b-matching:
Given: bipartite graph
\[ G = (U \cup V, E) \]
integer vector
\[ b \in \mathbb{Z}_{\geq 0} \]
Goal: find a set \( M \) of edges such that every vertex \( v \in U \cup V \) is incident to at most \( b_v \) edges in \( M \).
Maximize \( |M| \)

variant 0  \( M \) can be a multi-set
variant 1  \( M \) must be a set

1. Reduction to \( s-t \) maximum flow problem

Diagram:
- Graph \( G = (U \cup V, E) \)
- Integer vector \( b \in \mathbb{Z}_{\geq 0} \)
- Maximize \( |M| \)
- Reduction to \( s-t \) maximum flow problem
Assume $\sum_{v \in L} bv = \sum_{v \in R} bv$.

A perfect b-matching is a (multi-)set $M$ of edges, where every vertex $v \in L \cup R$ is incident to exactly $b_v$ edges.

Suppose $G$ does not have a perfect b-matching, what structure can we find in $G$?

Variant 1: multiset variant.

$X \in N(x)$

Lemma: If $G$ does not have a perfect b-matching, then there exists a $X \subseteq L$, s.t.

$\sum_{u \in N(x)} b_u < \sum_{v \in X} bv$.
suppose there is no perfect $b$-matching
then size of maximum $b$-matching

$$\leq \sum_{u \in L} bu$$

$\Rightarrow$ value of maximum flow

$$is \leq \sum_{u \in L} bu$$

$\Rightarrow$ $\exists$ cut of value $\leq \sum_{u \in L} bu$

There are no edges between

$SNL$ and $RNT$

the value of the cut $(S, T)$

$$= \sum_{v \in LT} bv + \sum_{v \in VESAR} bv$$

$$< \sum_{v \in LT} bv$$

$\Leftrightarrow$ $\sum_{V \in VESAR} bv \leq \sum_{V \in VELAS} bv$

$\Rightarrow$ variant (2): 

$$\begin{array}{c}
2 \\
2
\end{array}$$