Lecture 17 (10/25/2017): Uncapacitated facility location

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17.1 Uncapacitated facility location Problem

We had given an approximation algorithm in previous lectures using Dual Fitting. This time we will solve it with Local Search.

This problem is described as follows:

Given: C of clients
F of facilities
$d(i, j)$ as cost of assigning client j to factory i
f_i as cost of opening factory i

Goal: find a set $S \subseteq F$ to minimize

$$\sum_{i \in S} f_i + \sum_{j \in C} \min_{i \in S} d(i, j)$$

This time our solution is pretty straightforward. The main idea is to check the cost changing for each node operation, and pick the better one. Here's the algorithm:

Algorithm 1 localSearch(C, F)

Start from arbitrary $S \in F$
[**1**] **if** $\exists$i $\in S$ and $cost(S \cup \{i\}) < cost(S)$
then $S \leftarrow S \cup \{i\}$, goto [**1**]
[**2**] **else if** $\exists$i $\in S$ and $cost(S \setminus \{i\}) < cost(S)$
then $S \leftarrow S \setminus \{i\}$, goto [**1**]
[**3**] **else if** $\exists$i $\in S, i' \notin S$ and $cost(S \setminus \{i\} \cup \{i'\}) < cost(S)$
then $S \leftarrow S \setminus \{i\} \cup \{i'\}$, goto [**1**]
return S

However, it is not easy to proof that this algorithm gives us a satisfying result. First, we will prove that our algorithm is not too bad.

Lemma 17.1 *Result given by algorithm 1 is not too bad.*

To help proving this conclusion, let's define S as the result of our algorithm, and S^* as the optimum result, or say OPT. Further more, we have:

$j \in C$: $i(j) = arg \min_{i \in S} d(i, j)$
$C_j = d(j, i(j))$
$i^*(j) = arg \min_{i \in S^*} d(i, j)$
$C_j^* = d(j, i^*(j))$
$f = \sum_{i \in S} f_i$: facility cost of S

$f^* = \sum_{i \in S} f_i$: facility cost of S^*
$c = \sum_{j \in C} C_j$: connection cost of S
$c^* = \sum_{j \in C} C_j$: connection cost of S^*

Lemma 17.2 $C \leq C^* + f^*$

Proof: For every $i \in S^* \setminus S$, we have cost(S) $\leq$ cost(S $\cup \{i\}$), which equals to

$$\sum_{i \in S} f_i + \sum_{j \in C} C_j \leq \mathrm{cost}(S \cup \{i\}) \leq \sum_{i \in S \cup \{i\}} f_i + \sum_{j \in C \setminus C_i^*} C_i + \sum_{j \in C_i^*} C_i^*$$

Which leads to:

$$\sum_{j \in C_i^*} C_j \leq f_i + \sum_{j \in C_i^*} C_j^*$$

On the other hand, for $i \in S^* \cap S$, $\forall$j $\in C_i^*$, we have:

$$d(j, i(j)) = C_j \leq C_j^* = d(i, j)$$

Thus we can conclude that

$$\sum_{i \in S^*} \sum_{j \in C_i^*} C_j = \sum_{i \in S^* \setminus S} \sum_{j \in C_i^*} C_j + \sum_{i \in S^* \cap S} \sum_{j \in C_i^*} C_j$$

$$\leq \sum_{i \in S^* \setminus S} (\sum_{j \in C_i^*} C_j + f_i) + \sum_{i \in S^* \cap S} \sum_{j \in C_i^*} C_j^*$$

$$= \sum_{j \in C^*} C_j^* + \sum_{i \in S^* \setminus S} f_i \leq C^* + f^*$$

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