Problem 1. Indicate whether each of the following pairs \((E, \mathcal{I})\) is a matroid or not. If a \((E, \mathcal{I})\) is a matroid, list all the bases and circuits of the matroid. If a \((E, \mathcal{I})\) is not a matroid, state why it is not.

(1a) \(E = \{1, 2, 3\}, \mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}\}\).

(1b) \(E = \{1, 2, 3, 4\}, \mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}\).

(1c) \(E = \{1, 2, 3, 4\}, \mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}\}\).

Problem 2. List the bases and circuits of the following two matroids.

![Figure 1: Matroids for Problem 2.](image)

(2a) The graphic matroid defined by the graph in Figure 1(a).

(2b) The transversal matroid defined by the graph in Figure 1(b). That is, a subset of \(\{1, 2, 3, 4, 5\}\) is an independent set if and only if there is a matching covering in the graph covering the set.

Problem 3. Prove that the following operations on matroids result in matroids.

(3a) Let \(M_1 = (E_1, \mathcal{I}_1)\) and \(M_2 = (E_2, \mathcal{I}_2)\) be two matroids with \(E_1 \cap E_2 = \emptyset\). Define \(\mathcal{I}_1 \oplus \mathcal{I}_2 = \{A_1 \cup A_2 : A_1 \in \mathcal{I}_1, A_2 \in \mathcal{I}_2\}\). Prove that \(M := (E_1 \cup E_2, \mathcal{I}_1 \oplus \mathcal{I}_2)\) is a matroid.
(3b) Let $M = (E, \mathcal{I})$ be a matroid, and $X \subseteq E$. Define $\mathcal{I}_X = \{A \subseteq X : A \in \mathcal{I}\}$. Show that $M_X := (X, \mathcal{I}_X)$ is a matroid.

(3c) Let $M = (E, \mathcal{I})$ be a matroid, and $k \geq 0$ be an integer. Define $\mathcal{I}_k = \{A \in \mathcal{I} : |A| \leq k\}$. Show that $M_k := (E, \mathcal{I}_k)$ is a matroid.

**Problem 4**

(4a) Let $G = (V, E)$ be a graph. For every $S \subseteq V$, define $f(S)$ to be the number of edges in $G$ that are between $S$ and $V \setminus S$. Show that the function $f : 2^V \rightarrow \mathbb{Z}_{\geq 0}$ is a submodular function. (The function $f$ is called the cut function of $G$.)

(4b) Let $U$ be a finite ground set. Let $U_1, U_2, \ldots, U_m$ be subsets of $U$, and $w_1, w_2, \ldots, w_m$ be non-negative real numbers. For every $S \subseteq \{1, 2, 3, \ldots, m\}$, define $f(S) = \sum_{j \in S} \max_{i \in S, j \in U_i} w_i$. (If for some $j \in U$, we have no $i \in S$ with $j \in U_i$, then $\max_{i \in S, j \in U_i} w_i = 0$.) Show that the function $f : 2^{\{1, 2, 3, \ldots, m\}} \rightarrow \mathbb{R}_{\geq 0}$ is sub-modular. ($f$ is called a weighted coverage function.)