CSE 632: Analysis of Algorithms II: Randomized Algorithms — Fall 2019

Homework 1

Lecturer: Shi Li — **Deadline: Sep 26, 2019**

Problem 1 (10 points) Another way to decrease the error probability of Freivalds' matrix multiplication verification algorithm is to choose the vector r randomly from the set $\{0, 1, 2, 3, \cdots, M-1\}^n$ and check if $ABr = Cr$. Prove that the error probability of this modification of Freivalds' algorithm is at most $1/M$.

Problem 2 (15 points) Consider the following modification of the randomized algorithm for selecting the i-th smallest element from an array A of *distinct* numbers. In this modification, we repeatedly select the pivot x until we find a good one.

Algorithm 1 randomized_selection(A, i)

```
if |A| = 1 then return A[1]
repeat
    x ← a random number in A
    B ← array of numbers in A that are smaller than x
    C ← array of numbers in A that are bigger than x
until |B| ≤ 3|A|/4 and |C| ≤ 3|A|/4
if i ≤ |B| then
    return randomized_selection(B, i)
else if i = |B| + 1 then
    return x
else
    return randomized_selection(C, i − |B| − 1)
```

Prove that this variant also has expected running time $O(n)$, where $n = |A|$.

Problem 3 (9 points) For each of the following three variables X,

(a) give all the values X can take and the probability that X equals each of these values,

(b) compute $\mathbb{E}[X]$, and

(c) compute $\mathrm{Var}[X]$.

Variable (i): Toss a unbiased coin 3 times and let X be the number of times we get head-up.

Variable (ii): Throw a 4-sided dice (with values 1, 2, 3, 4 on the 4 sides) twice and let X be the product of the two values we obtained.

Variable (iii): Suppose there is a puzzle and there are 3 programs which we can run to try solving the puzzle. Program A takes 1 minute to solve the puzzle and program B takes 2 minutes to solve the puzzle. Program C can not solve the puzzle and it will report failure in 1 minute. Suppose we run the 3 programs sequentially using a uniformly random order of the 3 programs (that is, with probability 1/6 each, we use the order ABC, ACB, BAC, BCA, CAB, CBA). We stop when a program solved the puzzle. Let X be the total time (in minutes) we spent on running the programs. For example, suppose the order we used is CAB. We run C and fail, then we run B and succeed and then X is 2.

Problem 4 (12 points) Remember in the course you learned the geometric distribution: we keep tossing a biased coin with head-up probability p until we see a head-up. Let X be the total number of coin tosses we have. Then X is the geometric distribution with parameter p. We showed that $\mathbb{E}[X] = 1/p$ and $\mathrm{Var}[X] = (1-p)/p^2$. Now suppose we stop the process when we see 10 head-up's and let Y be the number of coin tosses we have in the process. What is $\mathbb{E}[Y]$ and $\mathrm{Var}[Y]$? You need to give proofs for your answers.

Problem 5 (14 points) Let X be a random variable from a geometric distribution with parameter p. Prove that the following two events E and F are independent:

- event E: $X \geq 50$,
- event F: X is an integer multiple of 7.