Problem 1 (10 points). Consider the Markov Chain on 4 vertices \{a, b, c, d\} defined by the following transition matrix.

\[
\begin{pmatrix}
  a & b & c & d \\
  \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\
  \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\
  \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
  \frac{1}{2} & 0 & \frac{1}{2} & 0
\end{pmatrix}
\]

Assume we start the Markov Chain with \(X_0\) be randomly chosen from \{a, b, c, d\}, each with 1/4 probability. Give the distribution for \(X_2\).

Problem 2 (20 points). Let \(p \in (0, 1)\) and \(a\) be an integer in \([0, n]\). Consider the Markov chain on \(\{0, 1, 2, \cdots, n\}\) defined using the following process.

1: \(X_0 = a\)
2: for \(t \leftarrow 0, 1, 2, \cdots\) do 
3: if \(X_t = 0\) then output “left” and exit 
4: if \(X_t = n\) then output “right” and exit 
5: Let \(X_{t+1} = \begin{cases} X_t - 1 & \text{with probability } p \\ X_t + 1 & \text{with probability } 1 - p \end{cases}\)

Compute the probability that the algorithm outputs “right”, as a function of \(a\) and \(p\). (Hint: let \(Q_a\) be the probability that the algorithm outputs “right” when we start from \(X_0 = a\). Let \(\Delta_a = X_a - X_{a+1}\) and then derive some equalities about \(\Delta_a\)’s.)

Problem 3 (30 points). Let \(G = (V, E)\) be an undirected connected graph with \(n = |V|\) vertices. Recall that the hitting time \(H_{u,v}\) from \(u\) to \(v\) is the expected number of steps for a random walk starting from \(u\) to reach \(v\). Let \(h_{\max} = \max_{u,v \in V} h_{u,v}\).

Let \(\tilde{C}_u\) be the number of steps for a random walk starting from \(u\) to visit all vertices in \(V\). (So, \(\tilde{C}_u\) is a random variable). Then recall that \(C_u = \mathbb{E}[\tilde{C}_u]\) is the covering time from \(u\).

(3a) For every integer \(t \geq 1\) and \(u \in V\), prove that \(\Pr[\tilde{C}_u > 2th_{\max} \log n] \leq 1/n^{t-1}\). (Hint: for every \(v \in V\), give an upper bound the probability that the random walk does not cover \(v\) in \(2th_{\max} \log n\) steps. Then apply the union bound.)

(3b) Using (3a) to show that \(C_u \leq O(\log n)h_{\max}\) for every \(u \in V\).

Thus, the covering time and maximum hitting time of a graph differ by at most a factor of \(O(\log n)\).