$M_1 = (E, I_1)$, $M_2 = (E, I_2)$

Bipartite Matching polytope.

$P_{M_1} = \text{convex-hull} \left\{ x^A : A \in I_1 \right\}$

$P_{M_2} = \text{convex-hull} \left\{ x^A : A \in I_2 \right\}$

$P_{M_1} \cap M_2 = \text{convex-hull} \left\{ x^A : A \in I_1 \cap I_2 \right\}$

$P_{M_1} \cap M_2 = P_{M_1} \cap P_{M_2}$

$= \left\{ x \in [0, 1]^E : \sum_{i \in E'} x_i \leq \text{rank}_{M_1} (E), \forall E' \right\}$

$= \left\{ x \in [0, 1]^E : \sum_{i \in E'} x_i \leq \text{rank}_{M_2} (E), \forall E' \right\}$

$G = (A(U \cup B), E)$

A matching is exactly a common independent set of $M_1$ and $M_2$.

$M_1$: partition matroid defined by left side vertices

$M_2$: right.
minimum-cost arborescence problem

Given: directed graph

\[ G = (V, E), \text{ root } r \in V \]

\[ W : \mathbb{R}_{\geq 0} \text{ weights on edges} \]

find an arborescence \( F \subseteq E \)

spanning all vertices, rooted at \( r \),

so, as to minimize \( \sum_{e \in F} W_e \)

spanning

what is an arborescence?

1. when ignoring directions, it must be a spanning tree

\[ r \]

1. \( F \) must be a base of the graphic matroid defined on \( G \), ignoring directions.
\( \{ \text{Sink}(v) : v \notin r \} \)

Thresholds are 1. for all groups in the partition.

\( \Rightarrow \) partition matroid.

\( F \) is a base of the partition matroid.

\( \mathcal{P} = \text{convex-hull} \left( \{ x^F : F \subseteq E \text{ is an arborescence} \} \right) \)

\[ \mathcal{P} = \mathcal{F}^{\text{base}}_{M_1} \cup \mathcal{F}^{\text{base}}_{M_2}. \]

\( M_1 : \text{graphic matroid} \)

\( M_2 : \text{partition matroid} \)

\( \forall x \in E_0, \bigcap E_x \in E : \)

\( \forall S \subseteq V : \sum_{x \in E(S)} x_e \leq |S| - 1 \)

\( \sum_{e \in E} x_e = n - 1 \)

\( \forall v \neq r : \sum_{e \in E_{\text{sink}}(v)} x_e = 1 \)
Problem 3: color-restricted spanning tree.

Given $G = (V, E)$, edge costs w.
k colors, each edge is colored with 1 of the k colors.

Goal: find a minimum-cost spanning tree where
the number of edges of color i is at most $N_i$, $\forall i \in [k]$.

A feasible solution:

1. is a base of the graphic matroid.
2. is an independent set of a partition matroid.
Matching in general graphs.

\[ G = (V, E) \]

\( M \subseteq E \) is a matching if every vertex is incident to at most one edge in \( M \).

Perfect matching: every vertex in \( M \) is incident to exactly one edge in \( M \).

\[ |V| \] must be even.

\[ P_{pm}(G) = \text{convex-hull} \]

\( \{ x^M : M \subseteq E \text{ is a perfect matching} \} \)

\( G \) is bipartite:

\[ P_{pm}(G) = \{ x \in [0, 1]^E : \sum_{e \in s(w)} x_e = 1 \ \forall u \in V \} \]

Q: What if \( G \) is not bipartite?
in a perfect matching, there is an edge in $\{a, b, c\} \times \{d, e, f\}$.

$S \subseteq V$, $|S|$ is odd.

$$\sum_{x \in E(S, V \setminus S)} x_e \geq 1$$

set of edges between $S$ and $V \setminus S$.

Thm.

$$P_{PM}(G) = \{ x \in \{0,1\}^E :$$

$$\sum_{e \in \delta(u)} x_e = 1 \quad \forall u \in V$$

$$\sum_{x \in E(S, V \setminus S)} x_e \geq 1, \quad \forall S \subseteq V, |S| \text{ odd} \}$$
Proof:
call the right side \( Q(G_t) \)
need to prove \( P_{pm}(G_t) = Q(G_t) \)
suppose \( P_{pm}(G_t) \neq Q(G_t) \) for
some \( G_t \), then we take
the graph \( G_t \) with the
smallest \( |V| + |E| \), that
satisfies \( P_{pm}(G_t) \neq Q(G_t) \).

\[
Q(G_t) : \sum_{x \in f(u)} x_a = 1, \quad \forall u \in V,
\sum_{x \in E(s,v)} x_e \geq 1, \quad \forall s \leq v,
\frac{|s|}{|s|} \text{ odd.}
\]

there is a vertex point \( x \) of
\( Q(G_t) \), that is not integral.

1. \( \forall e \in E, \ x_e \in (0, 1) \)
\( x_e \neq 0, \ x_e \neq 1 \)
2. X is defined by a set of linear equations.

\[ \sum_{e \in E(u)} x_e = 1 \quad \forall u \in V \]

for some odd-sized sets \( S \), we have

\[ (\star) \quad \sum_{e \in E(S \cup S)} x_e = 1 \]

Idea: at least one inequality of type (\( \star \)) is used in defining \( X \).
Theorem holds for $G_1$ and $G_2$.

$\Rightarrow$ Theorem holds for $G$. 