Problem 1 (10 points)  For each of the following recurrences, using the master theorem to give the asymptotically tight upper bound.
(a) \( T(n) = 4T(n/4) + O(n) \).
(b) \( T(n) = 3T(n/3) + O(n) \).
(c) \( T(n) = 4T(n/2) + O(n^3 \sqrt{n}) \).
(d) \( T(n) = 5T(n/2) + O(n) \).

Problem 2 (15 points)  Consider a sequence of numbers defined using the following recursion:
\[
F_n = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{if } n = 1 \\
2 & \text{if } n = 2 \\
F_{n-3} + 2F_{n-2} + F_{n-1} & \text{if } n \geq 3 
\end{cases}
\]

The first few numbers in the sequence is 0, 1, 2, 4, 9, 19, 41, 88, \ldots. Given an integer \( n \), you need to output \( F_n \). Assume you are given the implementation of the BigInteger class; each object of the class holds an integer as large as \( F_n \); the basic operations such addition, subtraction and multiplication for BigInteger class are also provided to you. Design an algorithm to compute \( F_n \) that uses \( O(\log n) \) basic operations over the BigInteger class.

Problem 3(15 points)  Given two sorted arrays \( A \) and \( B \) with total size \( n \), you need to design and analyze an \( O(\log n) \)-time algorithm that outputs the median of the \( n \) numbers in \( A \) and \( B \). You can assume \( n \) is odd and all the numbers are distinct. For example, if \( A = [3, 5, 12, 18, 50] \) and \( B = [2, 7, 11, 30] \), then you need to output 11 since the set of numbers are \( [2, 3, 5, 7, 11, 12, 18, 30, 50] \).
Problem 4 (40 points) We consider the following problem of counting strong inversions. Given an array $A$ of $n$ positive integers, a pair $i, j \in \{1, 2, 3, \ldots, n\}$ of indices is called a strong inversion if $i < j$ and $A[i] > 2A[j]$. The goal of the problem is to count the number of strong inversions for a given array $A$. Implement an $O(n \lg n)$-time divide-and-conquer algorithm that runs in $O(n \lg n)$ time to solve the problem. You need to read from the standard input (i.e., the terminal) and output to the standard output (i.e., the screen).

- **Input format:** The first line of the input contains one positive integer $n$, $1 \leq n \leq 10^6$. The next $n$ lines contain the $n$ integers $A[1], A[2], \ldots, A[n]$; every integer is between 0 and $10^8$.

- **Output format:** Just output 1 line, which is total number of strong inversions.

| Input: 6 7 3 20 16 5 8 | Output: 4 | The pairs are (7, 3), (20, 5), (20, 8), (16, 5). |