Problem 1 (30 points). Consider the procedure of throwing \( m \) balls into \( n \) bins, where each ball is thrown into 1 of the \( n \) bins, uniformly at random independently of the other balls. For any integer \( f \), let \( E_f \) be the event that some bin contains at least \( f \) balls. We are interested in the following question: how big should \( f \) be so that \( \Pr[E_f] \leq 0.1 \)? Prove the following statements for three different values of \( m \):

- If \( m = n \), then for some \( f = O\left(\frac{\log n}{\log \log n}\right) \), we have \( \Pr[E_f] \leq 0.1 \).
- If \( m = n \log_2 n \) (assuming \( \log_2 n \) is an integer), then for some \( f = O(\log n) \), we have \( \Pr[E_f] \leq 0.1 \).
- If \( m = n^2 \), then for some \( f = n + O(\sqrt{n} \log_2 n) \), we have \( \Pr[E_f] \leq 0.1 \).

Problem 2 (20 points). Suppose we have an array \( A \) of \( n \) distinct integers. We say an integer in \( A \) is an approximate median, if its rank in \( A \) is between \( 0.4n \) and \( 0.6n \) (the rank of \( x \) is the number of integers in \( A \) that are smaller than or equal to \( x \)). Consider the following algorithm for finding the approximate median of \( A \):

1. \textbf{for} \( i \leftarrow 1 \) to \( m \) \textbf{do}
2. \hspace{1em} let \( B[i] \) be a number chosen uniformly at random from the array \( A \)
3. \hspace{1em} output the median of array \( B \)

Show that for the algorithm to succeed with probability at least \( 1 - 1/n^2 \), it suffices to choose \( m = O(\log n) \). (Thus, the running time to find an approximate median is much smaller than the running time to find the (exact) median.)

Problem 3 (10 points). Consider the following game.

1. put \( a \) white balls and \( b \) black balls into a bin
2. \textbf{repeat} \( n \) times:
3. \hspace{1em} randomly pick a ball from the bin
4. \hspace{1em} put the picked ball back and additionally add another ball of the same color to the bin

So at the end of the game, we have in total \( a + b + n \) balls.

For every \( i = 1, \cdots, n \), let \( Y_i \) be the fraction of black balls in the bin after the \( i \)-th iteration of the loop. Show that \( Y_1, Y_2, \cdots, Y_i \) is a martingale sequence.