CSE 431/531: Algorithm Analysis and Design (Spring 2020)

Introduction and Syllabus

Lecturer: Shi Li

Department of Computer Science and Engineering
University at Buffalo
Outline

1 Syllabus

2 Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3 Asymptotic Notations

4 Common Running times
CSE 431/531: Algorithm Analysis and Design

- Course Webpage (contains schedule, policies, homeworks and slides):
  http://www.cse.buffalo.edu/~shil/courses/CSE531/

- Please sign up course on Piazza via link on course webpage
  - announcements, polls, asking/answering questions
CSE 431/531: Algorithm Analysis and Design

- **Time and location:**
  - MoWeFr, 9:00-9:50am
  - Knox 110.

- **Instructor:**
  - Shi Li, shil@buffalo.edu, Davis 328
  - Office hours: TBD via poll
You *should* already have/know:
You should already have/know:

- Mathematical Background
- Reasoning, inductions, probabilities
You should already have/know:

- **Mathematical Background**
  - Reasoning, inductions, probabilities

- **Basic data Structures**
  - Stacks, queues, linked lists
You should already have/know:

- **Mathematical Background**
  - Reasoning, inductions, probabilities

- **Basic data Structures**
  - Stacks, queues, linked lists

- **Some Programming Experience**
  - C, C++, Java or Python
You Will Learn

- Classic algorithms for classic problems
  - Sorting, shortest paths, minimum spanning tree, ···
You Will Learn

- Classic algorithms for classic problems
  - Sorting, shortest paths, minimum spanning tree, · · ·

- How to analyze algorithms
  - Correctness
  - Running time (efficiency)
  - Space requirement (occasionally)
You Will Learn

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  - Sorting, shortest paths, minimum spanning tree, · · ·
- How to analyze algorithms
  - Correctness
  - Running time (efficiency)
  - Space requirement (occasionally)
- Meta techniques to design algorithms
  - Greedy algorithms
  - Divide and conquer
  - Dynamic programming
  - · · ·
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  - Sorting, shortest paths, minimum spanning tree, · · ·

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- Meta techniques to design algorithms
  - Greedy algorithms
  - Divide and conquer
  - Dynamic programming
  - · · ·

- NP-completeness
Tentative Schedule (42 Lectures)

See the course webpage.
Textbook (Highly Recommended):

- **Algorithm Design**, 1st Edition, by Jon Kleinberg and Eva Tardos

Other Reference Books

Highly recommended: read the correspondent sections from the textbook (or reference book) before classes. Sections for each lecture can be found on the course webpage. Slides and example problems for recitations will be posted on the course webpage before class.
Grading

- 40% for homeworks
  - 6 points \times 5 \text{ theory homeworks}
  - 10 points for programming homework
- 60% for mid-term + final exams, score for two exams is
  \[
  \max\{M \times 20\% + F \times 40\%, M \times 30\% + F \times 30\%\} \\
  M, F \in [0, 100]
  \]
For Homeworks, You Are Allowed to

- Use course materials (textbook, reference books, lecture notes, etc)
- Post questions on Piazza
- Ask me or TAs for hints
- Collaborate with classmates
  - Think about each problem for enough time before discussions
  - **Must write down solutions on your own, in your own words**
  - Write down names of students you collaborated with
For Homeworks, You Are Not Allowed to

- Use external resources
  - Can’t Google or ask questions online for solutions
  - Can’t read posted solutions from other algorithm course webpages
- Copy solutions from other students
For Programming Problems

- Need to implement the algorithms by yourself
- Can not copy codes from others or the Internet
- We use Moss ([https://theory.stanford.edu/~aiken/moss/](https://theory.stanford.edu/~aiken/moss/)) to detect similarity of programs
Late Policy

- You have 1 “late credit”, using it allows you to submit an assignment solution for three days
- With no special reasons, no other late submissions will be accepted
- Mid-Term and Final Exam will be closed-book
- Per Departmental Policy on Academia Integrity Violations, penalty for AI violation is:
  - “F” for the course
  - lose financial support as TA/RA
  - case will be reported to the department and university
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Questions?
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   - What is an Algorithm?
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4. Common Running times
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What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.
What is an Algorithm?

- Donald Knuth: An algorithm is a finite, definite effective procedure, with some input and some output.

- Computational problem: specifies the input/output relationship.

- An algorithm solves a computational problem if it produces the correct output for any given input.
Examples

Greatest Common Divisor

**Input:** two integers \( a, b > 0 \)

**Output:** the greatest common divisor of \( a \) and \( b \)
Examples

Greatest Common Divisor

**Input:** two integers $a, b > 0$

**Output:** the greatest common divisor of $a$ and $b$

Example:
- Input: 210, 270
- Output: 30
Examples

Greatest Common Divisor

Input: two integers $a, b > 0$

Output: the greatest common divisor of $a$ and $b$

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- Input: 210, 270
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- Algorithm: Euclidean algorithm
Examples

Greatest Common Divisor

**Input:** two integers \( a, b > 0 \)

**Output:** the greatest common divisor of \( a \) and \( b \)

Example:

- Input: 210, 270
- Output: 30

- Algorithm: Euclidean algorithm
  
  \[
  \text{gcd}(270, 210) = \text{gcd}(210, 270 \mod 210) = \text{gcd}(210, 60)
  \]
Examples

Greatest Common Divisor

**Input:** two integers $a, b > 0$

**Output:** the greatest common divisor of $a$ and $b$

Example:

- **Input:** 210, 270
- **Output:** 30

- **Algorithm:** Euclidean algorithm
  - $\text{gcd}(270, 210) = \text{gcd}(210, 270 \mod 210) = \text{gcd}(210, 60)$
  - $(270, 210) \rightarrow (210, 60) \rightarrow (60, 30) \rightarrow (30, 0)$
Examples

Sorting

**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

**Output:** a permutation \((a'_1, a'_2, \cdots, a'_n)\) of the input sequence such that \(a'_1 \leq a'_2 \leq \cdots \leq a'_n\)
Examples

Sorting

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

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Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59
Examples

### Sorting

**Input:** sequence of \( n \) numbers \((a_1, a_2, \cdots, a_n)\)

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#### Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

- Algorithms: insertion sort, merge sort, quicksort, \ldots
Examples

Shortest Path

**Input:** directed graph $G = (V, E)$, $s, t \in V$

**Output:** a shortest path from $s$ to $t$ in $G$
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Shortest Path

**Input:** directed graph $G = (V, E)$, $s, t ∈ V$

**Output:** a shortest path from $s$ to $t$ in $G$

Algorithm: Dijkstra’s algorithm
Algorithm = Computer Program?

- Algorithm: “abstract”, can be specified using computer program, English, pseudo-codes or flow charts.
- Computer program: “concrete”, implementation of algorithm, using a particular programming language
Pseudo-Code:

Euclidean\((a, b)\)

1. while \(b > 0\)

2. \((a, b) \leftarrow (b, a \mod b)\)

3. return \(a\)

C++ program:

```cpp
int Euclidean(int a, int b){
  int c;
  while (b > 0){
    c = b;
    b = a % b;
    a = c;
  }
  return a;
}
```
Main focus: correctness, running time (efficiency)
Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage

Why is it important to study the running time (efficiency) of an algorithm?

1. Feasible vs. infeasible
2. Efficient algorithms: less engineering tricks needed, can use languages aiming for easy programming (e.g., python)
3. Fundamental
4. It is fun!
Theoretical Analysis of Algorithms

- Main focus: correctness, running time (efficiency)
- Sometimes: memory usage
- Not covered in the course: engineering side
  - extensibility
  - modularity
  - object-oriented model
  - user-friendliness (e.g, GUI)
  - ...

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### Sorting Problem

**Input:** sequence of $n$ numbers $(a_1, a_2, \cdots, a_n)$

**Output:** a permutation $(a'_1, a'_2, \cdots, a'_n)$ of the input sequence such that $a'_1 \leq a'_2 \leq \cdots \leq a'_n$

#### Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59
Insertion-Sort

- At the end of $j$-th iteration, the first $j$ numbers are sorted.

  iteration 1: 53, 12, 35, 21, 59, 15
  iteration 2: 12, 53, 35, 21, 59, 15
  iteration 3: 12, 35, 53, 21, 59, 15
  iteration 4: 12, 21, 35, 53, 59, 15
  iteration 5: 12, 21, 35, 53, 59, 15
  iteration 6: 12, 15, 21, 35, 53, 59
Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

**insertion-sort**(*A*, *n*)

1. **for** *j* ← 2 to *n*
2. \( key \leftarrow A[j] \)
3. \( i \leftarrow j - 1 \)
4. **while** *i* > 0 and \( A[i] > key \)
5. \( A[i + 1] \leftarrow A[i] \)
6. \( i \leftarrow i - 1 \)
7. \( A[i + 1] \leftarrow key \)
Example:

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**insertion-sort**\( (A, n) \)

1. for \( j \leftarrow 2 \) to \( n \)
2. \( key \leftarrow A[j] \)
3. \( i \leftarrow j - 1 \)
4. while \( i > 0 \) and \( A[i] > key \)
5. \( A[i + 1] \leftarrow A[i] \)
6. \( i \leftarrow i - 1 \)
7. \( A[i + 1] \leftarrow key \)

\( j = 6 \)
\( key = 15 \)

12 21 35 53 59 15 15

\( i \)

↑
Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

**insertion-sort(A, n)**

1. **for** $j \leftarrow 2$ **to** $n$
2. \hspace{2ex} $key \leftarrow A[j]$
3. \hspace{2ex} $i \leftarrow j - 1$
4. **while** $i > 0$ **and** $A[i] > key$
5. \hspace{2ex} $A[i + 1] \leftarrow A[i]$
6. \hspace{2ex} $i \leftarrow i - 1$
7. \hspace{2ex} $A[i + 1] \leftarrow key$

**Example:**

1. $j = 6$
2. $key = 15$
3. 12 21 35 53 59 59

\[\uparrow\]

\[i\]
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12 21 35 53 53 59

↑

\( i \)
Example:
- Input: 53, 12, 35, 21, 59, 15
- Output: 12, 15, 21, 35, 53, 59

insertion-sort($A, n$)

1. for $j \leftarrow 2$ to $n$
2. \hspace{1cm} $key \leftarrow A[j]$
3. \hspace{1cm} $i \leftarrow j - 1$
4. \hspace{1cm} while $i > 0$ and $A[i] > key$
5. \hspace{2cm} $A[i + 1] \leftarrow A[i]$
6. \hspace{2cm} $i \leftarrow i - 1$
7. \hspace{1cm} $A[i + 1] \leftarrow key$

- $j = 6$
- $key = 15$

12 21 35 53 53 59

↑

i
Example:

- **Input:** 53, 12, 35, 21, 59, 15
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**insertion-sort(A, n)**

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<table>
<thead>
<tr>
<th>12</th>
<th>21</th>
<th>35</th>
<th>35</th>
<th>53</th>
<th>59</th>
</tr>
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<tr>
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Example:

- **Input:** 53, 12, 35, 21, 59, 15
- **Output:** 12, 15, 21, 35, 53, 59

### insertion-sort($A, n$)

1. **for** $j \leftarrow 2$ **to** $n$
2.   
   - **key** $\leftarrow A[j]$
3.   
   - $i \leftarrow j - 1$
4.   
   - **while** $i > 0$ **and** $A[i] > key$
5.   
   - $A[i + 1] \leftarrow A[i]$
6.   
   - $i \leftarrow i - 1$
7.   
   - $A[i + 1] \leftarrow key$

- $j = 6$
- **key** $= 15$

```
12  21  35  35  53  59
```

↑

$i$
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insertion-sort($A, n$)

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Example:

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**insertion-sort**\((A,n)\)

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\[12 \quad 15 \quad 21 \quad 35 \quad 53 \quad 59\]
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Analysis of Insertion Sort

- Correctness
- Running time
Correctness of Insertion Sort


  after $j = 1$: $53, 12, 35, 21, 59, 15$
  after $j = 2$: $12, 53, 35, 21, 59, 15$
  after $j = 3$: $12, 35, 53, 21, 59, 15$
  after $j = 4$: $12, 21, 35, 53, 59, 15$
  after $j = 5$: $12, 21, 35, 53, 59, 15$
  after $j = 6$: $12, 15, 21, 35, 53, 59$
Analyzing Running Time of Insertion Sort

Q1: what is the size of input?
Q1: what is the size of input?
A1: Running time as the function of size
Analyzing Running Time of Insertion Sort

- Q1: what is the size of input?
- A1: Running time as the function of size

possible definition of size :
- Sorting problem: \# integers,
- Greatest common divisor: total length of two integers
- Shortest path in a graph: \# edges in graph
Q1: what is the size of input?
A1: Running time as the function of size
possible definition of size:
  - Sorting problem: \# integers,
  - Greatest common divisor: total length of two integers
  - Shortest path in a graph: \# edges in graph

Q2: Which input?
  - For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
Q1: what is the size of input?
A1: Running time as the function of size
possible definition of size:
- Sorting problem: \# integers,
- Greatest common divisor: total length of two integers
- Shortest path in a graph: \# edges in graph

Q2: Which input?
For the insertion sort algorithm: if input array is already sorted in ascending order, then algorithm runs much faster than when it is sorted in descending order.
A2: Worst-case analysis:
- Running time for size $n = \text{worst running time over all possible arrays of length } n$
Q3: How fast is the computer?
Q4: Programming language?
Q3: How fast is the computer?
Q4: Programming language?
A: They do not matter!
Q3: How fast is the computer?
Q4: Programming language?
A: They do not matter!

Important idea: asymptotic analysis

Focus on growth of running-time as a function, not any particular value.
Informal way to define \( O \)-notation:
- Ignoring lower order terms
- Ignoring leading constant
Asymptotic Analysis: $O$-notation

Informal way to define $O$-notation:

- Ignoring lower order terms
- Ignoring leading constant

$3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$
Asymptotic Analysis: \( O \)-notation

Informal way to define \( O \)-notation:

- Ignoring lower order terms
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\[
3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3
\]
\[
3n^3 + 2n^2 - 18n + 1028 = O(n^3)
\]
Informal way to define $O$-notation:

- Ignoring lower order terms
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$3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3$

$3n^3 + 2n^2 - 18n + 1028 = O(n^3)$

$n^2/100 - 3n^2 + 10 \Rightarrow n^2/100 \Rightarrow n^2$
Asymptotic Analysis: \( O \)-notation

Informal way to define \( O \)-notation:

- Ignoring lower order terms
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\[
3n^3 + 2n^2 - 18n + 1028 \Rightarrow 3n^3 \Rightarrow n^3
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- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
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Asymptotic Analysis: $O$-notation

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$O$-notation allows us to ignore

- architecture of computer
- programming language
- how we measure the running time: seconds or # instructions?
Asymptotic Analysis: $O$-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
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$O$-notation allows us to ignore
- architecture of computer
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- how we measure the running time: seconds or # instructions?

To execute $a \leftarrow b + c$:
- program 1 requires 10 instructions, or $10^{-8}$ seconds
- program 2 requires 2 instructions, or $10^{-9}$ seconds
Asymptotic Analysis: $O$-notation

- $3n^3 + 2n^2 - 18n + 1028 = O(n^3)$
- $n^2/100 - 3n^2 + 10 = O(n^2)$

$O$-notation allows us to ignore
- architecture of computer
- programming language
- how we measure the running time: seconds or # instructions?

- to execute $a \leftarrow b + c$:
  - program 1 requires 10 instructions, or $10^{-8}$ seconds
  - program 2 requires 2 instructions, or $10^{-9}$ seconds
- they only change by a constant in the running time, which will be hidden by the $O(\cdot)$ notation
Asymptotic Analysis: $O$-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$
Asymptotic Analysis: $O$-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$
- Does not tell which algorithm is faster for a specific $n$!
Algorithm 1 runs in time $O(n^2)$
Algorithm 2 runs in time $O(n)$

Does not tell which algorithm is faster for a specific $n$!
Algorithm 2 will eventually beat algorithm 1 as $n$ increases.
Asymptotic Analysis: $O$-notation

- Algorithm 1 runs in time $O(n^2)$
- Algorithm 2 runs in time $O(n)$
- Does not tell which algorithm is faster for a specific $n$!
- Algorithm 2 will eventually beat algorithm 1 as $n$ increases.
- For Algorithm 1: if we increase $n$ by a factor of 2, running time increases by a factor of 4
Asymptotic Analysis: \( O \)-notation

- Algorithm 1 runs in time \( O(n^2) \)
- Algorithm 2 runs in time \( O(n) \)

- Does not tell which algorithm is faster for a specific \( n \)!
- Algorithm 2 will eventually beat algorithm 1 as \( n \) increases.

- For Algorithm 1: if we increase \( n \) by a factor of 2, running time increases by a factor of 4
- For Algorithm 2: if we increase \( n \) by a factor of 2, running time increases by a factor of 2
Asymptotic Analysis of Insertion Sort

insertion-sort\((A, n)\)

1. for \(j \leftarrow 2\) to \(n\)
2. \(key \leftarrow A[j]\)
3. \(i \leftarrow j - 1\)
4. while \(i > 0\) and \(A[i] > key\)
5. \(A[i + 1] \leftarrow A[i]\)
6. \(i \leftarrow i - 1\)
7. \(A[i + 1] \leftarrow key\)
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- Worst-case running time for iteration \(j\) of the outer loop?
Asymptotic Analysis of Insertion Sort

```plaintext
insertion-sort(A, n)

1. for j ← 2 to n
2.    key ← A[j]
3.    i ← j - 1
4.    while i > 0 and A[i] > key
6.        i ← i - 1
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```

- Worst-case running time for iteration $j$ of the outer loop?
  Answer: $O(j)$
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- Worst-case running time for iteration $j$ of the outer loop?
  Answer: $O(j)$

- Total running time $= \sum_{j=2}^{n} O(j) = O(\sum_{j=2}^{n} j)$
  $= O\left(\frac{n(n+1)}{2} - 1\right) = O(n^2)$
Computation Model

Random-Access Machine (RAM) model

A \[ j \] takes \( O(1) \) time

Basic operations such as addition, subtraction and multiplication take \( O(1) \) time.

Each integer (word) has \( c \log n \) bits, \( c \geq 1 \) large enough.

Reason: often we need to read the integer \( n \) and handle integers within range \( [-n^c, n^c] \), it is convenient to assume this takes \( O(1) \) time.

What is the precision of real numbers?
Most of the time, we only consider integers.

Can we do better than insertion sort asymptotically? Yes: merge sort, quicksort and heap sort take \( O(n \log n) \) time.
Computation Model

- Random-Access Machine (RAM) model
- reading and writing $A[j]$ takes $O(1)$ time

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• Remember to sign up for Piazza.

Questions?
Outline

1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times
Asymptotically Positive Functions

Def. $f : \mathbb{N} \rightarrow \mathbb{R}$ is an asymptotically positive function if:

- $\exists n_0 > 0$ such that $\forall n > n_0$ we have $f(n) > 0$
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We only consider asymptotically positive functions. Why not (everywhere-)positive functions? Answer: for the sake of convenience.
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**O-Notation: Asymptotic Upper Bound**

**O-Notation**  For a function $g(n)$,

$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$
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- \( 3n^2 + 2n \in O(n^2 - 10n) \)

**Proof.**

Let \( c = 4 \) and \( n_0 = 50 \), for every \( n > n_0 = 50 \), we have,

\[
3n^2 + 2n - c(n^2 - 10n) = 3n^2 + 2n - 4(n^2 - 10n)
\]
\[
= -n^2 + 40n \leq 0.
\]
\[
3n^2 + 2n \leq c(n^2 - 10n)
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**O-Notation** For a function $g(n)$,

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Conventions

- We use "\( f(n) = O(g(n)) \)" to denote "\( f(n) \in O(g(n)) \)"

Analogy: Mike is a student. A student is Mike.
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“$=$” is asymmetric! Following equalities are wrong:
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**Ω-Notation**: Asymptotic Lower Bound

**O-Notation**  For a function $g(n)$,

$$O(g(n)) = \{ \text{function } f : \exists c > 0, n_0 > 0 \text{ such that } f(n) \leq cg(n), \forall n \geq n_0 \}.$$  

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\]
Again, we use “=” instead of $\in$.

- $4n^2 = \Omega(n - 10)$
- $3n^2 - n + 10 = \Omega(n^2 - 20)$
Ω-Notation: Asymptotic Lower Bound

- Again, we use “=” instead of ∈.
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**Theorem** \(f(n) = O(g(n)) \Leftrightarrow g(n) = \Omega(f(n))\).
**Θ-Notation: Asymptotic Tight Bound**

**Θ-Notation**  For a function $g(n)$,

$\Theta(g(n)) = \{\text{function } f : \exists c_2 \geq c_1 > 0, n_0 > 0 \text{ such that} \}

c_1 g(n) \leq f(n) \leq c_2 g(n), \forall n \geq n_0 \}$. 

$f(n) = \Theta(g(n))$, then for large enough $n$,

"$f(n) \approx g(n)$".
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- $f(n) = \Theta(g(n))$, then for large enough $n$, we have “$f(n) \approx g(n)$”.

![Diagram showing the relationship between $f(n)$, $c_1 g(n)$, and $c_2 g(n)$ for large $n$.](image-url)
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- $3n^2 + 2n = \Theta(n^2 - 20n)$
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- \( 3n^2 + 2n = \Theta(n^2 - 20n) \)
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**Theorem**  $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$. 
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### Trivial Facts on Comparison Relations

- $f \leq g \iff g \geq f$
- $f = g \iff f \leq g \text{ and } f \geq g$
- $f \leq g \text{ or } f \geq g$
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- \( f \leq g \iff g \geq f \)
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### Correct Analogies

- \( f(n) = O(g(n)) \iff g(n) = \Omega(f(n)) \)
- \( f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \) and \( f(n) = \Omega(g(n)) \)
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- $f(n) = \Theta(g(n)) \iff f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$

### Incorrect Analogy
- $f(n) = O(g(n))$ or $g(n) = O(f(n))$
Incorrect Analogy

- $f(n) = O(g(n))$ or $g(n) = O(f(n))$
Incorrect Analogy

\[ f(n) = O(g(n)) \text{ or } g(n) = O(f(n)) \]

\[ f(n) = n^2 \]

\[ g(n) = \begin{cases} 
1 & \text{if } n \text{ is odd} \\
3n^3 & \text{if } n \text{ is even}
\end{cases} \]
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2), 3n^2 - 10n - 5 = \Theta(n^2)$
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2), 3n^2 - 10n - 5 = \Theta(n^2)$
- In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \rightarrow 3n^2$
- ignoring leading constant: $3n^2 \rightarrow n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2), 3n^2 - 10n - 5 = \Theta(n^2)$

In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.

$3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, though weird
Recall: Informal way to define $O$-notation

- ignoring lower order terms: $3n^2 - 10n - 5 \to 3n^2$
- ignoring leading constant: $3n^2 \to n^2$
- $3n^2 - 10n - 5 = O(n^2)$
- Indeed, $3n^2 - 10n - 5 = \Omega(n^2), 3n^2 - 10n - 5 = \Theta(n^2)$

In the formal definition of $O(\cdot)$, nothing tells us to ignore lower order terms and leading constant.

- $3n^2 - 10n - 5 = O(5n^2 - 6n + 5)$ is correct, though weird
- $3n^2 - 10n - 5 = O(n^2)$ is the most natural since $n^2$ is the simplest term we can have inside $O(\cdot)$. 

Notice that $O$ denotes asymptotic upper bound

- $n^2 + 2n = O(n^3)$ is correct.
- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.
- We say: the running time of the insertion sort algorithm is $O(n^2)$ and the bound is tight.
Notice that $O$ denotes asymptotic upper bound

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- The following sentence is correct: the running time of the insertion sort algorithm is $O(n^4)$.
- We say: the running time of the insertion sort algorithm is $O(n^2)$ and the bound is tight.
- We do not use $\Omega$ and $\Theta$ very often when we talk about running times.
Exercise

For each pair of functions $f, g$ in the following table, indicate whether $f$ is $O, \Omega$ or $\Theta$ of $g$.

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Questions?
Outline

1. Syllabus

2. Introduction
   - What is an Algorithm?
   - Example: Insertion Sort
   - Analysis of Insertion Sort

3. Asymptotic Notations

4. Common Running times
Computing the sum of $n$ numbers

**sum**($A$, $n$)

1. $S \leftarrow 0$
2. for $i \leftarrow 1$ to $n$
3. \hspace{1em} $S \leftarrow S + A[i]$
4. return $S$
$O(n)$ (Linear) Running Time

- Merge two sorted arrays

```
3  8  12  20  32  48
5  7  9  25  29
```
Merge two sorted arrays

<table>
<thead>
<tr>
<th>3</th>
<th>8</th>
<th>12</th>
<th>20</th>
<th>32</th>
<th>48</th>
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</thead>
<tbody>
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<td>5</td>
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$O(n)$ (Linear) Running Time

- Merge two sorted arrays

```
3 8 12 20 32 48
5 7 9 25 29
3
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$O(n)$ (Linear) Running Time

- Merge two sorted arrays

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3 8 12 20 32 48
5 7 9 25 29
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Merge two sorted arrays

\[ \begin{align*}
&\text{3} \quad 8 \quad 12 \quad 20 \quad 32 \quad 48 \\
\end{align*} \]

\[ \begin{align*}
&\text{5} \quad 7 \quad 9 \quad 25 \quad 29 \\
\end{align*} \]

\[ \begin{align*}
&\text{3} \quad 5 \\
\end{align*} \]
Merge two sorted arrays

\[
\begin{array}{c}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5
\end{array}
\]
$O(n)$ (Linear) Running Time

- Merge two sorted arrays

```
3 8 12 20 32 48
5 7 9 25 29
3 5 7
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\(O(n)\) (Linear) Running Time

- Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 \\
3 & 5 & 7 \\
\end{array}
\]
$O(n)$ (Linear) Running Time

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\[
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3 & 8 & 12 & 20 & 32 & 48 \\
\end{array}
\]

\[
\begin{array}{cccccc}
5 & 7 & 9 & 25 & 29 \\
\end{array}
\]

\[
\begin{array}{cccccc}
3 & 5 & 7 & 8 \\
\end{array}
\]
$O(n)$ (Linear) Running Time

- Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 &
\end{array}
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\[
\begin{array}{cccc}
3 & 5 & 7 & 8 \\
\end{array}
\]
**$O(n)$ (Linear) Running Time**

- Merge two sorted arrays

<table>
<thead>
<tr>
<th>3</th>
<th>8</th>
<th>12</th>
<th>20</th>
<th>32</th>
<th>48</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>7</td>
<td>9</td>
<td>25</td>
<td>29</td>
<td></td>
</tr>
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<td>3</td>
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<td>7</td>
<td>8</td>
<td>9</td>
<td>12</td>
</tr>
</tbody>
</table>
**$O(n)$ (Linear) Running Time**

- Merge two sorted arrays

\[
\begin{array}{cccccc}
3 & 8 & 12 & 20 & 32 & 48 \\
5 & 7 & 9 & 25 & 29 & 3 & 5 & 7 & 8 & 9 & 12 & 20 & 25 & 29 & 32 & 48
\end{array}
\]
**$O(n)$ (Linear) Running Time**

merge($B, C, n_1, n_2$) \[ B \text{ and } C \text{ are sorted, with length } n_1 \text{ and } n_2 \]

1. $A \leftarrow []$; $i \leftarrow 1$; $j \leftarrow 1$
2. while $i \leq n_1$ and $j \leq n_2$
3. \hspace{0.5cm} if ($B[i] \leq C[j]$) then
4. \hspace{1.5cm} append $B[i]$ to $A$; $i \leftarrow i + 1$
5. \hspace{1.5cm} else
6. \hspace{2.5cm} append $C[j]$ to $A$; $j \leftarrow j + 1$
7. \hspace{0.5cm} if $i \leq n_1$ then append $B[i..n_1]$ to $A$
8. \hspace{0.5cm} if $j \leq n_2$ then append $C[j..n_2]$ to $A$
9. return $A$

Running time = $O(n)$ where $n = n_1 + n_2$. 
**O(n) (Linear) Running Time**

merge($B, C, n_1, n_2$) \ \ \ \ \ \ B and C are sorted, with length $n_1$ and $n_2$

1. $A \leftarrow []; i \leftarrow 1; j \leftarrow 1$
2. while $i \leq n_1$ and $j \leq n_2$
3. \hspace{1em} if ($B[i] \leq C[j]$) then
4. \hspace{2em} append $B[i]$ to $A$; $i \leftarrow i + 1$
5. \hspace{1em} else
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7. if $i \leq n_1$ then append $B[i..n_1]$ to $A$
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9. return $A$

Running time $= O(n)$ where $n = n_1 + n_2$. 
$O(n \log n)$ Running Time

merge-sort($A$, $n$)

1. if $n = 1$ then
2. return $A$
3. else
4. $B \leftarrow$ merge-sort($A[1..\lfloor n/2 \rfloor], \lfloor n/2 \rfloor$)
5. $C \leftarrow$ merge-sort($A[\lfloor n/2 \rfloor + 1..n], n - \lfloor n/2 \rfloor$)
6. return merge($B, C, \lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor$)
$O(n \log n)$ Running Time

- Merge-Sort

![Diagram of Merge-Sort algorithm](image)
$O(n \log n)$ Running Time

- **Merge-Sort**

```
A[1..8]
```

```
```

```
```

- Each level takes running time $O(n)$
$O(n \log n)$ Running Time

- Merge-Sort

Each level takes running time $O(n)$

There are $O(\log n)$ levels
**$O(n \log n)$ Running Time**

- **Merge-Sort**

![Merge-Sort Diagram]

- Each level takes running time $O(n)$
- There are $O(\log n)$ levels
- Running time $= O(n \log n)$
Closest Pair

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$

**Output:** the pair of points that are closest
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closest-pair\((x, y, n)\)

1. \( bestd \leftarrow \infty \)
2. for \( i \leftarrow 1 \) to \( n - 1 \)
3. for \( j \leftarrow i + 1 \) to \( n \)
4. \( d \leftarrow \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2} \)
5. if \( d < bestd \) then
6. \( besti \leftarrow i, bestj \leftarrow j, bestd \leftarrow d \)
7. return \((besti, bestj)\)
**Closest Pair**

**Input:** $n$ points in plane: $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$

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```plaintext
bestd ← ∞
for $i ← 1$ to $n - 1$
  for $j ← i + 1$ to $n$
    $d ← \sqrt{(x[i] - x[j])^2 + (y[i] - y[j])^2}$
    if $d < bestd$ then
      $besti ← i, bestj ← j, bestd ← d$
return $(besti, bestj)$
```

Closest pair can be solved in $O(n \log n)$ time!
Multiply two matrices of size $n \times n$

\[
\text{matrix-multiplication}(A, B, n)
\]

1. $C \leftarrow$ matrix of size $n \times n$, with all entries being 0
2. for $i \leftarrow 1$ to $n$
3. for $j \leftarrow 1$ to $n$
4. for $k \leftarrow 1$ to $n$
5. $C[i, k] \leftarrow C[i, k] + A[i, j] \times B[j, k]$
6. return $C$
Def. An independent set of a graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $u, v \in S$, we have $(u, v) \notin E$. 

$O(n^k)$ Running Time for Integer $k \geq 4$
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**Independent set of size \(k\)**

**Input:** graph \(G = (V, E)\)

**Output:** whether there is an independent set of size \(k\)
$O(n^k)$ Running Time for Integer $k \geq 4$

**Independent Set of Size $k$**

**Input:** graph $G = (V, E)$

**Output:** whether there is an independent set of size $k$

**independent-set($G = (V, E)$)**

1. for every set $S \subseteq V$ of size $k$
2. $b \leftarrow$ true
3. for every $u, v \in S$
4. \hspace{1em} if $(u, v) \in E$ then $b \leftarrow$ false
5. \hspace{1em} if $b$ return true
6. return false

Running time $= O\left(\frac{n^k}{k!} \times k^2\right) = O(n^k)$ (assume $k$ is a constant)
Maximum Independent Set Problem

Input: graph $G = (V, E)$
Output: the maximum independent set of $G$

max-independent-set($G = (V, E)$)

1. $R \leftarrow \emptyset$
2. for every set $S \subseteq V$
3. \hspace{0.5cm} $b \leftarrow \text{true}$
4. \hspace{0.5cm} for every $u, v \in S$
5. \hspace{1.2cm} if $(u, v) \in E$ then $b \leftarrow \text{false}$
6. \hspace{0.5cm} if $b$ and $|S| > |R|$ then $R \leftarrow S$
7. return $R$

Running time $= O(2^n n^2)$. 
Beyond Polynomial Time: \( n! \)

**Hamiltonian Cycle Problem**

**Input:** a graph with \( n \) vertices

**Output:** a cycle that visits each node exactly once, or say no such cycle exists
Hamiltonian Cycle Problem

**Input:** a graph with $n$ vertices

**Output:** a cycle that visits each node exactly once, or say no such cycle exists
Beyond Polynomial Time: $n!$

Hamiltonian($G = (V, E)$)

1. for every permutation $(p_1, p_2, \cdots, p_n)$ of $V$
2. $b \leftarrow$ true
3. for $i \leftarrow 1$ to $n - 1$
4. if $(p_i, p_{i+1}) \notin E$ then $b \leftarrow$ false
5. if $(p_n, p_1) \notin E$ then $b \leftarrow$ false
6. if $b$ then return $(p_1, p_2, \cdots, p_n)$
7. return “No Hamiltonian Cycle”

Running time = $O(n! \times n)$
**$O(\log n)$ (Logarithmic) Running Time**

**Binary search**

Input: sorted array $A$ of size $n$, an integer $t$;

Output: whether $t$ appears in $A$.

E.g., search 35 in the following array:
$O(\log n)$ (Logarithmic) Running Time

- Binary search
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- E.g, search 35 in the following array:

```
3 8 10 25 29 37 38 42 46 52 59 61 63 75 79
```
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  E.g, search 35 in the following array:

<table>
<thead>
<tr>
<th>3</th>
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<th>10</th>
<th>25</th>
<th>29</th>
<th>37</th>
<th>38</th>
<th>42</th>
<th>46</th>
<th>52</th>
<th>59</th>
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  $25 < 35$
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```plaintext
3  8  10  25  29  37  38  42  46  52  59  61  63  75  79
```

3 arrows pointing to the positions where 35 is searched.
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

E.g., search 35 in the following array:

| 3 | 8 | 10 | 25 | 29 | 37 | 38 | 42 | 46 | 52 | 59 | 61 | 63 | 75 | 79 |
|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|

37 > 35
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  - Input: sorted array $A$ of size $n$, an integer $t$;
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| 3  | 8  | 10 | 25 | 29 | 37 | 38 | 42 | 46 | 52 | 59 | 61 | 63 | 75 | 79 |
|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
Binary search

- Input: sorted array $A$ of size $n$, an integer $t$;
- Output: whether $t$ appears in $A$.

```python
binary-search(A, n, t)
1. $i \leftarrow 1, j \leftarrow n$
2. while $i \leq j$ do
3.   $k \leftarrow \lfloor (i + j)/2 \rfloor$
4.   if $A[k] = t$ return true
5.   if $t < A[k]$ then $j \leftarrow k - 1$ else $i \leftarrow k + 1$
6. return false
```

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Running time $= O(\log n)$
Comparing the Orders

- Sort the functions from smallest to largest asymptotically
  \( \log n, \ n, \ n^2, \ n \log n, \ n!, \ 2^n, \ e^n, \ n^n \)
- \( \log n = O(n) \)
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- \( e^n = O(n!) \)
- \( n! = O(n^n) \)
Terminologies

When we talk about upper bound on running time:

- Logarithmic time: $O(\log n)$
- Linear time: $O(n)$
- Quadratic time $O(n^2)$
- Cubic time $O(n^3)$
- Polynomial time: $O(n^k)$ for some constant $k$
- Exponential time: $O(c^n)$ for some $c > 1$
- Sub-linear time: $o(n)$
- Sub-quadratic time: $o(n^2)$
Goal of Algorithm Design

- Design algorithms to minimize the order of the running time.
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- Using asymptotic analysis allows us to ignore the leading constants and lower order terms.
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- Design algorithms to minimize the order of the running time.

  - Using asymptotic analysis allows us to ignore the leading constants and lower order terms
  - Makes our life much easier! (E.g., the leading constant depends on the implementation, compiler and computer architecture of computer.)
Q: Does ignoring the leading constant cause any issues?

- e.g., how can we compare an algorithm with running time $0.1n^2$ with an algorithm with running time $1000n$?
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- However, when $n$ is big enough, $1000n < 0.1n^2$
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- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
- For “natural” algorithms, constants are not so big!
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A:

- Sometimes yes
- However, when $n$ is big enough, $1000n < 0.1n^2$
- For “natural” algorithms, constants are not so big!
- So, for reasonably large $n$, algorithm with lower order running time beats algorithm with higher order running time.