CSE 632: Analysis of Algorithms II **Fall 2017**

Lecture 18 (10/27/2017): Uncapacitated facility location

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18.1 Proof for Uncapacitated facility location Problem

In last lecture, we proved that our local search alrogithm will give us a result that is not too bad. This time we will further prove its efficiency.

Lemma 18.1 $f \leq f^* + 2C^*$

It is much complicated to prove this lemma. We will divide it to several sub inequalities so that we can combine them to the first lemma.

Firstly, $\forall i \in S^*$, let $\phi(i)$ be the nearest facility in S to i. Then we will have:

Lemma 18.2 $d(j, \phi(i^*(j))) \leq C_j + 2C_j^*$

Proof:

$$\begin{aligned} d(j, \phi(i^*(j))) &\leq d(j, i^*(j)) + d(i^*(j), \phi(i^*(j))) \\ &\leq C_j^* + d(i^*(j), j) + d(j, i(j)) \\ &= C_j^* + C_j^* + C_j = C_j + 2C_j^* \end{aligned}$$

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Secondly, let's pretend that $\forall j \in C, i(j) \neq \Phi(i^*(j))$. Then we can prove the second inequality:

Lemma 18.3 $f^* \leq 2C^*$

Proof: $\forall i \in S$, we have:

$$f_i + \sum_{j \in C_i} C_j \leq \sum_{j \in C_i} (C_j + 2C_j^*)$$

Which leads to

$$\sum_{i \in S} f_i \leq \sum_{j \in C_i} 2C_j^*$$

So we have

$$f \leq 2C_j^*$$

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The third part takes more steps and assumptions. $\forall i \in S^*$, let $\phi^{-1}(i) = \{i' \in S^*, \phi(i') = i\}$. If $\phi^{-1}(i) \neq \emptyset$, define $\psi(i)$ be nearest facility of i in $\phi^{-1}(i)$.

Consider such operation: remove i, and add $\psi(i)$. To be more exact, for each $j \in C_i$, if $\phi(i^*(j)) \neq i$, then we connect customer j to $\phi(i^*(j))$. If $\phi(i^*(j)) = i$, then we connect customer j to $\psi(i)$.

So, if $\phi(i^*(j)) = i$, then we can have:

Lemma 18.4 $d(j, \psi(i)) \leq 2C_j + C_j^*$

Proof:

$$\begin{aligned} d(j, \psi(i)) &\leq d(j, i(j)) + d(i(j), \psi(i)) \\ &\leq C_j + d(i(j), i^*(j)) \\ &\leq C_j + d(i(j), j) + d(j, i^*(j)) \\ &= C_j + C_j + C_j^* = 2C_j + C_j^* \end{aligned}$$

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The last sections requires some more definitions. Let's define:
$C_j^{'}$: clients $j \in C_i$, s.t. $\phi(i^*(j)) \neq i$
$C_j^{''}$: clients $j \in C_i$, s.t. $\phi(i^*(j)) = i$
So, for $i \in S$, if $\phi(i)$ is not defined, we will have:

$$f_i + \sum_{j \in C_i} C_j \leq \sum_{j \in C_i} (C_j + 2C_j^*)$$

Which leads to

$$f_i \leq 2 \sum_{j \in C_i} C_j^*$$

On the other hand, for $i \in S$, if $\phi(i)$ is defined, we will have:

$$f_i + \sum_{j \in C_i} C_j \leq \sum_{j \in C_i} (C_j + C_j^*) + \sum_{j \in C_i^{'}} (2C_j + C_j^*) + f_{\psi(i)}$$

Which leads to

$$f_i \leq \sum_{j \in C_i^{'}} 2C_j^* + \sum_{j \in C_i^{''}} (C_j + C_j^*) + f_{\psi(i)}$$

So $\forall i \in S^*$, s.t. $\psi(\phi(i)) \neq i^{*'}$, we have:

Lemma 18.5 $\sum_{j \in C_i^{*'} : \phi(i') = i(j)} C_j \leq f_i^{'} + \sum_{j \in C_i^{*'} : \phi(i') = i(j)} C_j^*$

Finally, let's sum up all the inequalities we get by proofs described above. We will have:

$$\sum_{i \in S} f_i \leq \sum_{i \in S^*} f_i + 2 \sum_{j \in C} C_j^*$$

Lemma proved.