Today:
rank functions of a matroid
sub-modular functions.

Last time:
base of a matroid \( M = (E, \mathcal{I}) \)
a base = a maximal independent set
= a maximum
⇒ all bases have the same size

uniform matroid:
\( E, \mathcal{I} = \{X \subseteq E : |X| \leq k\} \)
assume \(|E| \leq k\)
a base \( X \): \(|X| = k\)
a circuit \( X \): \(|X| = k+1\)
(every proper subset of \( X \) is independent)

partition matroid:
\( E = \{1, 2, 3, 4, 5, 6\} \)
\( \leq 1 \)
\( \leq 2 \)
\( \leq 3 \)
\( \leq 4 \)
\( \leq 5 \)
\( \leq 6 \)

\( \{1, 3, 4\} \) is a base.
\( \{2, 4, 5\} \) is a base
\( \{1, 2\} \) is a circuit
\( \{3, 4, 5\} \) is a circuit
A base = a spanning tree
\{1, 2, 4, 6\} is a base
A circuit = a cycle.
\{1, 2, 3\} is a circuit
\{1, 2, 3, 5\} is not a circuit
because \{1, 2, 3\} ⊈ \{1, 2, 3, 5\} is dependent

\{1, 2, 3\} is not a circuit
\{1, 2, 3\} ⊈ \{1, 2, 3\} is dependent

\{1, 4, 6, 7, 2\} is a circuit

Graphic matroid.

\begin{figure}
\centering
\begin{tikzpicture}
\draw (0,0) -- (1,2) -- (2,1) -- (3,0) -- (0,0);
\end{tikzpicture}
\end{figure}

Rank function of a matroid
Given a matroid \( M = (E, \mathcal{I}) \),
the rank function \( r_M : 2^E \to \mathbb{Z}_{\geq 0} \)
is a function such that
\[ r_M(X) = \max \left\{ |Y| : Y \subseteq X, Y \in \mathcal{I} \right\}. \]
\( \forall X \subseteq E \).
即 \( r_M(X) \) 是集的 \( X \) 最大独立子集的大小。

\( 2^E = \{ X \subseteq E \} \)
\( 2^{\{1,2,3\}} = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \} \)
uniform matroid.

\[ \mathcal{I} = \{ X \subseteq E : |X| \leq k^2 \} \]

\[ \forall X \subseteq E : \gamma_m(X) = \min \{ |X|, k^2 \} \]

graphic matroid over \( G = (V, E) \).

\[ \gamma_m(X) = n - \# \text{ connected components of } (V, X) \]

partition matroid

\( E \) is partitioned into \( E_1, E_2, \ldots, E_m \)

\( k_1, k_2, \ldots, k_m \) are in \( \mathbb{Z}_{\geq 0} \)

\[ \mathcal{I} = \left\{ X \subseteq E : |X \cap E_t| \leq k_t \right\} \]

\[ \forall t = 1, 2, \ldots, m \]

\[ \gamma_m(X) = \sum_{t=1}^{m} \min \{ |X \cap E_t|, k_t \} \]

\( X = \{ 1, 2, 4, 5, 10 \} \)

\( (V, X) \) has \( 3 \) connected components

\[ \gamma_m(X) = 7 - 3 = 4. \]
start from $Y = \emptyset$
while $\exists e \in X \setminus Y$, s.t $Y \cup \{e\} \in \mathcal{I}$,
$Y \leftarrow Y \cup \{e\}$.
obs: initially $Y$ has $n$ connected components
each time we add an edge to $Y$, 
#CC is decreased by 1
eventually, $Y$ has the same CC as $X$

Linear matroid: $E = \{v_1, v_2, \ldots, v_n\}$
$v_1, v_2, \ldots, v_n \in \mathbb{R}^d$
$X \in \mathcal{I}$ iff $X$ is linearly independent

$\begin{align*}
\text{rank}(X) &= \text{dimension of span}(X) \\
&= \text{rank (Matrix for } X) \\
\end{align*}$

\[ \text{rank}(X) = r_m(x) \]

1. $r_m$ is monotone.
   $\forall X \subseteq Y \subseteq E$
   $r_m(X) \leq r_m(Y)$

2. $\forall X \subseteq Y \subseteq E$
   $r_m(Y) \leq r_m(x) + |Y \setminus X|$.

3. $r_m$ is sub-modular
\[ f: 2^E \rightarrow \mathbb{R} \text{ is said to be submodular, if and only if} \]

\[ \forall A, B \subseteq E, \quad f(A \cup B) + f(A \cap B) \leq f(A) + f(B). \]

equivalent definition:

\[ f: 2^E \rightarrow \mathbb{R} \text{ is said to be submodular, if and only if} \]

\[ \forall A \subseteq E, \quad i, j \in E \setminus A \]

we have

\[ f(A \cup \{i,j\}) - f(A \cup \{i\}) \leq f(A \cup \{i,j\}) - f(A) \]

\[ X: \text{subset of items I own.} \]

\[ f(X): \text{how happy I am.} \]

\[ f(\{\text{car, }$1000\}) - f(\{\text{car}\}) \]

\[ \geq f(\{\text{car, house, }$1000\}) - f(\{\text{car, house}\}) \]

"diminishing returns", game theory

machine learning
\[ f(X \cup \{i, j\}) + f(X) \leq f(X \cup \{i\}) + f(X \cup \{j\}) \leq f(X \cup \{i\}) + f(X \cup \{j\}) \]

Diagram:
- Set A
- Set B
- Intersection A \cap B
- Union A \cup B
- Set \{i, j\}
- Union X \cup \{i, j\}