CSE 632: Analysis of Algorithms II — Fall 2017

Homework 3

Lecturer: Shi Li — **Deadline: 11/13/2017**

Instructions and Policy. Each student should write their own solutions independently and needs to indicate the names of the people you discussed a problem with. It is highly recommended that you think about each problem on your own for enough time before you discuss with others. It is recommended that you type your solutions using latex and submit the pdf file via UBLearns.

Problem 1 (50 points) We learned two 3-approximation algorithms for facility location problem: one based on primal-dual and the other based on local-search. However, both algorithms give statements stronger than required by 3-approximation algorithms. Let f^* and c^* be the facility cost and connection cost of an optimum solution S^*. Let f and c be the facility cost and connection cost of the solution S^* our algorithm outputs. A 3-approximation only requires $f + c \leq 3f^* + 3c^*$. However, the primal-dual algorithm guarantees $3f + c \leq 3f^* + 3c^*$, and the local-search algorithm guarantees that $f + c \leq 2f^* + 3c^*$.

Show how to change each of the two algorithms to give a better than a 3 approximation. (Hint: you can consider scaling the down the facility costs of the original problem by a factor of $\gamma \geq 1$ and pretend you are working with the modified instance.) You do not need to repeat the proofs we had in our class. Just describe what changes in the analysis.

Problem 2 (50 points) To get a full 50 points, you just need to solve one of the following two problems.

Problem 2.1 Recall that a function $f : 2^V \to \mathbb{R}_{\geq 0}$ is sub-modular if for every subset $S \subseteq V$ and two distinct elements $i, j \in V \setminus S$, we have $f(S \cup \{i, j\}) - f(S \cup \{i\} \leq f(S \cup \{j\}) - f(S)$. A function $f : 2^V \to \mathbb{R}_{\geq 0}$ is called symmetric if $f(S) = f(V \setminus S)$ for every $S \subseteq V$.

(a) Show that a weighted cut function is a sub-modular function. Given a graph $G = (V, E)$ with weights $w \in \mathbb{R}^E$, the weighted cut function f of G is defined $f(S) = \sum_{e=(u,v)\in E:|\{u,v\}\cap S|=1} w_e$.

(b) Given a black-box oracle for a symmetric sub-modular function f on 2^V, this problem asks for the set $S \subseteq V$ with the maximum $f(S)$ (notice that $f(S)$ may not be monontone). Give a local-search $(2 + \epsilon)$-approximation algorithm for the problem, in time polynomial in $n = |V|$.

Problem 2.2: Exercise 7.2 of the WS book. Consider the multicut problem in trees. In this problem, we are given a tree $T = (V, E)$, k pairs of vertices s_i-t_i, and costs $c_e \geq 0$ for each edge $e \in E$. The goal is to find a minimum-cost set of edges F such that for all i, s_i and t_i are in different connected components of $G' = (V, E \setminus F)$.

Let P_i be the set of edges in the unique path in T between s_i and t_i. Then we can formulate the problem as the following integer program:

$$\begin{aligned} \min \quad & \sum_{e \in E} c_e x_e, \qquad \text{s.t.} \\ \sum_{e \in P_i} x_e \geq 1 \qquad & \forall 1 \leq i \leq k \\ x_e \in \{0,1\} \qquad & \forall e \in E \end{aligned}$$

The LP relaxation changes $x_e \in \{0,1\}$ to $x_e \geq 0$. First, write down the dual LP relaxation of the problem.

Suppose we root the tree at an arbitrary vertex r. Let $depth(v)$ be the number of edges on the path from v to r. Let $lca(s_i, t_i)$ be the vertex v on the path from s_i to t_i whose depth is minimum. Suppose we use the primal-dual method to solve this problem, where the dual variable we increase in each iteration corresponds to the violated constraint that maximizes $depth(lca(s_i, t_i))$.

Prove that this gives a 2-approximation algorithm for the multicut problem in trees.