Maximum weight bipartite matching.

Given:
- Bipartite graph $G = (L \cup R, E)$
- $w \in \mathbb{R}_{\geq 0}^E$

Goal: find a matching $M \subseteq E$ so as to maximize $\sum_{e \in E} w_e$.

LP is integral $\Rightarrow$ solve the problem directly.
Hungrian Algorithm

: primal-dual algorithm.

1: unweighted bipartite matching.

Given a matching $M$:

- a vertex is free if it is not matched.

augmenting path: a path in $G$ that

- alternates between matched edges and unmatched edges
- two end-vertices are free
- augmenting operation over a augmenting path $P$. 
change matched edges in $P$

to unmatched.

change unmatched edges in $P$

to matched.

algorithm for maximum
(unweighted) bipartite matching

$M \leftarrow \emptyset$

While $\exists$ augmenting path $P$

augment along $P$. 
auxiliary graph $\overrightarrow{G}[M]$: 

edges in $M$ are directed from $R \rightarrow L$.

edges in $E \setminus M$ are directed from $L \rightarrow R$.

augmenting path in $G$.

$\iff$ directed path between 2 free vertices in $\overrightarrow{G}[M]$.

primal LP:

$$\max \sum_{e \in E} w_e x_e$$

$$\sum_{e \in s(u)} x_e = 1 \quad \forall u \in LUR$$

$$x_e \geq 0 \quad \forall e \in E$$

dual LP:

$$\min \sum_{u \in LUR} y_u$$

$$y_u + y_v \leq w_{(u,v)} \quad \forall (u,v) \in E.$$
In algorithm:

maintain:

1. a feasible dual solution \((y_u)_{u \in L \cup R}\).

    an edge \((u, v)\) is tight
    if \(y_u + y_v = W(u, v)\)
    loose if \(y_u + y_v > W(u, v)\)

2. \(G_{tight} = (L \cup R, E_{tight})\)

\(E_{tight}\): set of tight edges.

3. matching \(M \subseteq E_{tight}\).

algorithm:

initialize \(y\) as follows:

\(y_u = \max_{e \in s(u)} \) \(e \in E\) \(u \in L\)

\(y_u = 0\). \(u \in R\).
While $M$ is not a perfect matching:

- If $\exists$ augmenting path in $G_{\text{tight}}$ w.r.t. $M$.
  - Augment along the path.

- Else:
  - Update the dual $y$.

Update the dual $y$:

$S$: set of vertices that can be reached from a free vertex in $L$ in $G_{\text{tight}}[M]$

$T: = (1UR) \setminus S$

$\Delta$: large enough

$u \in S \setminus L$: $y_u \leftarrow y_u - \Delta$

$u \in S \cap R$: $y_u \leftarrow y_u + \Delta$
\( \text{LAS} \leftrightarrow \text{RAS} \)

\( \text{LAT} \leftrightarrow \text{RAT} \)

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\( \text{W} + \text{W}_v \) does not change

\( \text{W} + \text{W}_v \) will increase by \( \Delta \)

\( \text{W} + \text{W}_v \) will decrease by \( \Delta \)