CSE 632: Analysis of Algorithms II **Fall 2017**

Lecture 19 (11/01/2017): Ellipsoid Method, Semidefinite Programming

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19.1 Ellipsoid Method

Consider the following example.

Example 1:
LP relaxation for travelling salesman problem
Given: a metric d over V,
Goal: find a shortest tour to visit all vertices in V

Let's say $x_{\{u,v\}} \in \{0,1\}$ denotes whether $\{u,v\}$ is used in the tour.
So, the integer programming for the problem will be the following:

$$\min \sum_{\{u,v\}} x_{\{u,v\}}.d(u,v)$$

$$\text{subject to}$$

$$\sum_{u \neq v} x_{\{u,v\}} = 2 \ (\forall v \in V)$$

$$\sum_{\{u,v\}:|\{u,v\}\cap S|=1} x_{\{uv\}} \geq 2 \ (\forall S \subseteq V, S \neq \emptyset)$$

$$x_{\{u,v\}} \in \{0,1\} \ (\forall u, v \in V)$$

For LP relaxation, we will only change the last condition into $x_{\{u,v\}} \geq 0$ $(\forall u, v \in V)$.

Seperation Oracle O:
Given x, O will
$\star$ accept x if x satisfies all the constraints,
$\star$ reject x if x does not satisfy all the constraints and return a constraint that x violates.

Efficient seperation oracle for LP relaxation of TSP:

Using max-flow-min-cut theorem(MFMC), fix $s \in V$, enumerate $t \in V \setminus \{s\}$, check if we can send 2 units flow from s to t, in the network with capacities $\{x_{u,v}\}_{u,v}$.

If for some t, we cannot send 2 units flow from s to t, then by MFMC theorem, we can find a cut $(s, v \setminus S)$, $s \in S, t \notin S$ such that

$$\sum_{\{u,v\}:|\{u,v\}\cap S|=1} x_{\{u,v\}} < 2$$

otherwise, all constraints are satisfied.

$$y^t_{(v,u)} + y^t_{(u,v)} \leq x_{\{u,v\}} \ (\forall u, v)$$
$$\sum_{u \neq v} y^t_{(v,u)} \ (\forall v \notin \{s,t\})\text{-Flow Conservation-}$$
$$\sum_{u \neq s} y^t_{(s,u)} = 2 \ (\forall s)$$
$$y^t_{(u,s)} = 0 \ (\forall u)$$

Example 2:
Multicut Problem
Given: $G = (V, E)$, cost $\{c_e\}_{e \in E}$
$(s_1, t_1), (s_2, t_2), ..., (s_k, t_k)$ pairs of vertices in V.
Goal: find a set E' of edges such that s_i and t_i are disconnected in $(V, E \setminus E'); (\forall i \in [k])$, minimize $\sum_{e \in E'} c_e$.

Now, let x_e denote whether $e \in E'$ or not, in other words whether we are removing edge e to obtain the multicut. Thus, the LP will be the following:

$$\min \sum_{e \in E} c_e$$
$$\sum_{e \in P} x_e \geq 1 \ (\forall \text{ path P connecting} s_i \text{ to } t_i, \text{for some } i)$$
$$x_e \geq 0 \ (\forall e \in E)$$

Efficient seperation oracle for LP relaxation of multicut:
For every i, find the shortest path from s_i to t_i, using $\{x_e\}$ as costs.
If distance from s_i to t_i is less than 1, then return P=shortest path, otherwise accept x.

Now, this is where we are
Given: seperation oracle O, $c \in R^n$,
Goal: min $C^T x$ such that x is accepted by O.

$\star$ Ellipsoid is obtained from a ball by scaling and rotation.

Algorithm 1 Ellipsoid Method

$P \leftarrow$ ellipsoid containing all feasible solutions
while P is "not small enough" **do**
  $\hat{x} \leftarrow$ center of P
  query O whether $\hat{x}$ is feasible
  if Yes **then**
    $P' \leftarrow \{x \in P : C^T x \leq C^T \hat{x}\}$
    $x^* \leftarrow \hat{x}$
  else
    let $ax \geq b$ be the constraint that $\hat{x}$ violated returned by O
    $P' \leftarrow \{x \in P : ax \geq b\}$
    $P \leftarrow$ small ellipsoid containing P'

Lemma 19.1 *We can guarantee that volume of P is at most $(1 - \frac{1}{2n})$ times volume of P in the previous iteration.*

Number of iterations $= O(n.lg \frac{\text{initial volume of P}}{\text{minimal possible value}})$