicate the arrays being sorted at each level.

- Each subproblem eventually contains a single element, which is considered sorted.

The running time at each level is $O(n)$, and since there are $\log n$ levels, the overall running time is $O(n \log n)$. 

Notes:
- Each level takes running time $O(n)$.
- There are $O(\log n)$ levels.
- Running time is $O(n \log n)$. 

Diagram:
**$O(n \log n)$ Running Time**

- Merge-Sort

```
A[1..8]  
  /     \  
 /     \      /     \  
 / \      / \      / \      / \  
```

- Each level takes running time $O(n)$
$O(n \log n)$ Running Time

- **Merge-Sort**

  ![Merge-Sort Diagram]

  - Each level takes running time $O(n)$
  - There are $O(\log n)$ levels
$O(n \log n)$ Running Time

- Merge-Sort

- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$
Closest Pair

Input: \( n \) points in plane: \((x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)\)

Output: the pair of points that are closest
Closest Pair

**Input:** \( n \) points in plane: \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

**Output:** the pair of points that are closest
Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

**Output:** the pair of points that are closest

/closest-pair(x, y, n)/

1: $bestd \leftarrow \infty$
2: for $i \leftarrow 1$ to $n - 1$ do
3:     for $j \leftarrow i + 1$ to $n$ do
4:         $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
5:         if $d < bestd$ then
6:             $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
7: return $(besti, bestj)$

Closest pair can be solved in $O(n^2)$ time!
Closest Pair

Input: $n$ points in plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

Output: the pair of points that are closest

\[
\text{closest-pair}(x, y, n)
\]

1: $bestd \leftarrow \infty$
2: for $i \leftarrow 1$ to $n - 1$ do
3: \hspace{1em} for $j \leftarrow i + 1$ to $n$ do
4: \hspace{2em} $d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
5: \hspace{2em} if $d < bestd$ then
6: \hspace{3em} $besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d$
7: return $(besti, bestj)$

Closest pair can be solved in $O(n \log n)$ time!
Multiply two matrices of size $n \times n$

\[
\text{matrix-multiplication}(A, B, n)
\]

1: $C \leftarrow$ matrix of size $n \times n$, with all entries being 0
2: \textbf{for} $i \leftarrow 1$ \textbf{to} $n$ \textbf{do}
3: \hspace{1em} \textbf{for} $j \leftarrow 1$ \textbf{to} $n$ \textbf{do}
4: \hspace{2em} \textbf{for} $k \leftarrow 1$ \textbf{to} $n$ \textbf{do}
5: \hspace{3em} $C[i, k] \leftarrow C[i, k] + A[i, j] \times B[j, k]$
6: \textbf{return} $C$
$O(n^k)$ Running Time for Integer $k \geq 4$

**Def.** An **independent set** of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$. 
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![Graph Diagram]
An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$. 

$O(n^k)$ Running Time for Integer $k \geq 4$
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**Def.** An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$.

Independent set of size $k$

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$
Independent Set of Size $k$

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$

```
\text{independent-set}(G = (V, E))
```

1: \textbf{for} every set $S \subseteq V$ of size $k$ \textbf{do}
2: \hspace{1em} $b \leftarrow \text{true}$
3: \hspace{1em} \textbf{for} every $u, v \in S$ \textbf{do}
4: \hspace{2em} if $(u, v) \in E$ then $b \leftarrow \text{false}$
5: \hspace{1em} if $b$ return true
6: \hspace{1em} \textbf{return} false

Running time $= O\left(\frac{n^k}{k!} \times k^2\right) = O(n^k)$ (assume $k$ is a constant)
Beyond Polynomial Time: $2^n$

### Maximum Independent Set Problem

**Input:** graph $G = (V, E)$

**Output:** the maximum independent set of $G$

### max-independent-set($G = (V, E)$)

1: $R \leftarrow \emptyset$
2: for every set $S \subseteq V$ do
3:   $b \leftarrow true$
4:   for every $u, v \in S$ do
5:       if $(u, v) \in E$ then $b \leftarrow false$
6:   if $b$ and $|S| > |R|$ then $R \leftarrow S$
7: return $R$

Running time = $O(2^n n^2)$. 
## Hamiltonian Cycle Problem

**Input:** a graph with \( n \) vertices  

**Output:** a cycle that visits each node exactly once,  
or say no such cycle exists

---

![Graph Diagram](attachment:image.png)
Hamiltonian Cycle Problem

**Input:** a graph with \( n \) vertices

**Output:** a cycle that visits each node exactly once,

or say no such cycle exists
Beyond Polynomial Time: $n!$

Hamiltonian($G = (V, E)$)

1: for every permutation $(p_1, p_2, \cdots, p_n)$ of $V$ do
2: \hspace{1em} $b \leftarrow$ true
3: \hspace{1em} for $i \leftarrow 1$ to $n - 1$ do
4: \hspace{2em} if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
5: \hspace{2em} if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
6: \hspace{1em} if $b$ then return $(p_1, p_2, \cdots, p_n)$
7: return “No Hamiltonian Cycle”

Running time = $O(n! \times n)$
$O(\log n)$ (Logarithmic) Running Time

**Binary search**

**Input:** sorted array $A$ of size $n$, an integer $t$;  
**Output:** whether $t$ appears in $A$.

E.g., search 35 in the following array:
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$. 

$O(\log n)$ (Logarithmic) Running Time
$O(\log n)$ (Logarithmic) Running Time

- Binary search
  - Input: sorted array $A$ of size $n$, an integer $t$;
  - Output: whether $t$ appears in $A$.
- E.g, search 35 in the following array:

<p>| | | | | | | | | | | | | | | |</p>
<table>
<thead>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>8</td>
<td>10</td>
<td>25</td>
<td>29</td>
<td>37</td>
<td>38</td>
<td>42</td>
<td>46</td>
<td>52</td>
<td>59</td>
<td>61</td>
<td>63</td>
<td>75</td>
<td>79</td>
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$O(\log n)$ (Logarithmic) Running Time

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E.g., search 35 in the following array:

\[
\begin{array}{cccccccccccccccc}
3 & 8 & 10 & 25 & 29 & 37 & 38 & 42 & 46 & 52 & 59 & 61 & 63 & 75 & 79 \\
\end{array}
\]

$42 > 35$
Binary search
- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.
- E.g, search 35 in the following array:

\[
\begin{array}{cccccccccccccccc}
3 & 8 & 10 & 25 & 29 & 37 & 38 & 42 & 46 & 52 & 59 & 61 & 63 & 75 & 79 \\
\end{array}
\]
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E.g, search 35 in the following array:

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25 < 35
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E.g, search 35 in the following array:
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O(\log n) (Logarithmic) Running Time

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```

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E.g, search 35 in the following array:
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

binary-search($A, n, t$)

1: $i \leftarrow 1, j \leftarrow n$
2: while $i \leq j$ do
3: $k \leftarrow \lfloor (i + j)/2 \rfloor$
4: if $A[k] = t$ return true
5: if $t < A[k]$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6: return false

$O(\log n)$ (Logarithmic) Running Time
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
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```
binary-search(A, n, t)

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4:     if $A[k] = t$ return true
5:     if $t < A[k]$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6: return false
```

Running time $= O(\log n)$
Comparing the Orders

- Sort the functions from smallest to largest asymptotically:
  \[ \log n, \ n, \ n^2, \ n \log n, \ n!, \ 2^n, \ e^n, \ n^n \]
- \[ \log n = O(n) \]
Comparing the Orders

- Sort the functions from smallest to largest asymptotically
  \( \log n, \ n, \ n^2, \ n \log n, \ n!, \ 2^n, \ e^n, \ n^n \)

- \( \log n = O(n) \)

- \( n = O(n^2) \)
Comparing the Orders

- Sort the functions from smallest to largest asymptotically:
  \( \log n, \ n, \ n^2, \ n \log n, \ n!, \ 2^n, \ e^n, \ n^n \)

- \( \log n = O(n) \)

- \( n = O(n \log n) \)

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- Sort the functions from smallest to largest asymptotically:
  \( \log n, \ n, \ n^2, \ n \log n, \ n!, \ 2^n, \ e^n, \ n^n \)
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- \( n = O(n \log n) \)
- \( n \log n = O(n^2) \)
- \( n^2 = O(n!) \)
Comparing the Orders

- Sort the functions from smallest to largest asymptotically
  \(\log n, \ n, \ n^2, \ n \log n, \ n!, \ 2^n, \ e^n, \ n^n\)

- \(\log n = O(n)\)

- \(n = O(n \log n)\)

- \(n \log n = O(n^2)\)

- \(n^2 = O(2^n)\)

- \(2^n = O(n!)\)
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Comparing the Orders

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- \( \log n = O(n) \)
- \( n = O(n \log n) \)
- \( n \log n = O(n^2) \)
- \( n^2 = O(2^n) \)
- \( 2^n = O(e^n) \)
- \( e^n = O(n!) \)
- \( n! = O(n^n) \)
Terminologies

When we talk about upper bound on running time:

- Logarithmic time: $O(\log n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant $k$
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$
Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.
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- Using asymptotic analysis allows us to ignore the leading constants and lower order terms.
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- Design algorithms to minimize the order of the running time.

- Using asymptotic analysis allows us to ignore the leading constants and lower order terms.

- Makes our life much easier! (E.g., the leading constant depends on the implementation, compiler and computer architecture of computer.)
Q: Does ignoring the leading constant cause any issues?

▶ e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?
Q: Does ignoring the leading constant cause any issues?

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A:
Q: Does ignoring the leading constant cause any issues?

▶ e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

▶ Sometimes yes
Q: Does ignoring the leading constant cause any issues?

▶ e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

▶ Sometimes yes
▶ However, when $n$ is big enough, $1000n < 0.1n^2$
Q: Does ignoring the leading constant cause any issues?

▶ e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

▶ Sometimes yes

▶ However, when $n$ is big enough, $1000n < 0.1n^2$

▶ For “natural” algorithms, constants are not so big!
Q: Does ignoring the leading constant cause any issues?

➤ e.g, how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?

A:

➤ Sometimes yes
➤ However, when $n$ is big enough, $1000n < 0.1n^2$
➤ For “natural” algorithms, constants are not so big!
➤ So, for reasonably large $n$, algorithm with lower order running time beats algorithm with higher order running time.