\[ P_{r} \left[ \sum_{i=1}^{k} Y_i \geq (1+s)c \right] \leq \left( \frac{e^s}{(1+s)^{1+s}} \right)^c \]

\[ s = \frac{2 \log N}{\log \log N} \]

\[ f = \left( \frac{2 \log N}{\log \log N} \right)^{2 \log N / \log \log N} \]

\[ = \log N \cdot \frac{\log N}{\log \log N} = N \]

Q: for what \( s \), do we have \( f^s = N \)
\[ f = \frac{\log N}{\log \log N} \]

\[ f^* = \left( \frac{\log N}{\log \log N} \right)^{\frac{\log N}{\log \log N}} \]

\[ \leq \left( \log N \right)^{\frac{\log N}{\log \log N}} \]

\[ = N. \]

A: \[ f = \Theta \left( \frac{\log N}{\log \log N} \right) \]

if we set \( f = \Theta \left( \frac{\log n}{\log \log n} \right) \) to be big enough,

then we have

\[ \Pr \left[ \sum_{i=1}^{k} Y_i \geq (1+\varepsilon)c \right] \leq \frac{1}{2^m} \]

the congestion of \( e \) is at least \( c(1+\varepsilon)c \).
If we consider all edges, using union bound:

\[ \Pr \left[ \exists e \in E, \text{ congestion of } e \text{ is at least } (1+\delta)C \right] \leq \frac{1}{2m} \cdot m = \frac{1}{2} \]

\[ \Pr \left[ \forall e \in E, \text{ congestion of } e \text{ is less than } (1+\delta)C \right] \geq 1 - \frac{1}{2} = \frac{1}{2} \]

Weight vertex cover:

\[ G = (V, E), \ w \in \mathbb{R}_{\geq 0}^V \]

Goal: find a vertex cover \( S \) with minimum \( \sum_{v \in S} w_v \).

Vertex cover:

2-approximation

1. LP rounding
2. Greedy algorithm
① can be easily extended to weighted vertex cover.

② cannot.

Today: more general greedy algorithm that gives a 2-approximation for weighted vertex cover.

Technique: primal-dual algorithm:

Construct a solution for the problem, and a dual solution for the dual LP, so that we can bound the cost of solution.
Primal: \[ \min \sum_{v \in V} w_v x_v \]
\[ x_u + x_v \geq 1, \forall (u,v) \in E \]
\[ x_u \geq 0, \forall u \in V \]

Dual: \[ \max \sum_{e \in E} y_e \]
\[ \sum_{e \in \delta(v)} y_e \leq w_v, \forall v \in V \]
\[ y_e \geq 0, \forall e \in E \]

How should we interpret the dual variables \((y_e)_{e \in E}\)?

\(y_e\): the price that \(e\) is willing to pay to get covered.
\( y_e \leftarrow 0 \ \forall e \in E \)

\( S \leftarrow \emptyset \)

while \( \exists (u, v) \in E \) that is not covered by \( S \): increase \( y_e \) until

\[ \sum_{e' \in S(u)} y_{e'} = Wu, \ OR \]

\[ \sum_{e' \in S(v)} y_{e'} = Wv. \]

add \( u \), and/or \( v \) to \( S \), depending on whether constraints are tight.

always maintain \( S \) is the set of tight vertices:

a vertex \( u \) is tight if

\[ \sum_{e \in S(u)} y_e = Wu. \]
Analysis of the primal-dual algorithm:

1. $S$ is a valid vertex cover when the algorithm terminates.
2. Algorithm runs in polynomial time.
3. Need to prove algorithm is a 2-approximation.

$S = \{a, b, d\}$

$y = (3, 0, 1, 0, 2)$

If all weights are 1, algorithm is the greedy algorithm.
(3a) $y$ is a valid dual solution.

(3b)

$$\sum_{vw} w_{vw} \leq \sum_{v \in S} e_{v} \leq \sum_{v \in S} y \cdot e_{v} \leq 2 \cdot \sum_{e \in E} y \cdot e \leq 2 \cdot (\text{dual value}) \leq 2 \cdot (\text{optimum cost})$$

$$\text{D} \leq p \leq \text{I}$$

Value of dual LP

Value of primal LP

Value of integer program