CSE 632: Analysis of Algorithms II: Randomized Algorithms	Fall 2019

Homework 4

Lecturer: Shi Li **Deadline: Nov 24, 2019**

Problem 1 (10 points). Consider the Markov Chain on 4 verticies $\{a, b, c, d\}$ defined by the following transition matrix.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix} \end{array}$$

Assume we start the Markov Chain with X_0 be randomly chosen from $\{a, b, c, d\}$, each with $1/4$ probability. Give the distribution for X_2.

Problem 2(20 points). Let $p \in (0, 1)$ and a be an integer in $[0, n]$. Consider the Markov chain on $\{0, 1, 2, \cdots, n\}$ defined using the following process.

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X_0 = a
for t ← 0, 1, 2, ··· do
    if X_t = 0 then output "left" and exit
    if X_t = n then output "right" and exit
    Let X_{t+1} = { X_t − 1   with probability p
                  { X_t + 1   with probability 1 − p
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Compute the probability that the algorithm outputs "right", as a function of a and p. (Hint: let Q_a be the probability that the algorithm outputs "right" when we start from $X_0 = a$. Let $\Delta_a = X_a - X_{a-1}$ and then derive some equalities about Δ_a's.)

Problem 3 (30 points). Let $G = (V, E)$ be an undirected connected graph with $n = |V|$ vertices. Recall that the hitting time $H_{u,v}$ from u to v is the *expected* number of steps for a random walk starting from u to reach v. Let $h_{\max} = \max_{u,v \in V} H_{u,v}$.

Let $\tilde{C}_u$ be the number of steps for a random walk starting from u to visit all verticies in V. (So, $\tilde{C}_u$ is a random variable). Then recall that $C_u = \mathbb{E}[\tilde{C}_u]$ is the covering time from u.

(3a) For every integer $t \geq 1$ and $u \in V$, prove that $\Pr[\tilde{C}_u > 2th_{\max} \log n] \leq 1/n^{t-1}$. (Hint : for every $v \in V$, give an upper bound the probability that the random walk does not cover v in $2th_{\max} \log n$ steps. Then apply the union bound.)

(3b) Using (3a) to show that $C_u \leq O(\log n) h_{\max}$ for every $u \in V$.

Thus, the covering time and maximum hitting time of a graph differ by at most a factor of $O(\log n)$.